The spectral reconstruction problem for thermal photon and dilepton rates

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XVIth Quark Confinement and the Hadron Spectrum

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partly based on: [2403.11647]

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The inverse problem challenge - a mismatch in information

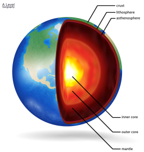
At this conference many talks addressing different QCD physics, but facing a similar core issue: An inverse problem.

In fact, inverse problems are common across science, in general:

- A mismatch between available and desired information
- Need of a robust map / transformation from one into the other
- Inverse problem: This transformation is (numerically) ill-posed or ill-conditioned

Examples outside and inside of particle physics:

available information	desired information
seismic waves	geological tomogram
electromagnetic waves	medical images
Euclidean matrix elements	time-like observables



Electromagnetic source imaging: Backus–Gilbert resolution spread function-constrained and functional MRI-guided spatial filtering

Xiaohong Wan,[1] Atsushi Sekiguchi,[2,3] Satoru Yokoyama,[3,4] Jorge Riera,[3,5] and Ryuta Kawashima[3,5]

Particle physics observables - inverse problems and lattice data

The following can be extracted from a **Euclidean expectation value** by solving an **inverse problem** at $T = 0$:

- $N \to N'$ scattering amplitude at any $s = E_{\mathrm{cm}}^2$ ($\pi\pi \to \pi\pi$, $N\pi \to N\pi\pi$,...)
- $N + j \to N'$ transitions at any s ($K \to \pi\pi$, $D \to \pi\pi, K\overline{K}$, $\gamma \to \pi\pi$,...)
- Non-local matrix elements (R-ratio, hadronic tensor, $D\overline{D}$ mixing)

  ⇝ see e.g. [Liu '94], [Hansen, Meyer, Robaina '17], [Bulava, Hansen '19], [ETMC '23]
- Inclusive processes (V_{us}, deep inelastic scattering, heavy semileptonic decays)

  ⇝ see e.g. [QCDSF '17], [Gambino et al '20], [Alexandrou et al '24]
- Parton distribution functions (PDFs, GPDs, TMDs)

  ⇝ see e.g. [Ji '13], [Ji et al '23], [HadStruc '24]

At finite temperature solving an inverse problem of this type gives access to transport coefficients, production rates and thermal broadening:

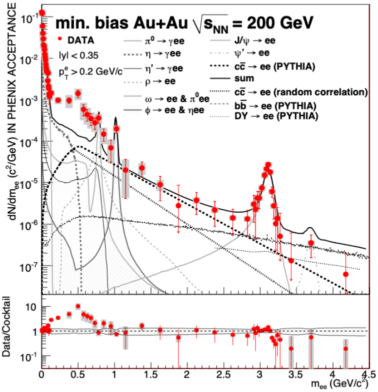
- Bulk and shear viscosities ⇝ see e.g. [Meyer '07], [Altenkort, AF et al '23]
- Electrical conductivity and heavy quark diffusion

  ⇝ see e.g. [Kaczmarek, Shu '22], [Brandt, AF et al '16]
- **Dilepton and photon production rates** ⇝ see e.g. [Cè et al '20], [Ali, AF et al '24]

Dilepton and photon production rates - identifying the problem

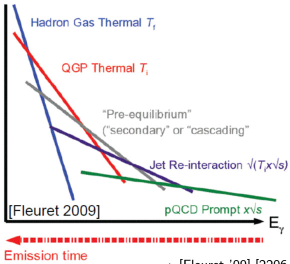
Probes of the quark gluon plasma

- Dileptons and photons are produced at all stages of Heavy-Ion Collisions.
- Weak coupling to QGP constituents
  $\rightarrow$ they decouple after production
  $\rightarrow$ probes of the full thermal medium evolution



[PHENIX '10]

Sketch of different photon sources



⇝ [Fleuret '09],[2206.1467]

Thermal QCD goals

- thermal dileptons: understand contribution for $m_{ee} \sim 0.5\,\mathrm{GeV}$
- thermal photons: understand dominant contribution for $p_T \in [1:2]\,\mathrm{GeV}$

