RIKEN iTHEMS 理化学研究所 数理創造プログラム RIKEN Interdisciplinary Theoretical and Mathematical Sciences Program

Rebuilding Dense Matter EoSs from Neutron Star Observations:

An Inverse Problem Perspective

Lingxiao Wang(王凌霄) (RIKEN)

Prog.Part.Nucl.Phys. 104084(2023)
JCAP08 (2022) 071, Phys. Rev. D 107, 083028 (with **S. Soma, S. Shi, H. Stoeker and K. Zhou**)

Aug 21, 2024
Aug 18–24, 2024, XVIth Quark Confinement and the Hadron Spectrum

DEEP-IN Working Group

Concept
Activities
Facilitators
Members
Contact

DEEP-IN SCIENCE

深入 科学

CONCEPT

"DEEP learning for INverse problems (DEEP-IN)" in Sciences Working Group

The essence of discovery in sciences has always been rooted in the reverse engineering of natural phenomena and observational data. This paradigm of deducing the underlying laws of nature from observable outcomes forms the cornerstone of our scientific inquiry. **The DEEP-IN working group is established with the recognition that the elucidation of such complex phenomena demands a fusion of physics insights and advanced deep learning methodologies.**

In response to the evolving landscape of scientific research, our objective is to integrate cutting-edge **deep learning techniques, alongside generative models and other advanced statistical learning methods**, into the toolkit of scientists.

The DEEP-IN working group at RIKEN-iTHEMS is dedicated to creating an interdisciplinary platform that harnesses the transformative potential of artificial intelligence(AI). This platform is designed to **tackle inverse problems that span a diverse spectrum of sciences, from biology to physics and more in the future.**

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DEEP-IN Working Group

"DEEP learning for INverse problems (DEEP-IN) in Sciences" working group (April 1st, 2024 -)

Lattice Computations

Gert Aarts, Swansea U.
Takumi Doi, iTHEMS
Andreas Ipp, TU Wien
Tetsuo Hatsuda, iTHEMS
Yan Lyu, iTHEMS

Now mostly physicists **->** **Future** more diverse scientists

BioPhysics: **Catherine Beauchemin,** iTHEMS
Condensed Matter Physics: **Steffen Backes,** iTHEMS
QCD Physics: **Kenji Fukushima**, UTokyo
Nuclear Physics: **Haozhao Liang**, UTokyo
Quantum Computing: **Enrico Rinaldi,** Quantinuum K.K./iTHEMS

Heavy-Ion Collisions

Long-Gang Pang, CCNU
Shuzhe Shi, THU
Kai Zhou, CUHK-ShenZhen

Astrophysics

Márcio Ferreira, Coimbra U.
Yuki Fujimoto, INT->iTHEMS
Akira Harada, NIT-Ibaraki
Zhenyu Zhu, TDLI->RIT

Machine Learning

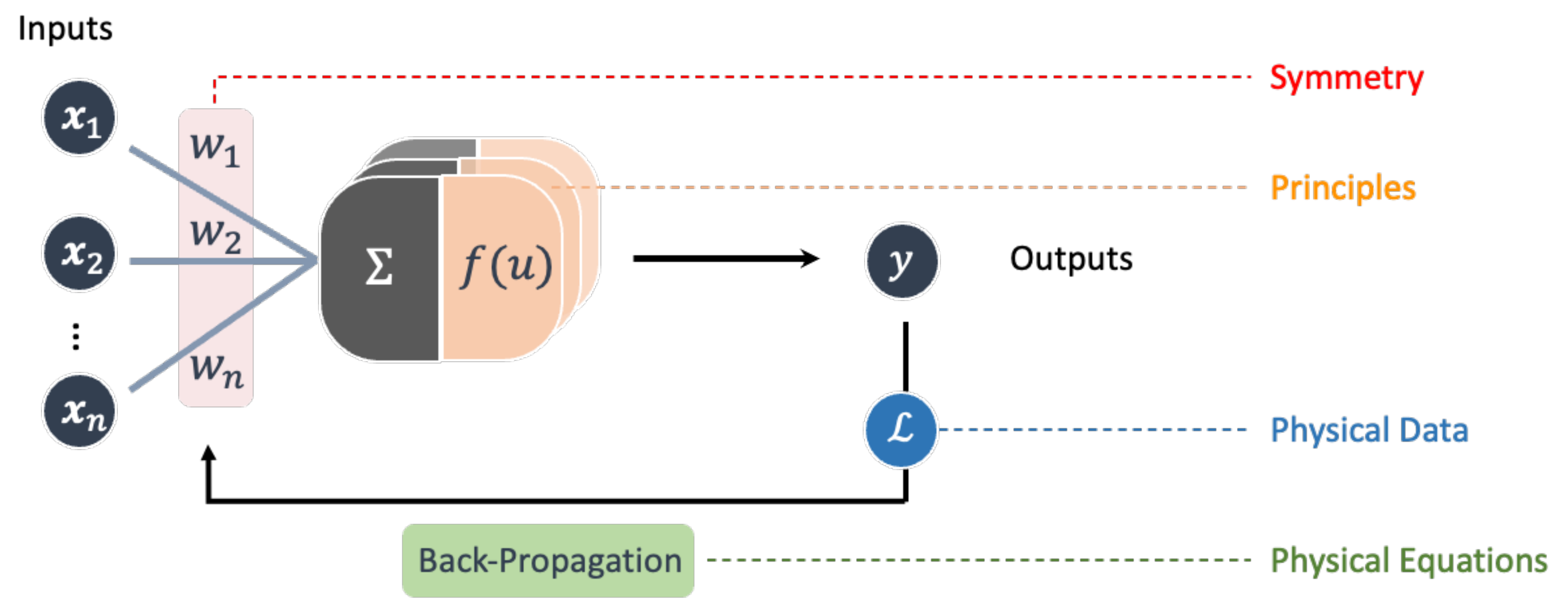
Akinori Tanaka, AIP/iTHEMS
Lingxiao Wang, iTHEMS

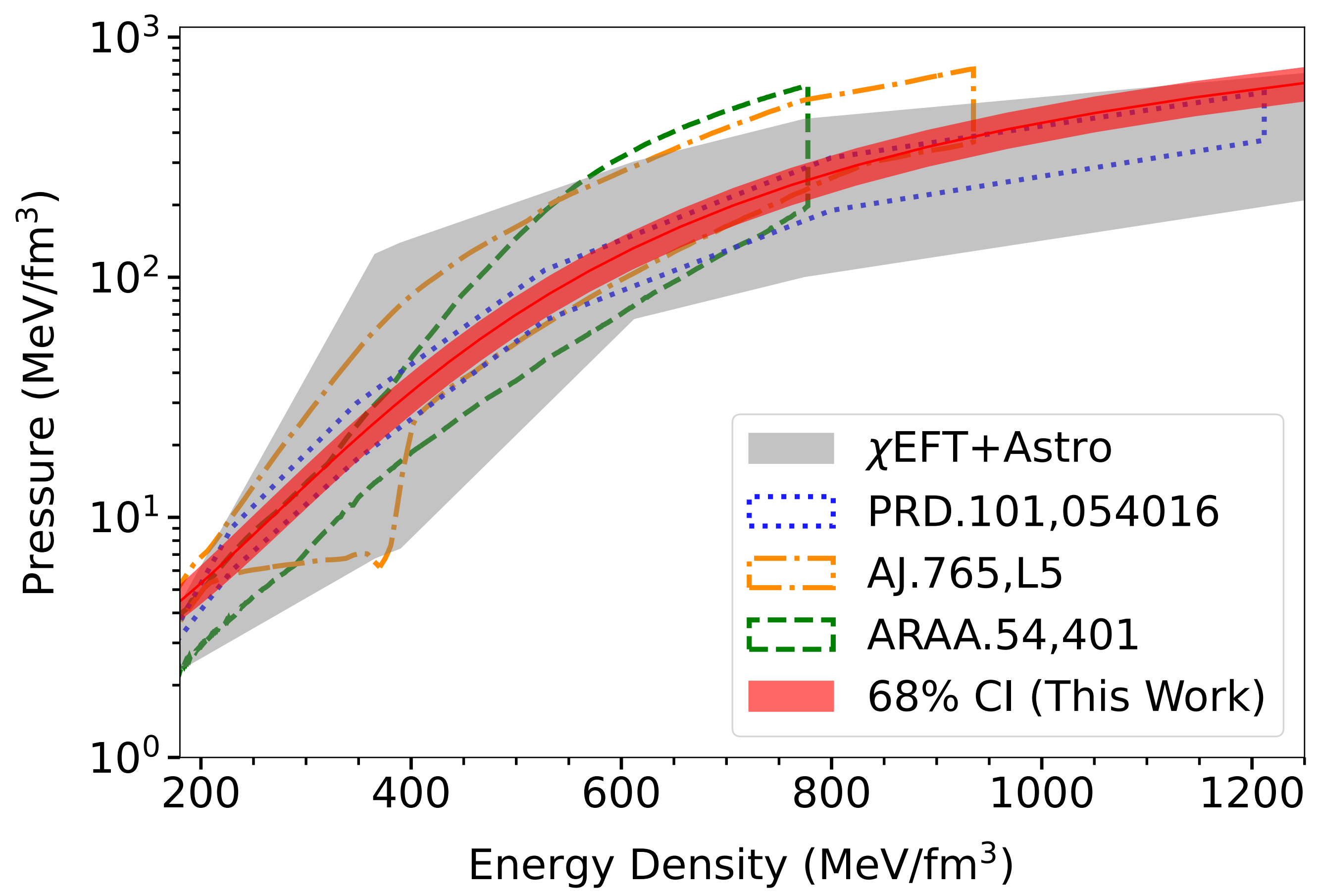
https://sites.google.com/view/deep-in-wg/homepage

Lingxiao Wang (RIKEN iTHEMS) *Contact at lingxiao.wang@riken.jp

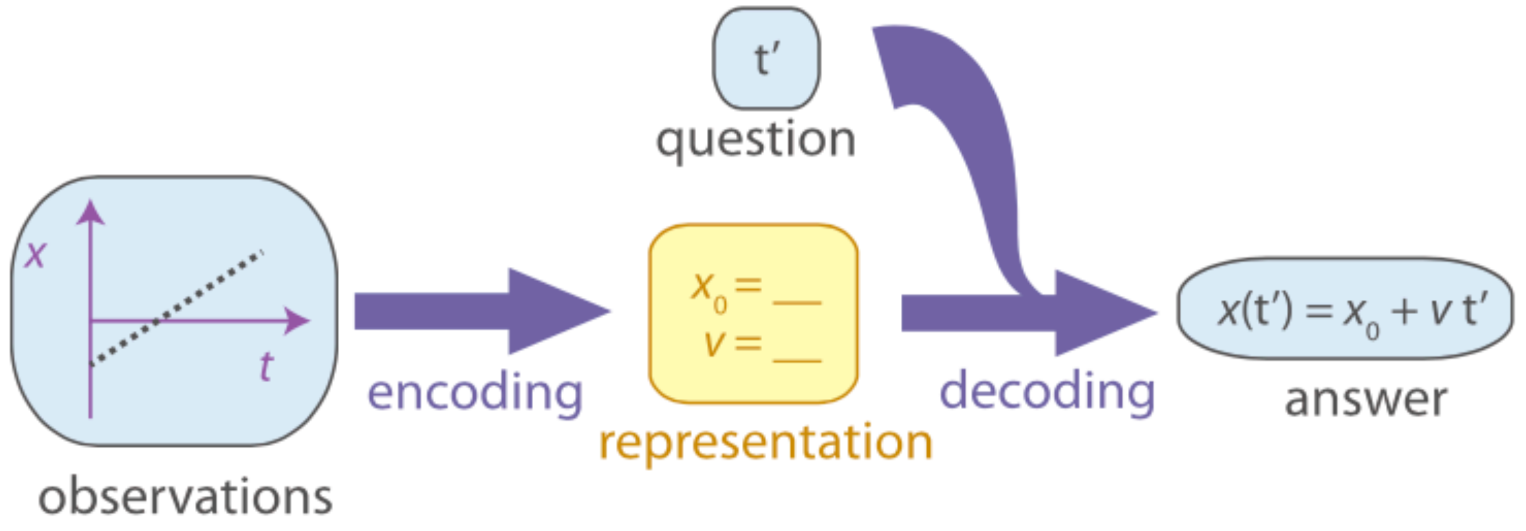
Outline

- **Inverse Problems**
  - Data-Driven Learning
  - Physics-Driven Learning
- **Hadron Forces**
- **Neutron Star EoSs**
- **Outlooks**

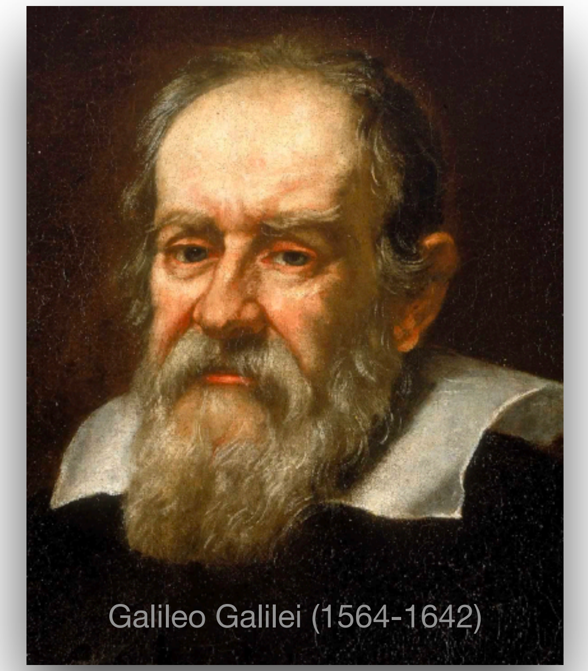




Machine Learning and Physics

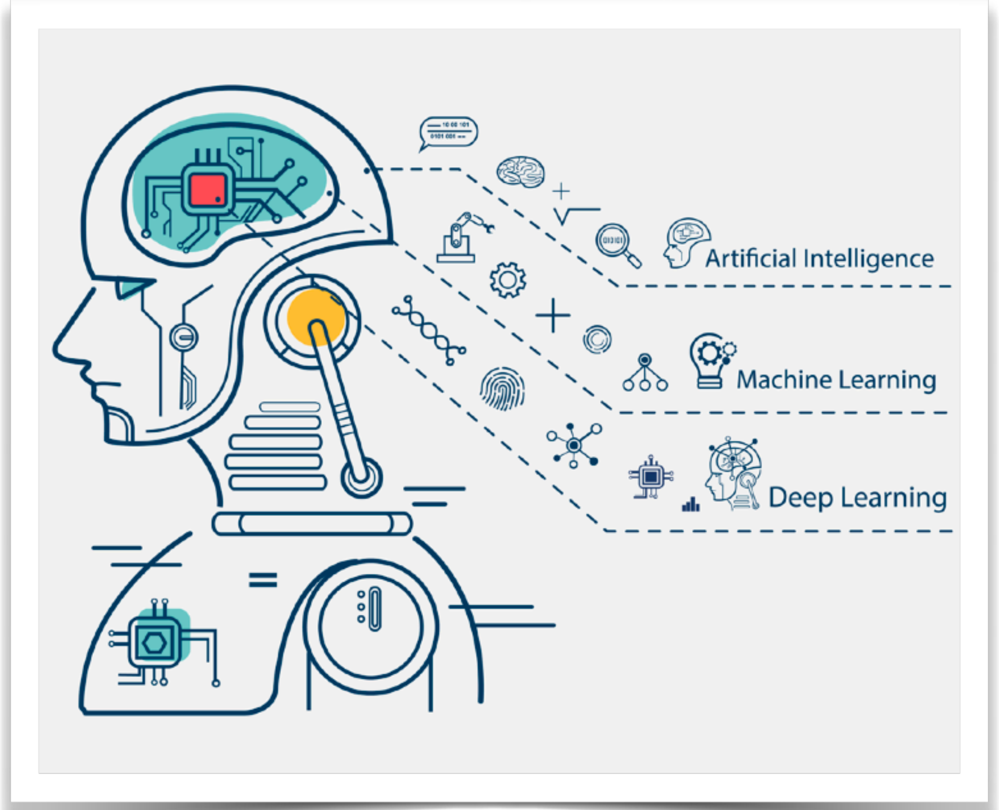


Phys.Rev. Lett. **124**, 010508 (2020)



An inverse problem in science is the process of inferring from a set of observations the causal factors that produced them.

Data, X



Machine, $\{\theta\}$

Prediction

Estimation

Machine Learning and Inference

Maximum Likelihood Estimation(MLE)

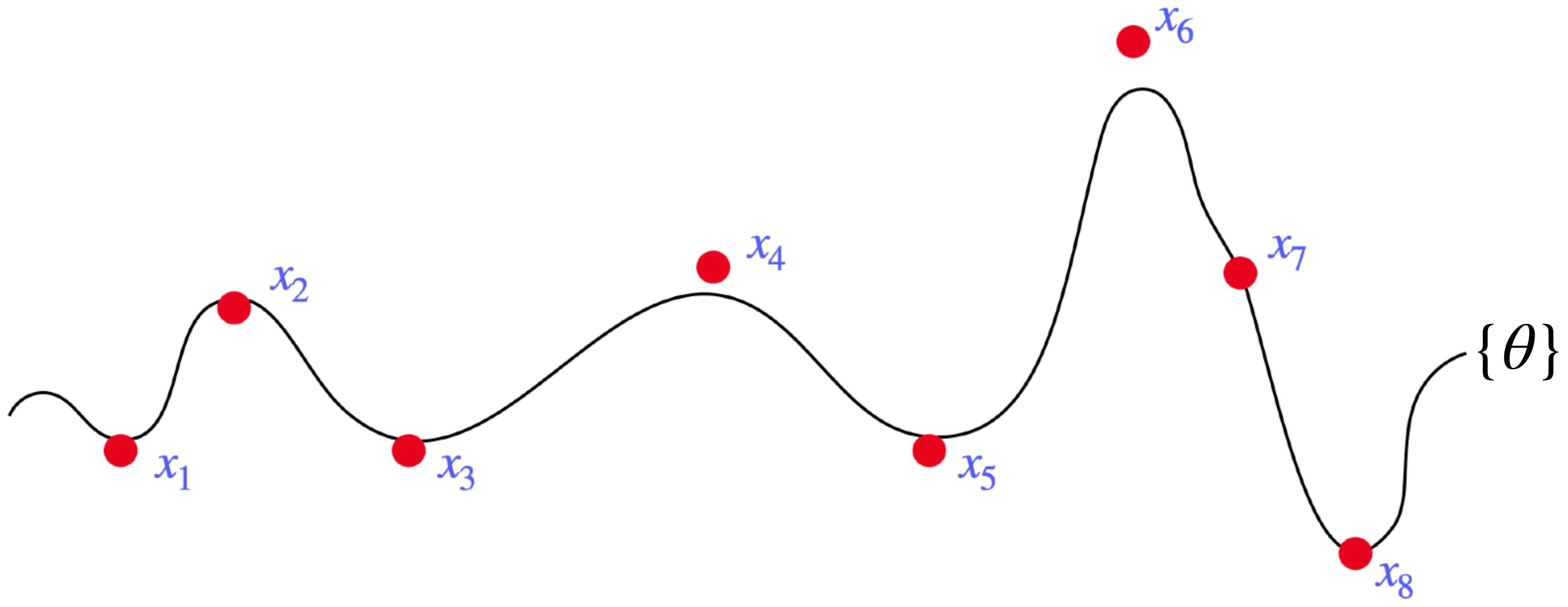
$$\max_{\theta} \prod_{i=1}^{N} p(\mathbf{x}_i \mid \theta)$$

Bayesian Inference

Maximum A Posterior(MAP)

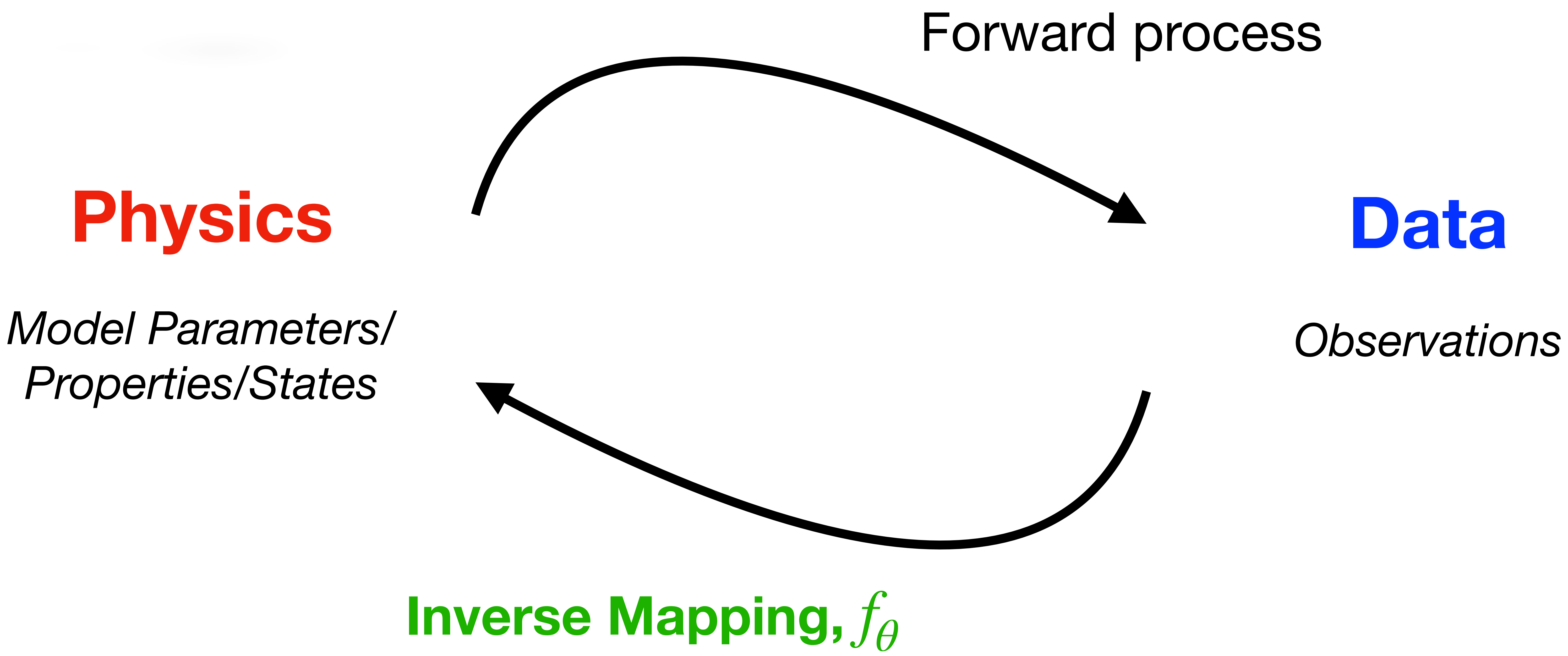
$$p(\theta \mid X) = \frac{p(X \mid \theta)\pi(\theta)}{p(X)}$$

Posterior $p(\theta \mid X)$, **Prior** $\pi(\theta)$, **Evidence** $p(X)$



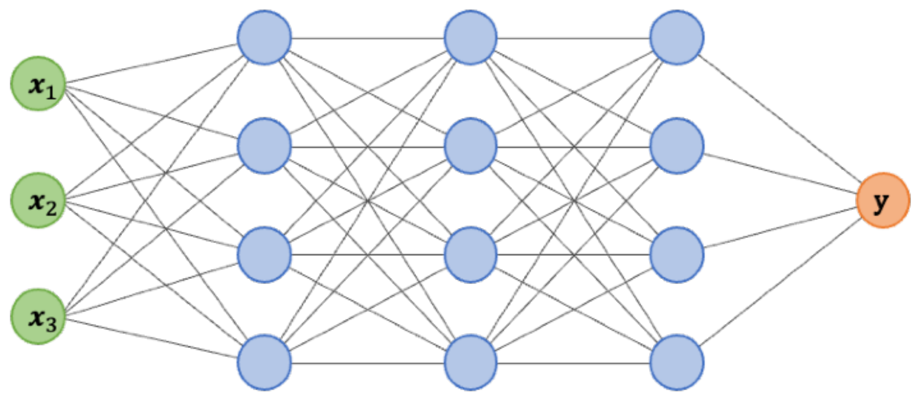
Data-Driven Learning

$$f_\theta : X \to Y$$

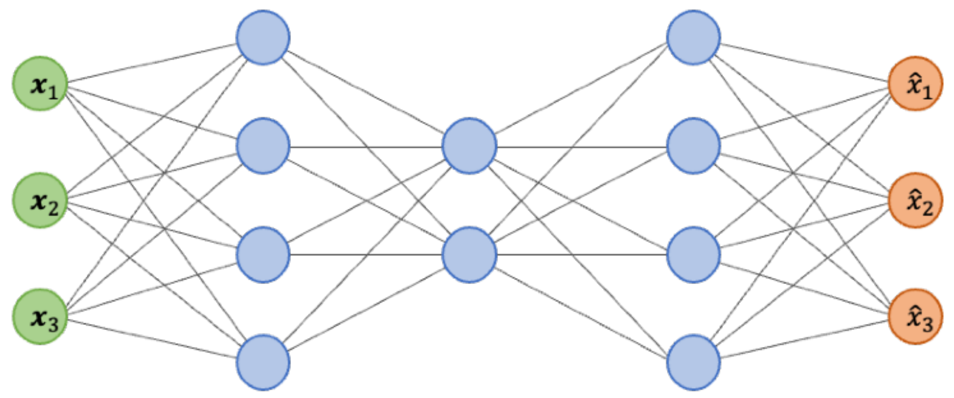


Data-Driven Learning

$$f_\theta : X \to Y$$



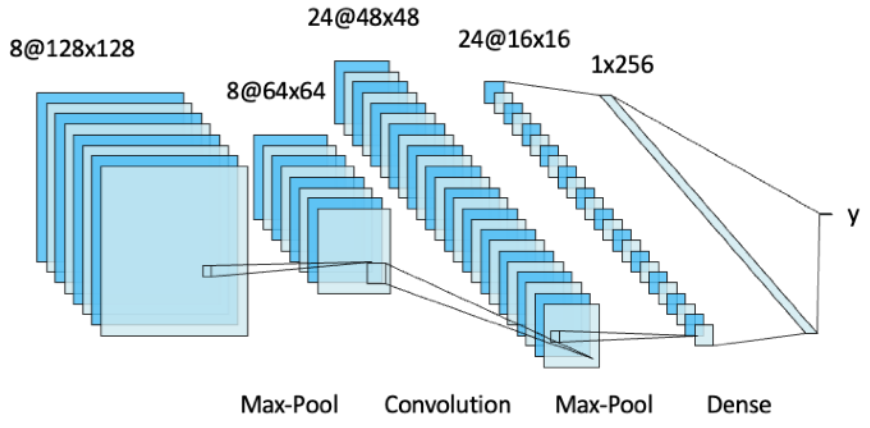
Deep Neural Network



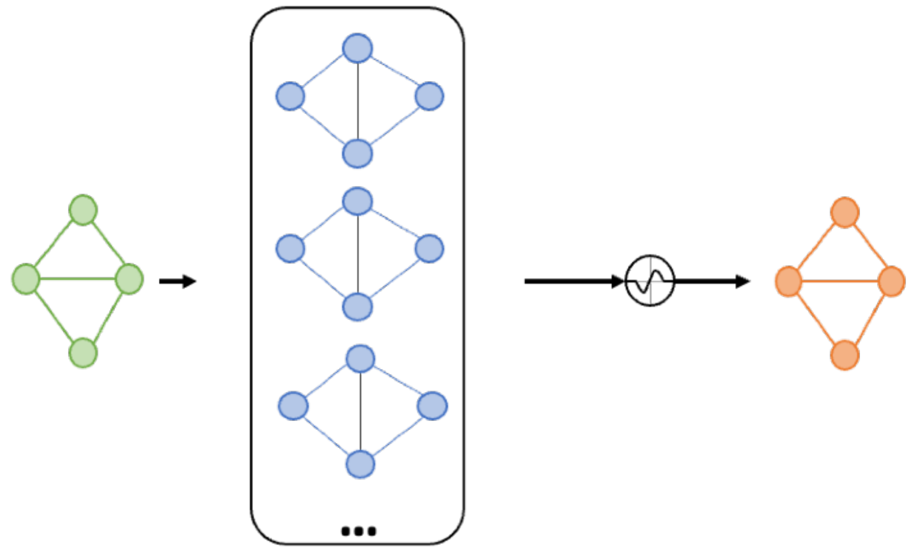
AutoEncoder

Universal Approximation Theorem (1989,1991)

A feed-forward network with *a single hidden layer* containing *a finite number of neurons* can approximate arbitrary continuous functions.



