Interacting p-form gauge theories: New developments

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Based on:

- C. Ferko, SMK, L. Smith and G. Tartaglino-Mazzucchelli, Phys. Rev. D **108**, 106021 (2023) [arXiv:2309.04253]
- C. Ferko, SMK, K. Lechner, D. P. Sorokin and G. Tartaglino-Mazzucchelli, JHEP **05**, 320 (2024) [arXiv:2402.06947]

Duality, confinement, gravity

- (Electromagnetic) duality and confinement are often interrelated, especially in supersymmetric Yang-Mills theories.

  Seiberg & Witten (1994)

- Patterns of duality invariance were observed in the late 1970s in extended supergravity.

  Ferrara, Scherk & Zumino (1977)
  Cremmer & Julia (1979)

  This triggered research into general aspects of duality invariance.

Impact of supergravity on theoretical physics

- Realisation of Einstein's dream to unify gravity & electromagnetism (1976, $\mathcal{N} = 2$ supergravity).
- New types of gauge theories (compared with Yang-Mills theories):
  1. open gauge algebra; and/or
  2. linearly dependent gauge generators (e.g., gauge p-forms in $d > p > 1$ dimensions).

  New quantisation methods (standard Faddeev-Popov approach is not applicable), including the Batalin-Vilkovisky formalism.
- New types of anomalies (e.g., superconformal anomalies).
- Modern Kaluza-Klein theories.
- Renaissance of electromagnetic duality (nonlinear self-duality).
- Gauge/gravity duality (AdS/CFT).

This talk is mainly devoted to deformations of U(1) duality-invariant models for nonlinear electrodynamics and their six-dimensional counterparts – interacting chiral form field theories, specifically:
New surprising results concerning these old subjects.

Outline

1. U(1) duality in nonlinear electrodynamics
2. Generating formalism for duality-invariant theories
3. $T\bar{T}$-like flows in four dimensions
4. $T\bar{T}$-like flows in $4n$ dimensions
5. $T\bar{T}$-like flows in $4n+2$ dimensions

Electromagnetic duality: Maxwell's theory

- Maxwell's electrodynamics is the simplest and oldest example of a duality-invariant theory in four spacetime dimensions.

$$L_{\text{Maxwell}}(F) = -\frac{1}{4} F^{ab} F_{ab} = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) , \qquad F_{ab} = \partial_a A_b - \partial_b A_a$$

- The Bianchi identity and the equation of motion are

$$\partial^b \widetilde{F}_{ab} = 0 , \qquad \partial^b F_{ab} = 0$$

with $\widetilde{F}_{ab} := \frac{1}{2}\, \varepsilon_{abcd}\, F^{cd}$ the Hodge dual of F.

- Since both differential equations have the same functional form, one may consider so-called duality rotations

$$F + \mathrm{i}\widetilde{F} \to \mathrm{e}^{\mathrm{i}\varphi}\big(F + \mathrm{i}\widetilde{F}\big) \quad \Longleftrightarrow \quad \vec{E} + \mathrm{i}\vec{B} \to \mathrm{e}^{\mathrm{i}\varphi}\big(\vec{E} + \mathrm{i}\vec{B}\big) , \quad \varphi \in \mathbb{R}$$

- Lagrangian $L_{\text{Maxwell}}(F)$ changes, but the energy-momentum tensor

$$T^{ab} = \frac{1}{2}\big(F + \mathrm{i}\widetilde{F}\big)^{ac}\big(F - \mathrm{i}\widetilde{F}\big)^{bd}\,\eta_{cd} = F^{ac} F^{bd} \eta_{cd} - \frac{1}{4}\eta^{ab} F^{cd} F_{cd}$$

is invariant under U(1) duality transformations.

Electromagnetic duality: Born-Infeld theory

- In 1934, Born & Infeld put forward a particular model for nonlinear electrodynamics

$$L_{\rm BI}(F) = \frac{1}{g^2}\Big\{ 1 - \sqrt{-\det(\eta_{ab} + g F_{ab})} \Big\} = -\frac{1}{4} F^{ab} F_{ab} + \mathcal{O}(F^4)$$

  as a new fundamental theory of the electromagnetic field (with g the coupling constant).

- In 1935, Schrödinger showed that the Born-Infeld theory possessed invariance under generalised U(1) duality rotations.
- Although the great expectations of Born and Infeld never came true, the Born-Infeld action has re-appeared in the spotlight since the 1980's as a low-energy effective action in string theory.

