Temperature dependence of fractional topological charge objects

Jackson Mickley

In collaboration with

Waseem Kamleh, Derek Leinweber

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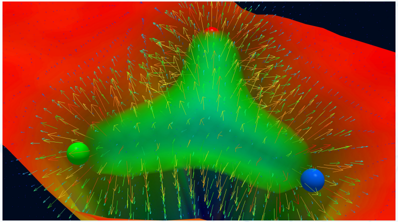
The XVIth Quark Confinement and the Hadron Spectrum Conference
Cairns Convention Centre, Queensland, 19-24 August 2024

Overview

- Confinement in Quantum Chromodynamics
  - Generated from topological structure
  - Brief history: instanton liquid, fractional charge



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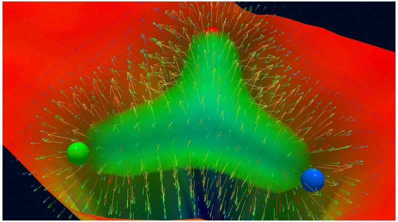

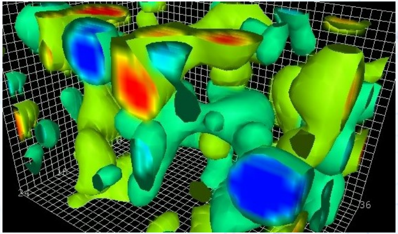

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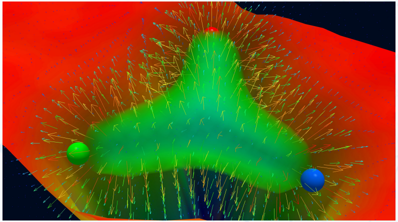
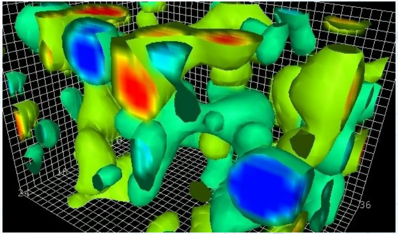
- Quantified by nontrivial holonomy
  - Instanton-dyon model for vacuum structure
  - Extension to general $\mathrm{SU}(N)$





Topological structure and confinement

- First models in terms of instantons
  - Classical solutions to equations of motion on $\mathbb{R}^4$
  - Possess integer topological charge

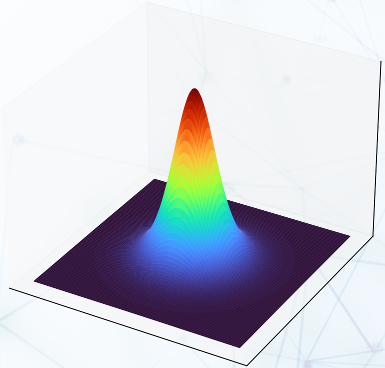
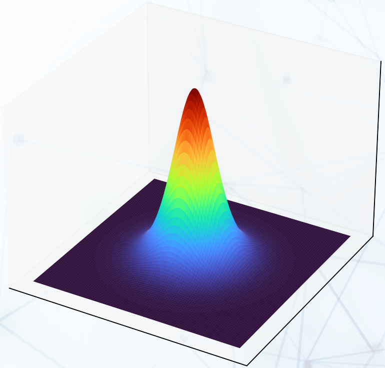


Figure: 2D slice of an instanton's topological charge profile

Topological structure and confinement

- First models in terms of instantons
  - Classical solutions to equations of motion on $\mathbb{R}^4$
  - Possess integer topological charge
- Instanton Liquid Model: QCD vacuum as ensemble of interacting semiclassical instantons
  - Generated chiral symmetry breaking
  - Unable to reproduce confinement (carry no flux)



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Topological structure and confinement

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  - Topological configurations $\sim 1/N$ in $\mathrm{SU}(N)$
  - Carry $\mathbb{Z}_N$ flux $\implies$ mechanism for confinement?

A. Gonzalez-Arroyo, On the fractional instanton liquid picture of the Yang-Mills vacuum and Confinement, arXiv:2302.12356 [hep-th]

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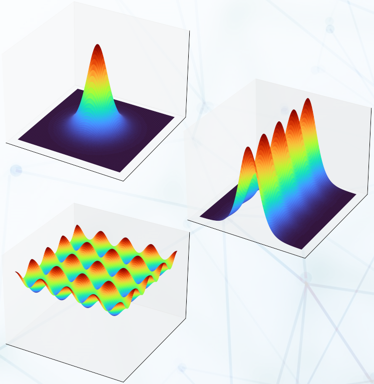

Figure: Profile of fractional instanton on $\mathbb{R}^2 \times \mathbb{T}^2$ in the various 2D planes

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  - Solutions exist on $\mathbb{R}^n \times \mathbb{T}^{4-n}$ for each n
- Calorons on $\mathbb{R}^3 \times \mathbb{T}^1$: natural generalisation of instanton to finite temperature
  - Caloron profile comprises N constituent dyons
  - Dyon topological charges depend on holonomy

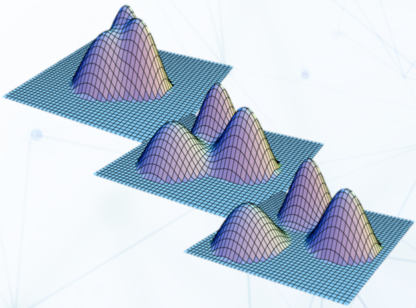

Figure: $\mathrm{SU}(3)$ caloron decomposes into its 3 dyons as the temperature increases

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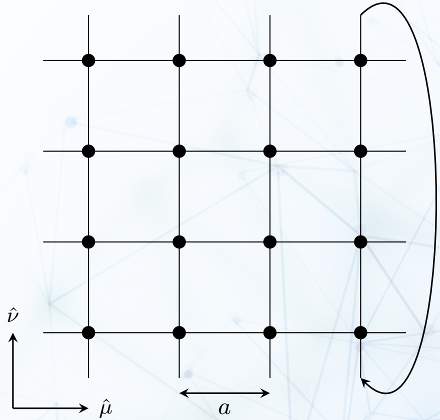
T. C. Kraan and P. van Baal, Monopole constituents inside $SU(n)$ calorons, Phys. Lett. B **435**, 389 (1998)

Lattice details

- Temperature determined by period of temporal dimension $T = (aN_t)^{-1}$
  - Control temperature through N_t (with a fixed)
  - We work with $32^3 \times N_t$ pure gauge ensembles at $a = 0.1$ fm



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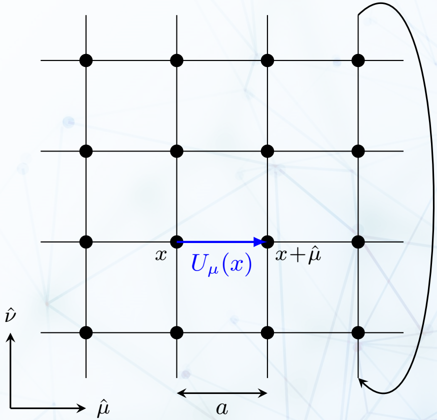
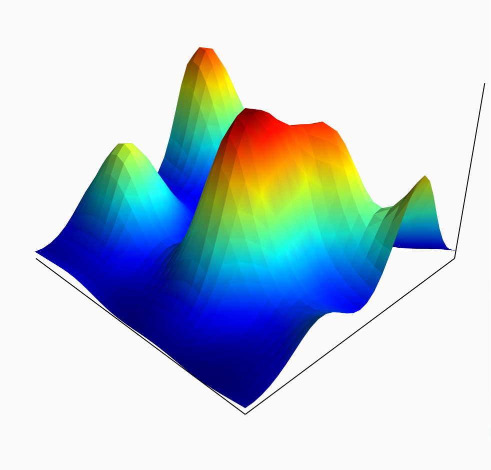
- Gauge field given by links $U_\mu(x) \in \mathrm{SU}(N)$
  - Related to continuum gauge field $A_\mu(x)$ by $U_\mu(x) \simeq e^{igaA_\mu(x)}$