Dilepton and photon production rates - identifying the problem II

The photon emissivity is related to thermal vector-vector current spectral functions

$$\rho^{\mu\nu}(\mathcal{K}) = \int d^4x e^{i\mathcal{K}\cdot x} \frac{1}{Z} \sum_n e^{-E_n/T} \langle n | [j^\mu(x), j^\nu(0)] | n \rangle$$

- Rate of dilepton production per unit volume plasma: ⇝ [McLerran, Toimela '85]

$$d\Gamma_{\ell^+\ell^-}(\mathcal{K}) = \alpha^2 \frac{d^4\mathcal{K}}{6\pi^3\mathcal{K}^2} \frac{-\rho^\mu{}_\mu(\mathcal{K})}{e^{\beta\mathcal{K}^0} - 1} \qquad (\mathcal{K}^2 \equiv \omega^2 - k^2)$$

- Rate of photon production per unit volume plasma:

$$d\Gamma_\gamma(\boldsymbol{k}) = \alpha \frac{d^3k}{4\pi^2 k} \frac{-\rho^\mu{}_\mu(k, \boldsymbol{k})}{e^{\beta k} - 1}.$$

- Electrical conductivity of the quark gluon plasma: ⇝ see e.g. [1104.3708]

$$\sigma_{el} = e^2 \sum_{f=1}^{N_f} Q_f^2 \lim_{k\to 0^+} \frac{\rho^i{}_i(k,0)}{k}$$

⇝ from Meyer, Lattice@CERN '24

Dilepton and photon production rates - identifying the problem III

We are interested in three properties of the vector spectral function:

- conductivity: $\sigma_{el} \sim \rho^{i}{}_{i}(0,0)$ note: $\rho^{0}{}_{0}(0,0) = const. =: \chi_q$
- dilepton rate: $d\Gamma_{\ell^+\ell^-}(\mathcal{K}) \sim -\rho^{\mu}{}_{\mu}(\mathcal{K}) \rightsquigarrow \rho^{\mu}{}_{\mu}(\omega, \boldsymbol{k}=0)$
- photon rate: $d\Gamma_{\gamma}(\boldsymbol{k}) \sim -\rho^{\mu}{}_{\mu}(k, \vec{k}) \rightsquigarrow \rho^{\mu}{}_{\mu}(\omega = k, \boldsymbol{k} \neq 0)$

The most interesting are in the low-energy, non-perturbative regime of QCD.

$\rightsquigarrow$ Where can we trust perturbative calculations?

$\rightsquigarrow$ Access through lattice QCD?

On the lattice and at $T > 0$:

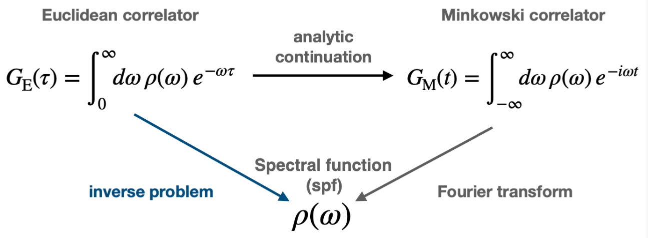
- We work in the imaginary-time path-integral representation of QFT (Matsubara formalism).
- Only Imaginary-time vector correlators are accessible ($j^{\mu} = \sum_f Q_f \bar{\psi}_f \gamma^{\mu} \psi_f$):

$$G^{\mu\nu}(x_0, \boldsymbol{k}) = \int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \langle j^{\mu}(x) j^{\nu}(0) \rangle_T = \int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \operatorname{Tr}\left\{ \frac{e^{-\beta H}}{Z(\beta)} j^{\mu}(x) j^{\nu}(0) \right\},$$

- Their spectral representation is: $\rightsquigarrow$ inverse problem(!)

$$G^{\mu\nu}(x_0, \boldsymbol{k}) = \int_0^{\infty} \frac{d\omega}{2\pi} \underbrace{\rho^{\mu\nu}(\omega, \boldsymbol{k})} \frac{\cosh[\omega(\beta/2 - x_0)]}{\sinh(\beta\omega/2)} \quad , \ \beta = 1/T.$$

Inverse problem vs. analytic continuation - a net of connections



From the Euclidean correlator calculated in lattice QCD we want to extract the spf
$\Rightarrow$ Problem can be seen as a **simultaneous Wick rotation and Fourier transform**.