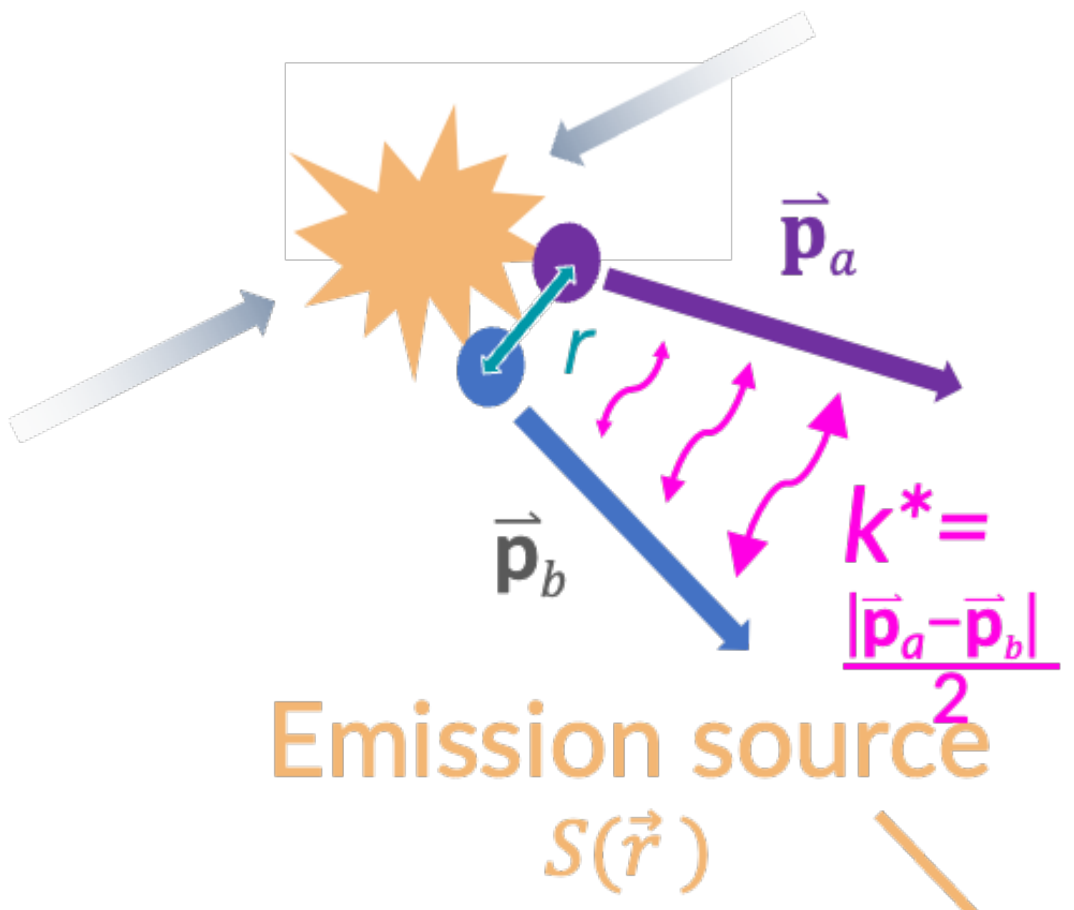
Convolutional Neural Network

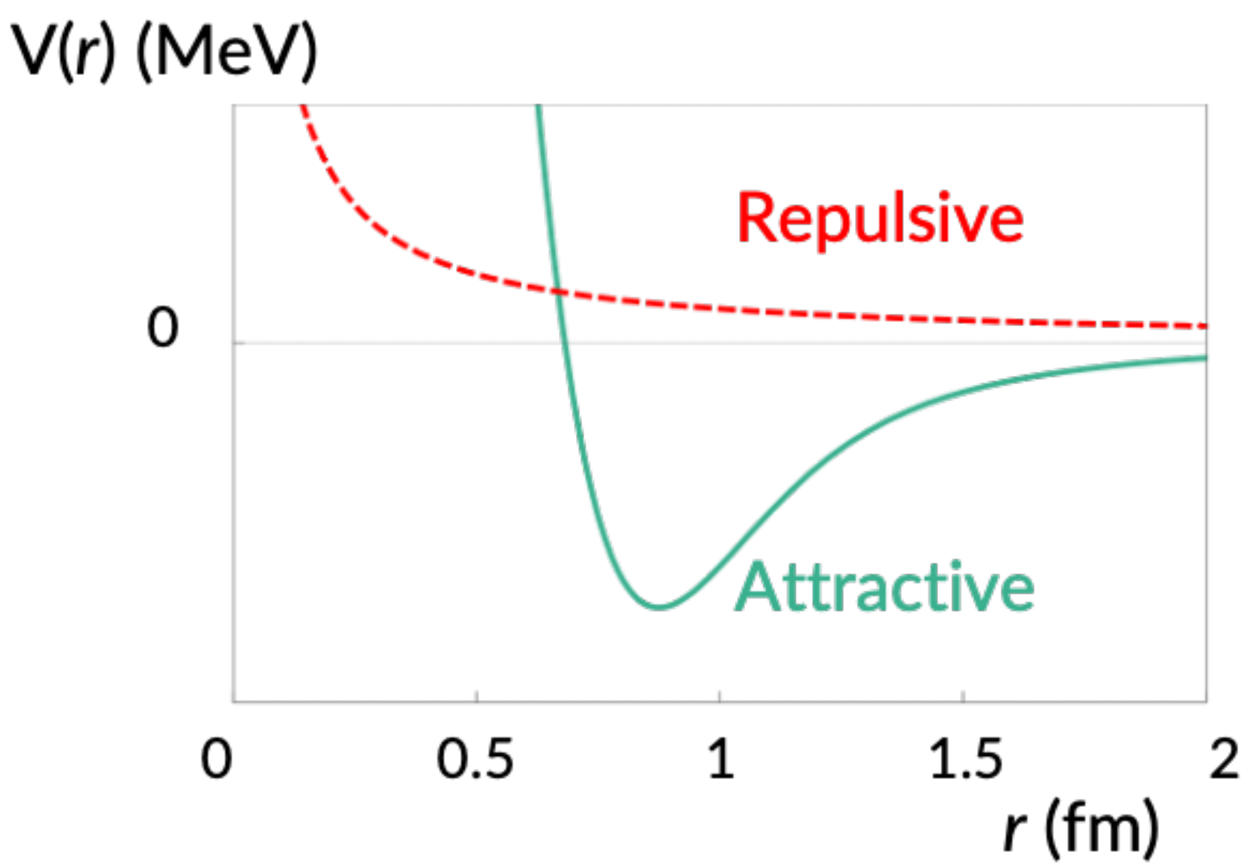


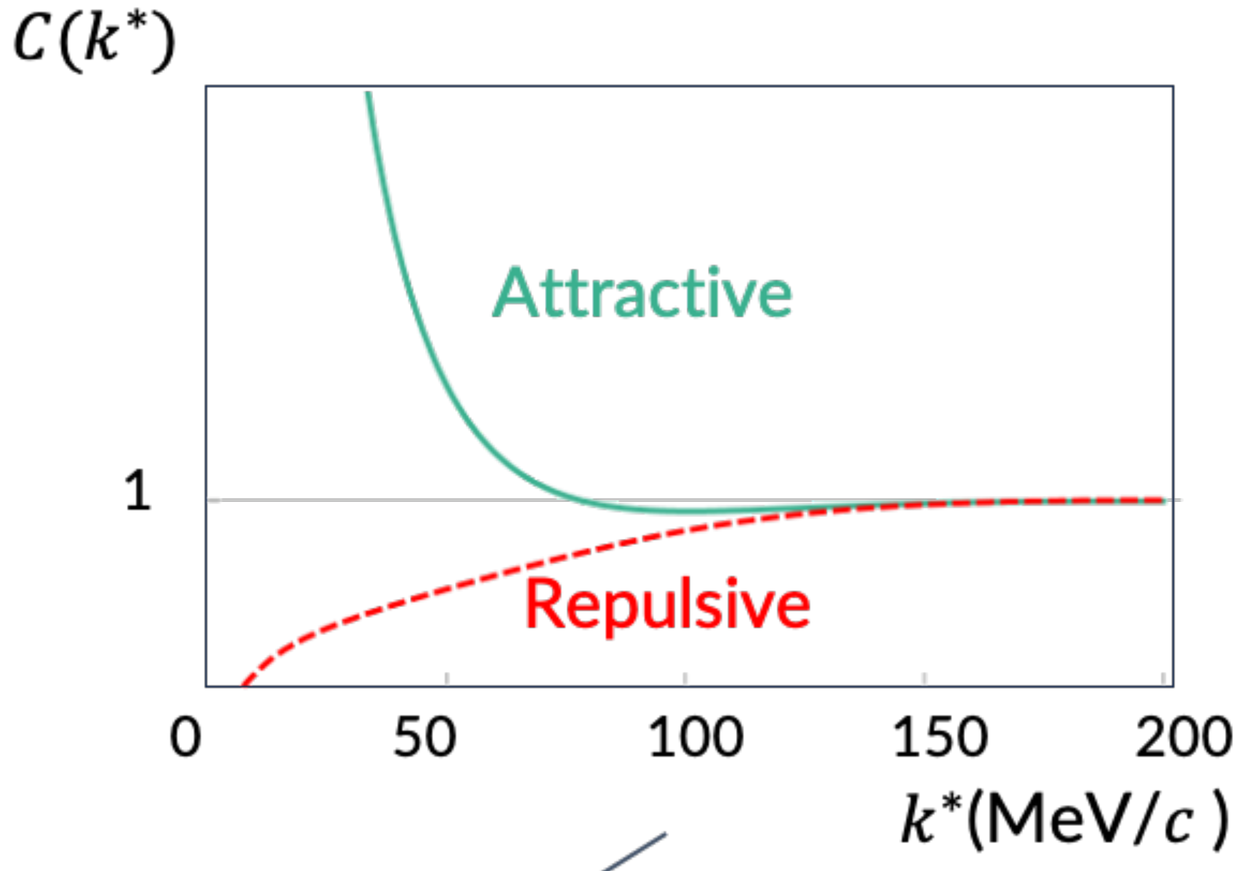
Graph Neural Network

Inverse Femtoscopy

in Preparation **with Jiaxing Zhao, Liang Zhang**







Two-particle wave function

$$C(k^*) = \int S(\vec{r}\,)\left|\psi(\vec{k}^*,\vec{r}\,)\right|^2 d^3\vec{r} = \mathcal{N}(k^*)\frac{N_{\text{same}}(k^*)}{N_{\text{mixed}}(k^*)}$$

Raffaele Del Grande | XQCD 2023

Correlations

↓ **?**

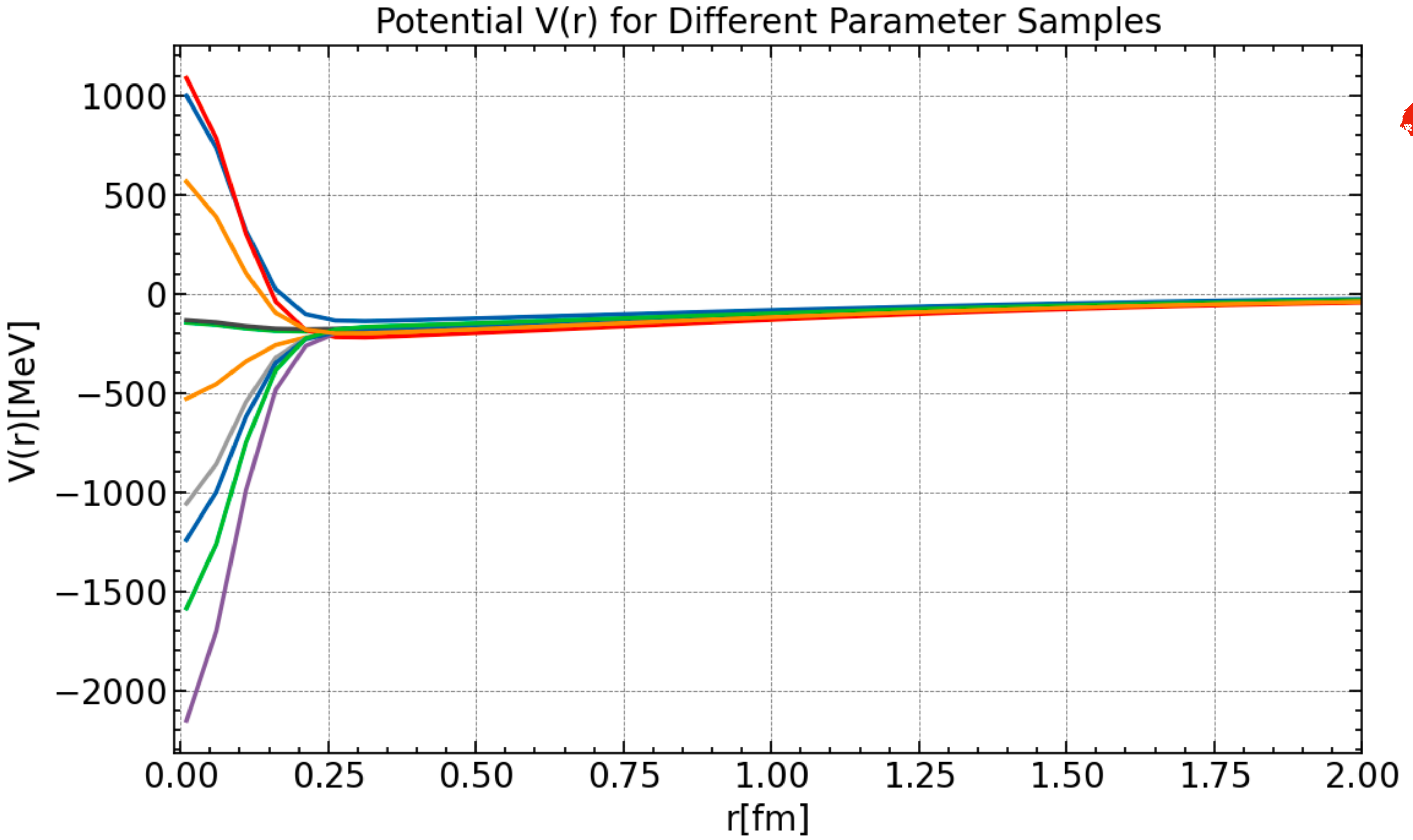
Potentials

Whether this inverse mapping exists?

Inverse Femtoscopy

in Preparation with Jiaxing Zhao, Liang Zhang

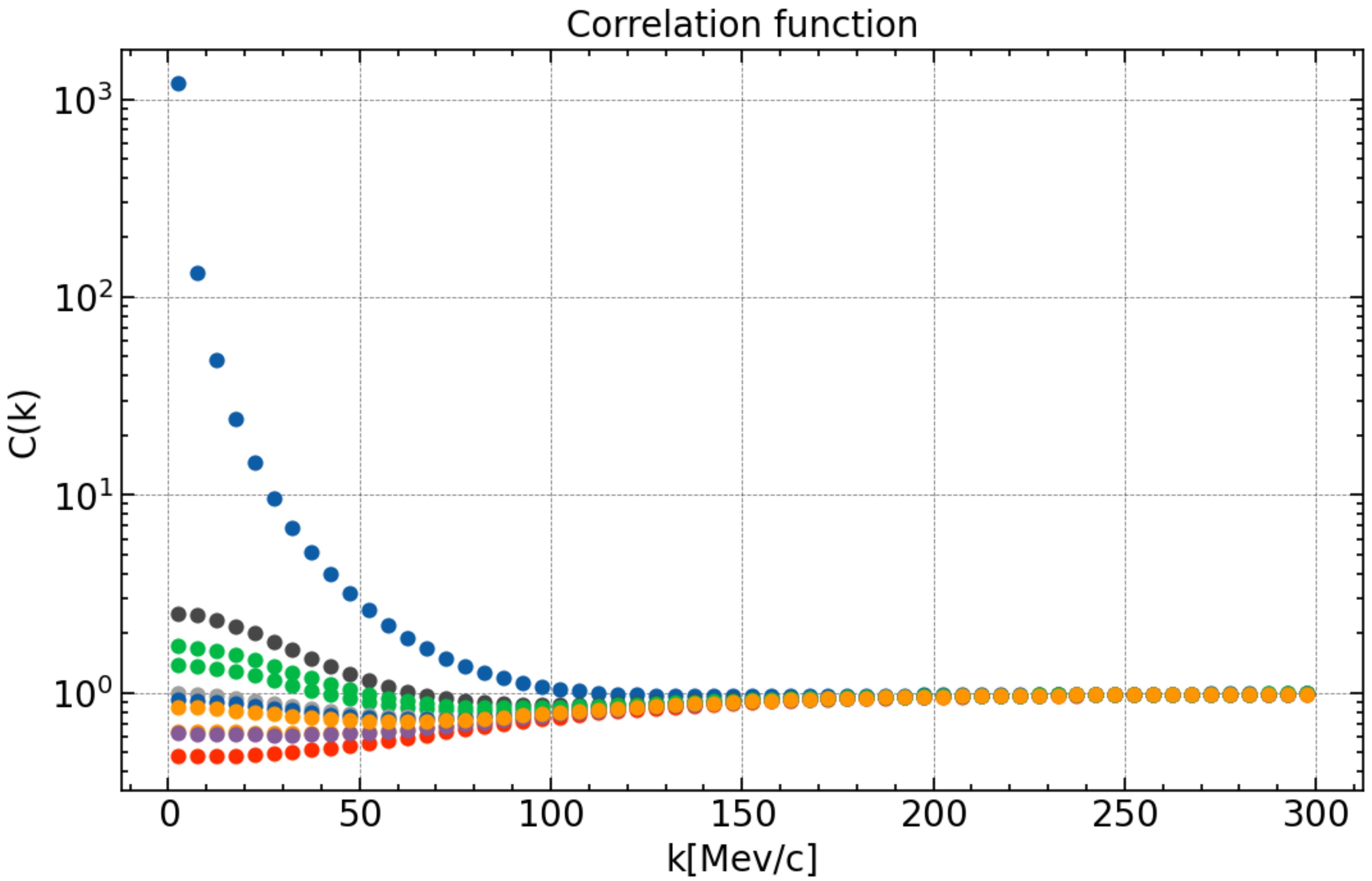
$$V(r) = b_1 e^{-b_2 r^2} + b_3 (1 - e^{-b_4 r^2})(\frac{e^{(-m_\pi r)}}{r})^{n_\pi}$$



?

Schrödinger eq.