Topological objects



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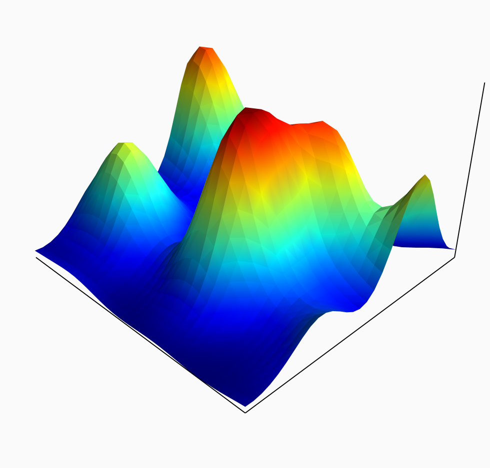

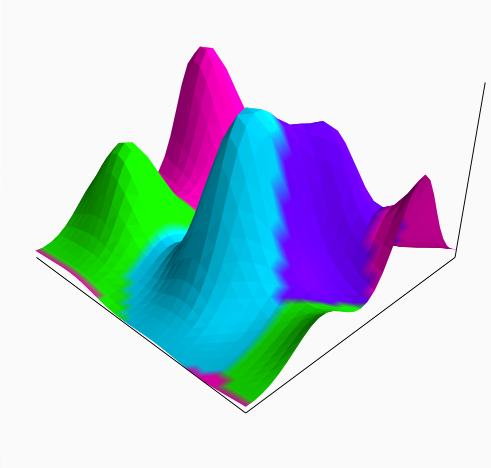

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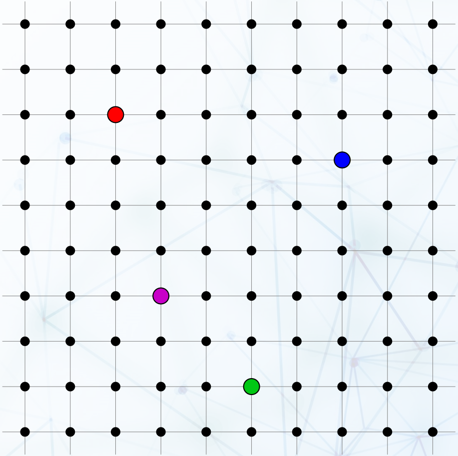

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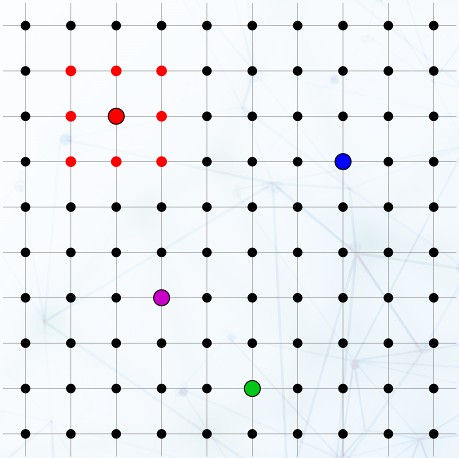

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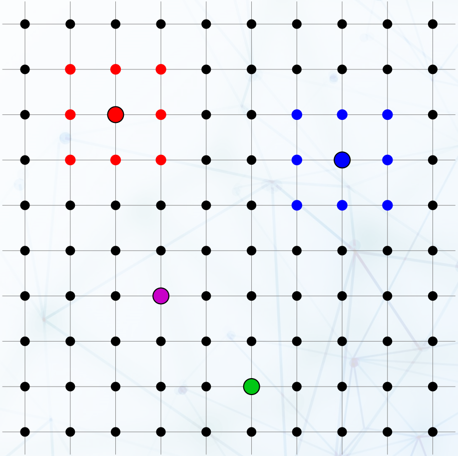

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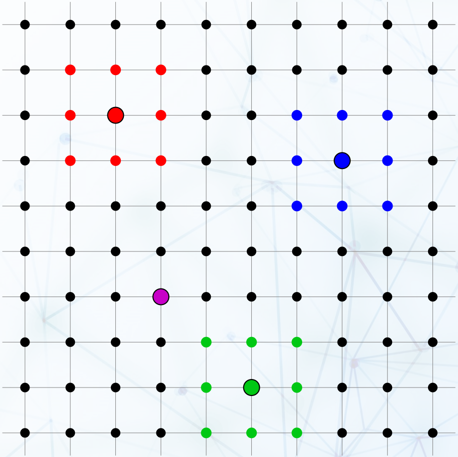

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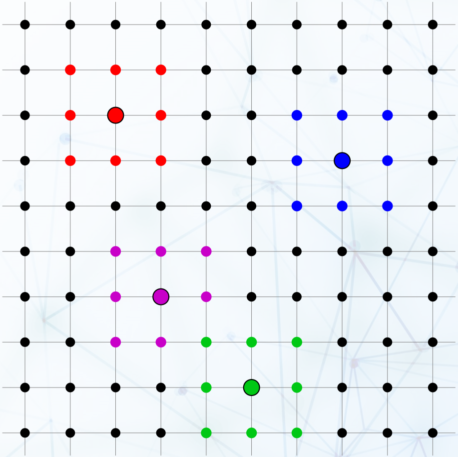

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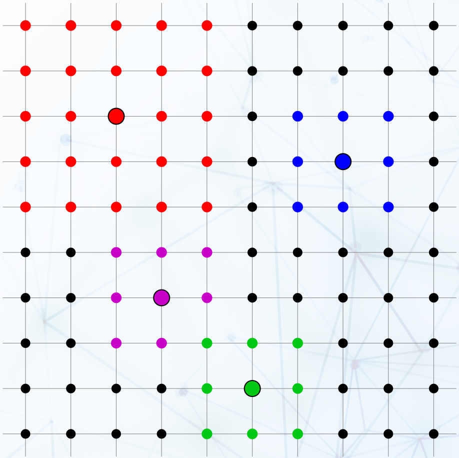

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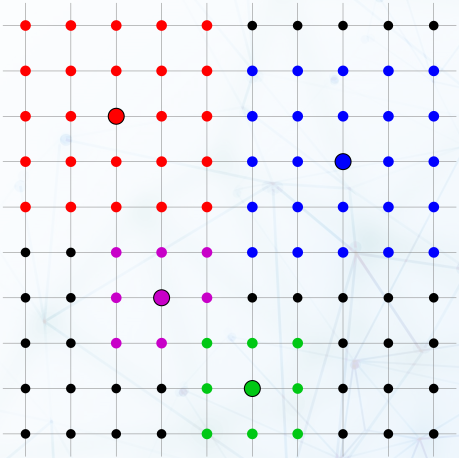

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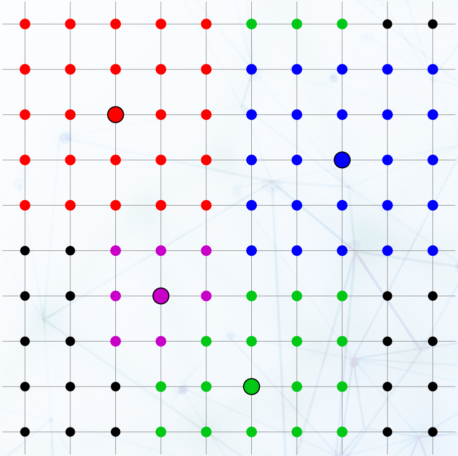

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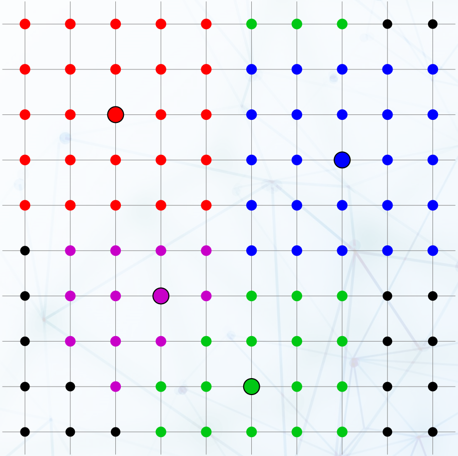