Also, formally:

$$\rho(\omega) = \mathcal{L}^{-1}\{G_E(\tau)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\omega\tau} G_E(\tau) d\tau \quad \text{and} \quad \rho(\omega) = \frac{1}{\pi}\text{Im}\left(G_M(-\omega)\right)$$

,

$\Rightarrow$ Problem is related to having only real data where complex information is required.

The role of the kernel - different physics, different problems

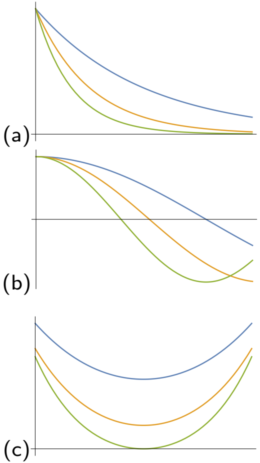
The specific problem sets the kernel entering the inverse problem

$$G_E(\tau) = \int_0^\infty d\omega\, \rho(\omega)\, \kappa(\omega, \tau)$$

(a) Zero-temperature quantities: $\kappa(\omega, \tau) = e^{-\omega\tau}$
$\rightsquigarrow$ Need to perform the inverse Laplace transform.

(b) qPDFs: $\kappa(\nu, x) = \cos(\nu x)\, \Theta(1 - x)$
$\rightsquigarrow$ Need to perform Fourier transform.

(c) Nonzero-temperature: $\kappa(\omega, \tau) = \frac{\cosh(\omega(\beta/2 - \tau))}{\sinh(\omega\beta/2)}$



Focusing on the $T > 0$ case and target spectral information at low energies:

- Laplace transform in limit $\lim_{T\to 0} \kappa(\omega, \tau) = e^{-\omega\tau}$
- Often $T = 1/N_\tau$ short in lattice calculations
  $\rightsquigarrow$ could indicate a benefit of anisotropic calculations
- low-ω contributions suppressed at short τ
- low-ω contributions compete with kernel T-effects at $\tau = N_\tau/2$

Nature of the inverse problem - setting expectations

Jacques Hadamard established three conditions for a well-posed problem

1. Existence
2. Uniqueness
3. Stability (solutions behavior changes continuously with the initial conditions)

The problems we consider fail in the sense of 3 and are thus **ill-posed**.

This is a problem due to **discrete sampling plus finite precision** and a method-independent statement.

Side remarks:

- Our problem can be shown to be **ill-conditioned** in principle [Cuniberti et al '01].
- A solution could be constructed from a finite number of Laguerre polynomials.
- But: It is not clear how many points at what precision are required.
- Qualitatively it was shown the precision must be drastically improved [Meyer '11]
- With often exponential signal-to-noise ratios this is difficult to achieve.

Nature of the inverse problem - illustrative example

- One approach to extracting $\rho(\omega)$ is to fit it to an Ansatz.
- Example: Motivated photon production rate Ansatz with 3 parameters a, b, ω_0, and minimized:

$$\chi_{fit} = \left[G(\tau, p) - \int d\omega \; \rho_{fit}\left(\omega, a, b, \omega_0\right) \frac{\cosh(\omega(\beta/2-\tau))}{\sinh(\omega\beta/2)} \right]$$

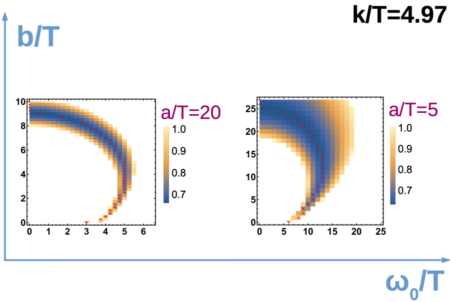


Figure from [1710.07050]

- no clear global minimum in χ^2-landscape
- each point on the map represents one "acceptable" pair of parameters $\rightarrow$ one spf solution each that describes the Euclidean data.
- often features are washed out after combining solutions into result.
- brute force accuracy increase to arrive at a more constrained result?