  Fradkin & Tseytlin (1985)

  D-branes — Polchinski (1995)

Born-Infeld action, duality and supersymmetry

There exist deep and mysterious connections between nonlinear duality invariance and supersymmetry.

- Maxwell-Goldstone multiplet model for partial $\mathcal{N} = 2 \to \mathcal{N} = 1$ supersymmetry breakdown in $\mathbb{M}^4$

  [Bagger & Galperin, 1996; Roček & Tseytlin, 1999]

  1. is $\mathcal{N} = 1$ supersymmetric extension of Born-Infeld action

     [Cecotti & Ferrara, 1987]
  2. is invariant under U(1) supersymmetric duality rotations.

     [Brace, Morariu & Zumino, 1999; SMK & Theisen, 2000]

- Maxwell-Goldstone multiplet model for partial $\mathcal{N} = 2 \to \mathcal{N} = 1$ SUSY breaking for curved maximally SUSY backgrounds: (i) $\mathbb{R} \times S^3$; (ii) $\mathrm{AdS}_3 \times \mathbb{R}$; and (iii) pp-wave spacetime

  [SMK & Tartaglino-Mazzucchelli, 2016]

  with analogous properties.

There exist only five maximally supersymmetric backgrounds in $d = 4$:

[Festuccia & Seiberg, 2011]

(i) $\mathbb{M}^4$; (ii) AdS_4; (iii) $\mathbb{R} \times S^3$; (iv) $\mathrm{AdS}_3 \times \mathbb{R}$; and (v) pp-wave spacetime.

Duality invariance and supersymmetry

- AdS/CFT correspondence provides the main evidence to believe in self-duality of the low-energy effective action for the $\mathcal{N} = 4$ SU(N) SYM theory on its Coulomb branch where the gauge group SU(N) is spontaneously broken to SU($N - 1$) $\times$ U(1).
- It predicts the $\mathcal{N} = 4$ SYM effective action (in the large-N limit) is related to the D3-brane action in $AdS_5 \times S^5$

$$S = T_3 \int \mathrm{d}^4 x \left(h^{-1} - \sqrt{-\det(g_{mn} + F_{mn})} \right) ,$$

$$g_{mn} = h^{-1/2} \eta_{mn} + h^{1/2} \partial_m X^I \partial_n X^I , \quad h = \frac{Q}{(X^I X^I)^2} ,$$

where X^I, $I = 1, \cdots, 6$, are transverse coordinates, $T_3 = (2\pi g_s)^{-1}$ and $Q = g_s(N - 1)/\pi$.

- The action S/T_3 possesses (deformed) conformal symmetry and is self-dual in the sense that it enjoys invariance under electromagnetic U(1) duality rotations.
- Self-duality of D3-brane action is a fundamental property related to the S-duality of type IIB string theory.

Tseytlin (1996), Green & Gutperle (1996)

Electromagnetic duality: Nonlinear electrodynamics

- General theory of duality invariance in four dimensions

  Gaillard & Zumino (1981)
  Gibbons & Rasheed (1995)
  Gaillard & Zumino (1997)

- General theory of duality invariance in higher dimensions

  Gibbons & Rasheed (1995)
  Araki &Tanii (1999)
  Aschieri, Brace, Morariu & Zumino (2000)

- General theory of duality invariance for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric nonlinear electrodynamics

  SMK & Theisen (2000)

  Partial SUSY breaking often implies U(1) duality invariance.

- Remarkable reformulation of duality-invariant nonlinear electrodynamics (manifest duality-invariant self-interaction).

  Ivanov & Zupnik (2001)

U(1) duality in nonlinear electrodynamics

- Nonlinear electrodynamics (effective field theory)

$$L(F_{ab}) = -\frac{1}{4}F^{ab}F_{ab} + \mathcal{O}(F^4)$$

- Using the definition

$$\widetilde{G}_{ab}(F) := \frac{1}{2}\,\varepsilon_{abcd}\,G^{cd}(F) = 2\,\frac{\partial L(F)}{\partial F^{ab}}\ , \qquad G(F) = \widetilde{F} + \mathcal{O}(F^3),$$

  the Bianchi identity (BI) and the equation of motion (EoM) read

$$\partial^b \widetilde{F}_{ab} = 0\ , \qquad \partial^b \widetilde{G}_{ab} = 0\ .$$

- The same functional form of BI and EoM provides a rationale to introduce a duality transformation

$$\begin{pmatrix} G'(F') \\ F' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} G(F) \\ F \end{pmatrix}\ , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$$

  For $G'(F')$ one should require

$$\widetilde{G}'_{ab}(F') \;=\; 2\,\frac{\partial L'(F')}{\partial F'^{ab}}$$

  Transformed Lagrangian, $L'(F)$, always exists.