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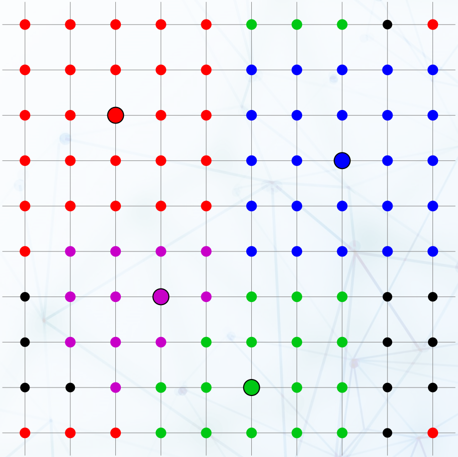

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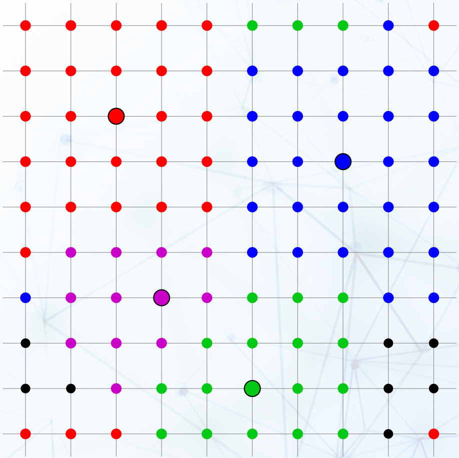

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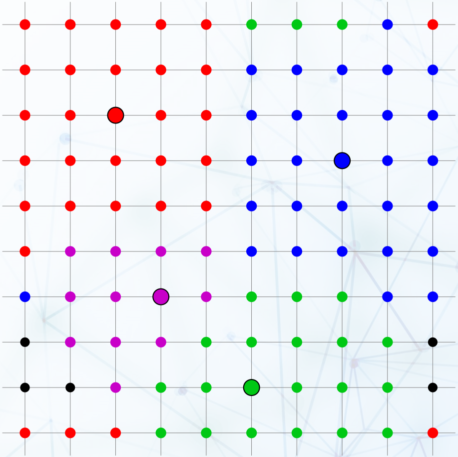

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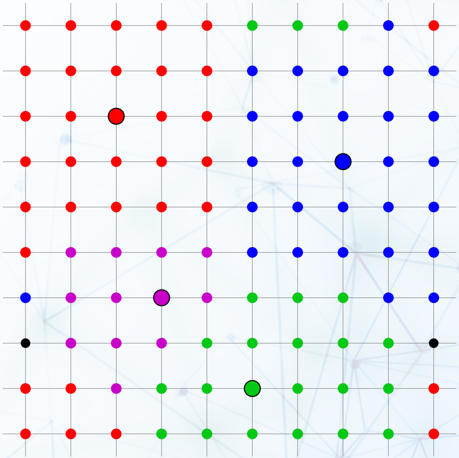

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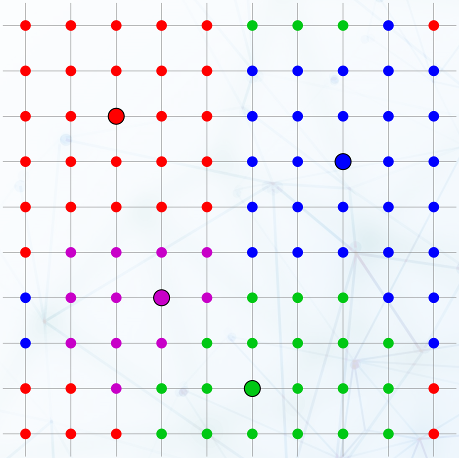

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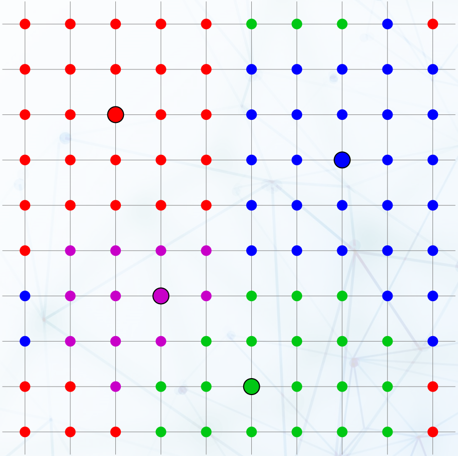
   - Nearby extrema, topological charge changes sign



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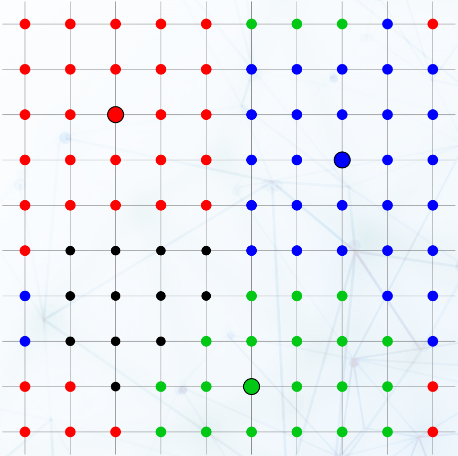

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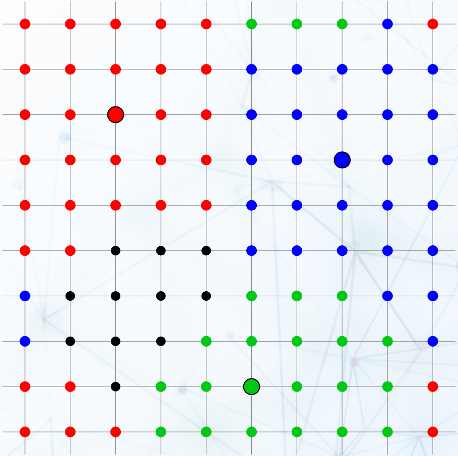

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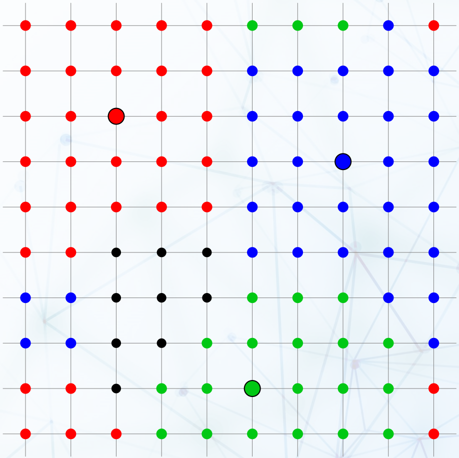

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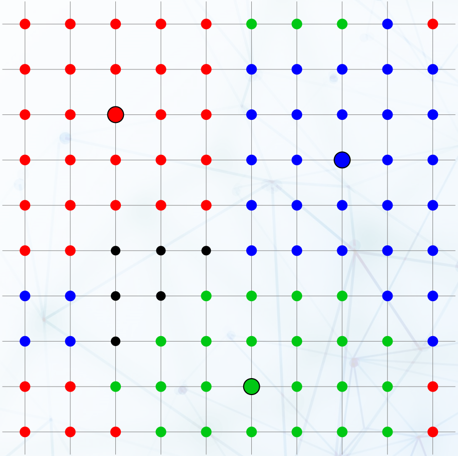

Classical limit

- Cooling iteratively minimises local action
  - Approaches ‘classical limit’ of instantons
  - Provides fertile ground for testing algorithm

