pQCD meets phase quenching: the pressure of cold quark matter

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In collaboration with R. Paatelainen and P. Navarrete

[2403.02180]

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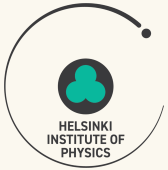
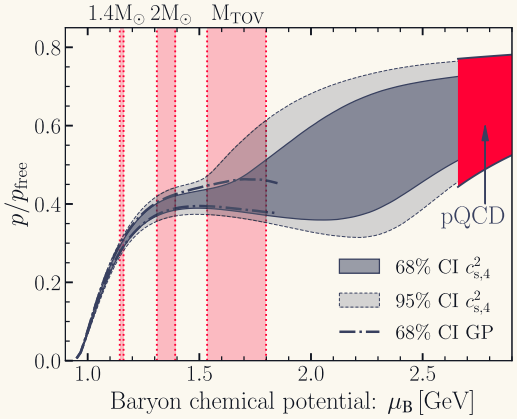


Cold and dense pQCD and phase quenching

QCD thermodynamics at high density

- Thermodynamics of cold quark matter ($T = 0$, $\mu_B \neq 0$) is largely unknown due to the Sign Problem of lattice field theory
- Perturbative QCD viable at high baryon chemical potential μ_B and zero temperature T as QCD coupling $\alpha_s \ll 1$
- **High-order corrections to pQCD pressure constrain the neutron-star equation of state** Komoltsev & Kurkela PRL '22



(adapted from Annala et al. Nat. Com. '23)

Framework for calculating cold and dense pQCD pressure

1. Generate Feynman diagrams from partition function:

$$p(\mu) \sim \ln Z = \ln \int \mathcal{D}\overline{\psi}\psi A e^{-S_{\mathrm{QCD}}}$$

$$\overset{\text{pQCD}}{=}$$  $$+ \ldots$$

- Fermionic 4-momenta at finite density: $P = (p_0 + i\mu, \vec{p})$

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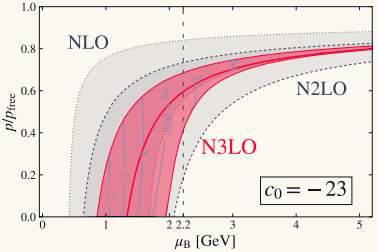
2. Calculate multi-loop integrals in dimensional regularization

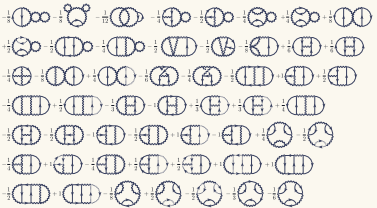
Resummations required due to IR divergences associated with in-medium screening $\Rightarrow \ln \alpha_s$ in perturbative expansion (Saga's talk)

Partial N^{3}LO pressure of cold and dense QCD

$$\frac{p}{p_{\text{free}}} = 1 + c_1\alpha_s + (c_{2,1}\ln\alpha_s + c_{2,0})\alpha_s^2 + (c_{3,2}\ln^2\alpha_s + c_{3,1}\ln\alpha_s + c_{3,0})\alpha_s^3 + O(\alpha_s^4)$$

- c_1, $c_{2,1}$, $c_{2,0}$: B. Freedman & L. McLerran PRD '77
- $c_{3,2}$: S. Säppi et al. PRL '18, S. Säppi et al. PRL '21
- $c_{3,1}$: T. Gorda, R. Paatelainen, S. Säppi, K.S., PRL '23
- $c_{3,0}$: 52 4-loops, work in progress...





Phase quenching

- Provides an alternative nonperturbative way of determining the cold and dense pQCD pressure Moore & Gorda JHEP '23, Fujimoto & Reddy PRD '24
- QCD partition function:

$$Z_{\mathrm{QCD}}(\vec{\mu}) = \int \mathcal{D}A \mathrm{e}^{-S[A]} \prod_{j=1}^{N_f} \det \underbrace{\mathcal{D}(\mu_j)}_{\not{D}+m+\mu_j\gamma^0}$$

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- Phase quenching:

$$\underbrace{\det \mathcal{D}(\mu_j)}_{\in\mathbb{C}} \longrightarrow \underbrace{|\det \mathcal{D}(\mu_j)|}_{\geq 0}$$

