Tackling the four-loop pressure of dense and hot quantum chromodynamics

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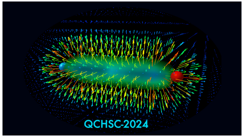
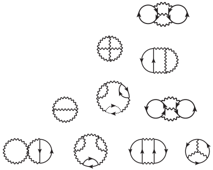
Based on earlier work with **York Schröder** (PoS 2207.10151)
and recent work with **R. Paatelainen**, **K. Seppänen**, **A. Vuorinen**, and others

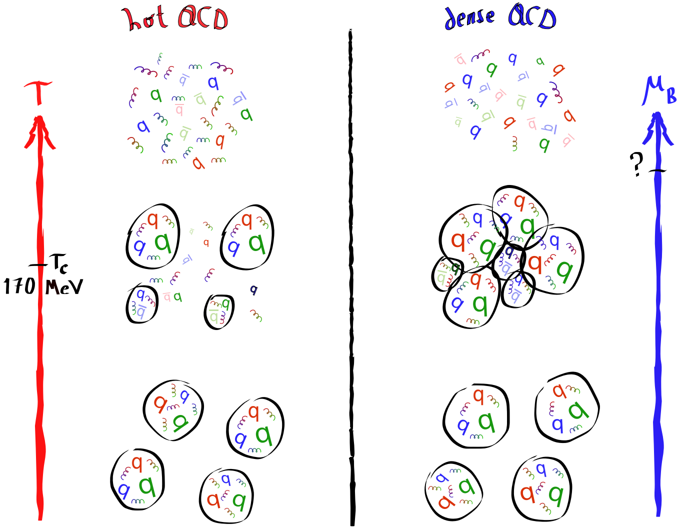
XVIth Quark Confinement and the Hadron Spectrum, Cairns

UNIVERSITY OF HELSINKI

August 21, 2024





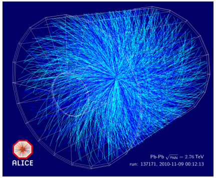

hot QCD
dense QCD
T
T_c
170 MeV
μ_B
?

hot QCD

dense QCD

T

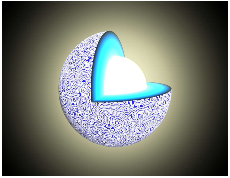
T_c 170 MeV



- HIC, early universe
- e.g. RHIC, LHC, GSI

μ_B

?



- Neutron stars
- e.g. NICER, LIGO/Virgo

Insist in perturbation theory: learn important constraints

Setting

- Grand canonical ensemble ($\mathcal{Z} = \mathrm{tr}\, e^{-(\hat{H}-\mu\hat{Q})/T}$; $t \to i\tau$ formalism)

$$\mathcal{Z}_{\mathrm{QCD}}(T, \{\mu_j\}) = \int \mathcal{D}[A_\mu^a, \bar{\psi}, \psi] \exp\left(-\int_0^{1/T} d\tau \int \mathrm{d}^d\mathbf{x}\, L_{\mathrm{QCD}}(\{\mu_j\})\right)$$

- Euclidean QCD

$$L_{\mathrm{QCD}} = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \sum_{j=1}^{N_\mathrm{f}} \bar{\psi}_j (\not{D} + \mu_j \gamma^0 + m_j) \psi_j$$

- **Periodic** bosonic fields → $P_0 = 2\pi T n$ ⇐ "Matsubara frequencies"
- **Antiperiodic** fermionic fields → $P_0 = \pi T(2n+1) + i\mu$ ⇐ Imaginary shift
- Momentum space measure: sum-integrals

$$\SumInt_P \equiv T \sum_{n\in\mathbb{Z}} \int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d}$$

Energy scales in hot QCD

- Non-zero modes & fermions → protected in the infrared
- Expansion parameter

$$g^2 n_{\mathrm{B}}(|p|) = \frac{g^2}{e^{|p|/T} - 1} \overset{|p| \lesssim T}{\simeq} \frac{g^2 T}{|p|}$$