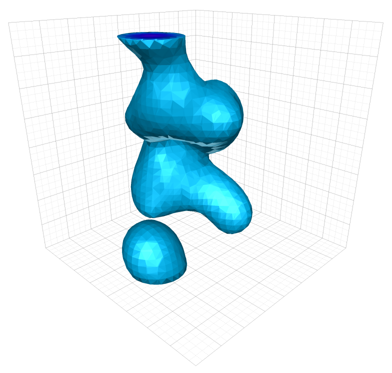

Figure: Topological charge density under extended cooling

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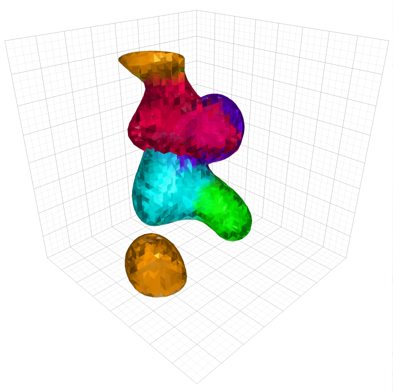

Figure: Topological charge density under extended cooling, coloured by object number

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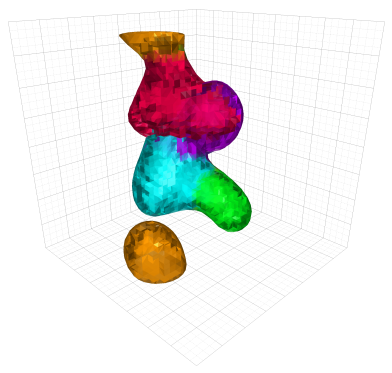

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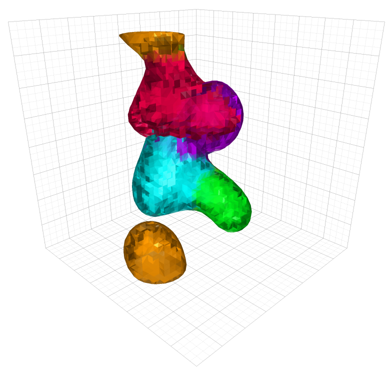
  - Demonstrates complexity in dividing up topological charge distribution



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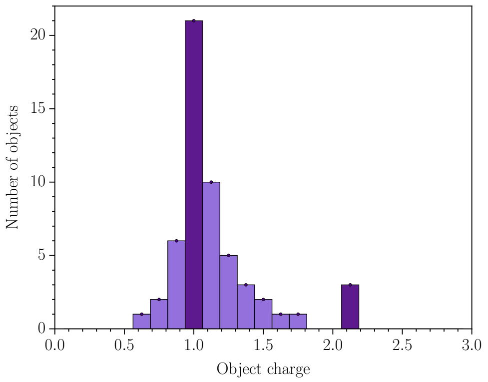
- Histogram of calculated topological charges
  - Strongly peaked at integer values!



Lattice topological charge density

- Topological charge density defined in terms of field strength tensor
  - Measures alignment/winding of gluon field lines

$$q(x) = \frac{g^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \, \mathrm{Tr} \left(F_{\mu\nu}(x) \, F_{\rho\sigma}(x) \right)$$

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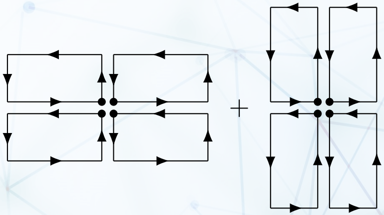
- Field strength tensor on the lattice
  - Imaginary part of $m \times n$ *Clover term*:

$$F_{\mu\nu}^{(m\times n)}(x) = \frac{1}{8} \, \mathrm{Im} \, C_{\mu\nu}^{(m\times n)}(x)$$

  - Associated topological charge density $q^{(m\times n)}(x)$



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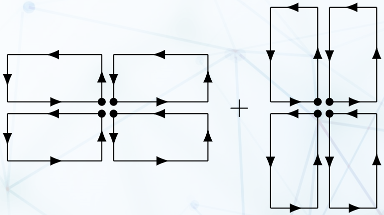
$$F_{\mu\nu}^{(m\times n)}(x) = \frac{1}{8}\, \mathrm{Im}\, C_{\mu\nu}^{(m\times n)}(x)$$

  - Associated topological charge density $q^{(m\times n)}(x)$



- Operators encounter severe renormalisations
  - Overcome by gauge smoothing
  - We implement gradient flow with $\epsilon = 0.005$

Mode matching

- Define two perturbatively-improved topological charge operators
  - Improved $F_{\mu\nu}(x)$: linear combination of $F_{\mu\nu}^{(m\times n)}$
  - Improved $q(x)$: linear combination of $q^{(m\times n)}$

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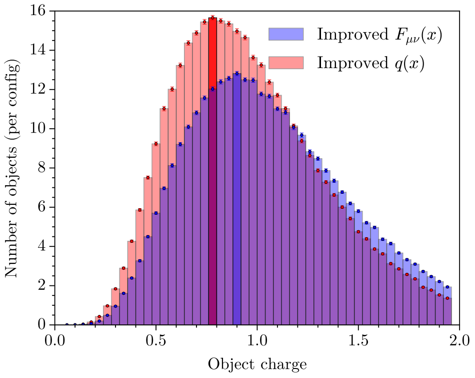

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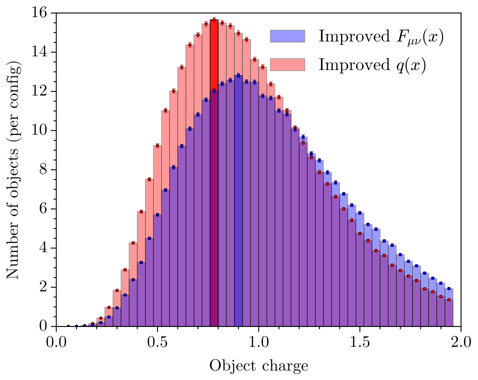

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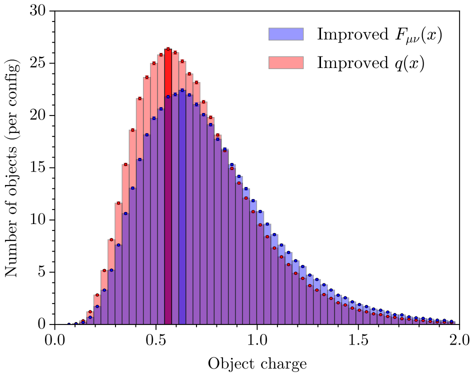

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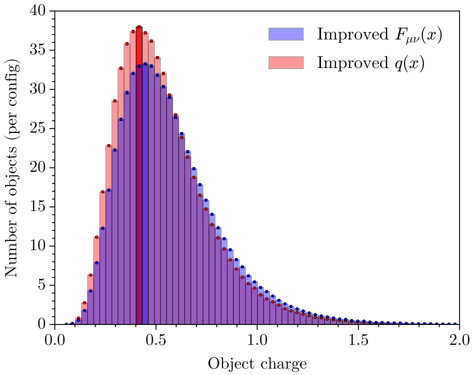

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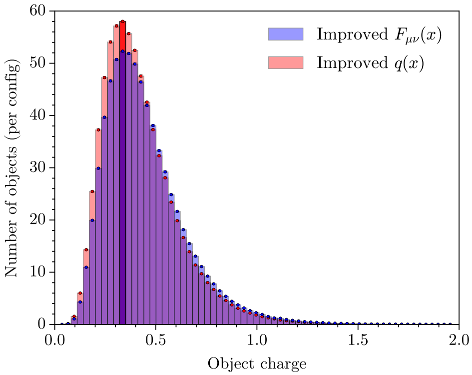

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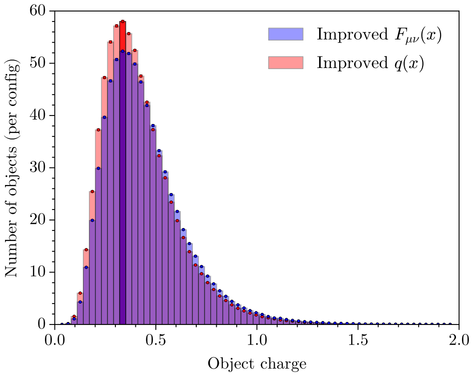