Sign Problem in lattice QCD ✗

Lattice QCD ok ✓

Upper bound for full QCD

- Upper bound for Z_{QCD} and pressure $p_{\mathrm{QCD}} \sim \ln Z_{\mathrm{QCD}}$

$$Z_{\mathrm{QCD}}(\vec{\mu}) \leq Z_{\mathrm{PQ}}(\vec{\mu}) \quad \Rightarrow \quad p_{\mathrm{QCD}}(\vec{\mu}) \leq p_{\mathrm{PQ}}(\vec{\mu})$$

- Turning the inequality into an equality

$$\underbrace{p_{\mathrm{PQ}}(\vec{\mu})}_{\text{lattice QCD}} - \underbrace{p_{\mathrm{QCD}}(\vec{\mu})}_{?} = \underbrace{\text{positive number}}_{\substack{\text{easier to compute} \\ \text{than } p_{\mathrm{QCD}}}}$$

- The difference between full QCD and phase-quenched QCD can be computed perturbatively at high densities where $\alpha_s \ll 1$

  $\Rightarrow$ Solve for p_{QCD}!

Phase-quenched Feynman rules

- γ^5-hermiticity of the Dirac operator:

$$\gamma^5 \mathcal{D}(\mu_j)\gamma^5 = \mathcal{D}^\dagger(-\mu_j) \quad \Rightarrow \quad [\det \mathcal{D}(\mu_j)]^* = \det \mathcal{D}(-\mu_j)$$

- Phase-quenched determinant:

$$|\det \mathcal{D}(\mu_j)| = \sqrt{\det \mathcal{D}(-\mu_j) \det \mathcal{D}(\mu_j)} = \exp\left\{\frac{1}{2}\Big[\mathrm{tr}\, \ln \mathcal{D}(-\mu_j) + \mathrm{tr}\, \ln \mathcal{D}(\mu_j)\Big]\right\}$$

$\Rightarrow$ average over $\pm\mu_j$ in quark loops

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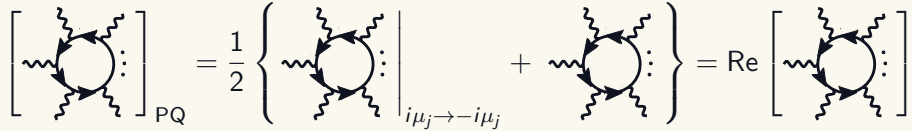
  $\Rightarrow$ average over $\pm\mu_j$ in quark loops

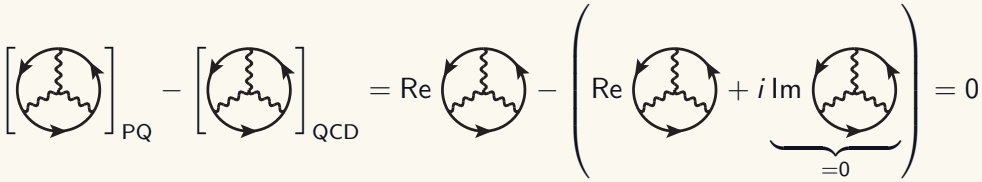
- PQ Feynman rule (recall $P = (p^0 + i\mu_j, \vec{p})$ for fermions):



$p_{\mathrm{PQ}} - p_{\mathrm{QCD}}$ perturbatively

- Vacuum-type diagrams are real due to symmetry under $i\vec{\mu} \to -i\vec{\mu}$
- Single-quark-loop diagrams are the same in phase-quenched and full QCD, e.g.

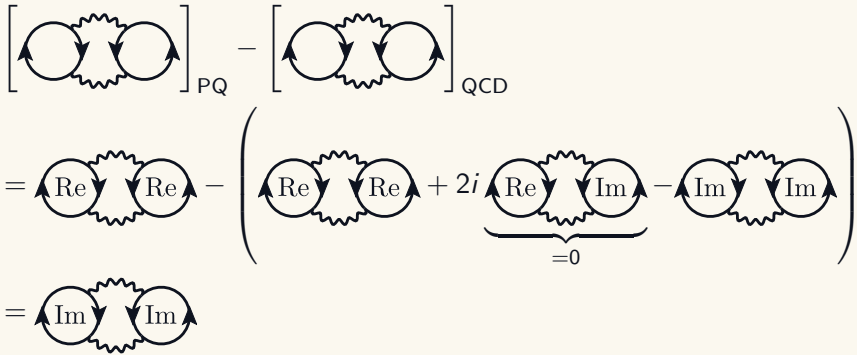
$$\left[\quad\right]_{\mathrm{PQ}} - \left[\quad\right]_{\mathrm{QCD}} = \mathrm{Re}\quad - \left(\mathrm{Re}\quad + i\,\mathrm{Im}\underbrace{\quad}_{=0}\right) = 0$$