Nature of the inverse problem - general approach

All methods to perform a spectral reconstruction can be understood as a master function

$$\mathcal{F}[\mathbf{G}, \mathbf{C}_G] = \left(\boldsymbol{\rho}, \mathbf{C}_\rho\right)$$

where

- $\mathbf{G} \quad = \quad$ discrete samples of $G(\tau)$
- $\mathbf{C}_G \quad = \quad$ covariance of $\mathbf{G}$
- $\boldsymbol{\rho} \quad = \quad$ discrete estimator of $\rho(\omega)$
- $\mathbf{C}_\rho \quad = \quad$ covariance of $\boldsymbol{\rho}$

$\rightarrow$ We want to understand the properties and limitations of the master function $\mathcal{F}$

$\rightarrow$ Crucial to be data focused: The best $\mathcal{F}$ will depend in detail on G, number of slices, properties of C, etc.

General difficulties

- For $\boldsymbol{\rho}_i = \rho(\omega_i)$

$$\left|\mathcal{F}[\mathbf{G} + \delta\mathbf{G}, \mathbf{C}_G + \delta\mathbf{C}_G] - \mathcal{F}[\mathbf{G}, \mathbf{C}_G]\right| \qquad \text{and thus} \qquad \left|\mathbf{C}_\rho\right|$$

  explode.
- For cases where $|\mathbf{C}_\rho|$ is under control, relation between $\rho(\omega) \Leftrightarrow \boldsymbol{\rho}$ may be obscured.

Approaching the inverse problem - three basic strategies

Strategy I: Accept the premise and focus on information in the data

Challenges

- Optimal bases for spare modeling / correlation structures for Gaussian processes?
- Constraints (positivity, sum rules)?

Methods - (non)linear

- Sparse modeling
- Neural Networks
- Gaussian processes

Strategy II: Accept the premise and try to supply as much extra information as possible

Challenges

- Effectively encode more specific information in priors/ Ansätze?
- Control over prior bias and systematics?

Methods - (non)linear

- χ^2-fits
- Maximum entropy methods
- Stochastic inference / optimization

Strategy III: Reject the premise and focus on smeared spf's or Euclidean quantities

Challenges

- Identify the physics observables that ...
  - can benefit from smeared spf input.
  - simplify / avoid inverse problems altogether.

Methods - linear

- Backus-Gilbert / Hansen-Lupo-Tantalo methods
- Gaussian processes

Each method has its own problem-dependent pros and cons.
Unlikely there is a single best solution.

Results

Roadmap:

1. **dilepton rates / electrical conductivity:** *brief overview of community status*
2. **photon rates:** *new research shown [2403.11647]*

Our lattice setup (HotQCD ensembles)

- $n_f = 0$ quenched QCD, $a^{-1} = 9.4, 11.3, 14.1\,\text{GeV}$, Wilson-Clover
- $n_f = 2 + 1$ full QCD, $a^{-1} = 7.04\,\text{GeV}$, Wilson-Clover on HISQ sea.
- $m_\pi = 320\,\text{MeV}$, $m_s = 5 \cdot m_\ell$

Lattice data uncertainties and systematics

- **Statistical uncertainty:** $\delta G(\tau)$ limits resolution
- **T-extent:** Inverse becomes solvable when $\delta G(\tau) = 0$ and $N_\tau = \infty$
- **Finite volume:** Spectral support is limited to lattice momenta
- **Quark masses:** Spectral properties can be very different based on quark mass, need to be in relevant regime
- **Discretization effects:** Cut-off region in spf can be too close to separate the scale (bottomonium)

These effects can be and are being enhanced or distorted when performing the inverse.

Ideally the data should be:

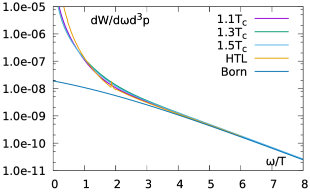
- extrapolated to the continuum ✓
- extrapolated to infinite volume
- at the correct quark masses ✓
- have very large time extents (✓)
- have very small statistical errors (✓)

Dilepton rate and electrical conductivity

- Dilepton rates available in quenched and full QCD from multiple groups (last update 2019).
- Methods used:
  - χ^2-fits with additional constraints
  - MEM
  - BG method with Tikhonov regulator
- Fits: Transport (+ BW) + Asymptotic
  $\Rightarrow$ Data well described by Ansatz
  $\Rightarrow$ Difficult to distinguish Transport and BW
- MEM prior: Positivity + default model

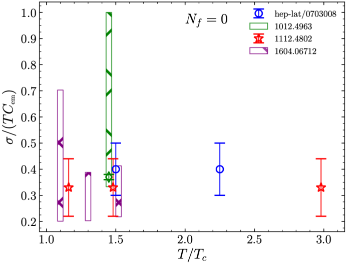
Consistent picture emerging.