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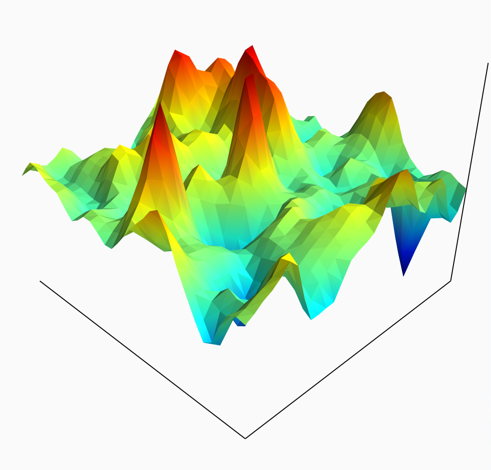

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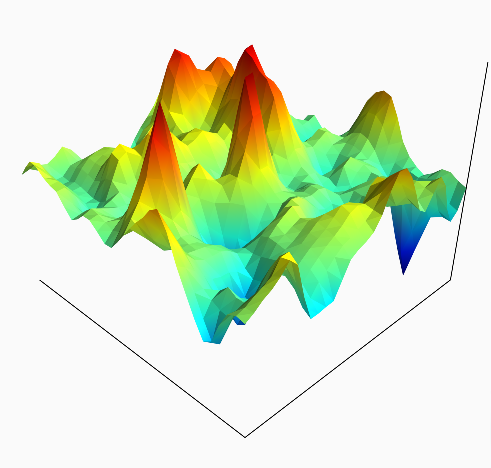

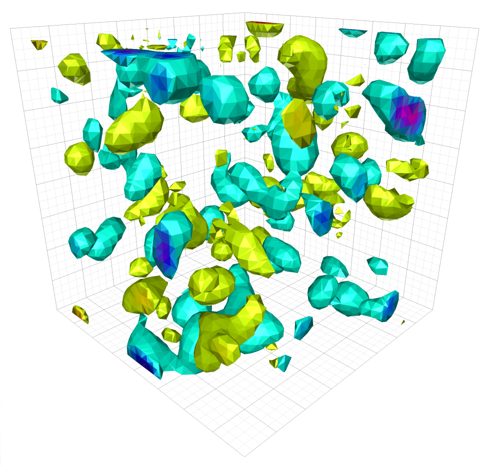

Finite temperature results

- Evolution of topological structure with temperature in SU(3)

N_t	T/T_c	Object charge mode
64	≈ 0	0.336
12	0.609	0.338
8	0.913	0.312
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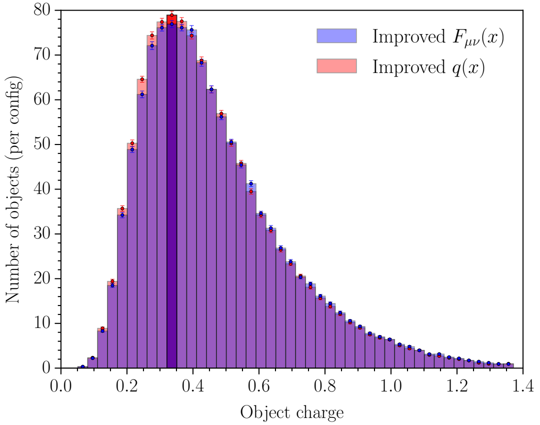

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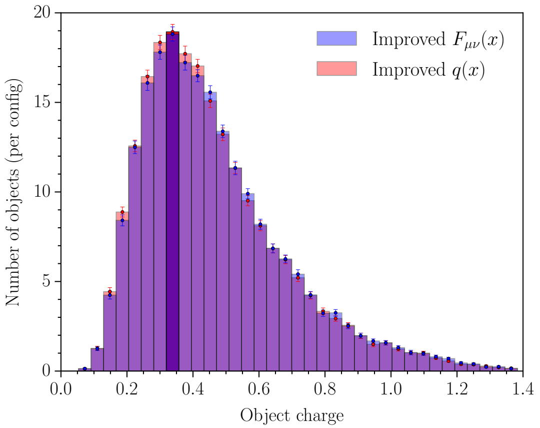

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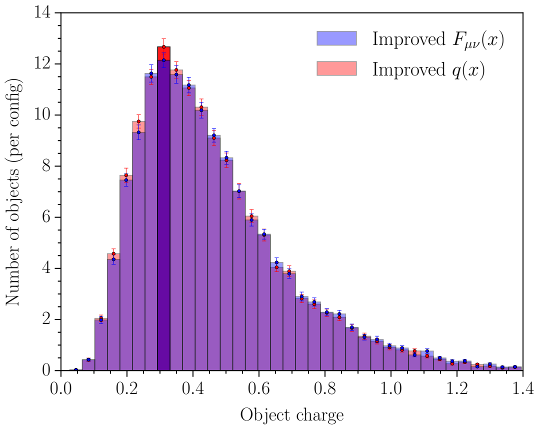

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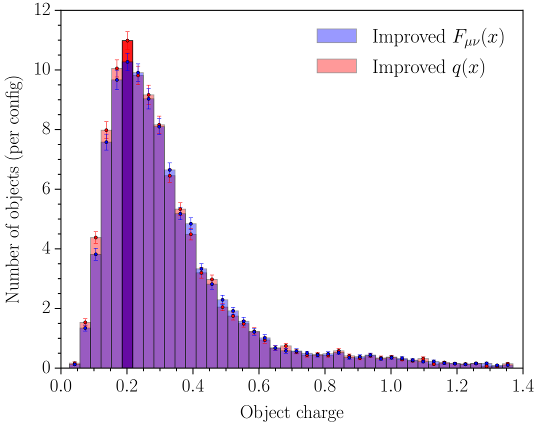

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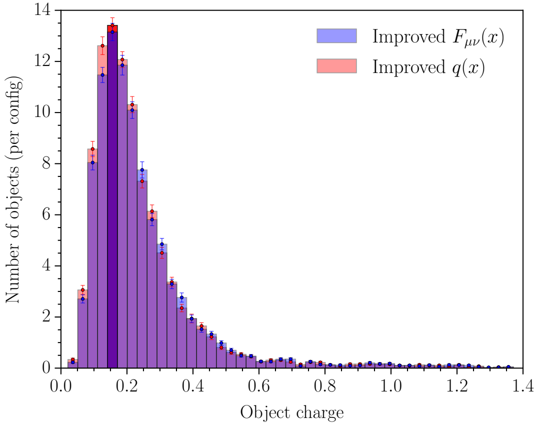

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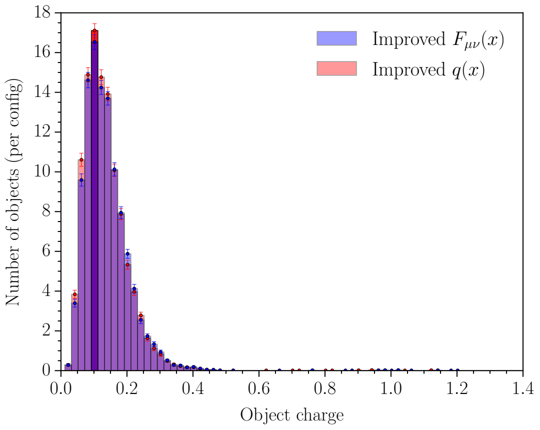

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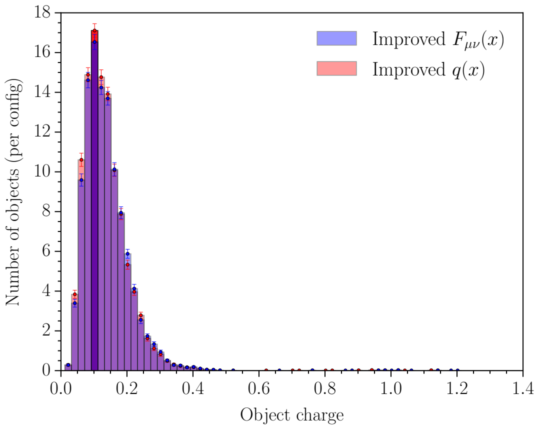
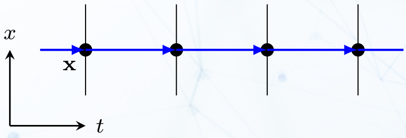