$\Rightarrow$ The difference has to contain at least **two quark loops**

$p_{\mathrm{PQ}} - p_{\mathrm{QCD}}$ perturbatively

- First two-quark-loop diagram appears at three loops:



$p_{\mathrm{PQ}} - p_{\mathrm{QCD}}$ perturbatively

- But the one-loop two-gluon function is real ($P^0 = p^0 + i\mu$):

$$\Gamma^{(2)}_{\alpha\beta}(\mu, Q) \equiv \int_{\{P\}} \frac{\mathrm{tr}\,[\not{P}\gamma_\alpha(\not{P} - \not{Q})\gamma_\beta]}{P^2(P-Q)^2} \overset{P\to Q-P}{=} \Gamma^{(2)}_{\alpha\beta}(\mu, Q)^* \quad \Rightarrow$$

$= 0$

$\Rightarrow$

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- No such symmetries for one-loop three-gluon function $\Rightarrow$

$\neq 0$

$\Rightarrow$ The difference has to contain at least **two three-gluon functions**

$p_{\mathrm{PQ}} - p_{\mathrm{QCD}}$ perturbatively at $O(\alpha_s^3)$

- First nonzero contribution to $p_{\mathrm{PQ}} - p_{\mathrm{QCD}}$ at four-loop order, "Bugblatter" Moore & Gorda JHEP '23:

$$\underbrace{p_{\mathrm{PQ}}(\mu_{\mathrm{B}})}_{\text{lattice QCD}} - \underbrace{p_{\mathrm{QCD}}(\mu_{\mathrm{B}})}_{?} = \text{Im} \quad \text{Im} + O(\alpha_s^4)$$

$$\propto \alpha_s^3 d^{abc} d^{abc} N_f^2 \int_{QS} \frac{[\mathrm{Im}\,\Gamma^{(3)}_{\mu\nu\sigma}(\mu_{\mathrm{B}}, Q, S)]^2}{Q^2 S^2 (Q-S)^2} + O(\alpha_s^4)$$

- Beta-equilibrium at high densities $\Rightarrow \mu_j = \mu_{\mathrm{B}}/N_c$ for $j \in \{u, d, s\}$
- UV and IR finite, positive, and gauge-invariant number

$\Rightarrow$ Full $O(\alpha_s^3)$ pressure from single four-loop diagram + PQ lattice result

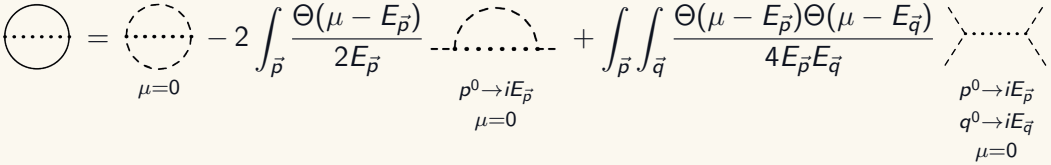
Multiloop integration techniques at finite density

Traditional multiloop techniques at $\mu = 0$

- Significant advancements in multiloop integration techniques for collider physics at $\mu = 0$ over the last decades
- Many traditional state-of-the-art methods rely on Lorentz symmetry:
  - Integration by parts for a reduction to master integrals
  - Differential equations
  - Sector decomposition
  - etc.
- Major obstacle at finite μ: **broken Lorentz symmetry** due to $p^0 \to p^0 + i\mu$ for fermions

Traditional multiloop techniques at $\mu > 0$

- Brute-force calculations used up to three loops
- Main approach: computing master integrals using finite-density cutting rules
- Basic idea: residue theorem **only for fermionic momenta** $P = (p^0 + i\mu, \vec{p})$



- Three-dimensional phase-space integrals over on-shell amplitudes at $\mu = 0$
- Each term can be divergent even if the integral is finite $\Rightarrow$ complicated integrals in dimensional regularization (especially at four loops)

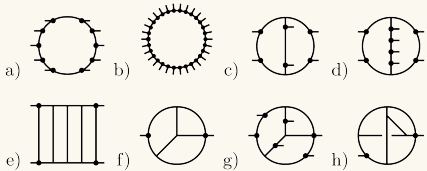
Seems like we are stuck...