U(1) duality in nonlinear electrodynamics

The above considerations become nontrivial if the model is required to be duality invariant (self-dual)

$$L'(F) = L(F) \ .$$

The requirement of self-duality implies the following:

- Only U(1) duality transformations can consistently be defined in the nonlinear case.

$$\begin{pmatrix} G(F') \\ F' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} G(F) \\ F \end{pmatrix}$$

  Maxwell's theory also allows scale duality transformations which, however, are forbidden if the energy-momentum tensor is required to be duality invariant.

- The Lagrangian is a solution of the self-duality equation

$$G^{ab}\,\widetilde{G}_{ab} + F^{ab}\,\widetilde{F}_{ab} = 0 \ , \qquad \widetilde{G}_{ab}(F) = 2\,\frac{\partial L(F)}{\partial F^{ab}}$$

Bialynicki-Birula (1983) (remained unnoticed for many years)
Gibbons & Rasheed (1995) Gaillard & Zumino (1997)

Fundamental properties of U(1) duality-invariant models

- Duality invariance of the energy-momentum tensor.
- $\mathsf{SL}(2,\mathbb{R})$ duality invariance in the presence of dilaton and axion.
- Self-duality under Legendre transformation.

Duality invariance of the energy-momentum tensor

- Given a duality-invariant parameter g in the self-dual theory, $\partial L(F,g)/\partial g$ is duality invariant.

$$\delta \frac{\partial}{\partial g} L = \frac{\partial}{\partial g} \delta L = \frac{1}{2} \lambda \frac{\partial}{\partial g} \left(\tilde{G} \cdot G \right) = \frac{1}{2} \lambda \frac{\partial}{\partial g} \left(\tilde{G} \cdot G + \tilde{F} \cdot F \right) = 0 \ ,$$

since F is g-independent. Gaillard & Zumino (1997)

- In particular, the energy-momentum tensor T_{ab} is duality invariant.

Non-compact duality: Coupling to dilaton and axion

- Given a U(1) duality-invariant model, $L(F_{mn}) = L(\omega, \bar{\omega})$, its compact duality group U(1) is enhanced to the non-compact $\mathsf{SL}(2,\mathbb{R})$ group by coupling F_{ab} to dilaton φ and axion $\mathfrak{a}$ by replacing

$$L(F) \;\to\; L(F,\tau,\bar{\tau}) \;=\; L(\mathrm{e}^{-\varphi/2}\,F) + \frac{1}{4}\,\mathfrak{a}\,F\cdot\tilde{F}\;, \qquad \tau = \mathfrak{a} + \mathrm{i}\,\mathrm{e}^{-\varphi}$$

Gibbons & Rasheed (1996) Gaillard & Zumino (1997)

- The duality group acts by transformations

$$\begin{pmatrix} G' \\ F' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} G \\ F \end{pmatrix}, \quad \tau' = \frac{a\tau + b}{c\tau + d}\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}(2,\mathbb{R})$$

- Maxwell's theory coupled to the dilaton and axion

$$L(F,\tau,\bar{\tau}) = -\frac{1}{4}\mathrm{e}^{-\varphi}F^{mn}F_{mn} + \frac{1}{4}\mathfrak{a}\tilde{F}^{mn}F_{mn}$$

is Weyl invariant (conformal in flat space), with τ being Weyl inert.
τ & $\bar{\tau}$ local couplings.
Non-minimal operator $\Longrightarrow$ generalised heat kernel techniques.