CATS Framework: D. Mihaylov et al., Eur. Phys. J. C78 (2018) 394

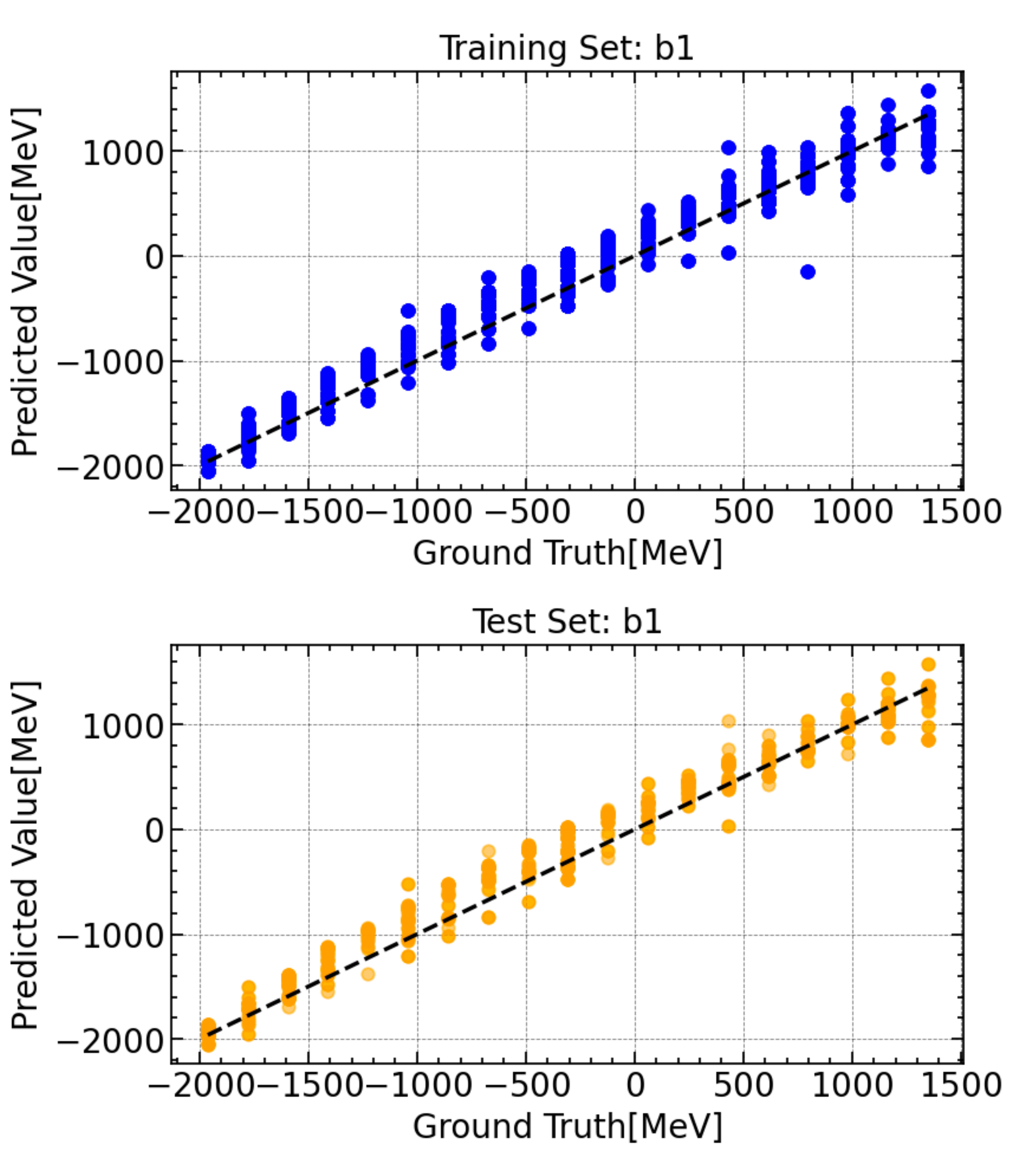


$$S(r) = (4\pi r_0^2)^{-3/2} e^{-\frac{r^2}{4r_0^2}}$$

$$r_0 = 1.3 \text{ fm}$$

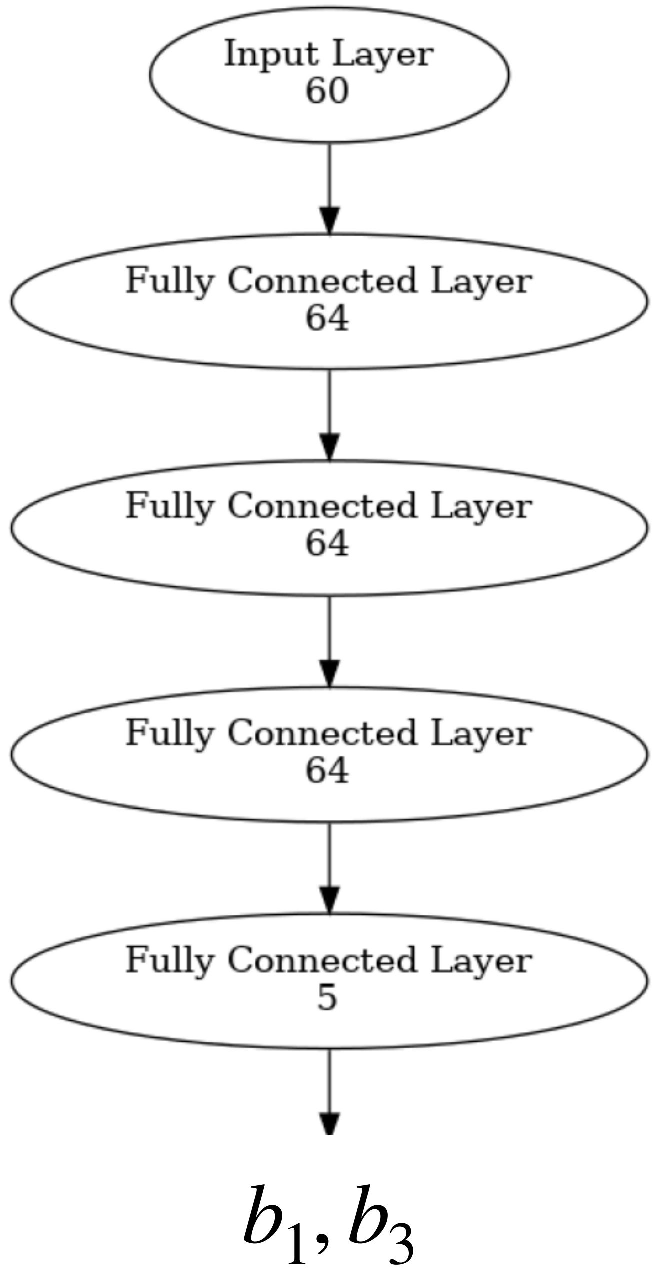
60 points(k), 10000 correlations

Inverse Femtoscopy



in Preparation **with Jiaxing Zhao, Liang Zhang**

60 points

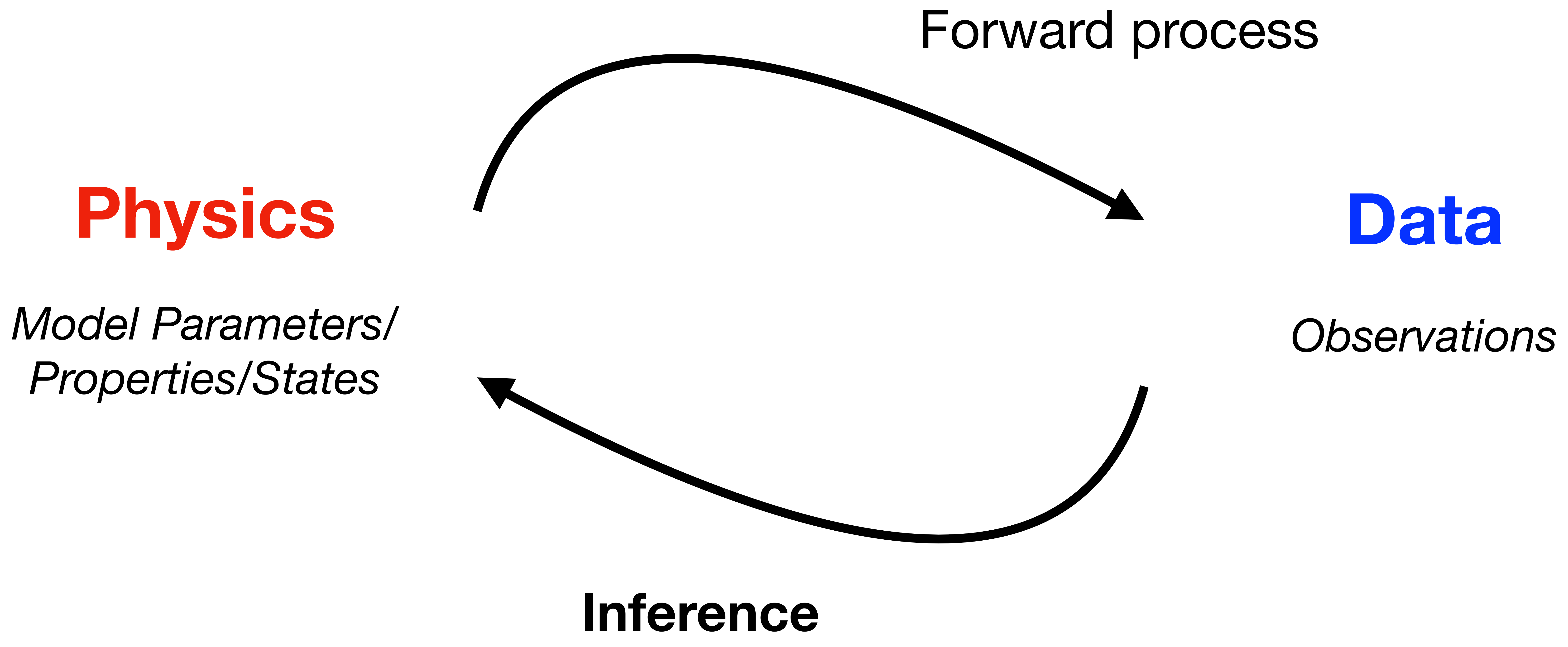


R-squared	b1	b3
Training	0.96	0.99
Testing	0.96	0.99

Neural Networks for Femtoscopy

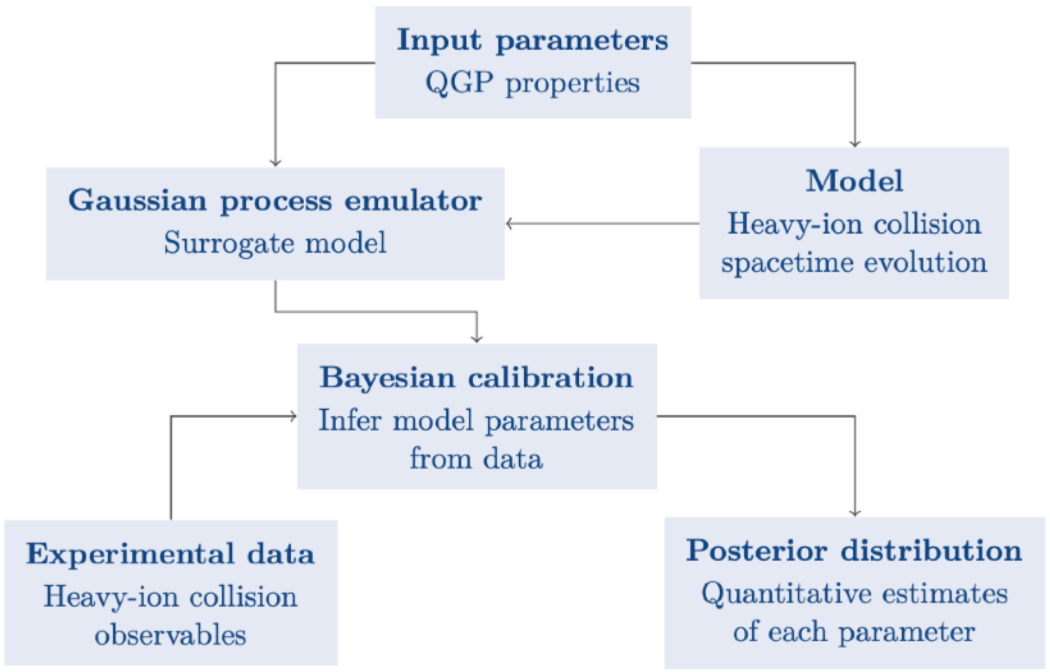
$$b_2 = 73.9, b_4 = 0.78, n_\pi = 2$$

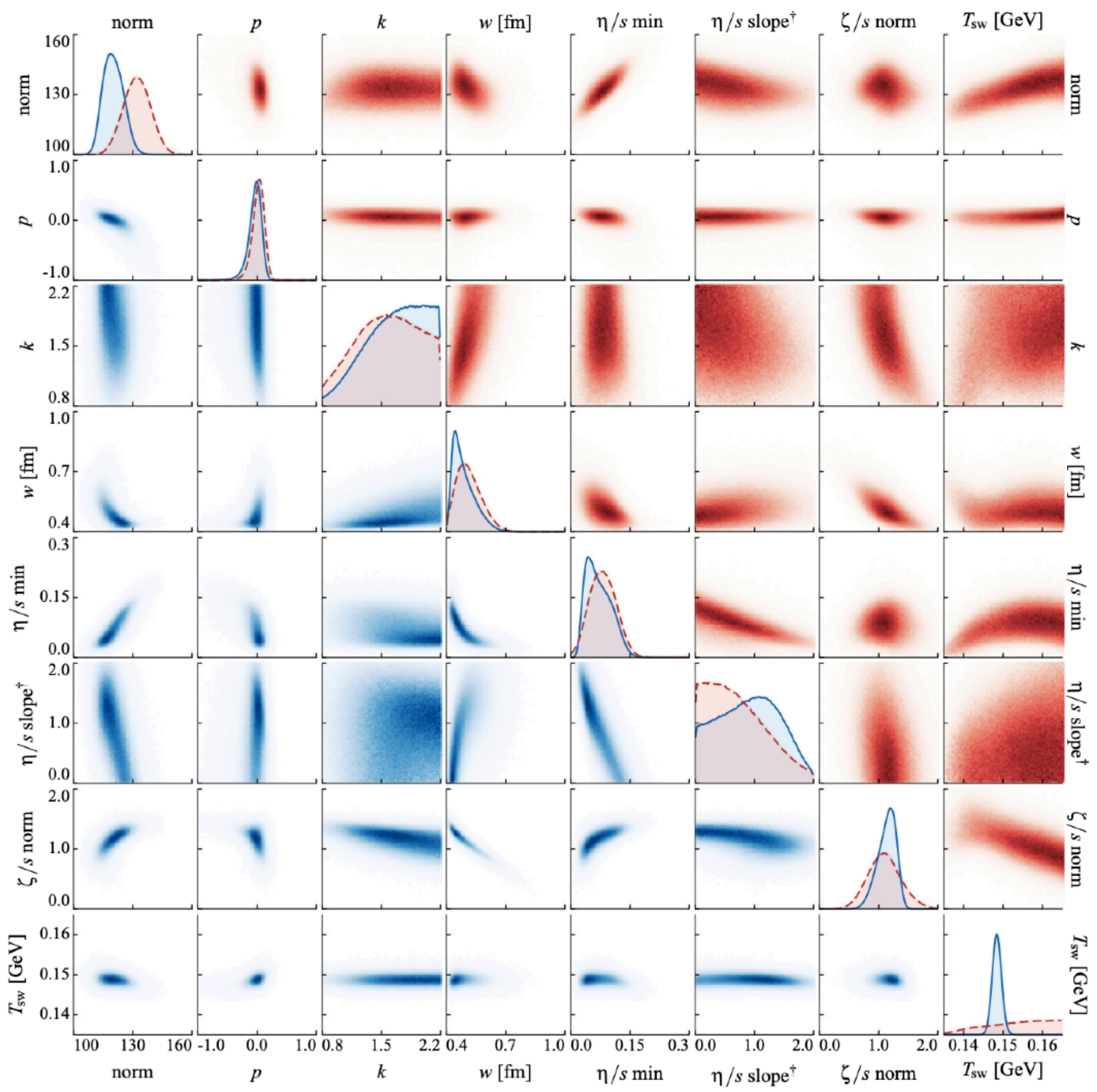
$$\hat{\theta} = \textbf{arg max}_{\theta}\{p(X \mid \theta)\}$$