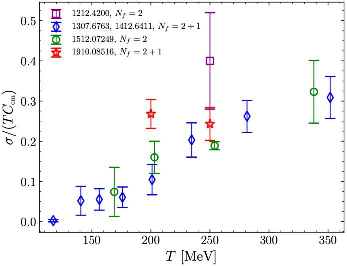
Dilepton rate in quenched QCD



Electrical conductivity

[2008.12326]





Photon production rate from the lattice - a better estimator

⇝ see [Cè et al '20], [Ali, AF et al '24], [Meyer, Lattice@CERN '24]

Back to:

$$\rho^{\mu\nu}(\mathcal{K}) = \int d^4x e^{i\mathcal{K}\cdot x} \frac{1}{Z} \sum_n e^{-E_n/T} \langle n | [j^\mu(x), j^\nu(0)] | n \rangle$$

at $T > 0$, there are two independent components (longitudinal and transverse):

$$\rho_L(\omega, k) \equiv \left(\hat{k}^i \hat{k}^j \rho^{ij} - \rho^{00} \right), \quad \text{and} \quad \rho_T(\omega, k) \equiv \frac{1}{2} \left(\delta^{ij} - \hat{k}^i \hat{k}^j \right) \rho^{ij} ,$$

where $\left(k \equiv |\boldsymbol{k}|, \quad \hat{k}^i = k^i / k \right)$

- Due to current conservation: $\omega^2 \rho^{00}(\omega, k) = k^i k^j \rho^{ij}(\omega, k)$
- ρ_L vanishes at light like kinematics, $\mathcal{K}^2 = 0$

→ We can rewrite the photon rate when introducing the estimator:

$$\rho(\omega, k, \lambda) = 2\rho_T + \lambda \rho_L \quad \overset{\lambda=1}{=} -\rho^\mu{}_\mu$$

→ For any value of λ we have

$$\boxed{d\Gamma_\gamma(\boldsymbol{k}) = \alpha \frac{d^3k}{4\pi^2 k} \frac{\rho(k, k, \lambda)}{e^{\beta k} - 1}}$$

→ Choose λ such that the reconstruction becomes particularly easy.

Lattice photon production rate - estimator and perturbative result

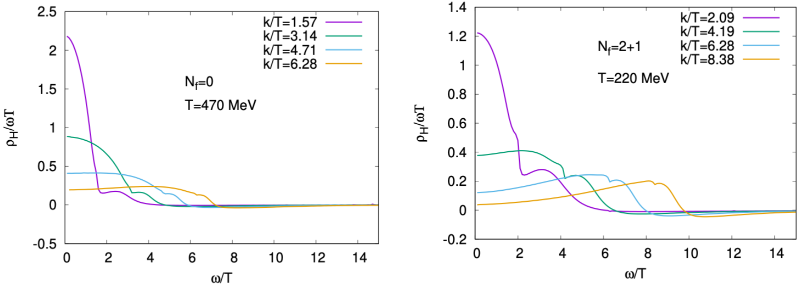
⇝ [Ali, AF et al '24]

We choose $\lambda = 2$ such that the spf we want to reconstruct becomes:

$$\rho_H(\omega, \vec{k}) = 2\left\{\rho_T(\omega, \vec{k}) - \rho_L(\omega, \vec{k})\right\}$$

- → At $T = 0$ this estimator is $\rho_H(\omega, \vec{k}) = 0$. (due to restoration of Lorentz symmetry)
- → Purely thermal effects contribute.
- → Aysmptotically $\rho_H(\omega, \vec{k}) \sim 1/\omega^4$ for $\omega \gg k$ (πT) .
- → Sum rule $\int_0^\infty \mathrm{d}\omega\omega\rho_H(\omega) = 0$ is a possible extra constraint.

Can be worked out in perturbation theory at NLO + $\mathrm{LPM}^{\mathrm{LO}}$. ⇝ [1910.09567]

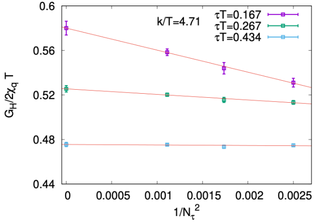


$\mathrm{LPM}^{\mathrm{LO}}$ means that near the light cone LPM resummation is performed at leading order.