Are there any powerful $\mu = 0$ methods that generalize to $\mu > 0$?

Inspiration from Loop-Tree Duality at $\mu = 0$

- Direct numerical integration in momentum space using the Loop-Tree Duality:
  1. **All** energy integrals computed analytically with residue theorem
  2. Remaining spatial momentum integrals computed numerically using Monte Carlo integration

- 0-components singled out $\Rightarrow$ naturally generalizes to finite μ where $p^0 \to p^0 + i\mu$



G	Reference	Numerical LTD	N $[10^6]$	$[\mu s]$
a)*	[33] $i\,4.31638 \cdot 10^{-7}$	$i\,4.31637(19) \cdot 10^{-7}$	110	1.1
b)	[33] $i\,0.358640$	$i\,0.358646(29)$	210	5.9
c)	[7] $1.1339(5) \cdot 10^{-4}$	$1.133719(58) \cdot 10^{-4}$	5500	2.5
c)*	[7] $4.398(1) \cdot 10^{-8}$	$4.39825(17) \cdot 10^{-8}$	5500	2.5
d)*	[7] $2.409(1) \cdot 10^{-8}$	$2.40869(27) \cdot 10^{-8}$	5500	3.5
e)	[34] $-1.433521 \cdot 10^{-6}$	$-1.4338(18) \cdot 10^{-6}$	1500	27.4
f)	[35] $i\,5.26647 \cdot 10^{-6}$	$i\,5.236(38) \cdot 10^{-6}$	7000	3.3
g)*	[7] $i\,1.7790(6) \cdot 10^{-10}$	$i\,1.77648(48) \cdot 10^{-10}$	22000	11
h)	[35] $-8.36515 \cdot 10^{-8}$	$-8.309(31) \cdot 10^{-8}$	7000	15.8

Z. Capatti et al. PRL '19

Representations for the spatial integrands

- 3d integrands obtained after energy integrations can be written using different representations:
  - Loop-Tree Duality (LTD), Cross-Free Family (CFF), Time-Ordered Pert. Theory,...
  - Choice of the representation affects the performance of numerical integration
- We generalize the CFF representation by Z. Capatti PRD '23 to finite μ
  - We call it the **dense Loop-Tree Duality (dLTD)** K.S., R. Paatelainen, P. Navarrete [2403.02180]
  - No spurious singularities unlike in the finite-density cutting rules

dLTD derivation

l-loop vacuum-type integral at finite chemical potential μ with n propagators:

$$I \equiv \int \left[\prod_{i=1}^{l} \frac{\mathrm{d}p_i^0}{2\pi}\right] \frac{\mathcal{N}(\mathbf{Q}^0)}{\prod_{j=1}^{n}\left[(Q_j^0)^2 + E_j^2\right]}$$

- Bold symbols = n-dimensional vectors
- $\mathbf{Q}^0 = \sum_{i=1}^{l} \mathbf{S}_i p_i^0 + i\boldsymbol{\tau}\mu$ = frequencies of the propagators
- $\mathbf{S}_i \in \{\pm 1, 0\}^n$ = vectors that fix the loop momentum basis
- $\boldsymbol{\tau} = \sum_{i=1}^{l} \mathbf{S}_i \tilde{\tau}_i \in \{\pm 1, 0\}^n$ with $\tilde{\tau}_i \in \mathbb{Z}$ = fermion signature vector
- $\mathcal{N}$ = multilinear polynomial

dLTD derivation: Performing the 0-component integrals

Insert the following unity for each propagator j in I:

$$1 = \int \mathrm{d}\tilde{p}_j^0 \delta\left(\tilde{p}_j^0 - \sum_{i=1}^{I} S_{ji} p_i^0\right)$$

and write the delta functions as integrals over α_j (Fourier representation)