Bayesian Inference

Inferring Dynamical Information

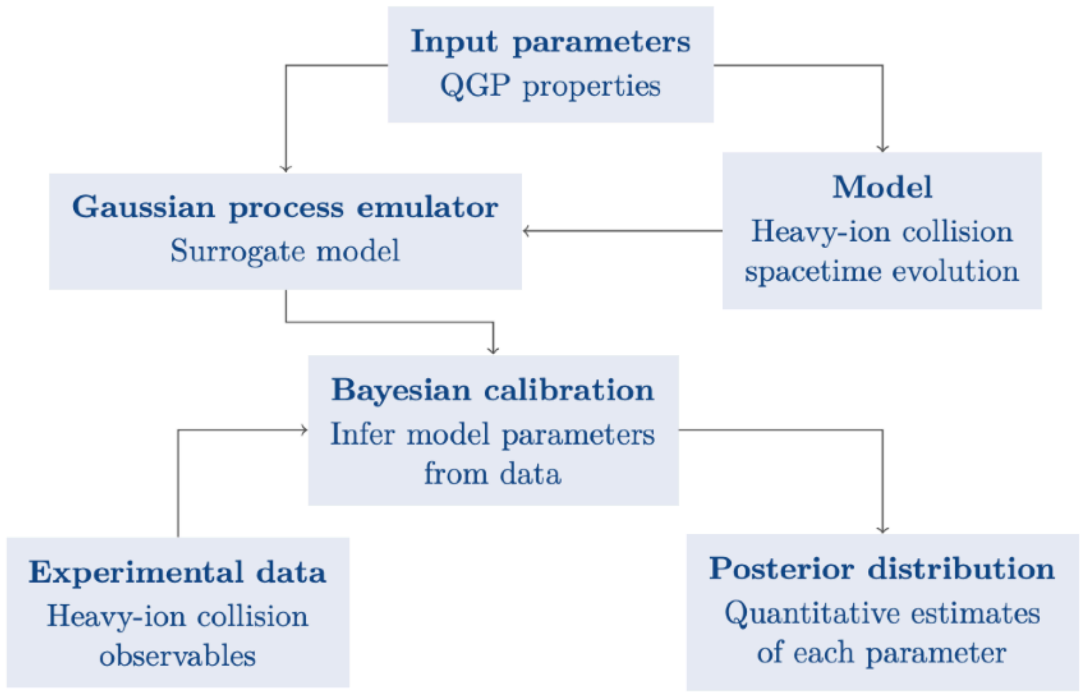




J. E. Bernhard, etc.(Duke Group), arXiv:1804.06469 and Nat Phy **15**, 1113 (2019).

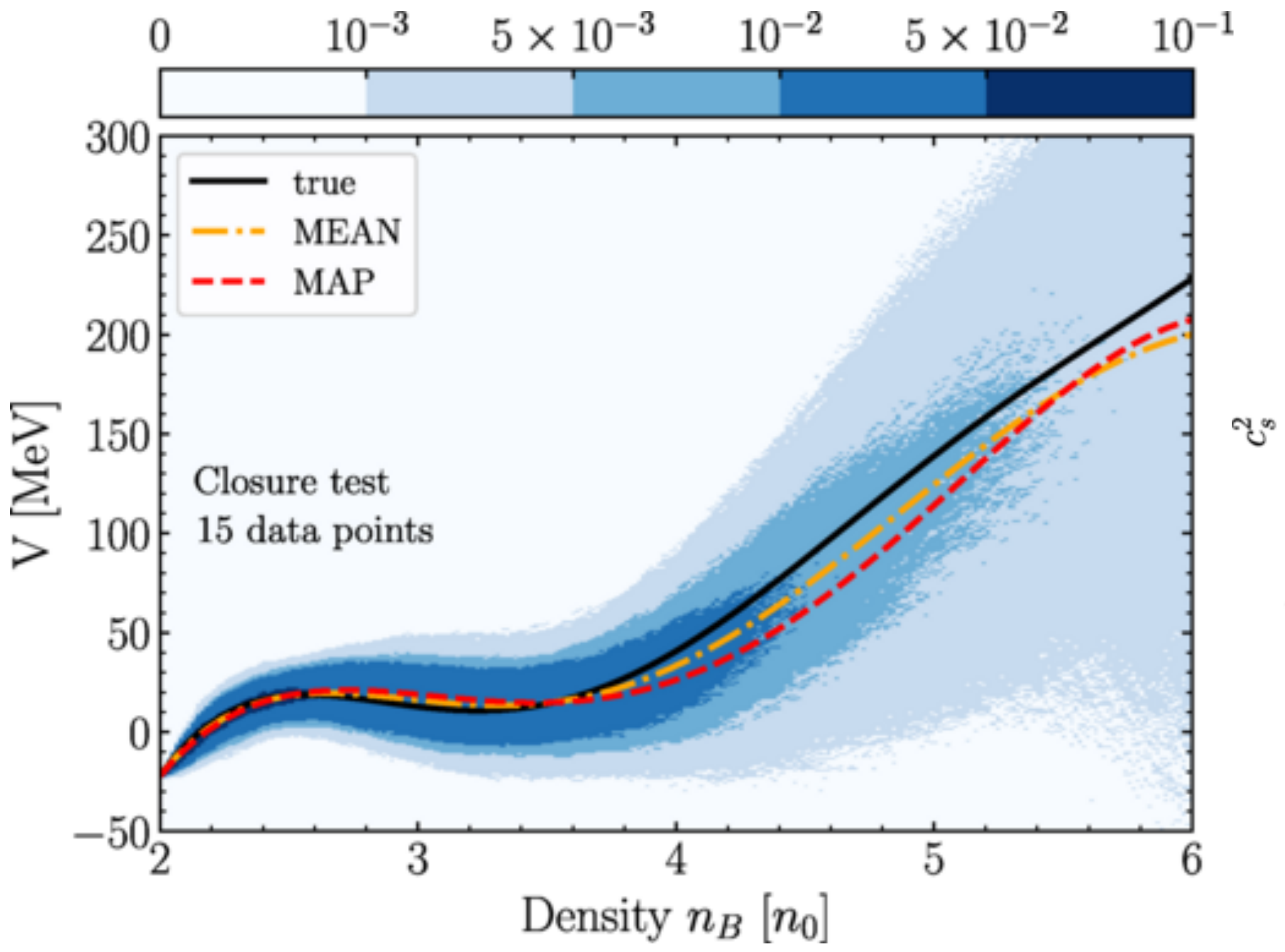
Bayesian Inference

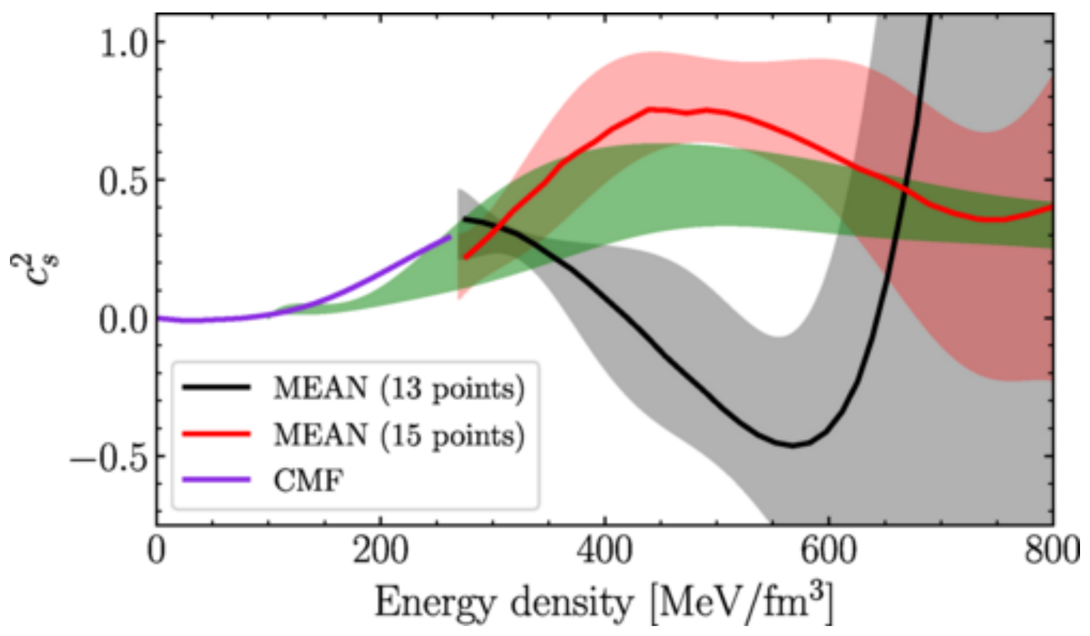
Inferring Dynamical Information



J. E. Bernhard, etc.(Duke Group), arXiv:1804.06469

Ectracting Dense Matter EoSs

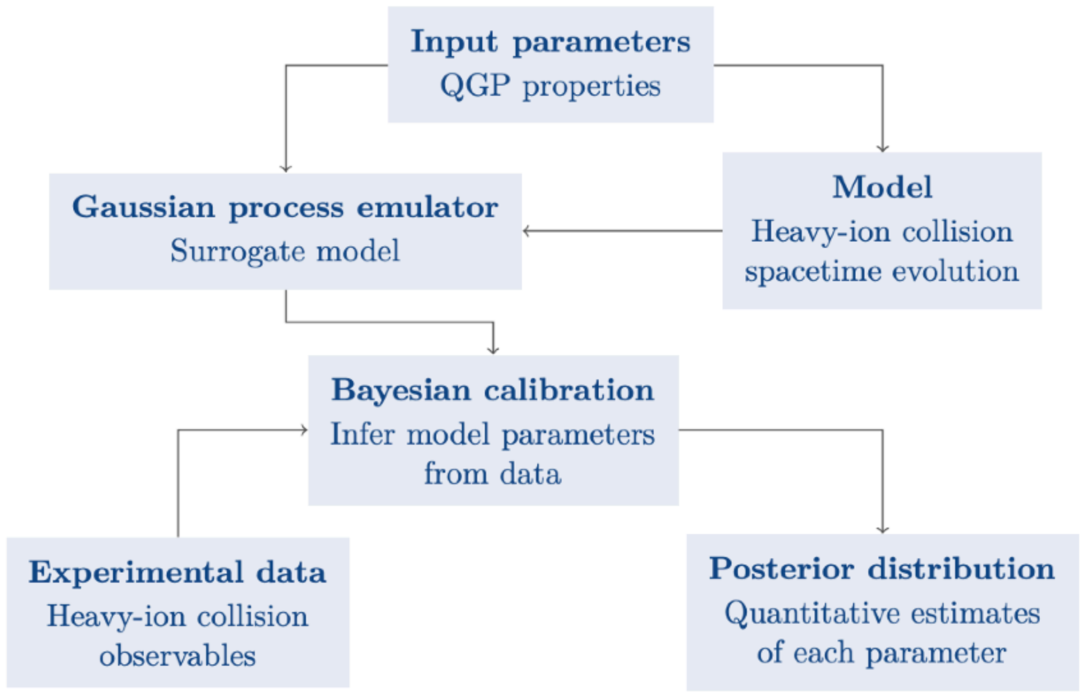




M. Omana Kuttan, J. Steinheimer, K. Zhou, and H. Stoecker, Phys. Rev. Lett. **131**, 202303 (2023).

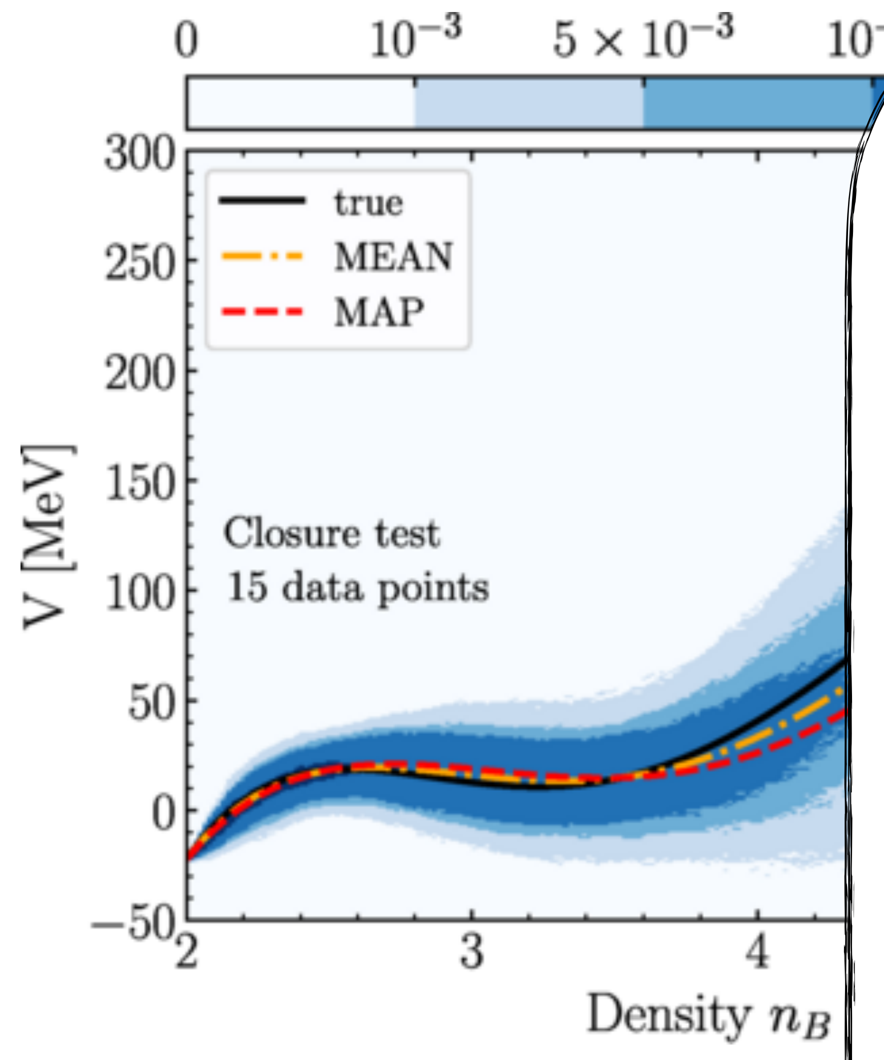
Bayesian Inference

Inferring Dynamical Information



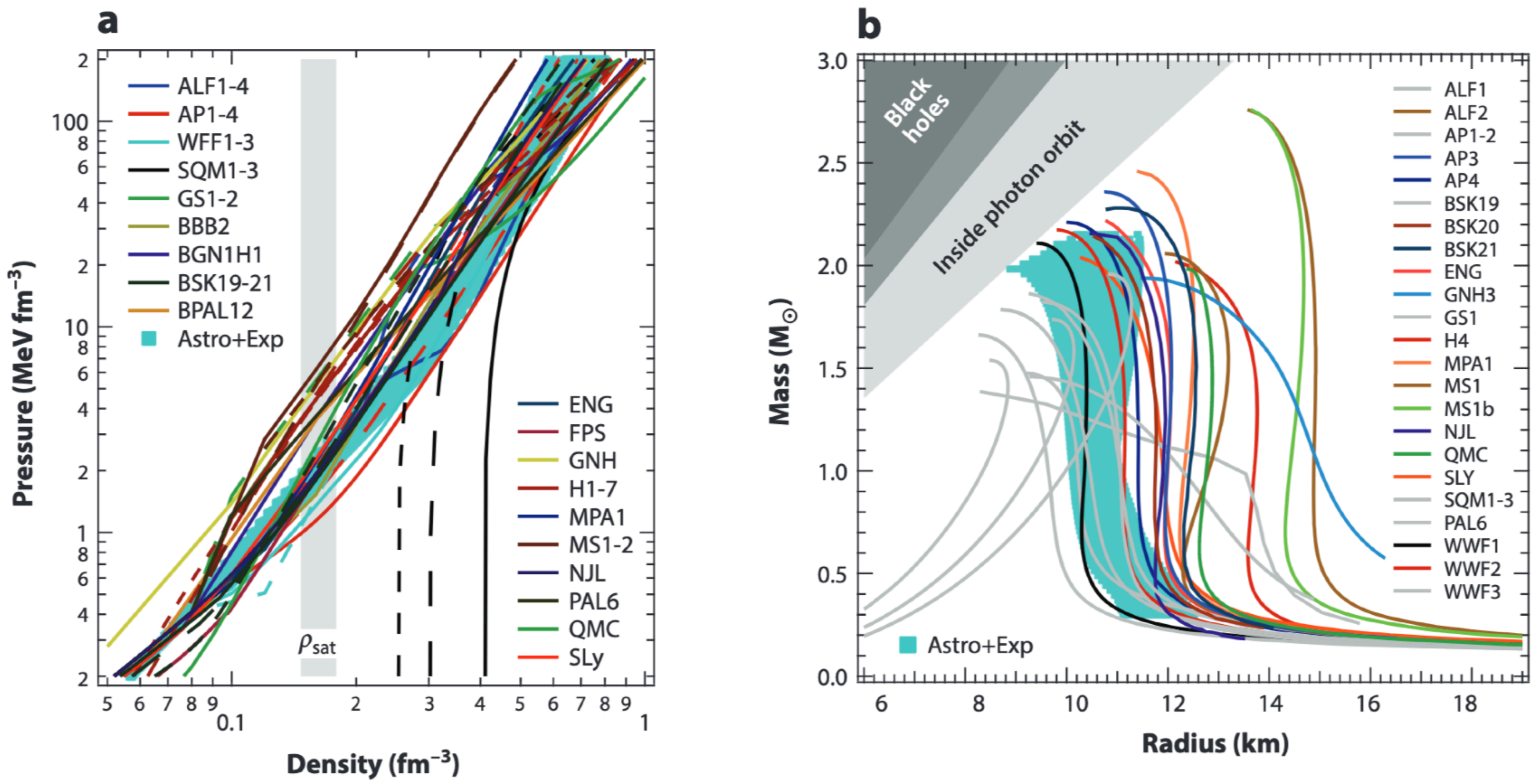
J. E. Bernhard, etc.(Duke Group), arXiv:1804.06469

Ectracting Dense Matter EoSs



M. Omana Kuttan, J. Ste

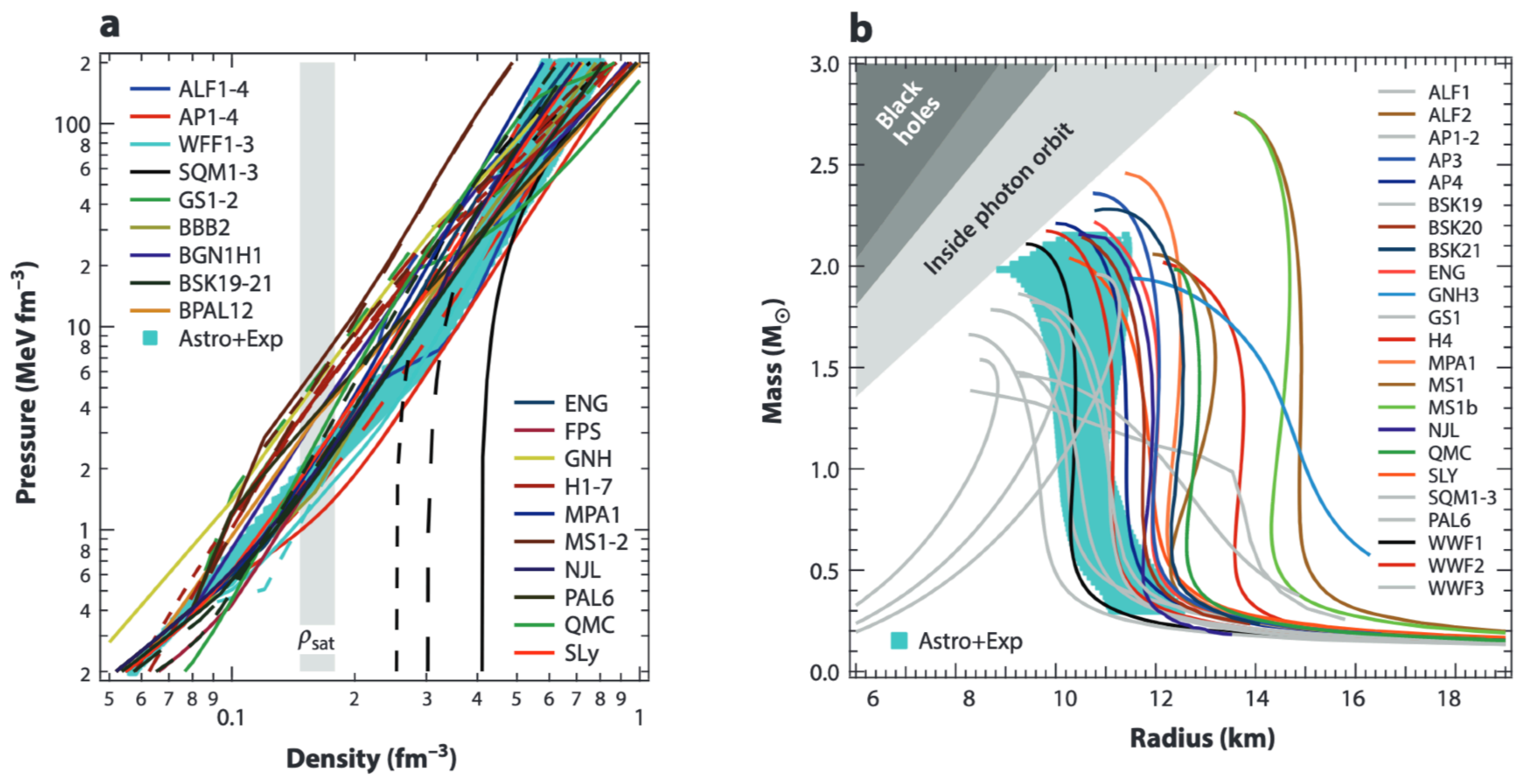
Building Nuclear Matter EoS



F. Özel and P. Freire, Annu. Rev. Astron. Astrophys. **54**, 401 (2016)

Bayesian Inference

Building Nuclear Matter EoS



F. Özel and P. Freire, Annu. Rev. Astron. Astrophys. **54**, 401 (2016)

Physics Parameters are Finite

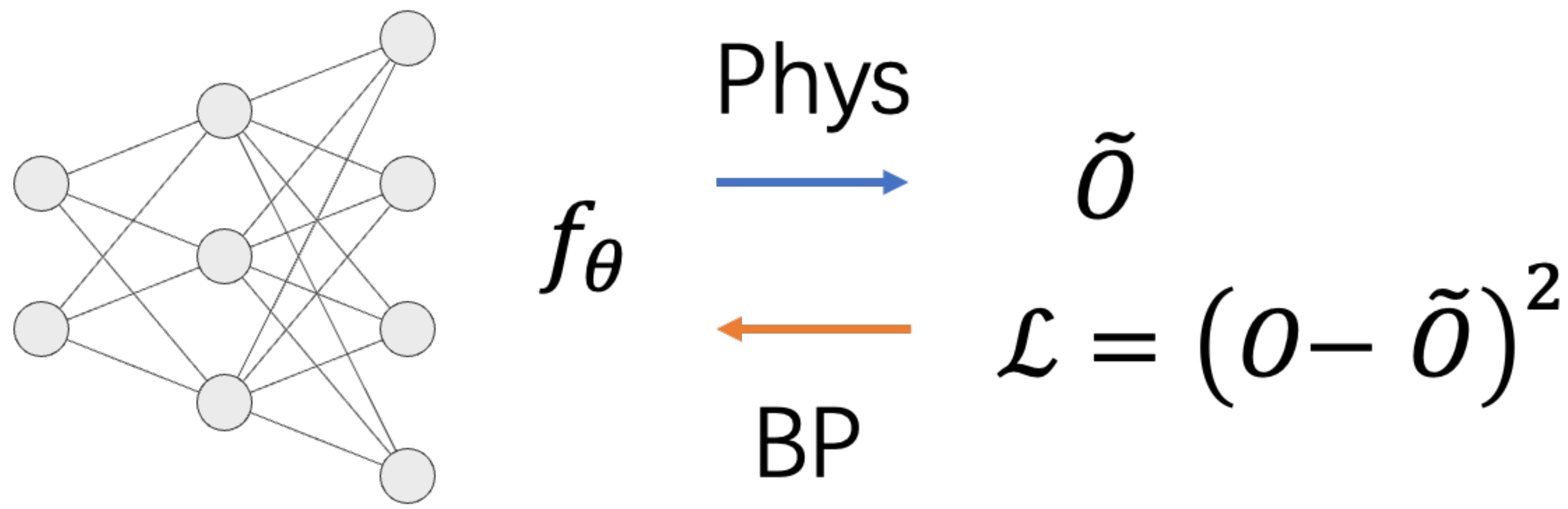
EoS, Wave-Function, **Potential**, ✗

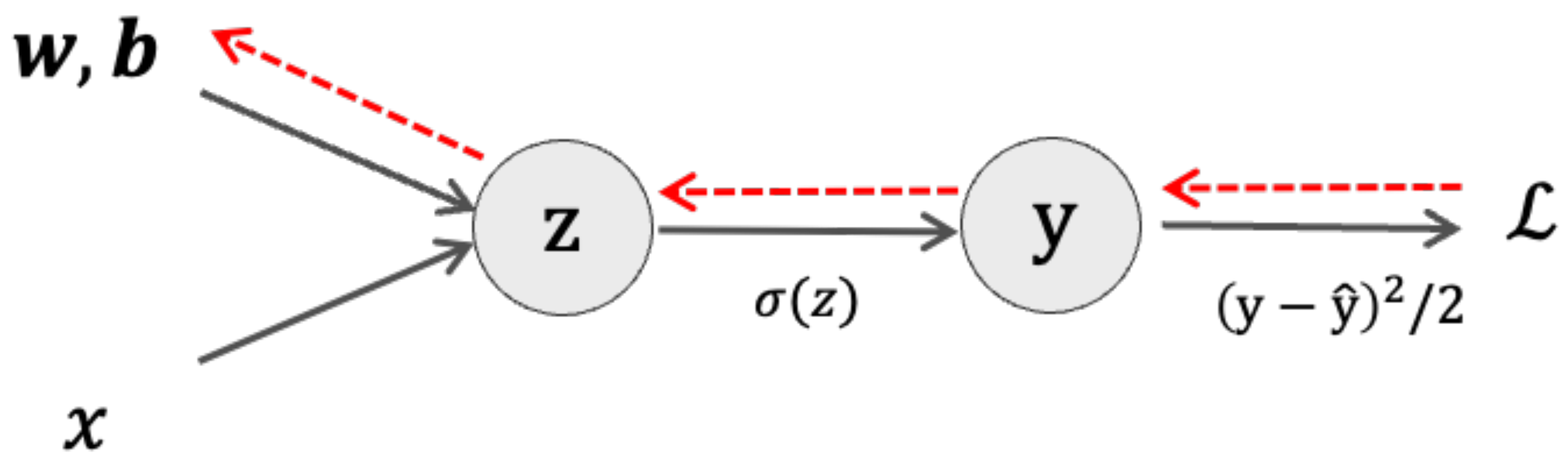
Inference is Easy-To-Compute

ODEs, PDEs, Simulations, … ✗

Physics-Driven Deep Learning

$$\hat{\theta} = \textbf{arg max}_{\theta}\{p(X \mid \theta)\}$$





Deep Neural Network represented Physics, f_θ

Flexible Representation

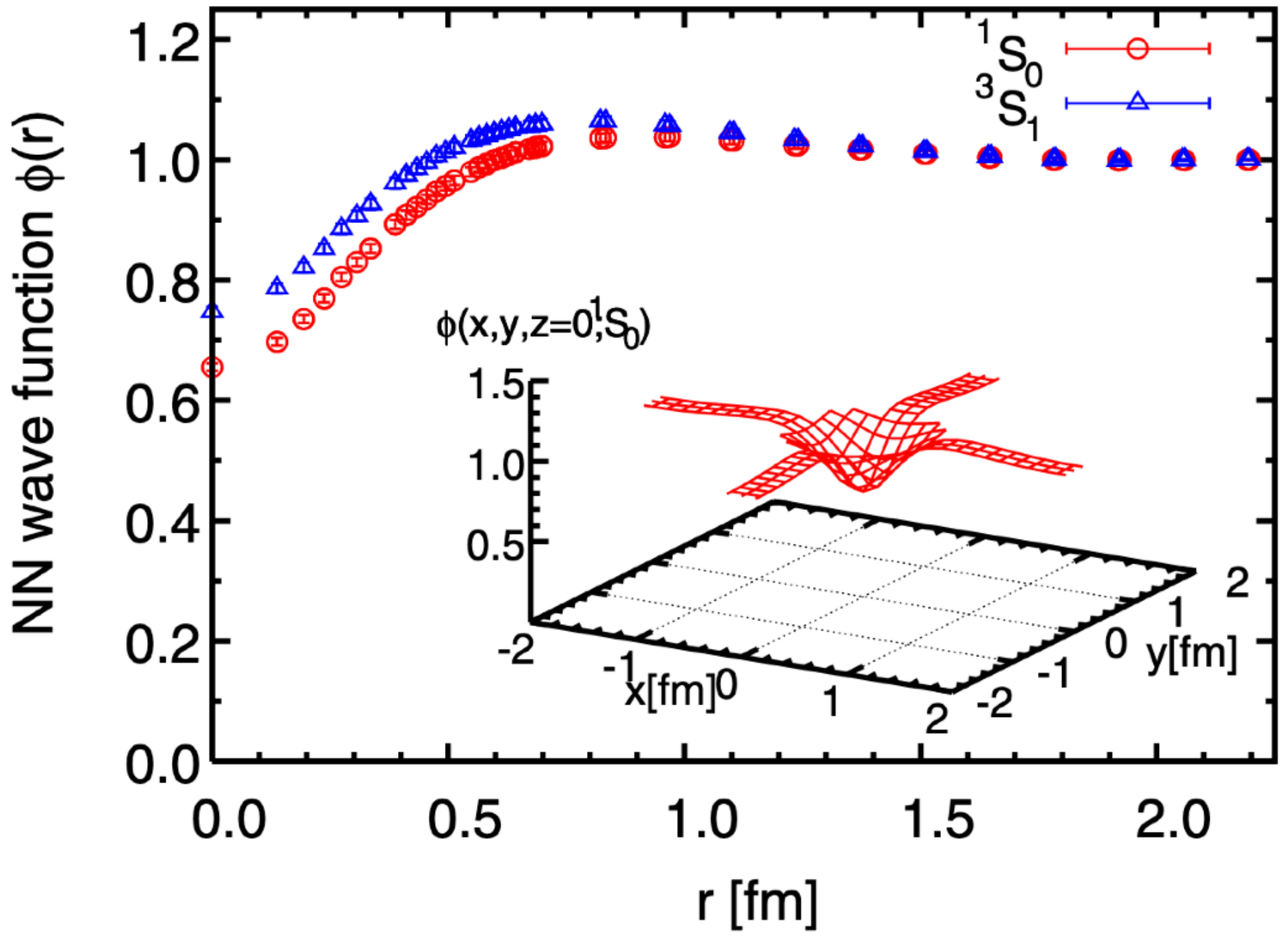
Back-Propagation

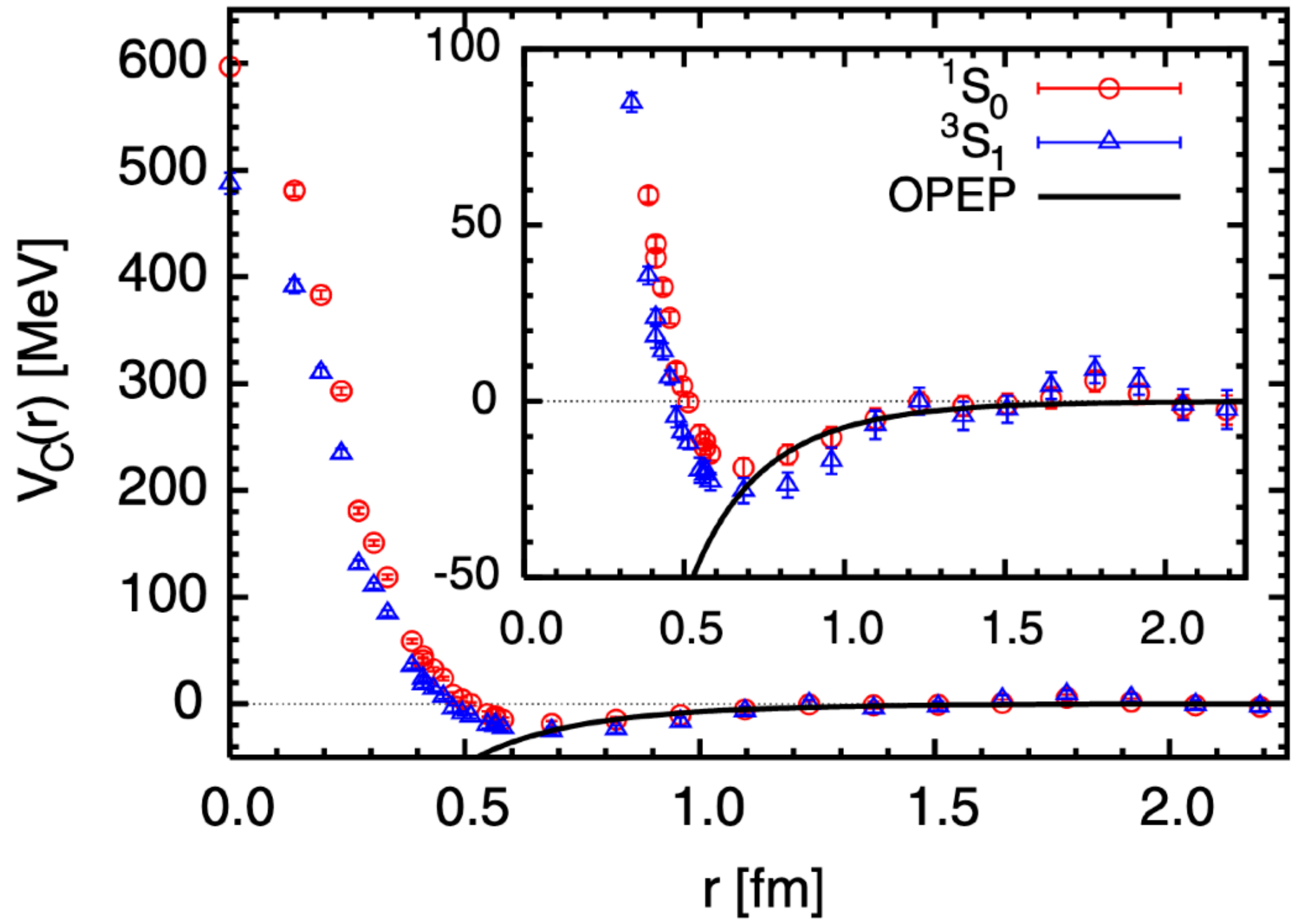
Easy-To-Compute on GPUs

Physics-Driven Deep Learning

1. Extracting Nuclear Force

N. Ishii, S. Aoki, and T. Hatsuda, Phys. Rev. Lett. **99**, 022001 (2007)





Nambu-Bethe-Salpeter (NBS) wave function

$$\psi_{NBS}(\vec{r}) = \langle 0|N(\vec{r})N(\vec{0})|N(\vec{k})N(-\vec{k}), in\rangle$$
$$\simeq e^{i\delta_l(k)}\sin(kr - l\pi/2 + \delta_l(k))/(kr)$$

(at asymptotic region)

Local Approx.
Gradient Expansion

HAL QCD method

Nulcear Force

$$(k^2/m_N - H_0)\,\psi_{NBS}(\vec{r}) = \int d\vec{r}'\, U(\vec{r},\vec{r}')\psi_{NBS}(\vec{r}')$$

(Schrodinger eq.)