- **Typical** plasma scale: $|p| \sim 2\pi T$ ⇐ perturbative (hard scale)
- Electric **screening**: $|p| \sim gT$ ⇐ barely perturbative (soft scale)
- Magnetic (non) **screening**: $|p| \sim g^2 T$ ⇐ non-perturbative (ultrasoft scale)

High T: $g^2 T \ll gT \ll 2\pi T$ ⇒ effective field theory

- Zero modes are **static** ⇒ physics effectively 3-dimensional (dimensional reduction)

The hot QCD pressure

- Main bulk quantity (= free energy density):

$$p_{\mathrm{QCD}}(T) = \lim_{V\to\infty} \frac{T}{V} \log \mathcal{Z}(T) = \sum (\text{conn. vac. Feynman diags})$$

- **Non-trivial** weak-coupling expansion:

$$p_{\mathrm{QCD}}(T) = p_0 + p_2 g^2 + p_3 g^3 + g^4 (p_4' \log g + p_4) + p_5 g^5 + g^6 (p_6' \log g + p_6) + O(g^7)$$

- Long history: p_2 [Shuryak '78], ..., p_6' [Kajantie et al. '03]
- Non-analytic behaviour in $\alpha_s \sim g^2 \Rightarrow$ screening of static gluons
- $p_6 \Rightarrow$ leading **non-perturbative** effects $\Rightarrow$ lattice input

Electrostatic QCD

- Integrate out $|p| \sim 2\pi T \Rightarrow$ non-zero Matsubara modes & fermions

$$p_{\text{QCD}}(T) = p_{\text{E}}(T) + \lim_{V\to\infty} \frac{T}{V} \log \int \mathscr{D}[A_i^a, A_0^a] \exp\left(-\int \mathrm{d}^d\mathbf{x}\, L_{\text{E}}\right)$$

- Effective **3d Yang-Mills + adj. $\mathbf{A_0}$** (EQCD)

$$L_{\text{E}} = \frac{1}{2}\text{Tr}\, F_{ij}F_{ij} + \text{Tr}\,[D_i, A_0]^2 + m_{\text{E}}^2 \text{Tr}\, A_0^2 + \lambda_{\text{E}}^{(1)}(\text{Tr}\, A_0^2)^2 + \lambda_{\text{E}}^{(2)}(\text{Tr}\, A_0)^4 + \delta L_{\text{E}}$$

- $m_{\text{E}}^2 \sim T^2(g^2 + g^4 + g^6 + \dots)$ $\Rightarrow$ **static** A_0 propagator in QCD and EQCD
- $g_{\text{E}}^2 \sim T(g^2 + g^4 + g^6 + g^8 + \dots)$ $\Rightarrow$ extracted from A_0 propagator (BF method)
- $\lambda_E^{(1,2)} \sim T(g^4 + g^6 + \dots)$ $\Rightarrow$ 4 point functions of soft A_0
- $p_{\text{E}} \sim T^4(1 + g^2 + g^4 + g^6 + \dots)$ $\Rightarrow$ **vacuum diagrams in naive QCD**

Matching coefficients: reduction

- Example: m_{E}^2 from **static** A_0 propagator [Braaten/Nieto '96]

$$P^2 + \Pi_{00}(P) = 0 \quad \text{at} \quad P_0 = 0,\ |\mathbf{p}| = i m_{\mathrm{E}} \sim g$$

- Taylor expand in **p** ⇒ **vacuum** sum-integrals
- State of the art: **three** loops ⇒ **447** diagrams in QCD

- **10 million** vacuum sum-integrals ⇒ project to basis ⇒ reduction algorithms
- Graphs' symmetries + integration by parts (IBP) ⇒ collider physics
- **10** "master" sum-integrals ⇒ **3** non-trivial [Ghisoiu/Möller/Schröder '15]

Magnetostatic QCD

- Integrate out $|p| \sim gT \Rightarrow$ massive A_0 field (MQCD)

$$p_{\mathrm{QCD}}(T) = p_{\mathrm{E}}(T) + p_{\mathrm{M}}(T) + \lim_{V\to\infty} \frac{T}{V} \log \int \mathcal{D}A_i^a \exp\left(-\int \mathrm{d}^d\mathbf{x}\, L_{\mathrm{M}}\right)$$