Euclidean correlators - NLO+LPM vs. lattice data

Quenched results

- Continuum limit with $N_\tau = 20, 24, 30$
- non-pert. tuned WCF



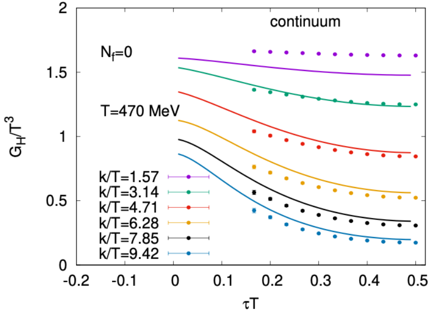
- General behavior reproduced, apart from lowest k/T

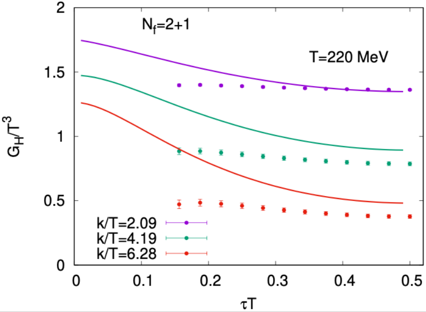
Full QCD results

→ Single lattice spacing

→ Differences more visible

→ Improvement after continuum limit?





Spectral reconstruction - a 3-pronged approach

Strategy

We employ all three basic strategies:

- χ^2-fits (strategy II)
- Backus-Gilbert method (strategy III)
- Gaussian processes (strategy I)

This makes visible and enables the study of:

- ⇒ Different systematics in all approaches.
- ⇒ Maximal view of possible outcomes.
- ⇒ Aim for robust, conservative final result.

Gaussian process regression

- Gaussian kernel (related to NN)
- Simultaneous reconstruction in $(\omega,\ k)$
- Continuity only constraint
- → Use mildest possible constraints

χ^2-fits

Two sets of Ansätze

(a) Polynomial fit to reproduce IR and UV results (OPE)

(b) Padé fit with sum rule incorporated (OPE and AdS/CFT)

→ Include as much extra info as possible

BGM

We stabilize/improve the reconstruction by rescaling the spf by asymptotic behavior

$$\frac{\rho_H^{\mathrm{BG}}(\omega,\vec{k})}{f(\omega,\vec{k})} = \sum_i q_i(\omega,\vec{k}) G_H\left(\tau_i,\vec{k}\right)$$

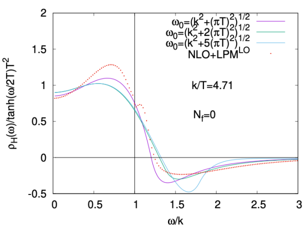
Rescaling function:

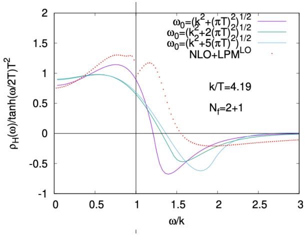
$$f(\omega,\vec{k}) = \left(\frac{\omega_0}{\omega}\right)^4 \tanh\left(\frac{\omega}{\omega_0}\right)^5$$