Physics-Driven Deep Learning

1. Extracting Nuclear Force

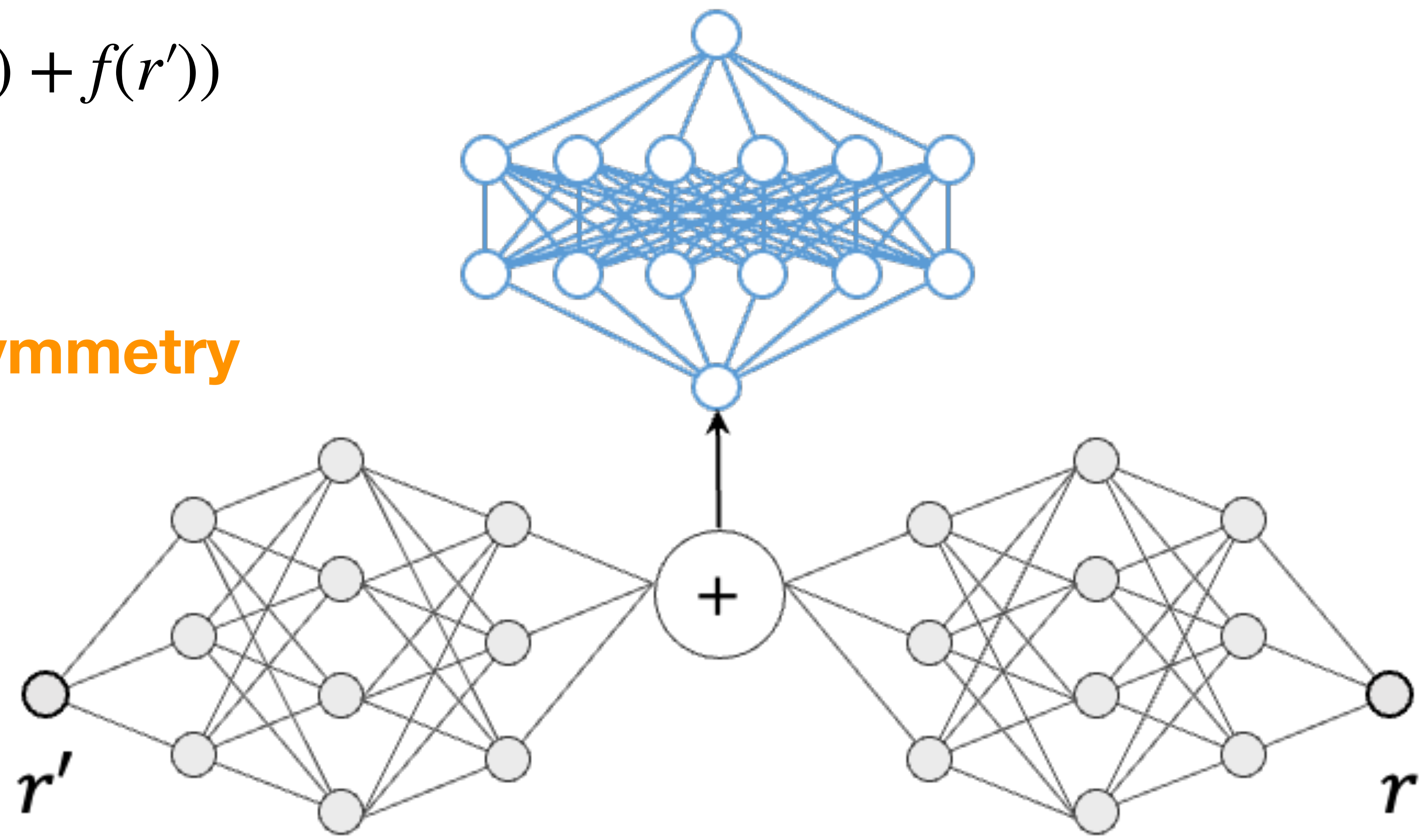
in preparation (with HAL QCD)

Two identical-particle interactions

$$U_\theta(r, r') \equiv g(f(r) + f(r'))$$

a. Exchange Symmetry

$U_\theta\,(r, r')$



BP

Residual of Schrödinger Eq.

$$(E_k - H_0)\phi_{\mathbf{k}}(\mathbf{r}) = \int d^3 r' U(\mathbf{r}, \mathbf{r}')\phi_{\mathbf{k}}(\mathbf{r}')$$

$$E_k = \frac{k^2}{2m}, \quad H_0 = -\frac{\nabla^2}{2m}, \quad m = \frac{m_N}{2}$$

$\phi_k\,(r)$

or

$R_t\,(r)$

Phys. Lett. B **712**, 437 (2012)

$$\left\{ \frac{1}{4m_N}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0 \right\} R(t, r) = \int 4\pi r'^2 dr'\, U(r, r')R(t, r')$$

b. Asymptotic Behaviour $\lim_{\mathbf{r}>R, \mathbf{r}'>R} U_\theta(\mathbf{r}, \mathbf{r}') \to 0$

Physics-Driven Deep Learning

1. Extracting Nuclear Force

in preparation (with HAL QCD)

Separable Potential

$$U(\mathbf{r},\mathbf{r}') \equiv \omega\nu(\mathbf{r})\nu(\mathbf{r}'), \qquad \nu(\mathbf{r}) \equiv e^{-\mu r}$$

$$\phi_k^0(r) = \frac{e^{i\delta_0(k)}}{kr}\left[\sin\{kr+\delta_0(k)\} - \sin\delta_0(k)e^{-\mu r}\left(1+\frac{r(\mu^2+k^2)}{2\mu}\right)\right]$$

$$k\cot\delta_0(k) = -\frac{1}{4\mu^2}\left[2\mu(\mu^2-k^2) - \frac{3\mu^2+k^2}{4\mu^3}(\mu^2+k^2)^2 + \frac{(\mu^2+k^2)^4}{8\pi m\omega}\right]$$

Nambu-Bethe-Salpeter (NBS) wave function

$$\phi_{\mathbf{k}}(\mathbf{r})e^{-W_{\mathbf{k}}t} \equiv \langle 0 | N(\mathbf{x}+\mathbf{r},t)N(\mathbf{x},t) | NN, W_k\rangle$$

$$(E_k - H_0)\phi_{\mathbf{k}}(\mathbf{r}) = \int d^3r' U(\mathbf{r},\mathbf{r}')\phi_{\mathbf{k}}(\mathbf{r}')$$

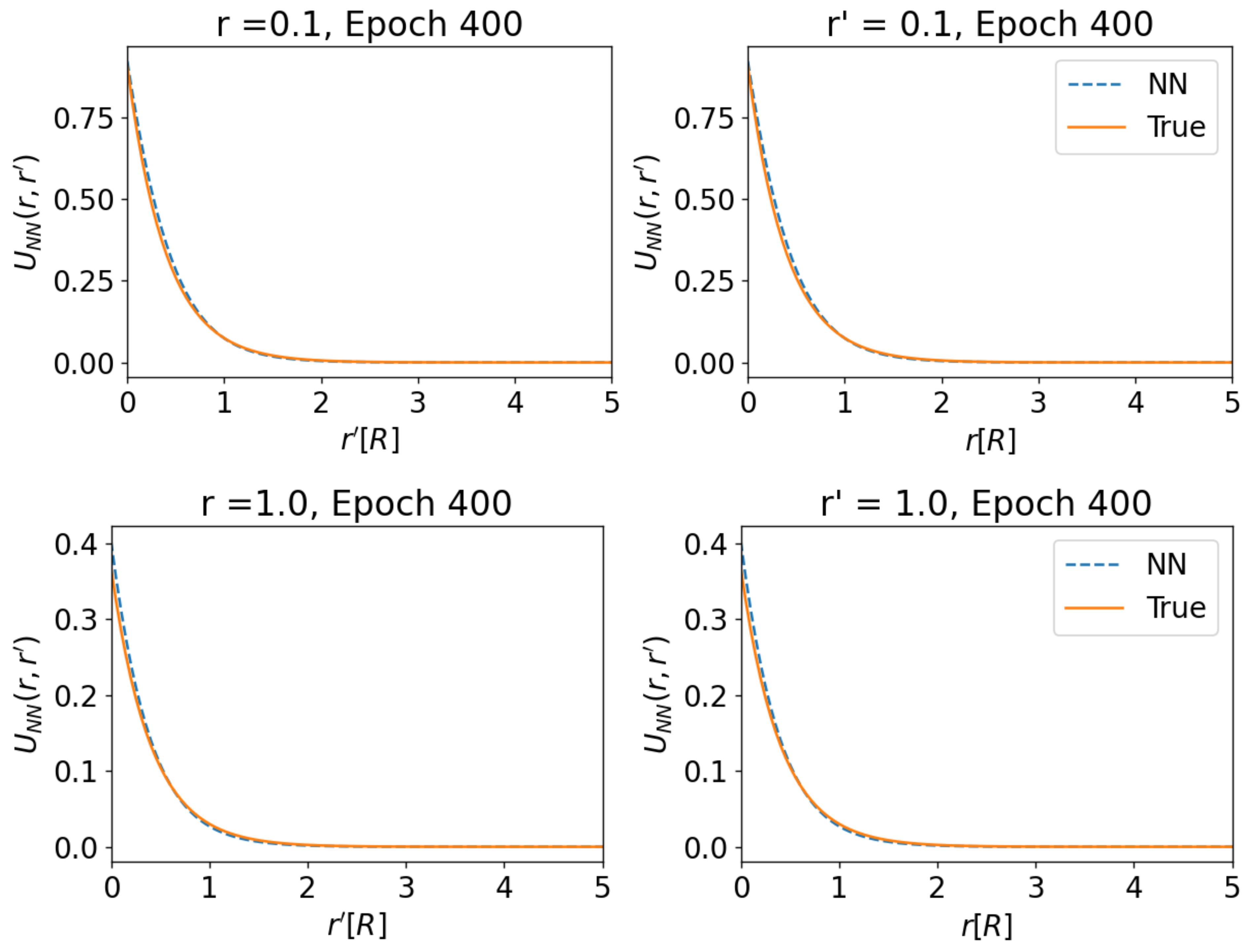
$$E_k = \frac{k^2}{2m}, \quad H_0 = -\frac{\nabla^2}{2m}, \quad m = \frac{m_N}{2}$$

No Approx. →

$$\mathscr{L} = \sum_k \int d^3r \left[\left(E_k - H_0\right)\phi_{\mathbf{k}}(\mathbf{r}) - \int d^3r' U_\theta(\mathbf{r},\mathbf{r}')\phi_{\mathbf{k}}(\mathbf{r}')\right]^2$$

Neural Network Hadron Force

$$U_{\mathbf{NN}}(r,r') = \omega f_\theta(r,r')$$



Physics-Driven Deep Learning

1. Extracting Nuclear Force

in preparation (with HAL QCD)

$\Omega_{ccc}\Omega_{ccc}$ Interaction

$$\left\{\frac{1}{4m_N}\frac{\partial^2}{\partial t^2}-\frac{\partial}{\partial t}-H_0\right\}R(t,r)=\int 4\pi r'^2 dr'\, U(r,r')R(t,r')$$

Nambu-Bethe-Salpeter (NBS) wave function

$$R2 = R_{t+1} - 2R_t + R_{t-1}, R1 = (R_{t+1} - R_{t-1})/2, Rr = \nabla^2 R(t,r)$$

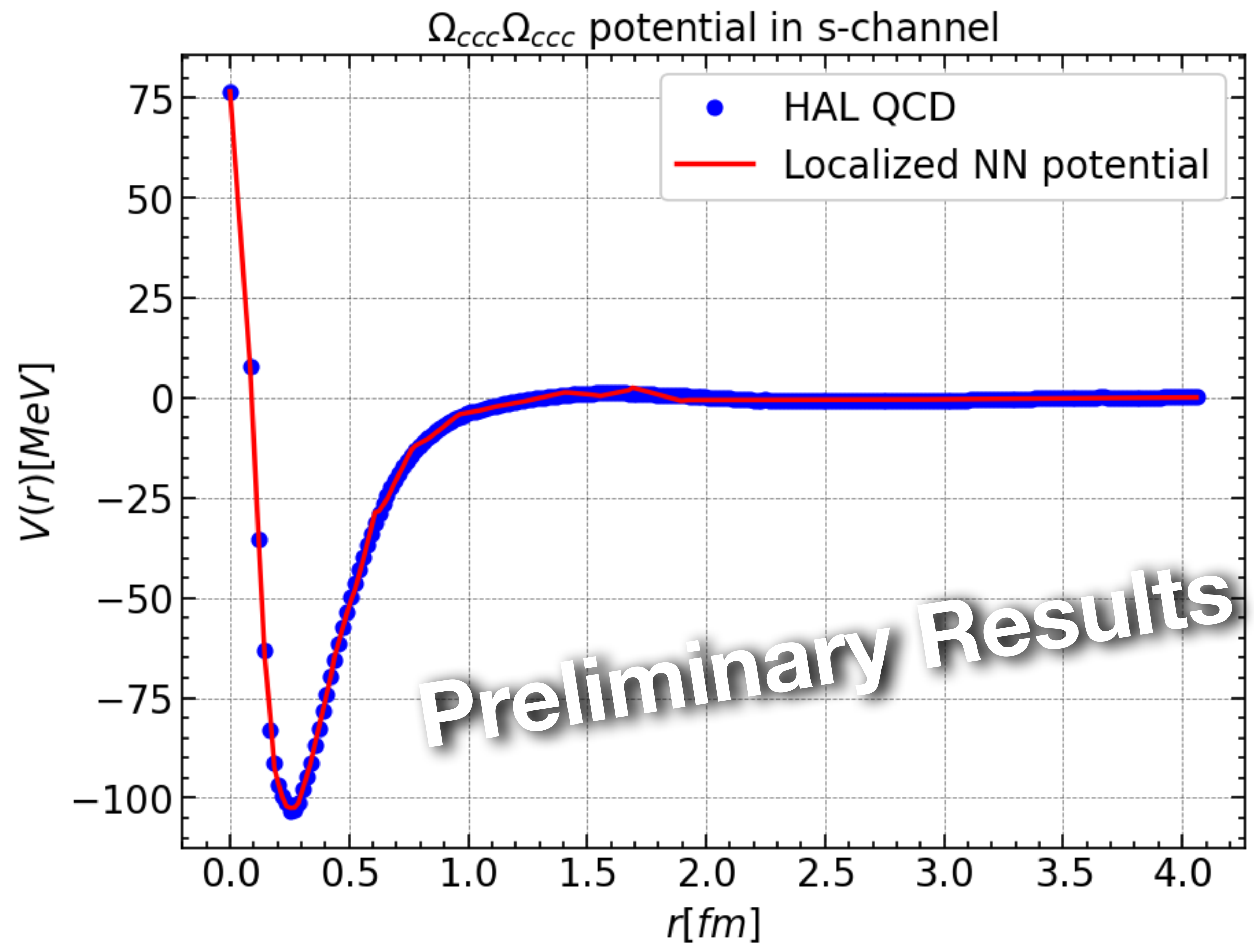
$$m_N = 2.073, a^{-1} = 2333.0\,\textbf{MeV}$$

$$t = 26$$

Phys. Rev. Lett. 127, 072003 (2021)

No Approx.

Neural Network Hadron Force



$$\mathcal{L} = \sum_t \left\{\frac{1}{4m_N}R2(t,r) - R1(t,r) + \frac{1}{m_N}Rr(t,r) - \int 4\pi r'^2 dr'\, U_\theta(r,r')R(t,r')\right\}$$

$$V_\theta(r) \equiv \frac{\sum_{r'} \Delta r'\, U_\theta(r,r')R(t,r')}{R(t,r)}$$

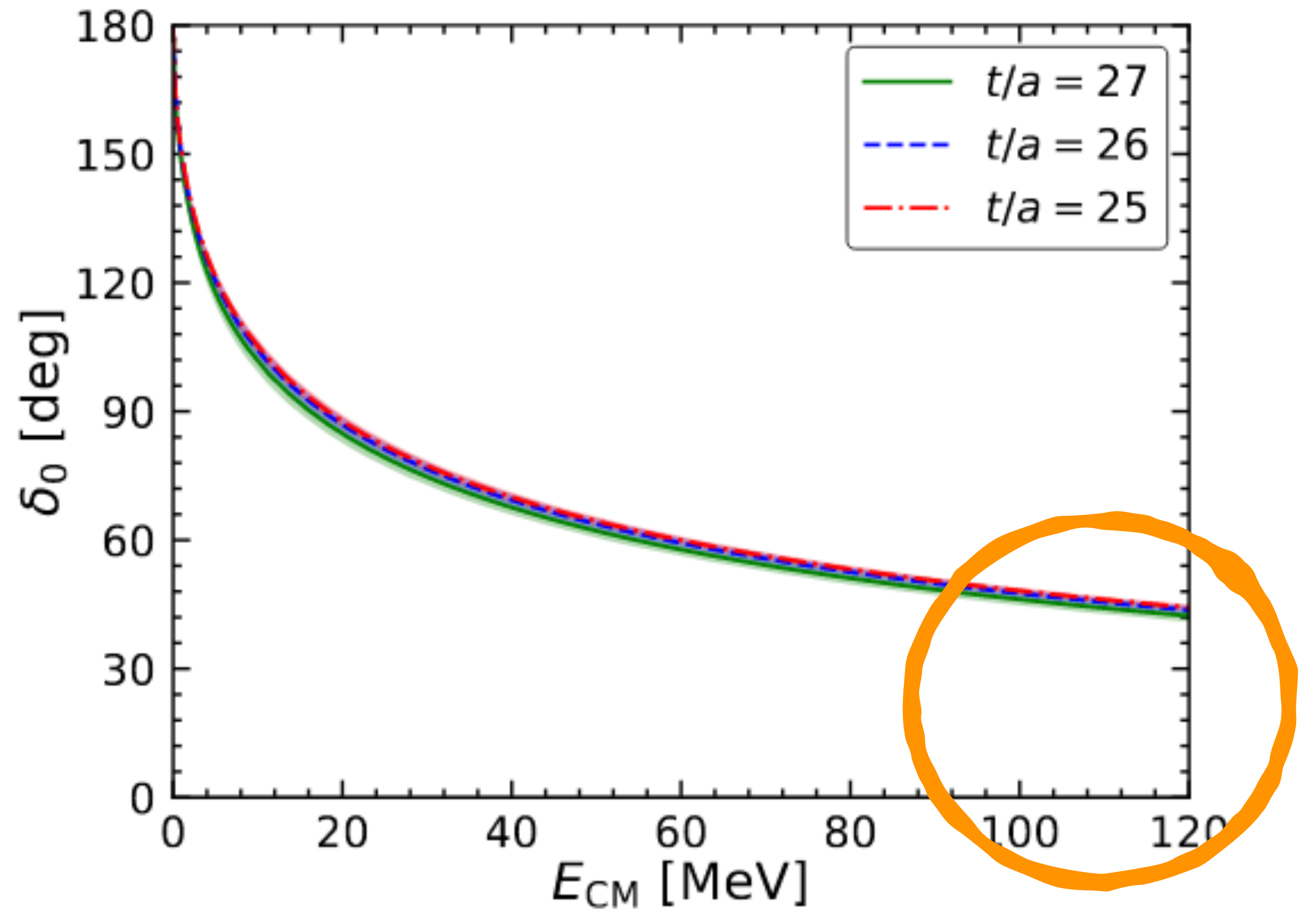
Physics-Driven Deep Learning

1. Extracting Nuclear Force

in preparation (with HAL QCD)

$\Omega_{ccc}\Omega_{ccc}$ **Interaction**

L. Meng & Epelbaum (2023)
K.Murakami+(2024)



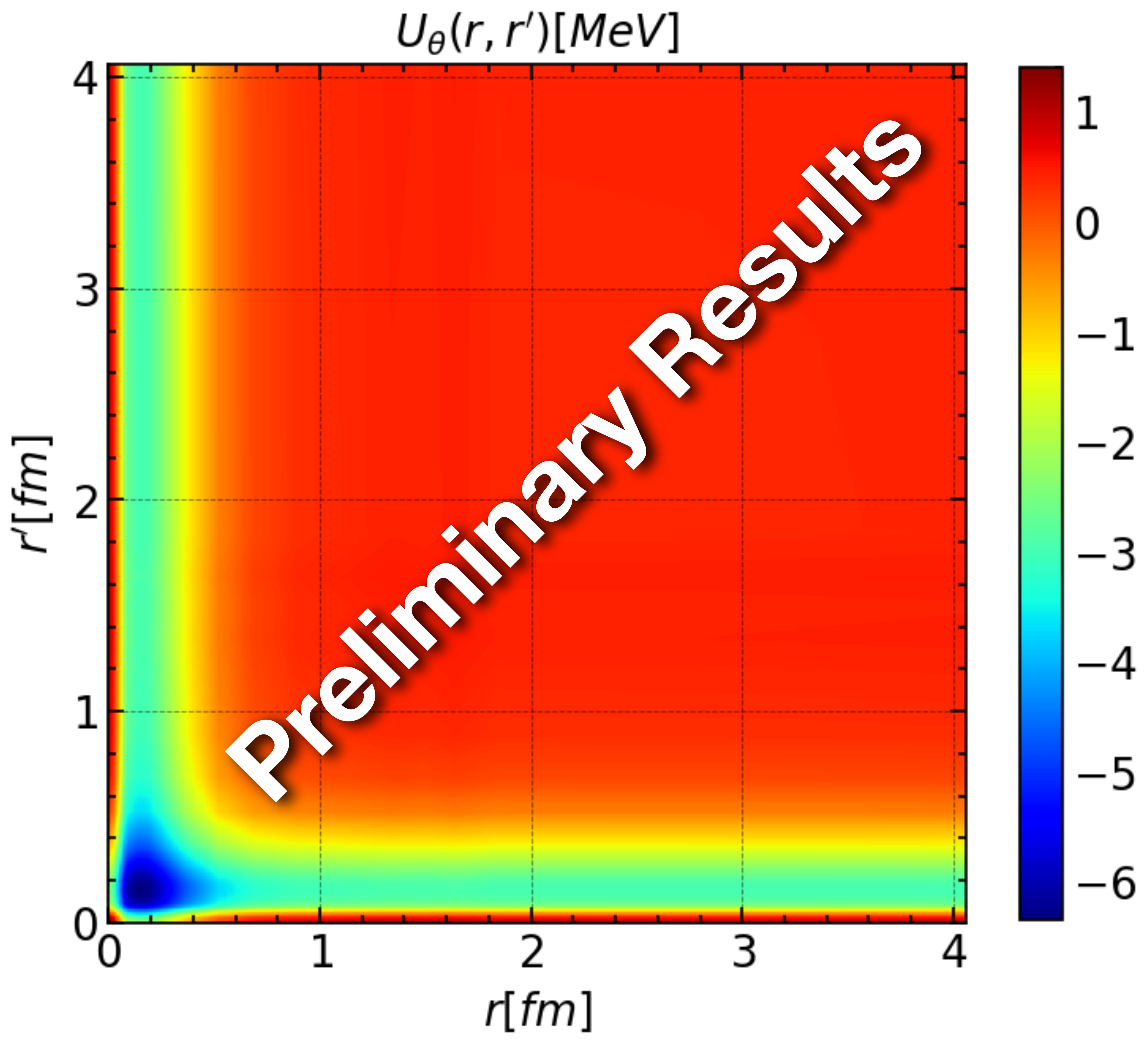
Y.Lyu+(2021)

$$(E_k - H_0)\phi_{\mathbf{k}}(\mathbf{r}) = \int d^3 r' U_\theta(\mathbf{r}, \mathbf{r}')\phi_{\mathbf{k}}(\mathbf{r}')$$

$$E_k = \frac{k^2}{2m}, \quad H_0 = -\frac{\nabla^2}{2m}, \quad m = \frac{m_N}{2}$$

Phase Shifts

Neural Network Hadron Force



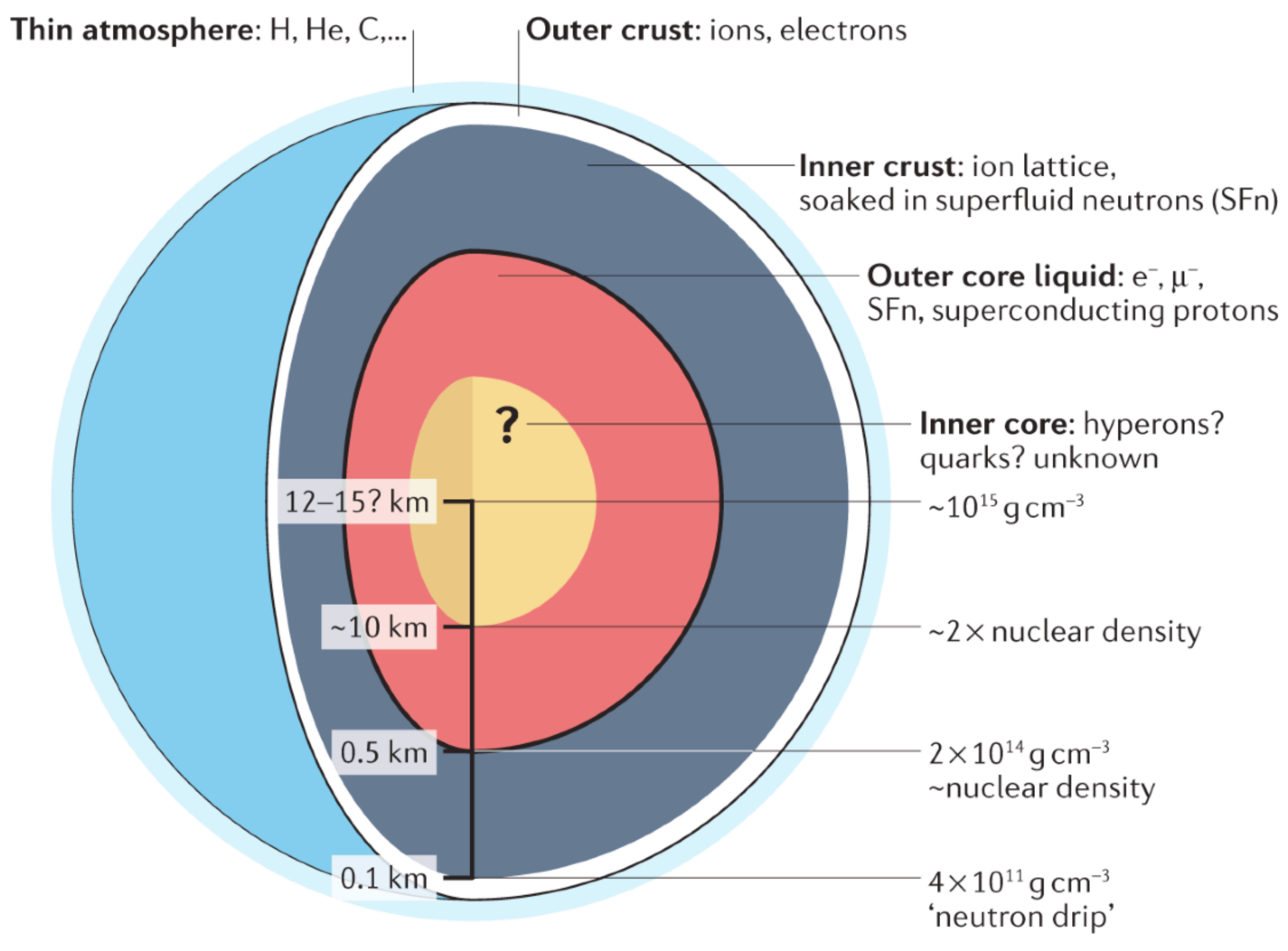
2. Building Neutron Star EoS

Tolman–Oppenheimer–Volkoff equations

$$\frac{dP}{dr} = -G\frac{m(r)\varepsilon(r)}{r^2}\left(1+\frac{P(r)}{\varepsilon(r)}\right)\left(1+\frac{4\pi r^3 P(r)}{m(r)}\right)\left(1-\frac{2Gm(r)}{r}\right)^{-1}$$

$$\frac{dm(r)}{dr} = 4\pi r^2 \epsilon(r)$$

Hydrostatic condition in each shell (dr)