- Mode near $|Q| \approx 1/3$ in confining phase, but decreases in the deconfined phase

Polyakov loop and holonomy

- Polyakov loop: product of links along temporal axis
  - Free energy of single quark: $\langle \mathrm{Tr}\, P \rangle \sim e^{-F_q/T}$
  - Order parameter for confinement: $\langle \mathrm{Tr}\, P \rangle = 0$ in confined phase, but $\neq 0$ in deconfined phase



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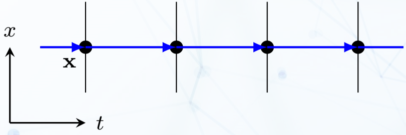
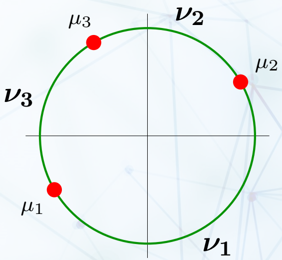


- Eigenvalues lie on unit circle
  - Phases written as $2\pi\mu_i$ with $\sum \mu_i = 0$
  - Holonomy parameters defined as $\nu_i = \mu_{i+1} - \mu_i$



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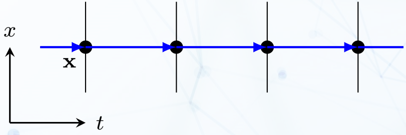

- Eigenvalues lie on unit circle
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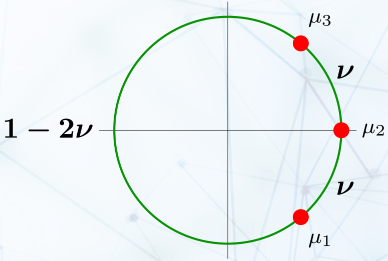
  - Holonomy parameters defined as $\nu_i = \mu_{i+1} - \mu_i$
- One unspecified parameter in $\mathrm{SU}(3)$
  - Reduces to $\langle \mathrm{Tr}\, P \rangle = 1 + 2\cos(2\pi\nu)$
  - Connection between holonomy and charges?



D. DeMartini and E. Shuryak, Deconfinement phase transition in the $SU(3)$ instanton-dyon ensemble, Phys. Rev. D **104**, 054010 (2021)

Holonomy and topological charges

N_t	T/T_c	$\frac{1}{3}\langle \mathrm{Tr}\, P\rangle$	ν
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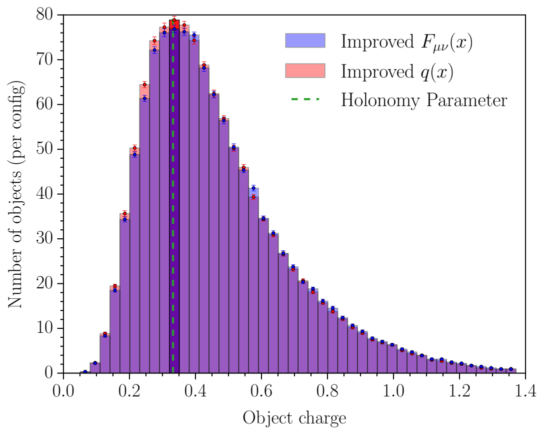

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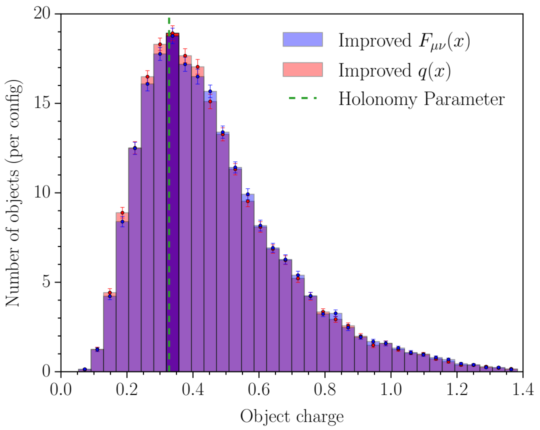

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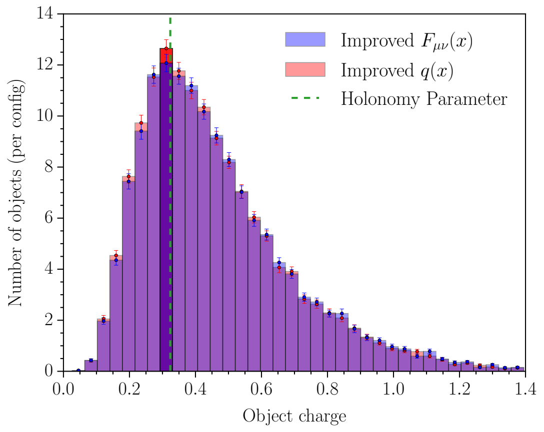

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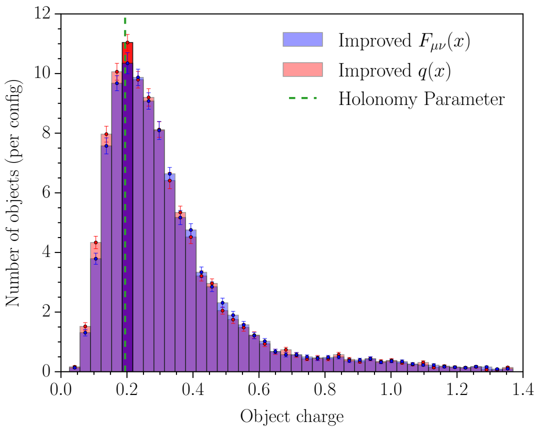

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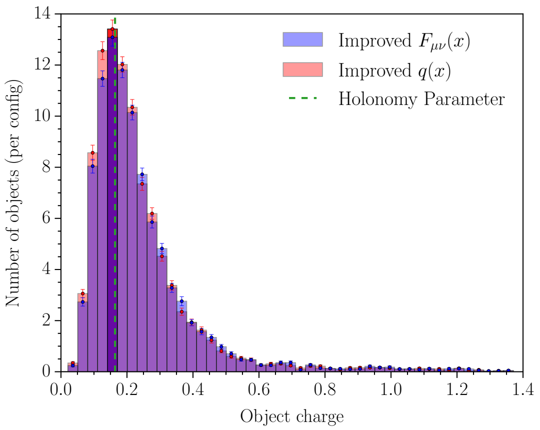

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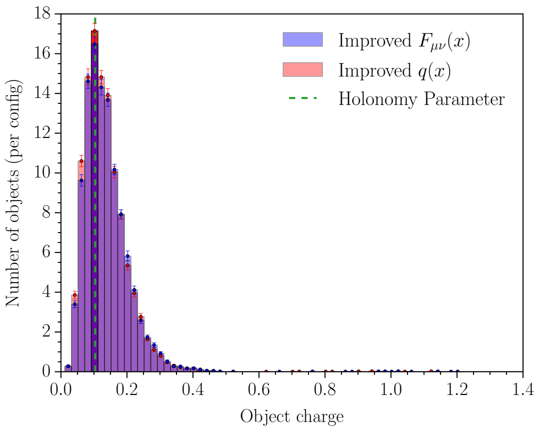