$$I = \int \prod_{i=1}^{I} \frac{\mathrm{d}p_i^0}{2\pi} \int \prod_{j=1}^{n} \mathrm{d}\alpha_j \, \mathrm{e}^{-i\alpha_j \sum_{i=1}^{I} S_{ji} p_i^0} \int \left[\prod_{j=1}^{n} \frac{\mathrm{d}\tilde{p}_j^0}{2\pi} \frac{\mathrm{e}^{i\alpha_j \tilde{p}_j^0}}{\left(\tilde{p}_j^0 + i\tau_j\mu\right)^2 + E_j^2}\right] \mathcal{N}\left(\tilde{\mathbf{p}}^0 + i\boldsymbol{\tau}\mu\right)$$

$\Rightarrow$ 0-component integrals over $\tilde{p}_j^0$ can be performed independently for each propagator

dLTD derivation: Final result

Using the residue theorem, we obtain the final result

$$I = \sum_{\boldsymbol{\rho},\boldsymbol{\sigma}\in\{\pm 1\}^n} \mathcal{N}(i\boldsymbol{\sigma}\odot\boldsymbol{\rho}\odot\mathbf{E}) \prod_{j=1}^{n} \frac{\Theta(\rho_j E_j - \sigma_j \tau_j \mu)}{2\rho_j E_j} \underbrace{\int_{\mathbb{R}_+^n} \mathrm{d}\boldsymbol{\alpha}\, \mathrm{e}^{-\boldsymbol{\alpha}\cdot(\boldsymbol{\rho}\odot\mathbf{E})} \prod_{i=1}^{l} \delta\big(\boldsymbol{\alpha}\cdot(\boldsymbol{\sigma}\odot\mathbf{S}_i)\big)}_{=\text{ some rational function of } E_j\text{'s, no } \mu\text{-dependence!}}$$

- Difference to the $\mu = 0$ case:
  - Step functions with a chemical potential μ
  - Additional sum over sign vectors $\boldsymbol{\rho}$
- $\boldsymbol{\alpha}$-integral is a Laplace transform of a nonsimplicial convex cone
  - The same integral already appears in the $\mu = 0$ case
  - Closed-form solution using diagrammatic algorithm by Z. Capatti PRD '23

Example: two-loop sunset

$$\int \left[\prod_{i=1}^{l} \frac{\mathrm{d}p_i^0}{2\pi}\right] \frac{\mathcal{N}(\mathbf{Q}^0)}{\prod_{j=1}^{n}[(Q_j^0)^2 + E_j^2]} = \sum_{\boldsymbol{\rho},\boldsymbol{\sigma}\in\{\pm 1\}^n} \mathcal{N}(i\boldsymbol{\sigma}\odot\boldsymbol{\rho}\odot\mathbf{E}) \prod_{j=1}^{n} \frac{\Theta(\rho_j E_j - \sigma_j\tau_j\mu)}{2\rho_j E_j} \int_{\mathbb{R}_+^n} \mathrm{d}\boldsymbol{\alpha}\, \mathrm{e}^{-\boldsymbol{\alpha}\cdot(\boldsymbol{\rho}\odot\mathbf{E})} \prod_{i=1}^{l} \delta(\boldsymbol{\alpha}\cdot(\boldsymbol{\sigma}\odot\mathbf{S}_i))$$

- $l = 2$, $n = 3$, $\mathbf{S}_1 = (1,0,1)$, $\mathbf{S}_2 = (0,1,-1)$, $\boldsymbol{\tau} = (1,1,0)$, $\mathbf{E} = (|\vec{p}_1|, |\vec{p}_2|, |\vec{p}_1 - \vec{p}_2|)$
- Performing the $\boldsymbol{\alpha}$-integrals for the sunset yields four distinct nonvanishing terms

$$\propto \int \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3} \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3} \left\{ \frac{(E_1E_2 + \vec{p}_1\cdot\vec{p}_2)\Theta(E_1-\mu)}{2E_1 2E_2 2E_3(E_1+E_2+E_3)} - \frac{(-E_1E_2 + \vec{p}_1\cdot\vec{p}_2)\Theta(E_1-\mu)\Theta(-E_2+\mu)}{2E_1 2E_2 2E_3(E_1-E_2+E_3)} + (1\leftrightarrow 2) \right\}$$