Nat. Rev. Phys. 4, 237–246 (2022)

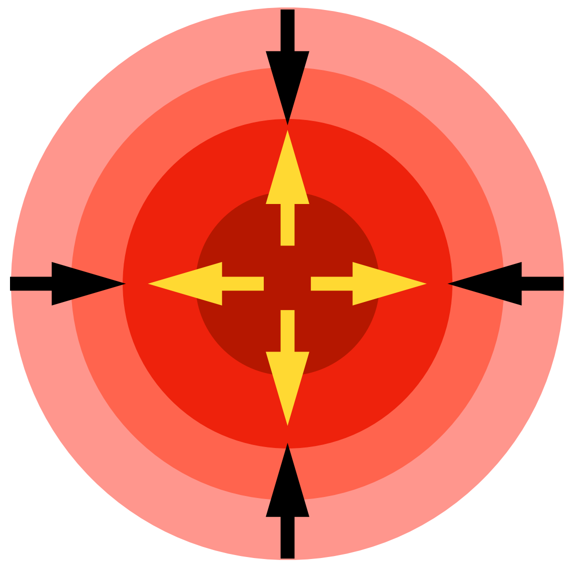
EoS $P(\varepsilon) = 0$

Core → Surface

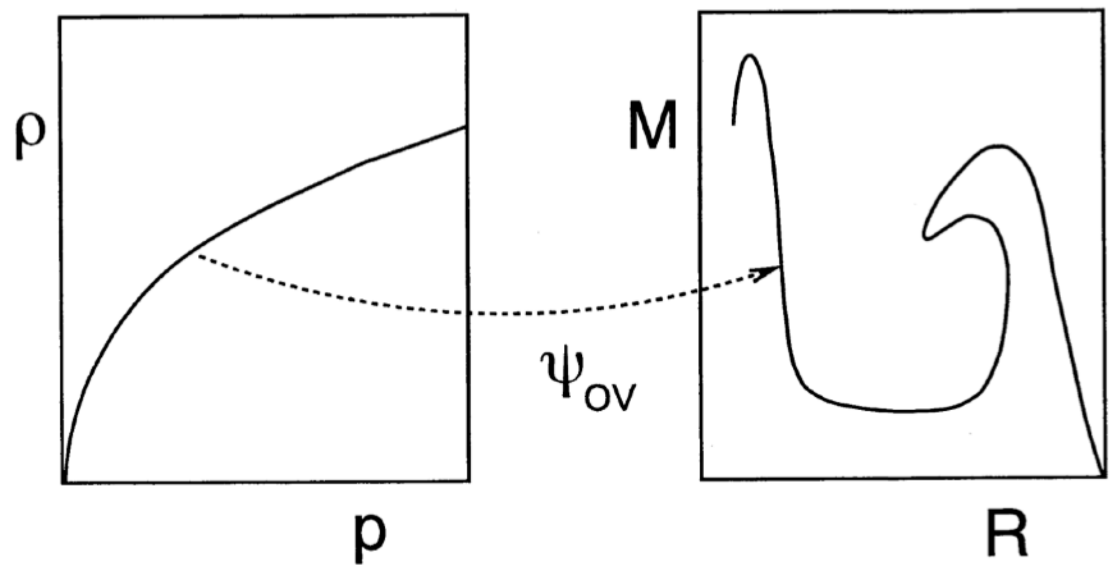
$r = 0, \varepsilon(r=0) = \varepsilon_c, P(r=0) = P(\varepsilon_c)$

$r = R, \varepsilon(r=R) \simeq 0, M = \int 4\pi r^2 \varepsilon(r) dr$

M, R



Pressure → ← Gravity

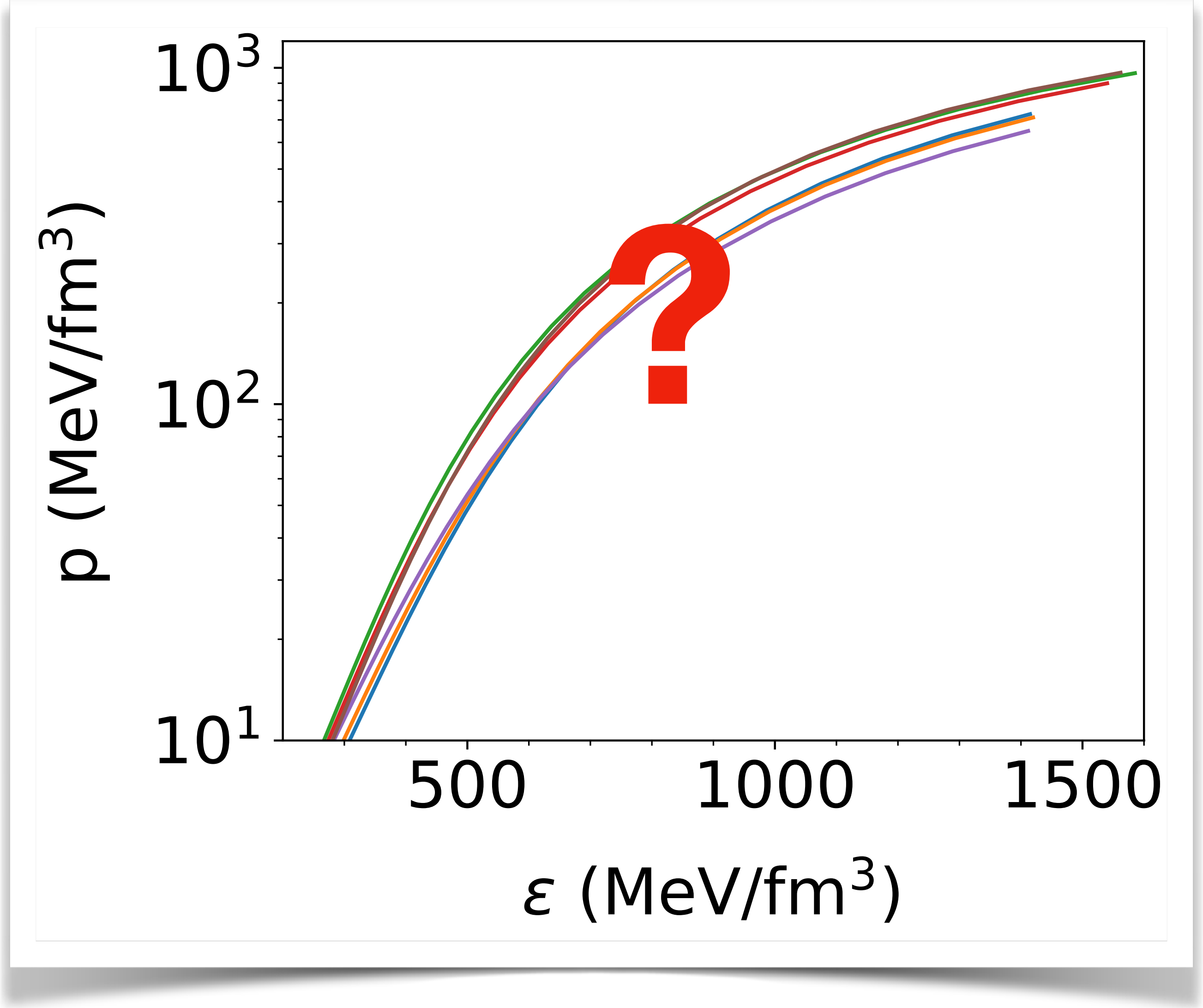


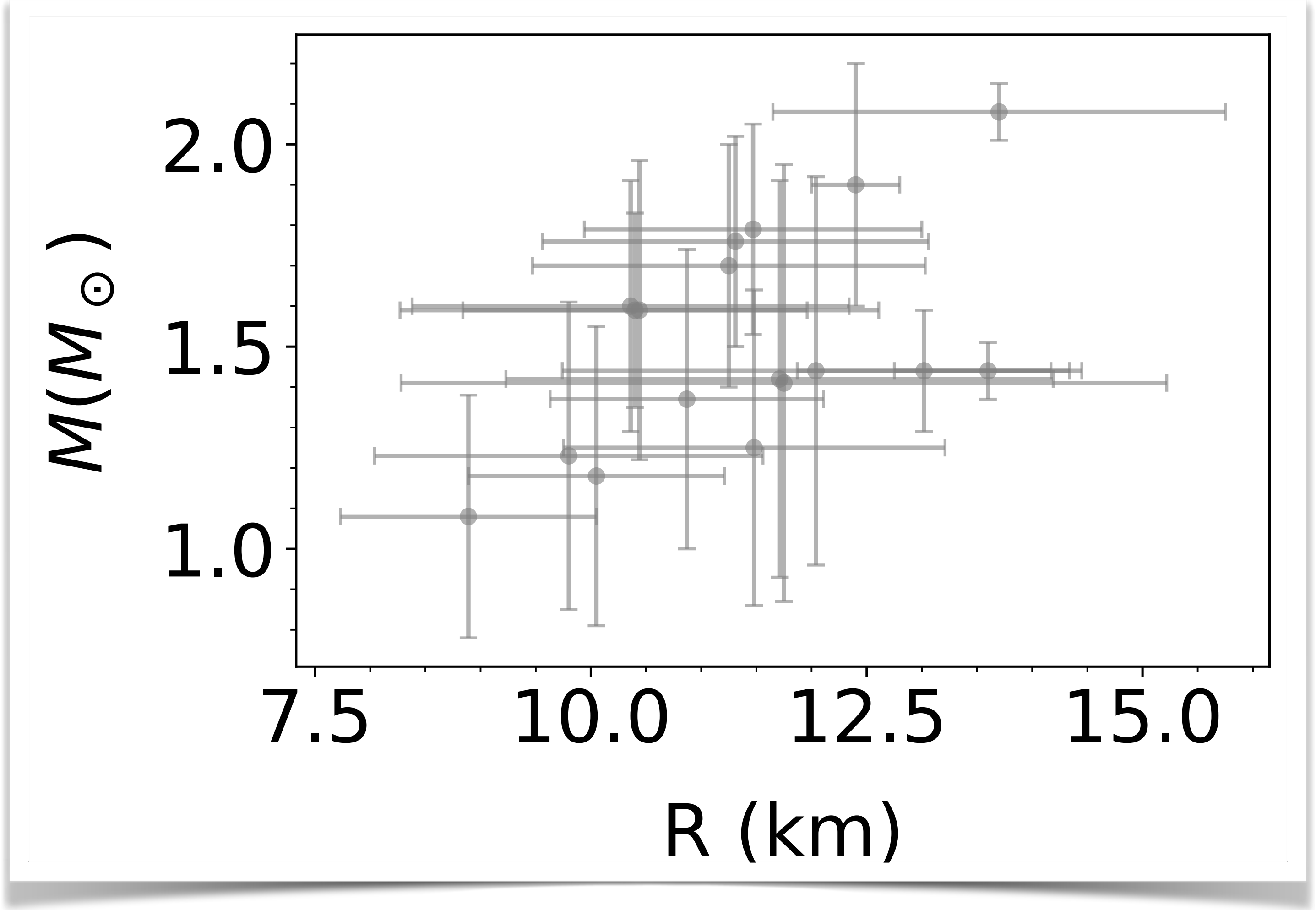
L. Lindblom, *A.J., 398, 569 (1992).*
If the **whole M(R)** is known,it's well-defined problem.

Physics-Driven Deep Learning

2. Building Neutron Star EoS

Tolman–Oppenheimer–Volkoff equations





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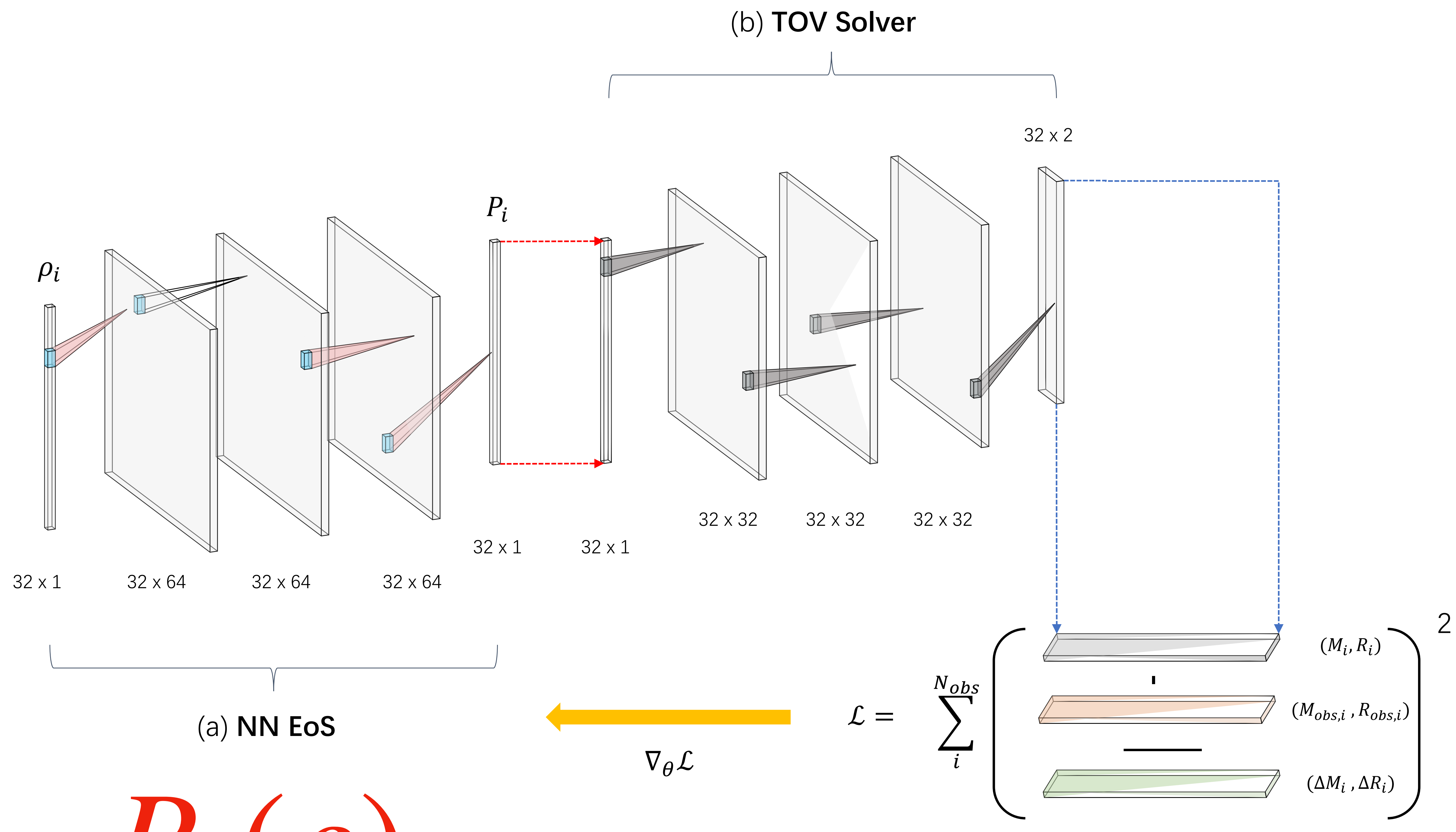
2. Building Neutron Star EoS

Phys. Rev. D 107, 083028; JCAP08 (2022) 071



$P_\theta(\rho)$

$$\mathscr{L} = \chi^2 = \sum_{i=1}^{N_{\text{obs}}} \frac{\left(M_i - M_{\text{obs},i}\right)^2}{\Delta M_i^2} + \frac{\left(R_i - R_{\text{obs},i}\right)^2}{\Delta R_i^2}$$

Loss function

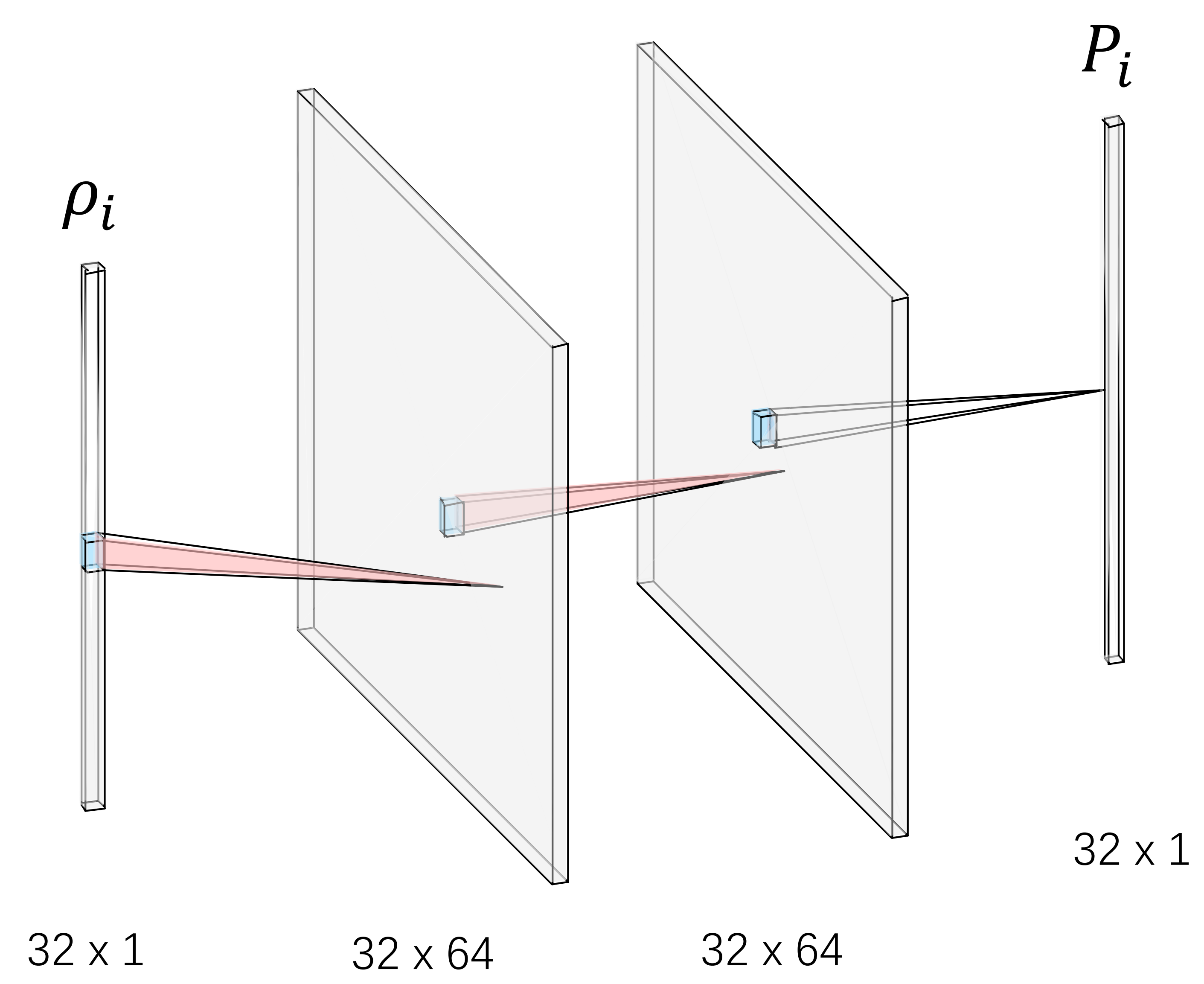
- (M_i, R_i) : predicted Mass-Radius
- $(M_{obs,i}, R_{obs,i})$: observed Mass-Radius
- $(\Delta M_i, \Delta R_i)$: uncertainties

Physics-Driven Deep Learning

2. Building Neutron Star EoS

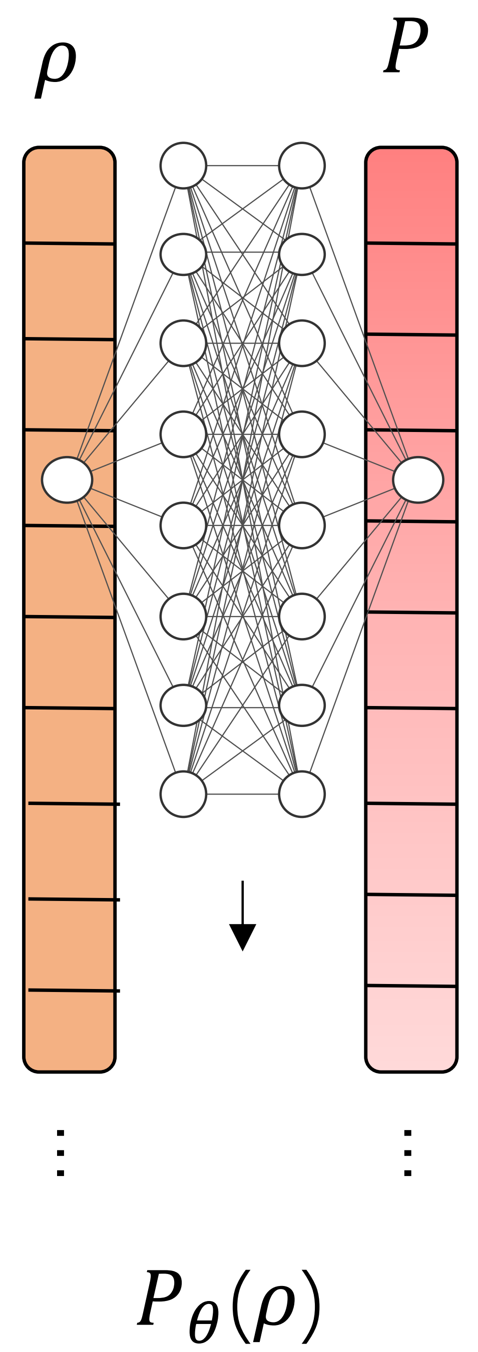
Phys. Rev. D 107, 083028; JCAP08 (2022) 071