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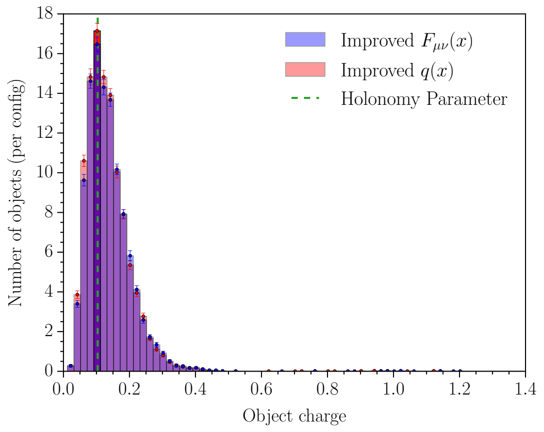
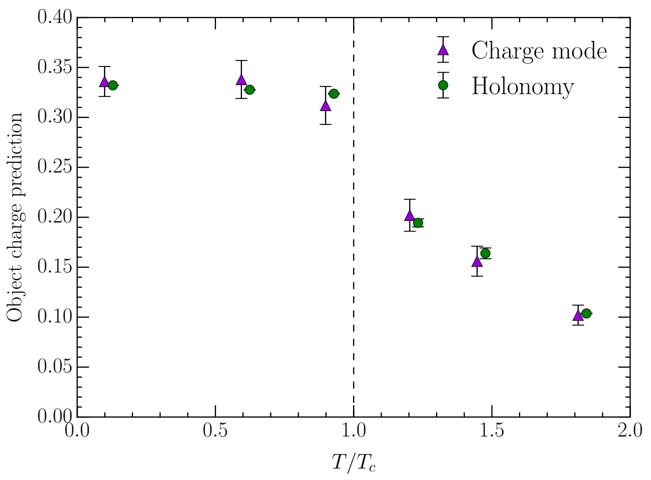


- Topological charge mode accurately described by free holonomy parameter!

Holonomy and topological charges



Instanton-dyons

- Dyon actions/topological charges divided up according to holonomy parameters
  - In $\mathrm{SU}(3)$, these are $|Q_1| = |Q_2| = \nu$ and $|Q_3| = 1 - 2\nu$
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- Third dyon, with larger topological charge?
  - Weighted by $e^{-S_i} \implies$ exponentially suppressed by its action
  - Topological index of zero maintained by equal amounts of dyons/antidyons

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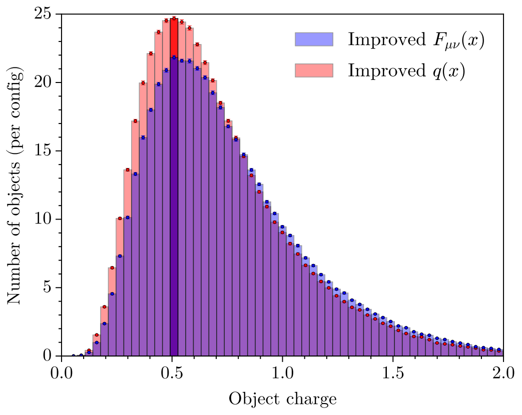

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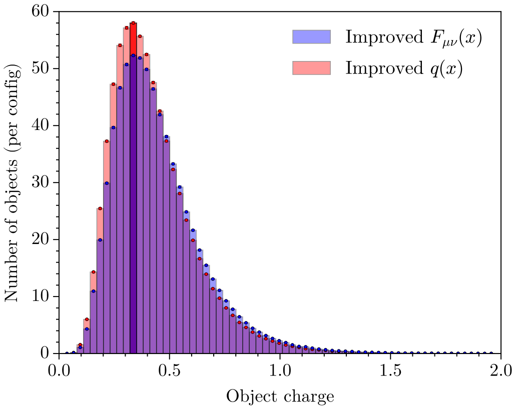

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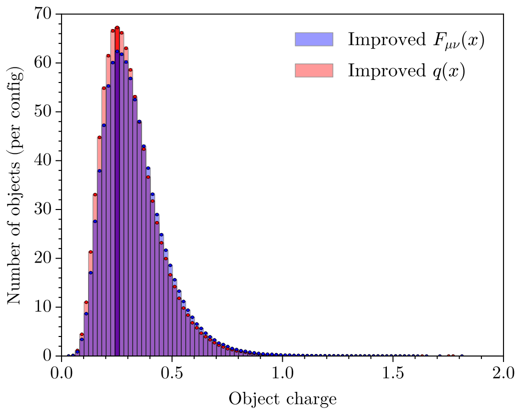

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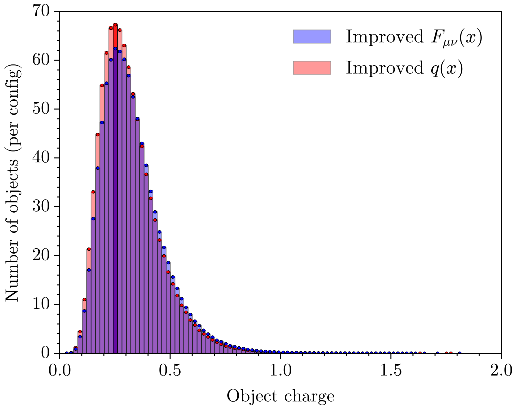

- So far, so good!

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- Numerical algorithm to break down topological structure of QCD vacuum
  - Below T_c, distinct topological objects $\sim 1/3$
  - Above T_c, topological charges shift towards zero

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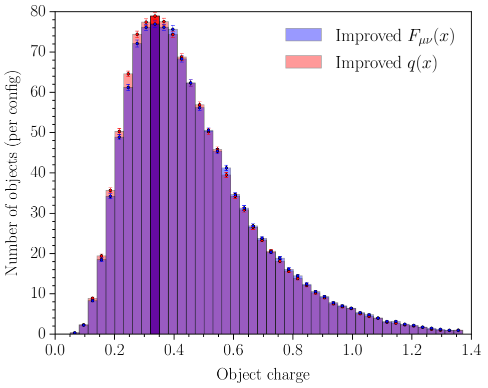
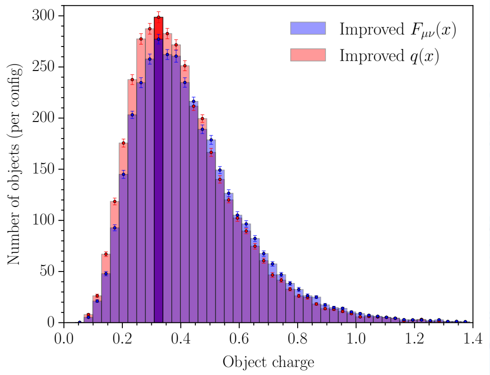
Thank you!

Continuum limit: fixed lattice dislocation filter

- $32^3 \times 64$ lattice at $a = 0.100\,\mathrm{fm}$
  - Hypercubic dislocation filter
  - Flow time $\tau = 1.45$



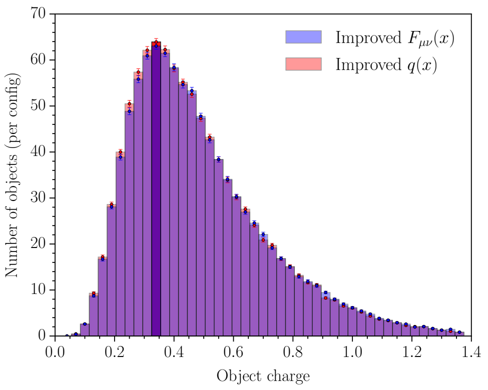
- $48^3 \times 96$ lattice at $a = 0.067\,\mathrm{fm}$
  - Hypercubic dislocation filter
  - Flow time $\tau = 1.25$

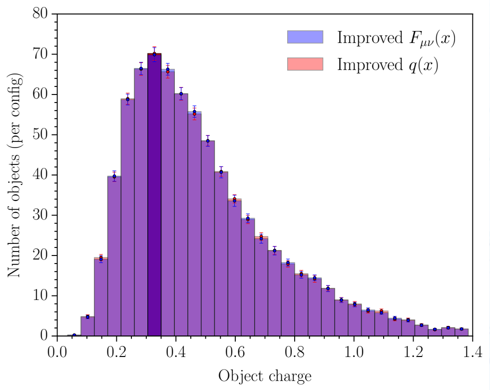


Continuum limit: fixed scale

- $32^3 \times 64$ lattice at $a = 0.100\,\mathrm{fm}$
  - 2-unit radial filter
  - Flow time $\tau = 1.65$