- Effective **3d pure Yang-Mills** (MQCD) $\Rightarrow$ confining

$$L_{\mathrm{M}} = \frac{1}{2}\mathrm{Tr}\, F_{ij}F_{ij} + \delta L_{\mathrm{M}}$$

- $g_{\mathrm{M}}^2 \sim g_{\mathrm{E}}^2 T(1 + \frac{g_{\mathrm{E}}^2}{m_{\mathrm{E}}} + \frac{g_{\mathrm{E}}^4}{m_{\mathrm{E}}^2})$ $\Rightarrow$ 2-pt fct [Giovannangeli '04, Laine/Schröder '05]
- $p_{\mathrm{M}} \sim m_{\mathrm{E}}^3 T(1 + \frac{g_{\mathrm{E}}^2}{m_{\mathrm{E}}} + \frac{g_{\mathrm{E}}^4}{m_{\mathrm{E}}^2} + \frac{g_{\mathrm{E}}^6}{m_{\mathrm{E}}^3})$ $\Rightarrow$ vac diags in EQCD; note g^{odd} [Kajantie et al. '03]

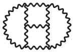
$$p_{\mathrm{G}} \sim \frac{T}{V} \log \int \mathcal{D}A_i^a \mathrm{e}^{-\int \mathrm{d}^d\mathbf{x}\, L_{\mathrm{M}}} \sim g_{\mathrm{M}}^6 T \times (\text{non-pert. coeff.})$$

Progress report: the g^6 term of hot QCD

- The pressure at order $O(g^6) \Rightarrow$ physical leading order

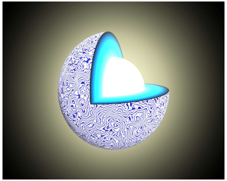
$$p_{\mathrm{QCD}}(T) = p_{\mathrm{E}}(T) + p_{\mathrm{M}}(T) + p_{\mathrm{G}}(T)$$

- $p_{\mathrm{G}}(T) \Rightarrow$ **plaquette in MQCD** ✓ [Di Renzo et al. '04-'06]
- $p_{\mathrm{M}}(T) \Rightarrow$ **404 vac. diags. in EQCD** ✓ [Kajantie et al. '03]
- $p_{\mathrm{E}}(T) \Rightarrow$ **117 vac. diags. in full QCD** χ [**PN**/Schröder '22-]

$$p_E(T)\Big|_{g^6} =$$  $+$  $+ (115 \text{ diags})$

- Focus on gauge sector (65 diags); $\mathrm{SU}(N)$ & $m_q = 0$; R_ξ-gauge
- **Algorithmic reduction** $\Rightarrow$ exploit graphs' symmetries $\Rightarrow$ momentum shifts
- **176k** sum-ints. $\longrightarrow$ **21** "master" sum-ints. ✓ $\Rightarrow$ ξ drops out ✓
- **11/21** factorized structures $\Rightarrow$ known (IBP) ✓
- **10 unknown four-loop sum-integrals !** [**PN**/Schröder '24]

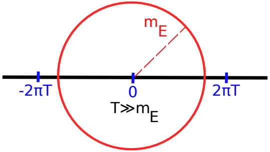
Cold and dense QCD

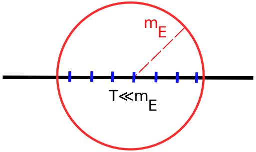


- $T = 0$, $\mu_B \gtrsim$ few GeV $\Rightarrow$ few tens of n_{sat}
- **Quark matter** cores in neutron stars **!**
- Finite $\mu_B \Rightarrow$ Sign Problem of lattice QCD
- Energy scales:

$$\text{hard scale } |p| \sim \mu, \qquad \text{soft scale } |p| \sim g\mu$$

- No $g^2\mu$ ultrasoft scale, but gap $\Delta \sim \mu g^{-5} e^{-c/g} \Rightarrow$ subleading in $p_{QCD}(\mu_B)$
- Soft physics $\Rightarrow$ **hard-thermal-loop (HTL) effective theory**