Module A. NN EoS



A Trainable Neural Network

$\simeq$



$\{\theta\}$: weights and bias of the neural network
Size of $\{\theta\}$ = 4353

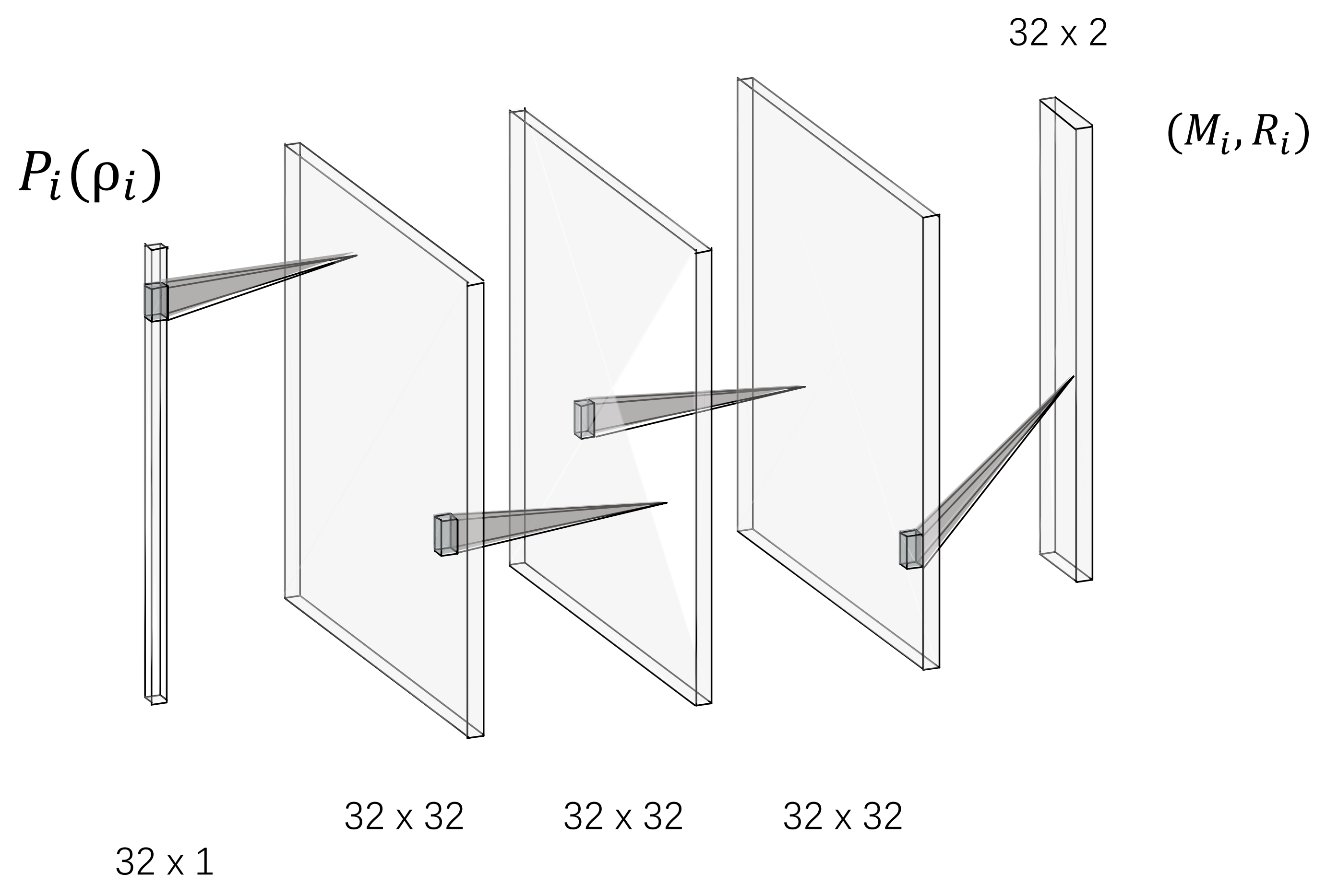
NS crust: **DD2**, inner: $P_\theta(1.1\rho_{\rm sat} \leq \rho)$

Physics-Driven Deep Learning

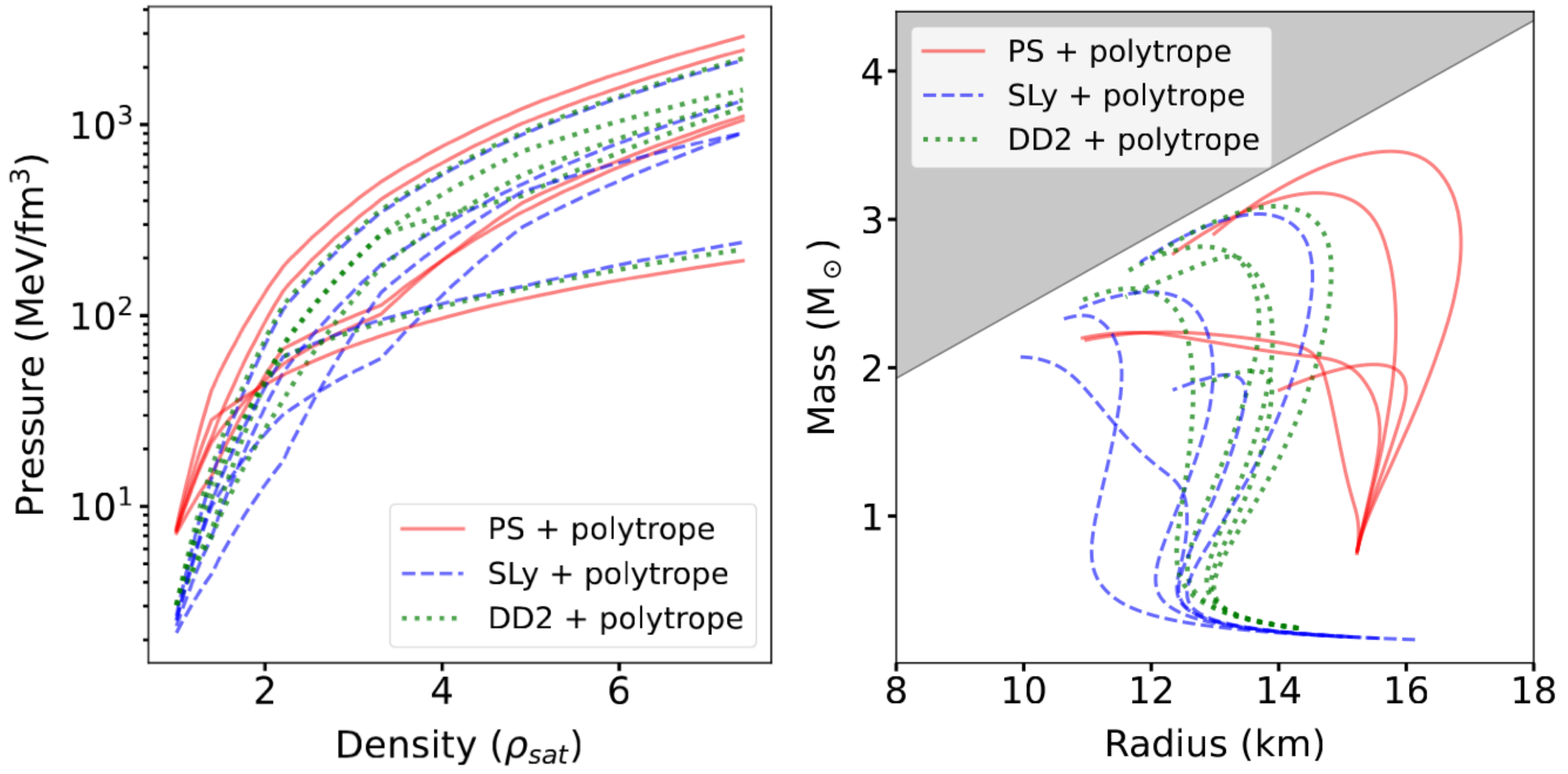
2. Building Neutron Star EoS

Phys. Rev. D 107, 083028; *JCAP08 (2022) 071*

Module B. TOV eq. Solver



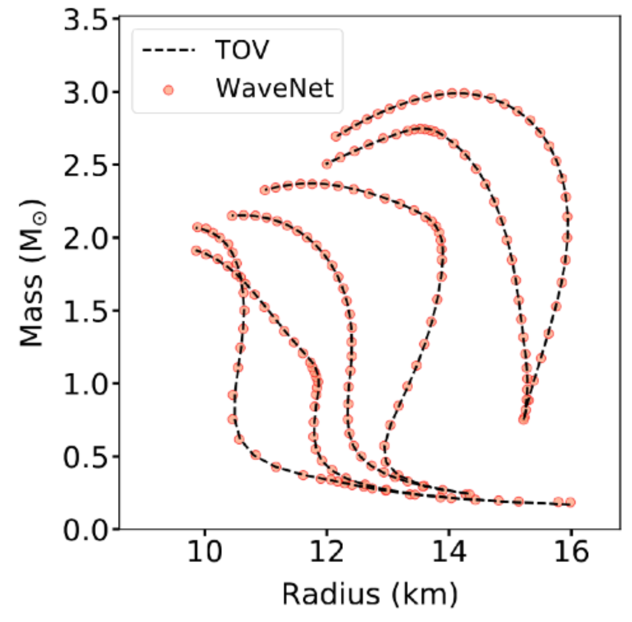
A **Pre-Trained** Neural Network



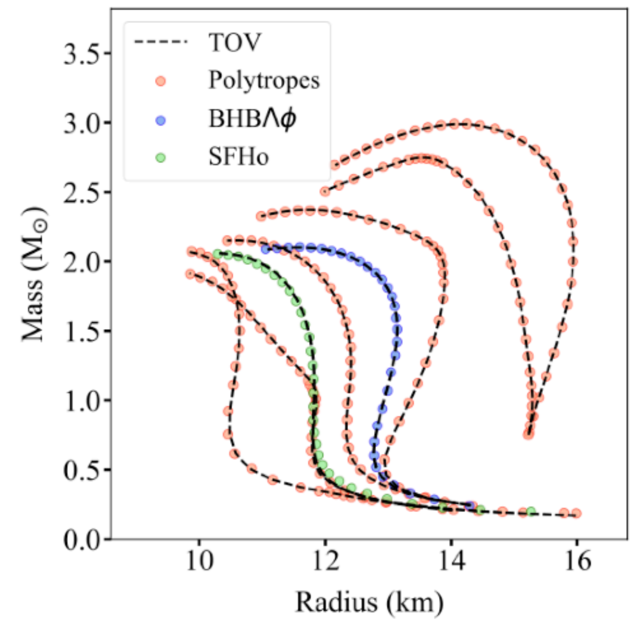
Training data-set
300,000 polytropic EoS functions with 3 low density models

$$P = K_i \rho^{\Gamma_i}, \quad i = [1,5], \quad 1.1\rho_{sat} \le \rho \le 7.4\rho_{sat}$$

In-set testing data-set



Out-set testing data-set

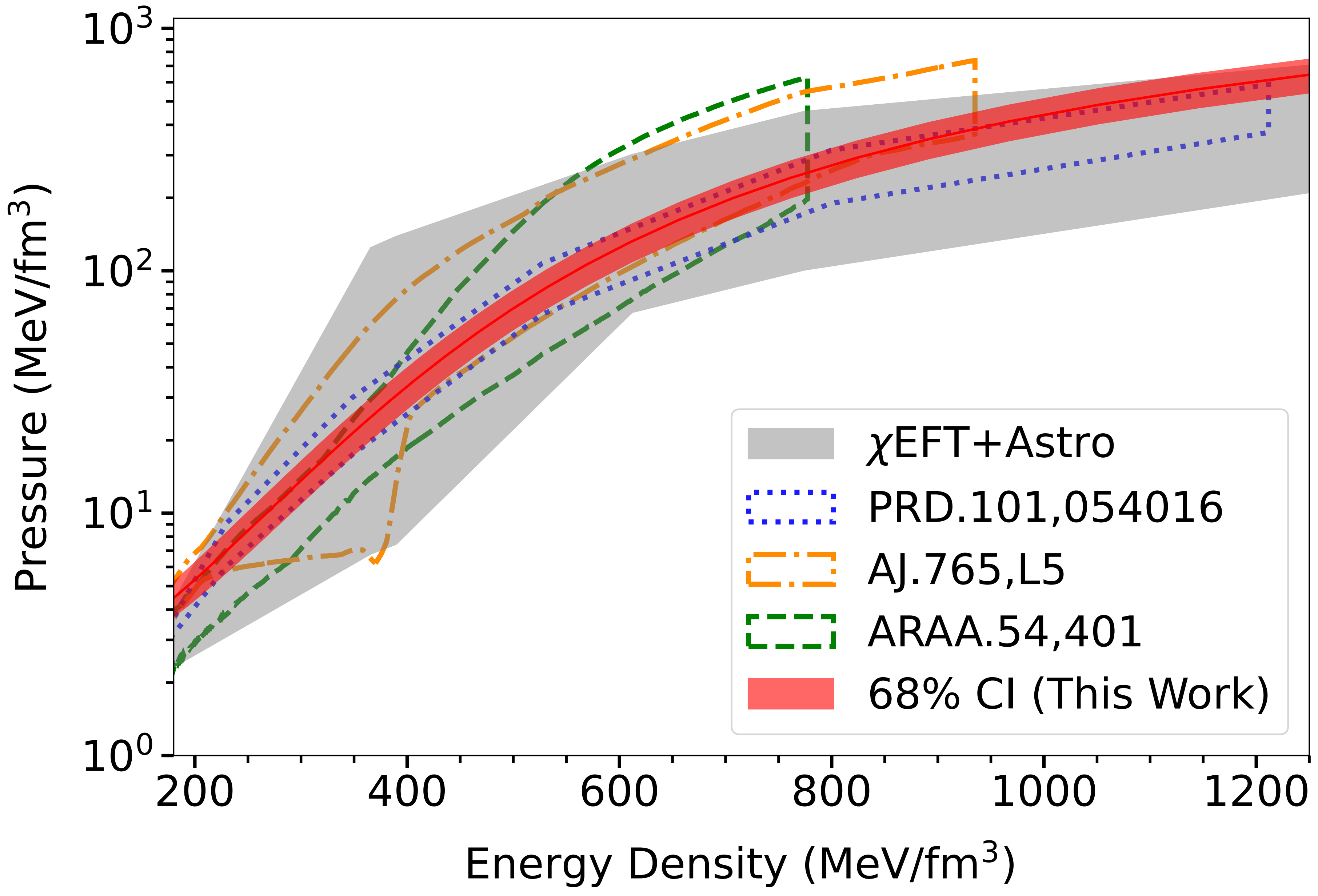


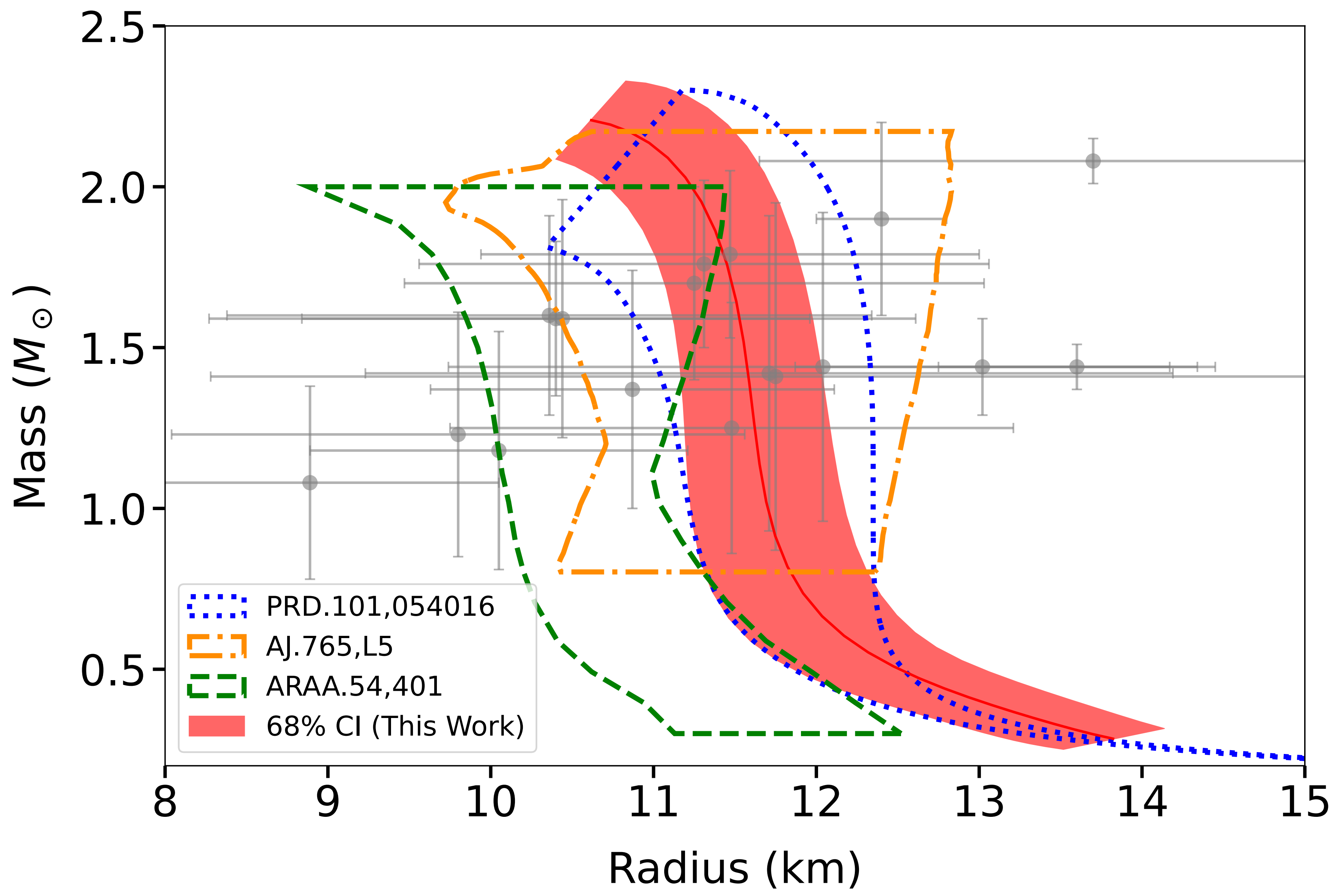
Physics-Driven Deep Learning

2. Building Neutron Star EoS

Phys. Rev. D 107, 083028

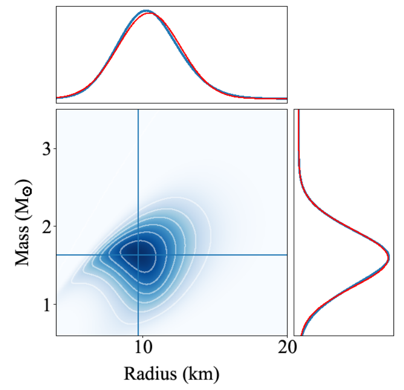
Blue dots: NN results, Fujimoto-Fukushima-Murase
Yellow and Green dashed lines: Bayesian Approaches





Our results: $R_{1.4} = 11.6 \pm 0.43$ **km (68% CI)**

18 (M_i, R_i) , sample size = **10k**
causality $(d\epsilon/dp < 1)$
Maximum mass $\geq 1.9\, M_\odot$

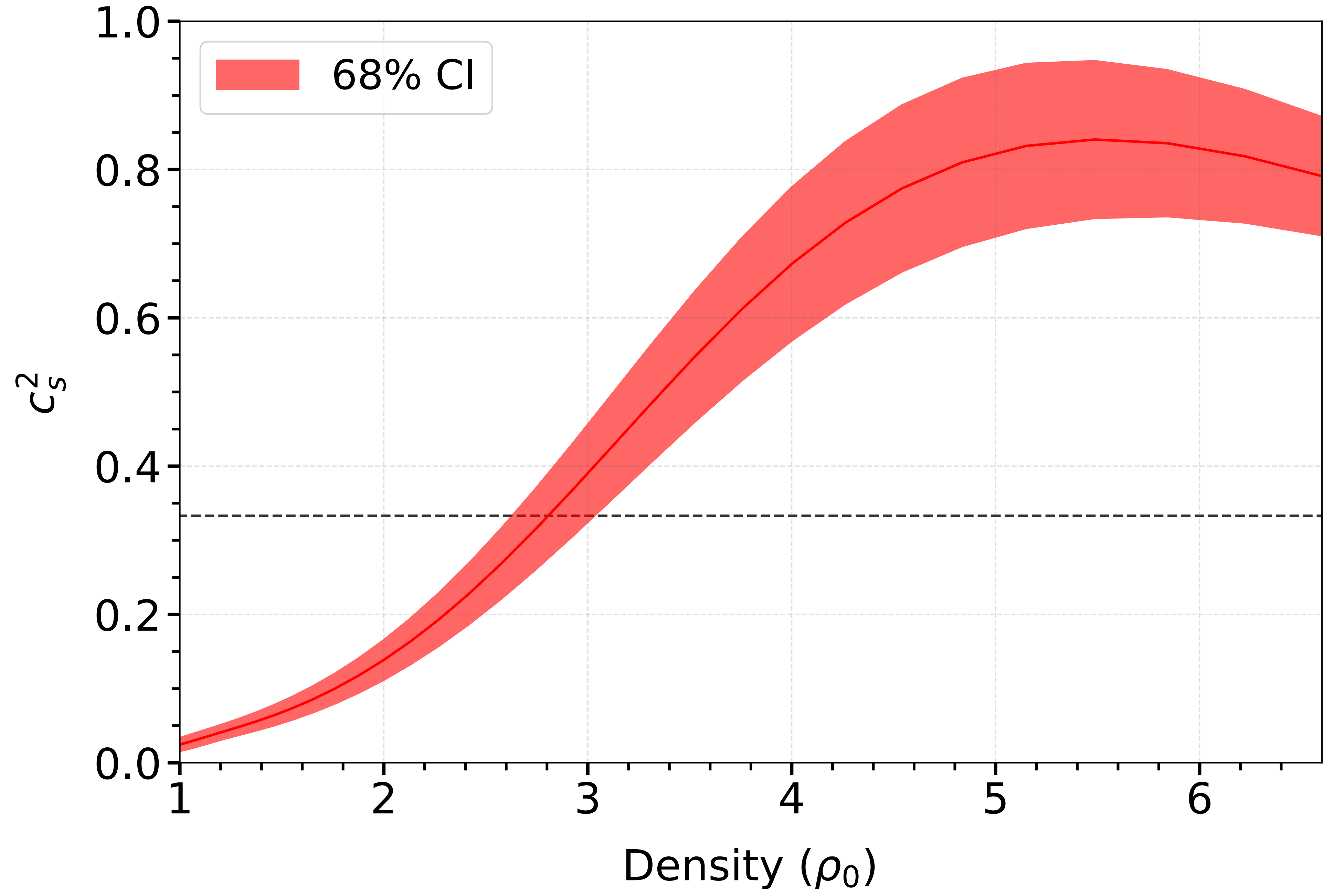


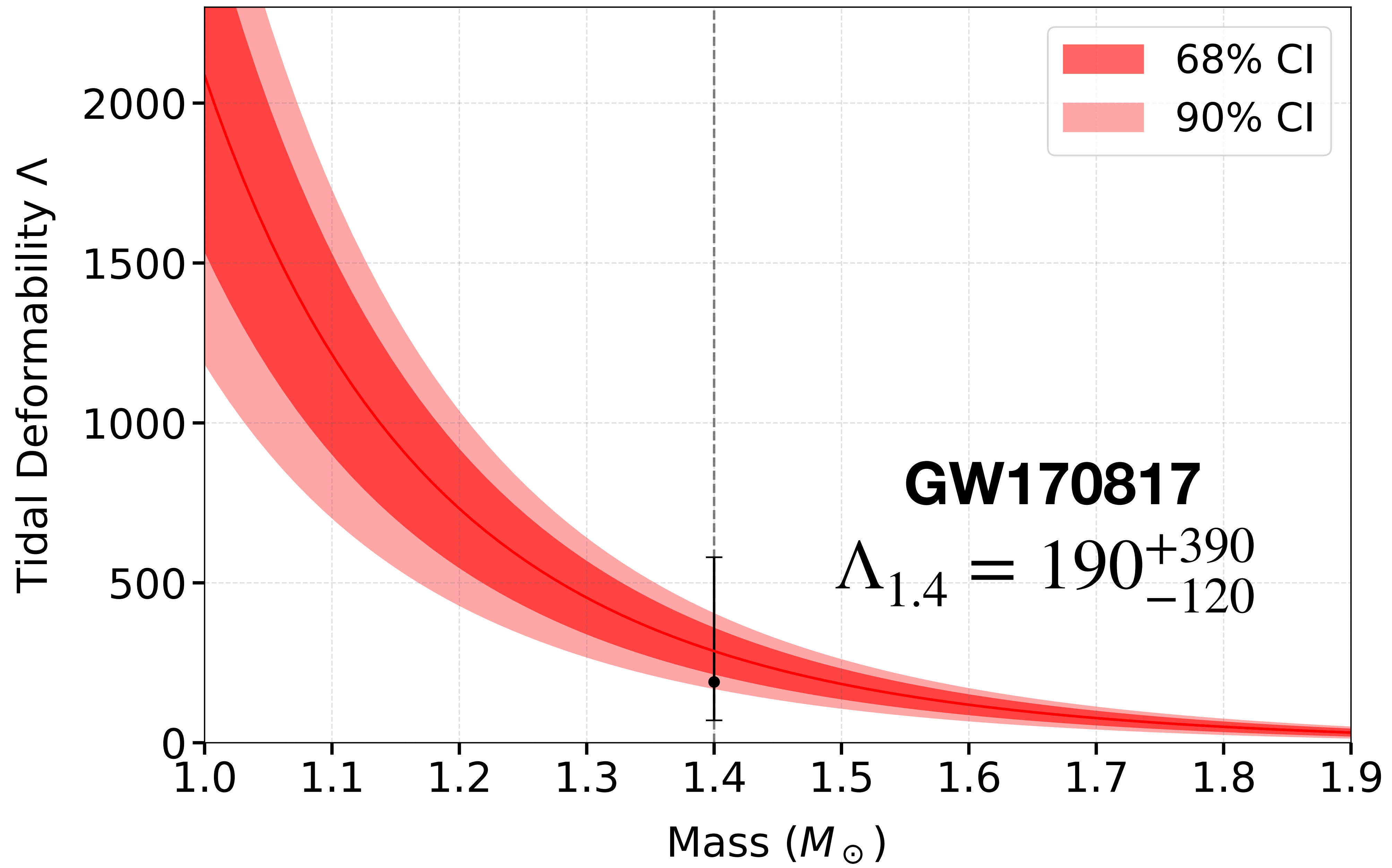
Observable	Mass($M_\odot$)	Radius(km)
M13	1.42±0.49	11.71±2.48
M28	1.08±0.30	8.89±1.16
M30	1.44±0.48	12.04±2.30
NGC 6304	1.41±0.54	11.75±3.47
NGC 6397	1.25±0.39	11.48±1.73
ωCen	1.23±0.38	9.80±1.76
4U 1608-52	1.60±0.31	10.36±1.98
4U 1724-207	1.79±0.26	11.47±1.53
4U 1820-30	1.76±0.26	11.31±1.75
EXO 1745-248	1.59±0.24	10.40±1.56
KS 1731-260	1.59±0.37	10.44±2.17
SAX J1748.9-2021	1.70±0.30	11.25±1.78
X5	1.18±0.37	10.05±1.16
X7	1.37±0.37	10.87±1.24
4U 1702-429	1.90±0.30	12.40±0.40
PSR J0437–4715	1.44±0.07	13.60±0.85
PSR J0030+0451	1.44±0.15	13.02±1.15
PSR J0740+6620	2.08±0.07	13.70±2.05