→ Work with a smeared spf

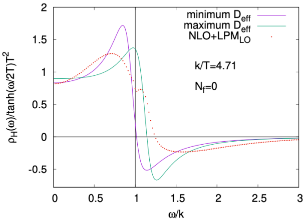
Spectral reconstruction - χ^2-fits

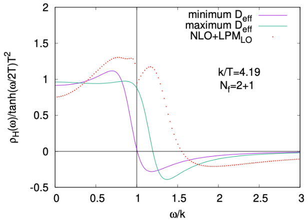
(a) Polynomial fits





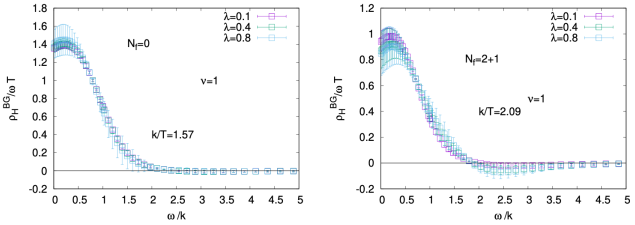
(b) Padé fits



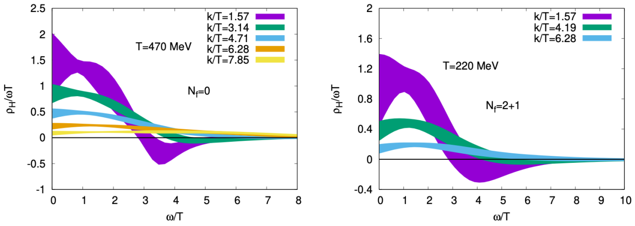


Spectral reconstruction - BGM and GPR

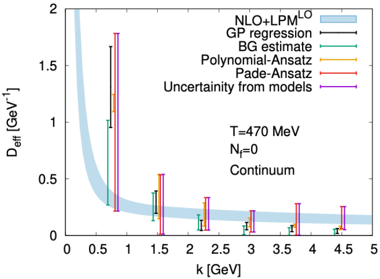
Backus-Gilbert method

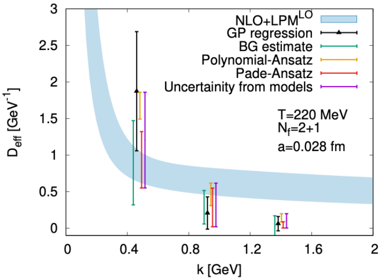


Gaussian processes



Photon rate from the lattice - final results





Cè et al [2205.02821]

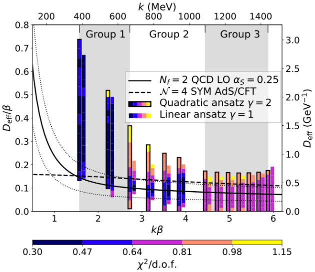
Plotted:

$$D_{\text{eff}}(k) \equiv \frac{\rho_H(\omega = k, k)}{2\chi_q k}$$

Connection to full photon rate:

$$\frac{d\Gamma_\gamma(\boldsymbol{k})}{d^3k} = \frac{\alpha}{4\pi^2 k} \frac{\rho(k, k, \lambda = 2)}{e^{\beta k} - 1}$$

$$= \frac{\alpha n_b(k)\chi_q}{\pi^2} \left(\sum_{i=1}^{N_f} Q_i^2\right) D_{\text{eff}}(k)$$



compatible results, see further material

Summary - inverse problem for thermal spfs from the lattice

I. Inverse problem for thermal spectral functions

- Presented in general terms the inverse problem to obtain (thermal) spectral functions from lattice correlators
- Broadly highlighted applications and connection to dilepton / photon rates
- Gave general approach strategies and their challenges

II. Dilepton rates and electrical conductivity

- Collected a brief overview (last update 2019). Consistent picture emerging.

III. Photon rate

- Showed the improved estimator for photon rate
- Applied all basic strategies to inverse problem to arrive at new result

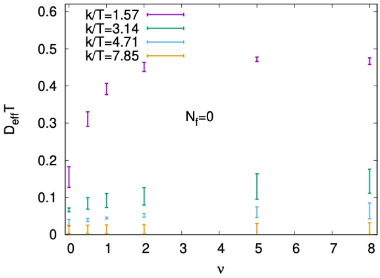
IV. Future questions

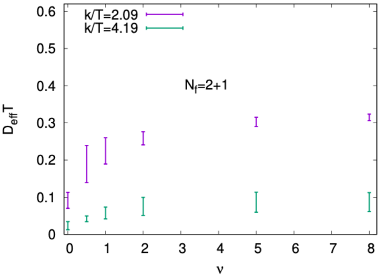
- Finding improved estimator has made the study of the photon rate (and heavy quark diffusion) tractable. *Are there more observables like this?*
- In the $T = 0$ community there is a lot of work on using smeared spf's and clarifying the role of the finite volume. *Possibilities also at* $T > 0$*?*
- Gaussian processes related to NN. Connections between BG-type methods worked out. *New methods that combine the best of all strategies possible?*

Thank you for your attention.

Further material

Photon rate from the lattice - $D_{eff}\,T$ comparison





Cè et al [2205.02821]