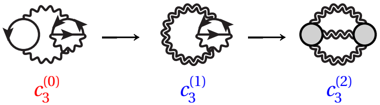


The cold and dense QCD pressure

- Unpaired quark matter

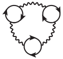
$$p_{\rm QCD}(\mu_{\rm B}) = p_0 \quad \text{[free quarks]}$$
$$+ c_1 \alpha_s \quad \text{[Freedman/McLerran '77]}$$
$$+ \alpha_s^2 (c_2^{(1)} \log \alpha_s + c_2^{(0)}) \quad \text{[Freedman/McLerran '77]}$$
$$+ \alpha_s^3 (c_3^{(2)} \log^2 \alpha_s + c_3^{(1)} \log \alpha_s + c_3^{(0)}) \quad \text{[see below]}$$

- State of the art: $(\text{next-to})^3$ leading order α_s^3 (**N^{3}LO**)
- $c_3^{(2)} \Rightarrow$ 2-loop **full** HTL diags. [Säppi et al. '21]
- $c_3^{(1)} \Rightarrow$ 3-loop **mixed** HTL/QCD diags. [Säppi, Seppänen et al. '23]
- $c_3^{(0)} \Rightarrow$ 4-loop **full** QCD diags. [**in progress**] **!**



[see S. Säppi's talk]

Progress report: N^{3}LO pressure

$$c_3^{(0)} \sim (41 \text{ diags})\, N_f + (10 \text{ diags})\, N_f^2 + \Bigg($$  $$\Bigg) N_f^3$$

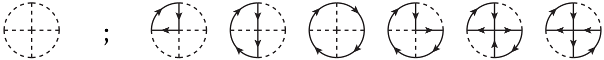
[Kärkkäinen, **PN**, Nurmela, Paatelainen, Seppänen, Vuorinen; **work in progress**]

- General SU(N) & N_f ($m_q = 0$); use R_ξ-gauge ⇒ ξ drops out ✓
- Reduction: **156k** ints. ⟶ **114** "master" ints. ✓

# Ints.	N_f^3	$N_f^2 C_A$	$N_f^2 C_F$	$N_f C_A^2$	$N_f C_A C_F$	$N_f C_F^2$
ξ^0	132	2229	958	5975	2841	890
ξ^1	205	7428	2054	34554	11507	2209
ξ^2	173	9461	2452	72831	17340	2949
ξ^3	125	5507	1080	75344	10951	1300
ξ^4	-	2632	-	44618	3491	-
ξ^5	-	-	-	20036	-	-
ξ^0	18	50	48	65	55	45

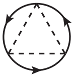
Progress report: N^{3}LO pressure

- $\sim$ **70** four-loop integrals $\Rightarrow$ more integral topologies than gauge sector **!**



YM sector ; fermion sector

- New results; e.g. $(d = 3 - 2\varepsilon)$ [Kärkkäinen, **PN**, et al.; **in progress**]



$$= \left(\frac{e^{\gamma_E}\bar{\Lambda}^2}{4\pi\mu^2}\right)^{\varepsilon} \frac{\mu^4}{(4\pi)^8}\left(\frac{\beta_1}{\varepsilon^2} + \frac{\beta_2}{\varepsilon} + \beta_3 + O(\varepsilon)\right)$$

- **Alternative approach?** [**PN**, Paatelainen, Seppänen '24]

see his Talk!

Outlook

- Pressure main equilibrium observable
- Hot QCD in heavy-ion collisions and early cosmology
- Cold and dense QCD in neutron-star environments
- Weak-coupling hot/cold QCD: $O(g^6)$/N^{3}LO $\Rightarrow$ **four loops**
- Large-scale **multiloop** thermal field theory
- **Need <u>new</u> tools to tackle (sum-)ints at the four-loop level !**

Thank You