Physics-Driven Deep Learning

2. Building Neutron Star EoS

Phys. Rev. D 107, 083028





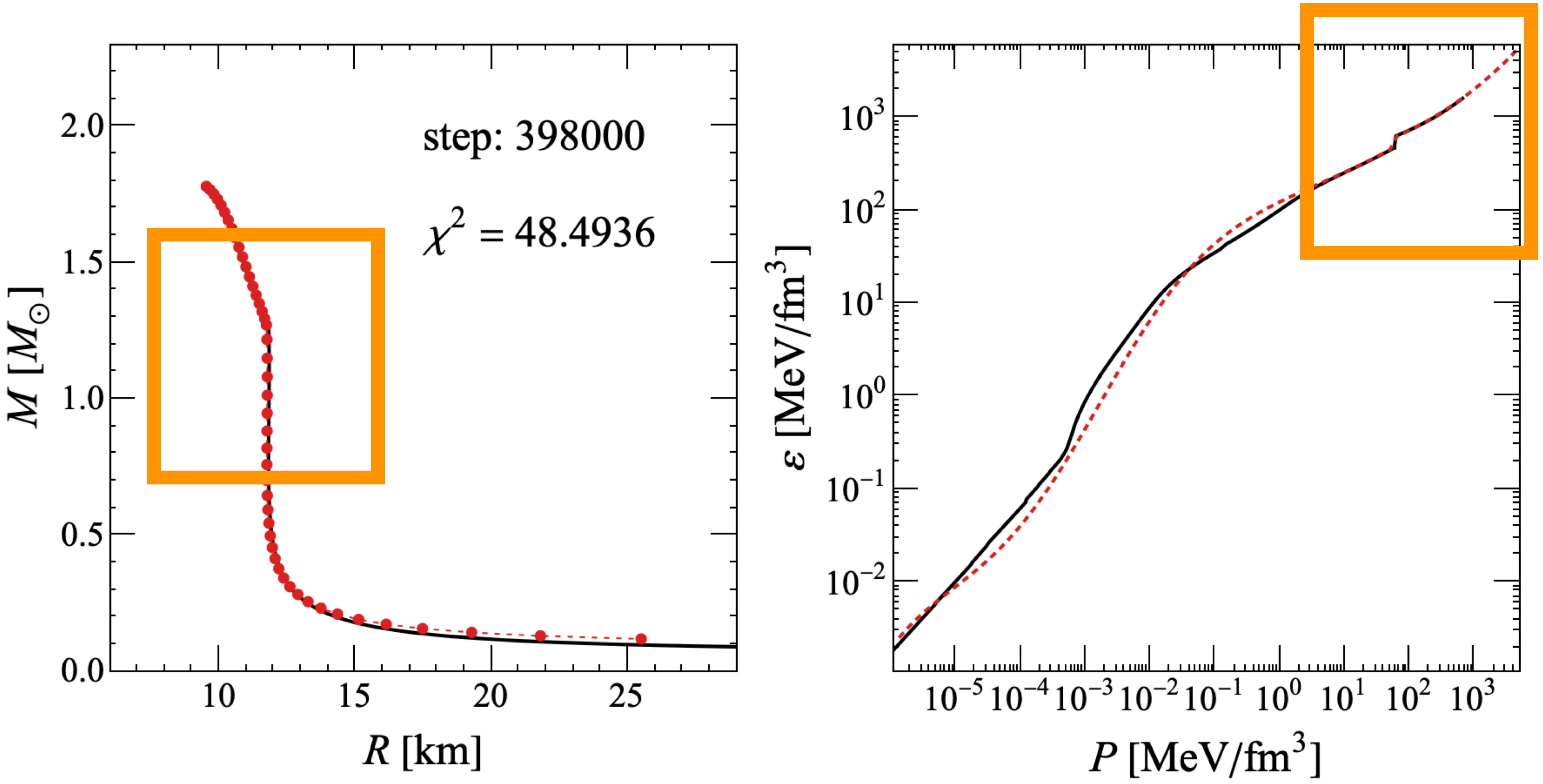
Our results, $\tilde{\Lambda}_{1.4} = 286.47^{+115.9}_{-115.9}$

Physics-Driven Deep Learning

2. Building Neutron Star EoS

in Preparation

with Shuzhe Shi, Zidu Lin, etc.



Able to capture first-order phase transition !

48 Neutron Stars

(24 in $M > M_\odot$)

Represent Speed of Sound

1.Microscopicly stable condition, $\frac{dp}{d\epsilon} \geq 0$

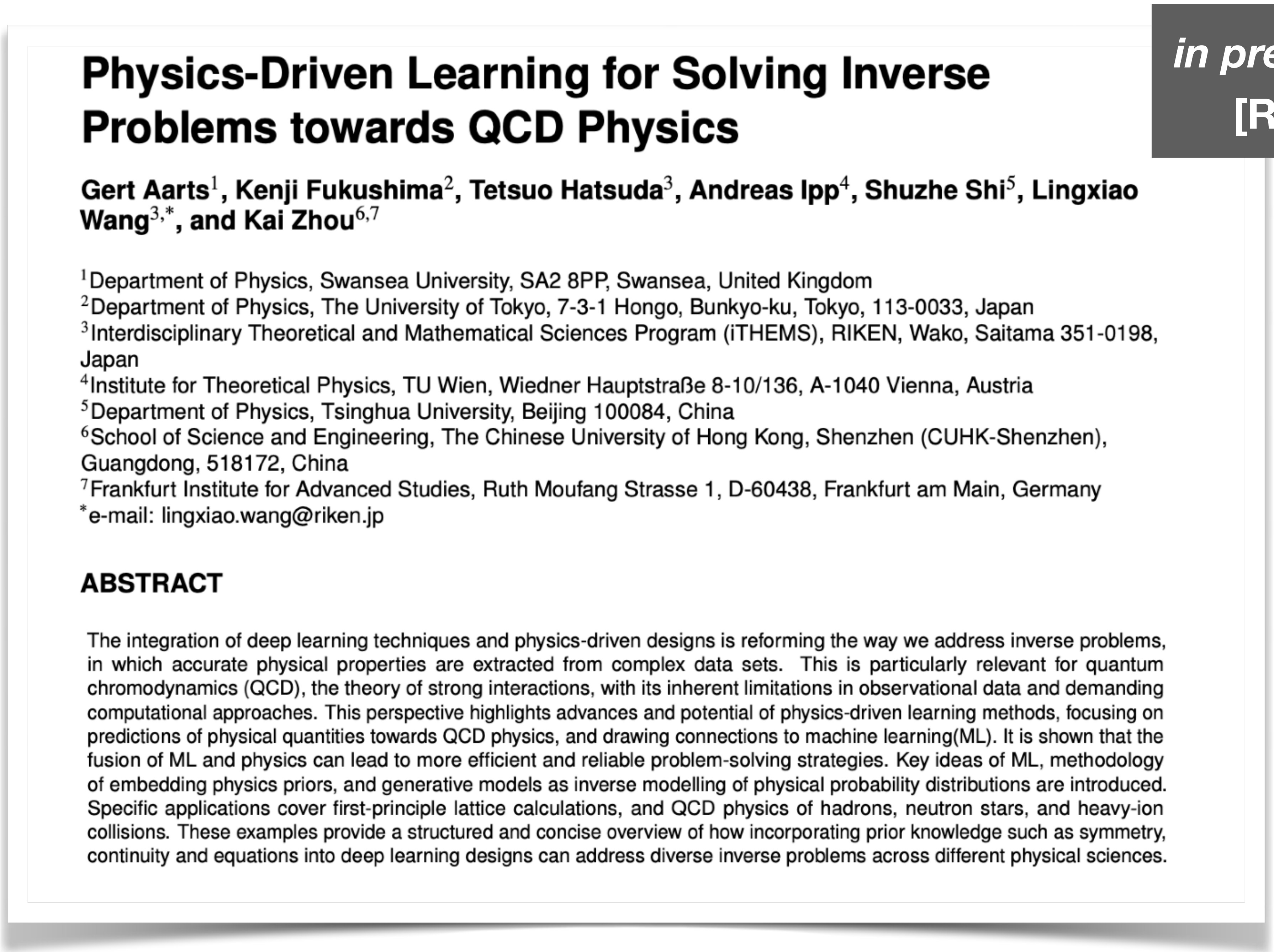
2.Causality condition, $\frac{dp}{d\epsilon} = \frac{c_s^2}{c^2} < 1$

Representing $c_s(\varepsilon) = \sigma(y(\varepsilon))$ with neural netowrks$y(\varepsilon)$, above two conditions can be naturally met with the ***sigmoid*** activation function, $\sigma(x) = 1/(1 + e^{-x})$.

Summary

- **Inverse Problems**
  - Data-driven learning **Supervised**
  - Physics-driven learning **Unsupervised!**
    - Neural network representations
    - Embed physics priors explicitly
      - **Exchange Symmetry** and **Asymptotic Behavior** for HH interactions
      - **Continuity** and **Causality** for EoSs
- **Future works**
  - PTs in Nuclear Matter EoS
  - More Hadron Forces

in preparation
[Review]



Physics-Driven Learning for Solving Inverse Problems towards QCD Physics

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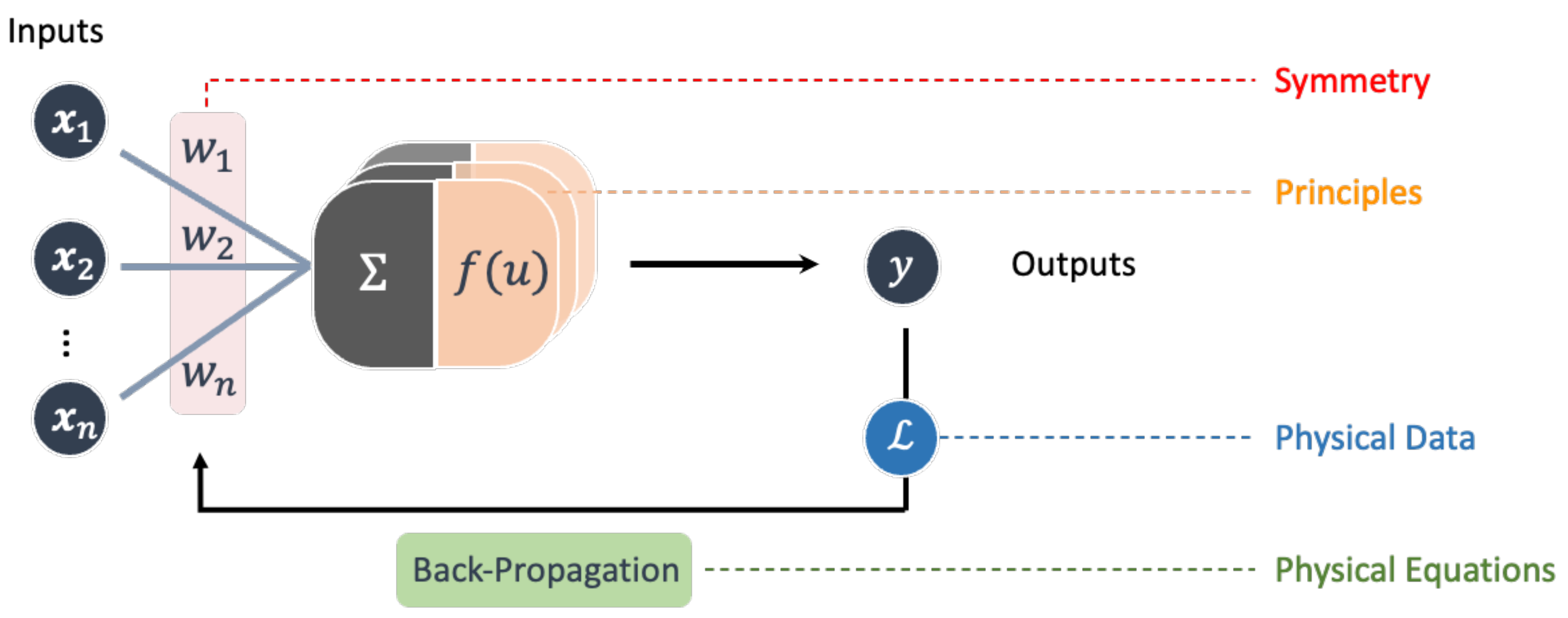
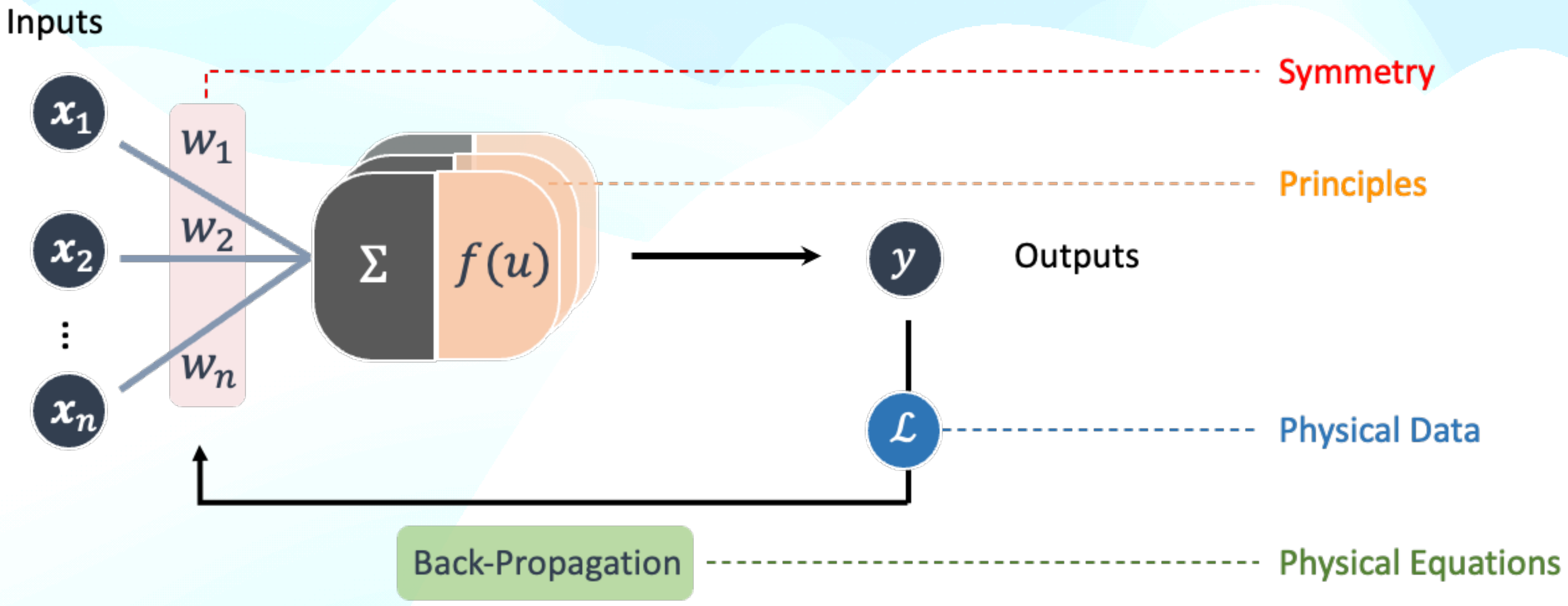
ABSTRACT

The integration of deep learning techniques and physics-driven designs is reforming the way we address inverse problems, in which accurate physical properties are extracted from complex data sets. This is particularly relevant for quantum chromodynamics (QCD), the theory of strong interactions, with its inherent limitations in observational data and demanding computational approaches. This perspective highlights advances and potential of physics-driven learning methods, focusing on predictions of physical quantities towards QCD physics, and drawing connections to machine learning(ML). It is shown that the fusion of ML and physics can lead to more efficient and reliable problem-solving strategies. Key ideas of ML, methodology of embedding physics priors, and generative models as inverse modelling of physical probability distributions are introduced. Specific applications cover first-principle lattice calculations, and QCD physics of hadrons, neutron stars, and heavy-ion collisions. These examples provide a structured and concise overview of how incorporating prior knowledge such as symmetry, continuity and equations into deep learning designs can address diverse inverse problems across different physical sciences.



Thank you!



Physics-Driven Deep Learning

Backups

2. Building Neutron Star EoS

Phys. Rev. D 107, 083028

Uncertainty estimations

- x : reconstructed EoSs given a sample
- $O(x)$: observables, M, R, P

Variance $\sigma(O)^2 = \langle \hat{O}^2 \rangle - \bar{O}^2$

Mean $\bar{O} = \langle \hat{O} \rangle = \sum_j^N w^{(j)} O^{(j)}$

Recall: Importance Sampling

- x : random variables
- $f(x)$: observables
- n : number of samples
- $p(x)$: original(true) distribution
- $q(x)$: reference distribution

$$E[f(x)] = \int f(x)p(x)dx = \int f(x)\frac{p(x)}{q(x)}q(x)dx$$

Weights

$$w^{(j)} = \frac{\text{Posterior}(\boldsymbol{\theta}^{(j)}_{\text{EoS}} \,|\, \text{data})}{\text{Proposal}(\boldsymbol{\theta}^{(j)}_{\text{EoS}})} \propto \frac{p(\text{data} \,|\, \boldsymbol{\theta}^{(j)}_{\text{EoS}}) \cdot \text{Prior}(\boldsymbol{\theta}^{(j)}_{\text{EoS}})}{p(\boldsymbol{\theta}^{(j)}_{\text{EoS}} \,|\, \text{samples}^{(j)}) \cdot p(\text{samples}^{(j)} \,|\, \text{data}) \cdot \text{Prior}(\text{data})}$$

L2 : $exp(-\lambda||\theta||_2^2) \sim 1, \quad \lambda = 10^{-8}$

$p(\text{data} \,|\, \boldsymbol{\theta}^{(j)}_{\text{EoS}}) \propto \exp(-\chi^2(M_{\theta^{(j)}_{\text{EoS}}}, R_{\theta^{(j)}_{\text{EoS}}}))$

$p(\boldsymbol{\theta}^{(j)}_{\text{EoS}} \,|\, \text{samples}^{(j)}) = 1$

deterministic

$p(\text{samples} \,|\, \text{data}) = \mathcal{N}(M_{\text{obs}}, \Delta M^2)\mathcal{N}(R_{\text{obs}}, \Delta R^2)$

$\tilde{w}^{(j)} = w^{(j)} / \sum_j w^{(j)}$

normalization

HAL QCD method

The equal-time Nambu-Bethe-Salpeter (NBS) wave function

N. Ishii, S. Aoki, and T. Hatsuda, Phys. Rev. Lett. 99, 022001 (2007)
Aoki, S., Doi, T., Front. in Phys. 8, 307 (2020)

$$\phi_{\mathbf{k}}(\mathbf{r})e^{-W_{\mathbf{k}}t} \equiv \langle 0 | N(\mathbf{x}+\mathbf{r}, t)N(\mathbf{x}, t) | NN, W_k \rangle$$

In the HAL QCD method, the NBS wave function is calculated from the non-local but energy independent potential, $U(\mathbf{r})$, as

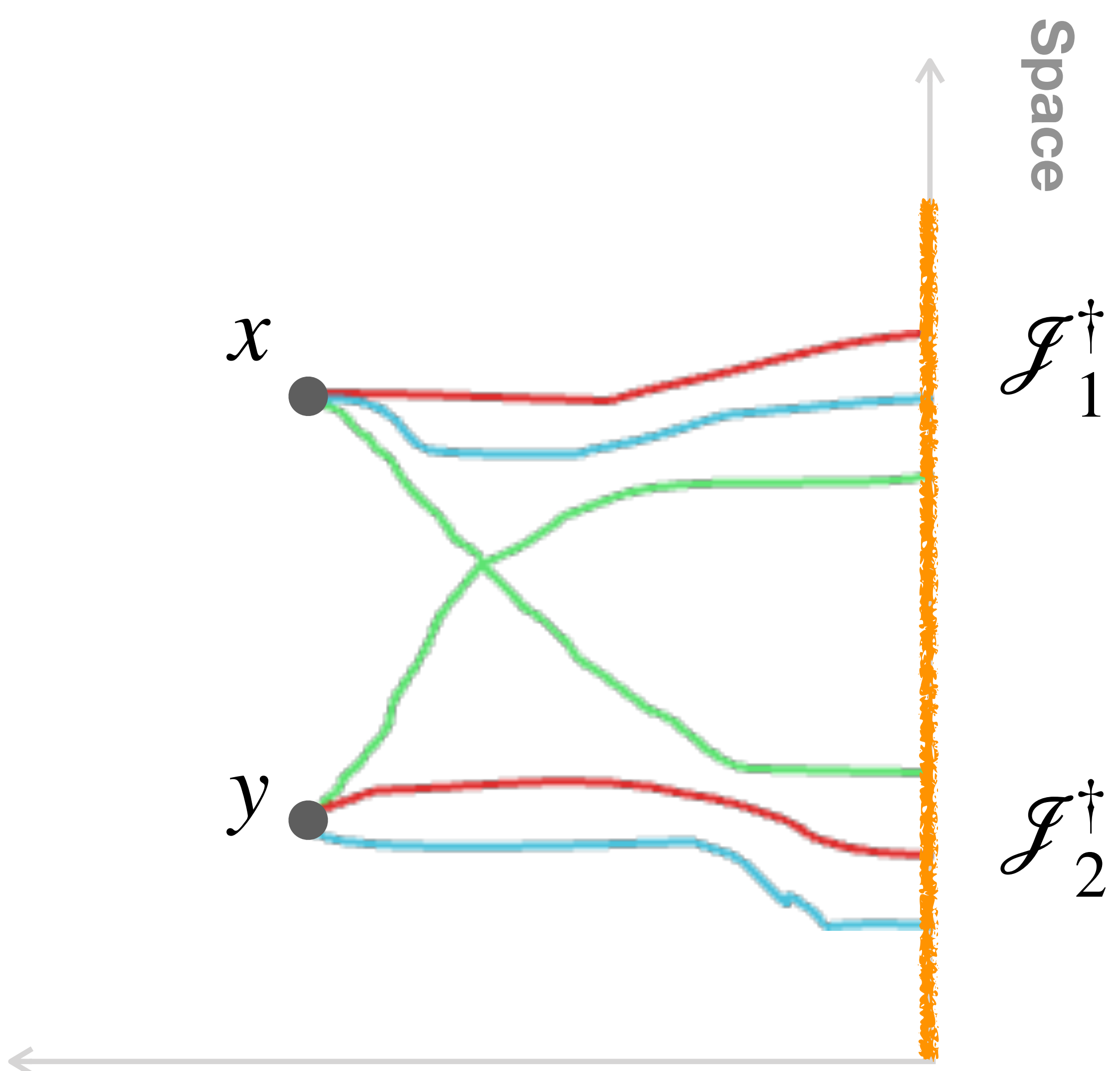
$$(E_k - H_0)\phi_{\mathbf{k}}(\mathbf{r}) = \int d^3 r' U(\mathbf{r}, \mathbf{r}')\phi_{\mathbf{k}}(\mathbf{r}'), \quad E_k = \frac{k^2}{2m}, \quad H_0 = -\frac{\nabla^2}{2m}, \quad m = \frac{m_N}{2}.$$

Extract the potential $U(\mathbf{r}, \mathbf{r}')$

HAL QCD method

Scattering

N. Ishii, S. Aoki, and T. Hatsuda, Phys. Rev. Lett. 99, 022001 (2007),
S. Aoki, T. Hatsuda, and N. Ishii, Prog. Theor. Phys. 123, 89 (2010).
N. Ishii, etc.(HAL QCD), Phys. Lett. B 712, 437 (2012)



$$\langle N_1(\mathbf{x},t)N_2(\mathbf{y},t)\mathcal{J}_1^\dagger(0)\mathcal{J}_2^\dagger(0)\rangle$$

$$= \sum_n \langle 0|N_1(\mathbf{x})N_2(\mathbf{y})|n\rangle a_n e^{-E_n t}$$

$$\xrightarrow{t>t^*} \phi(\mathbf{r},t) = \sum_{n<n^*} b_n \phi_n(\mathbf{r}) e^{-E_n t}$$

$\phi(\mathbf{r},t) \rightarrow$ **2 PI Kernel**

$$(E_k - H_0)\phi_{\mathbf{k}}(\mathbf{r}) = \int d^3r' U(\mathbf{r},\mathbf{r}')\phi_{\mathbf{k}}(\mathbf{r}'), \quad r < R$$

Consider the wave function at "interacting region"
⟶Phase shift, Binding energy