- For numerical integration, UV divergences are subtracted with local counterterms using BPHZ R-operation

Numerical integration & benchmarks

- Spatial loop momenta $\vec{p}_i$ parametrized using spherical coordinates
- Numerical integration of spatial momentum integrals using `Vegas`
- Multichannel approach implemented for flattening integrable singularities (see Z. Capatti et al. JHEP '20)

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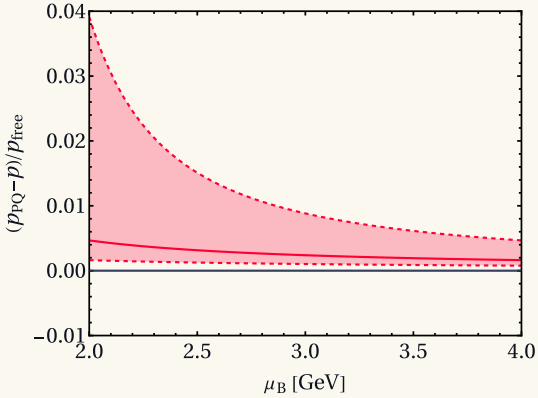
	Diagram	Analytic	dLTD	$N\,[10^6]$	$[\mu s]$
Massive scalars at $\mu = 0$		$2.16928 \cdot 10^{-9}$	$2.16931(4) \cdot 10^{-9}$	3000	5.2
QCD at $\mu > 0$		-0.00128325	$-0.00128338(23)$	150	5.8
		-0.000256547	$-0.00025654(9)$	800	9.6

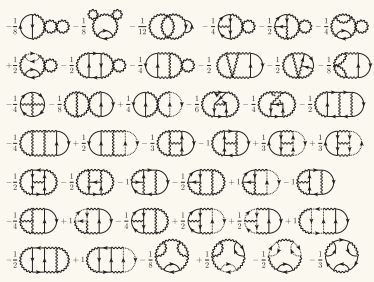
Result for $p_{\mathrm{PQ}} - p_{\mathrm{QCD}}$ using dLTD

$$p_{\mathrm{PQ}}(\mu_{\mathrm{B}}) - p_{\mathrm{QCD}}(\mu_{\mathrm{B}}) = 3.368(7) \cdot \left(\frac{\alpha_s}{\pi}\right)^3 p_{\mathrm{free}} + O(\alpha_s^4)$$

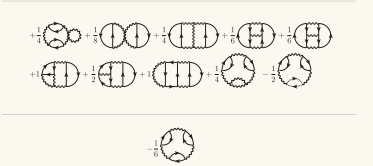


- dLTD representation of the Bugblatter integrated using $6 \cdot 10^{11}$ MC samples
- First ever computation of 2PI four-loop diagram at finite μ_{B}
- Small number compared to the known perturbative coefficients at $O(\alpha_s^3)$
- **Conclusion**: Phase-quenched lattice can provide a complementary method for determining the pressure of cold quark matter at high μ_{B}

Next: Applying dLTD to computing $c_{3,0}$ directly



IR finite
$\Rightarrow$ dLTD



IR div.

- Recall the 4-loop pressure:

$$\frac{p}{p_{\text{free}}} = 1 + c_1\alpha_s + (c_{2,1}\ln\alpha_s + c_{2,0})\alpha_s^2 + (c_{3,2}\ln^2\alpha_s + c_{3,1}\ln\alpha_s + c_{3,0})\alpha_s^3$$

- Plan: apply dLTD to IR finite diagrams contributing to $c_{3,0}$
- Completion of the $O(\alpha_s^3)$ pressure seems feasible for the first time

Summary

- pQCD constrains the neutron-star equation of state
- $O(\alpha_s^3)$ cold and dense QCD pressure can be determined from phase-quenched lattice simulations + our new four-loop result
- Powerful numerical tool from collider physics generalized to finite μ: direct numerical integration of Feynman diagrams in momentum space using the dense Loop-Tree Duality

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- Powerful numerical tool from collider physics generalized to finite μ: direct numerical integration of Feynman diagrams in momentum space using the dense Loop-Tree Duality

Thanks for your attention!