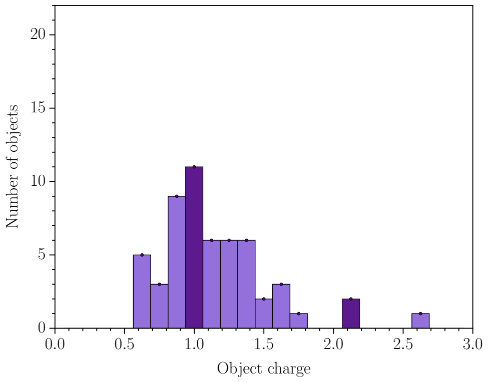
- $48^3 \times 96$ lattice at $a = 0.067\,\mathrm{fm}$
  - 3-unit radial filter
  - Flow time $\tau = 3.71$





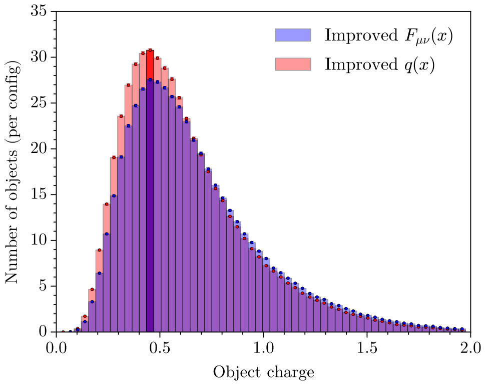
Object growing order

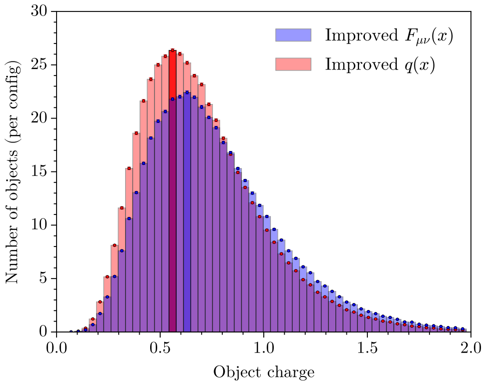
- Allocate sights in ascending peak value
  - Points near edge of distribution have larger relative weight
  - For net charge of the same order, must have a broader distribution
- Very important in classical limit
  - Descending order histogram visibly less clustered near integers
- Mostly inconsequential at the deduced flow time
  - ‘Boundary sites’ redistributed: some charges slightly smaller, some larger
  - Mode remains unchanged



SU(N) smoothing comparison

N	2	3	4
u_0	0.9022	0.8863	0.8808





Topological charge density improvement schemes

- $\mathcal{O}(a^4)$-Improved $F_{\mu\nu}(x)$

$$F_{\mu\nu}(x) = \sum_{m,n} k^{(m\times n)} F_{\mu\nu}^{(m\times n)}(x)$$

  - $k^{(1\times1)} = 19/9 - 55\,k^{(3\times3)}$
  - $k^{(2\times2)} = 1/36 - 16\,k^{(3\times3)}$
  - $k^{(1\times2)} = -32/45 + 64\,k^{(3\times3)}$
  - $k^{(1\times3)} = 1/15 - 6\,k^{(3\times3)}$

- 3-loop improved: $k^{(3\times3)} = 1/90$
- 5-loop improved: $k^{(3\times3)} = 1/180$

S. O Bilson-Thompson, D. B. Leinweber and A. G. Williams, Highly improved lattice field-strength tensor, Ann. Phys. **304**, 1 (2003)

- $\mathcal{O}(a^4)$-Improved $q(x)$

$$q(x) = \sum_{m,n} c^{(m\times n)} q^{(m\times n)}(x)$$

  - $c^{(1\times1)} = 19/9 - 55/9\,c^{(3\times3)}$
  - $c^{(2\times2)} = 1/9 - 64/9\,c^{(3\times3)}$
  - $c^{(1\times2)} = -64/45 + 128/9\,c^{(3\times3)}$
  - $c^{(1\times3)} = 1/5 - 2\,c^{(3\times3)}$

- 3-loop improved: $c^{(3\times3)} = 1/10$
- 5-loop improved: $c^{(3\times3)} = 1/20$

P. de Forcrand, M. García Pérez and I.-O. Stamatescu, Topology of the $\mathrm{SU}(2)$ vacuum: a lattice study using improved cooling, Nucl. Phys. B **499**, 409 (1997)

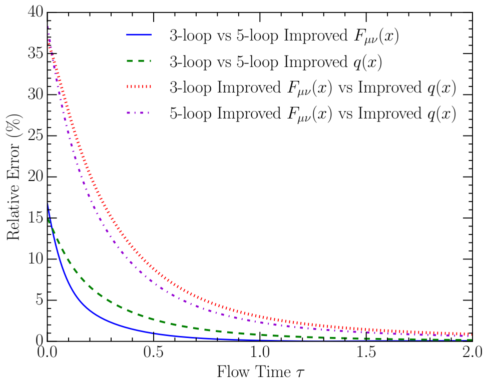
Improvement schemes comparison

- Investigate relative error between improvement schemes

$$\mathrm{RE} = \frac{|Q_1 - Q_2|}{\frac{1}{2}(Q_1 + Q_2)}$$

- Larger discrepancy between improvement schemes compared to varying the number of loops within each improvement scheme

- Utilise 3-loop definition for each improvement scheme



Holonomy

- Formally, the holonomy is the Polyakov loop at spatial infinity: $P_\infty = \lim_{|\mathbf{x}|\to\infty} P(\mathbf{x})$

- It is a topological invariant, and up to gauge symmetry can be written

$$P_\infty = \exp\left[2\pi i\,\mathrm{diag}(\mu_1, \ldots, \mu_N)\right],$$

$$\mu_1 < \ldots < \mu_N < \mu_{N+1} \equiv \mu_1 + 1\,, \qquad \sum_{i=1}^{N} \mu_i = 0\,.$$

- Translation invariance: $\langle L(\mathbf{x})\rangle = \langle \mathrm{Tr}\, P(\mathbf{x})\rangle$ is independent of $\mathbf{x}$

- On the lattice, we accordingly calculate L_∞ as the expectation $\langle \bar{L}\rangle$ of the spatially-averaged Polyakov loop

$$\bar{L} = \frac{1}{V}\sum_{\mathbf{x}} L(\mathbf{x})$$

Centre symmetry

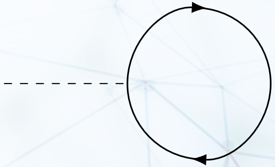
- Pure gauge theory features an additional complication in the form of a centre symmetry
  - Action invariant under centre transformations: $U_4(\mathbf{x}, x_4) \to z\, U_4(\mathbf{x}, x_4)$ for fixed x_4
- Polyakov loop transforms nontrivially as $P(\mathbf{x}) \to z\, P(\mathbf{x})$
  - Below T_c, centre symmetry preserved
  - Above T_c, one of the N centre phases is selected in spontaneous breaking of the symmetry
  - In Full QCD, it is always the positive real phase selected
- Overcome by performing centre transformations on a per-configuration basis
  - Rotate phase of Polyakov loop to bring dominant phase of each configuration to a phase of zero
  - Take real part to discard remnant imaginary part
  - Calculate ensemble average

Renormalisation effects

- Lattice operators encounter severe renormalisations
  - Link expansion: $U_\mu(x) = 1 + igaA_\mu(x) - g^2a^2A_\mu^2(x)$
  - Perturbative connection ruined at second order:

$$\langle A_\mu^2 \rangle \sim 1/a^2 \implies \text{2nd order term} \sim \mathcal{O}(g^2)$$

- Deviate significantly from continuum counterparts



Interactive 3D graphics

- Rendered in AVS Express Visualisation Edition
  - http://www.avs.com/solutions/express/
- Exported as VRML
- Converted to U3D format via PDF3D ReportGen
  - https://www.pdf3d.com/products/pdf3d-reportgen/
- Imported into LaTeX via media9 package
  - Viewed in Adobe Reader