Extra slides

Future directions for dLTD

Finite temperature

- Already benchmarked at 2-loops
- $\Theta \to n_{B/F}$

UV/IR subtractions

- All-orders automatization of UV-subtraction nearly complete
- IR HTL-counterterms at $T = 0$, $\mu > 0$

Performance

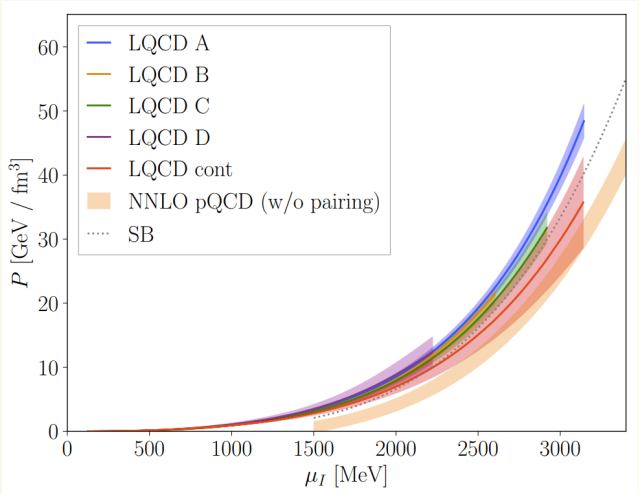
- More efficient sampling methods
- Optimizations

External legs

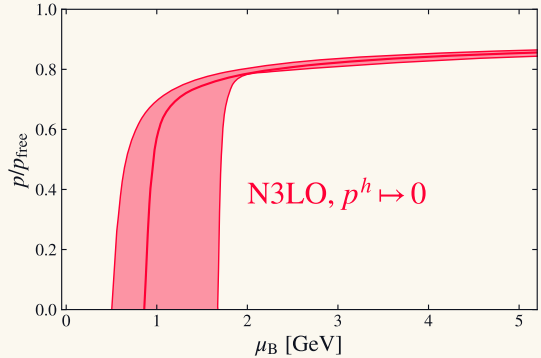
- Real-time n-point correlators
- Production rates, etc.

Lattice results for $2+1$ flavors at finite μ_I

R. Abbott, W. Detmold et al. [2406.09273]:

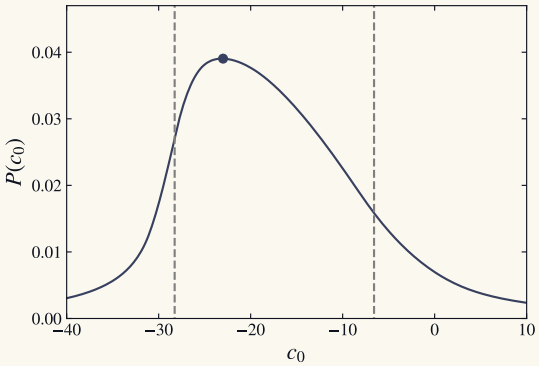


Results for N^{3}LO pressure with only soft and mixed sectors

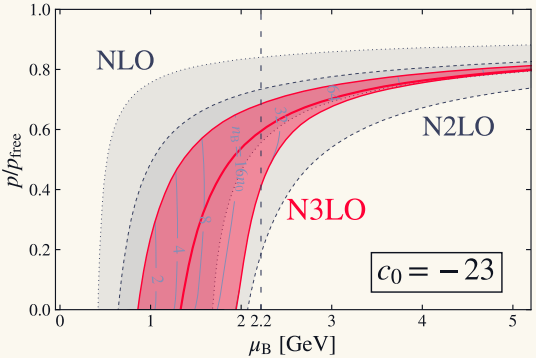


- N^{3}LO result including only screened gluonic sectors (soft+mixed) incredibly well-behaved with nearly vanishing renormalization-scale dependence
- Conclusion: hard sector main source for uncertainty
- Stark contrast to the high-T case

Full N^{3}LO pressure with fixed c_0



- Full N^{3}LO with $c_0 = -23$ most consistent with lower-order results according to Bayesian analysis



- Actual computation of c_0 may lead to a significantly improved EOS usable even at $\mu_{\rm B} = 2.2\text{GeV}$ ($n = 27n_0$), cf. N^{2}LO at $\mu_{\rm B} = 2.7\text{GeV}$ ($n = 40n_0$)