Non-compact duality and quantum theory

- Let $\Gamma(\tau, \bar{\tau})$ be the effective action obtained by integrating out the gauge field in the path integral.
- Both Weyl and rigid $\mathsf{SL}(2, \mathbb{R})$ duality transformations are anomalous at the quantum level. However, the logarithmically divergent part of $\Gamma(\tau, \bar{\tau})$ is invariant under these transformations.
- General structure of the logarithmic divergence of $\Gamma(\tau, \bar{\tau})$

$$
\begin{aligned}
\mathfrak{L} \quad = \quad & \frac{1}{2(\mathrm{Im}\,\tau)^2}\Big[\mathcal{D}^2\tau\mathcal{D}^2\bar{\tau} - 2(R^{mn} - \frac{1}{3}g^{mn}R)\nabla_m\tau\nabla_n\bar{\tau}\Big] \\
& + \frac{1}{12(\mathrm{Im}\,\tau)^4}\Big[\alpha\nabla^m\tau\nabla_m\tau\nabla^n\bar{\tau}\nabla_n\bar{\tau} + \beta\nabla^m\tau\nabla_m\bar{\tau}\nabla^n\tau\nabla_n\bar{\tau}\Big]
\end{aligned}
$$

where $\mathcal{D}^2\tau := \nabla^m\nabla_m\tau + \frac{\mathrm{i}}{\mathrm{Im}\,\tau}\nabla^m\tau\nabla_m\tau$, and α and β are numerical parameters.

Osborn (2003)

- $\int \mathrm{d}^4x\,\sqrt{-g}\,\mathfrak{L}$ is $\mathsf{SL}(2, \mathbb{R})$ invariant.
- $\int \mathrm{d}^4x\,\sqrt{-g}\,\mathfrak{L}$ is invariant under Weyl transformations

$$
g_{mn}(x) \to \mathrm{e}^{2\sigma(x)}g_{mn}(x)\ , \qquad \tau(x) \to \tau(x)
$$

- Contribution to the Weyl anomaly: $\delta_\sigma\Gamma(\tau, \bar{\tau}) \propto \int \mathrm{d}^4x\,\sqrt{-g}\,\sigma\,\mathfrak{L}$

Self-duality under Legendre transformation

Legendre transformation for nonlinear electrodynamics $L(F)$.

- Associate with $L(F)$ an equivalent auxiliary model defined by

$$L(F, F_{\rm D}) = L(F) - \frac{1}{2}\, F \cdot \widetilde{F}_{\rm D} \ , \qquad\qquad F_{\rm D}{}^{ab} = \partial^a A_{\rm D}{}^b - \partial^b A_{\rm D}{}^a \ ,$$

  in which F_{ab} is an unconstrained two-form (auxiliary field).

- Eliminate F_{ab} using its equation of motions $G(F) = F_{\rm D}$ to yield

$$L_{\rm D}(F_{\rm D}) := \Big(L(F) - \frac{1}{2} F \cdot \widetilde{F}_{\rm D} \Big) \Big|_{F = F(F_{\rm D})} \ .$$

- If $L(F)$ solves the self-duality equation $G \cdot \widetilde{G} + F \cdot \widetilde{F} = 0$, then

$$L_{\rm D}(F) = L(F) \ .$$

  Self-dual electrodynamics

General structure of self-dual electrodynamics

- Given a model for nonlinear electrodynamics, its Lagrangian $L(F_{ab})$ can be realised as a real function of one complex variable,

$$L(F_{ab}) = L(\omega, \bar{\omega}) \ , \qquad \omega = \alpha + \mathrm{i}\,\beta \ ,$$

with $\alpha = \frac{1}{4} F^{ab} F_{ab}$ and $\beta = \frac{1}{4} F^{ab} \widetilde{F}_{ab}$ the EM invariants.

$$L(\omega, \bar{\omega}) = -\frac{1}{2} \Big(\omega + \bar{\omega} \Big) + \omega \, \bar{\omega} \, \Lambda(\omega, \bar{\omega}) \ .$$

- Self-duality equation (SDE), $G \cdot \widetilde{G} + F \cdot \widetilde{F} = 0$, turns into

$$\mathrm{Im} \left\{ \frac{\partial(\omega \, \Lambda)}{\partial \omega} - \bar{\omega} \, \left(\frac{\partial(\omega \, \Lambda)}{\partial \omega} \right)^2 \right\} = 0 \ .$$

- Assuming $\Lambda(\omega, \bar{\omega})$ to be real analytic at $\omega = 0$ (existence of weak-field limit), the general solution of SDE involves a real function of one real argument $f(\omega\bar{\omega})$

$$\Lambda(\omega, \bar{\omega}) = \sum_{n=0}^{\infty} \sum_{p+q=n} c_{p,q} \, \omega^p \bar{\omega}^q \ , \qquad c_{p,q} = c_{q,p} \in \mathbb{R}$$

SDE uniquely fixes the level-n coefficients $c_{p,q}$ with $p \neq q$ through those at lower levels, while $c_{r,r}$ remain undetermined.

Functional freedom: Real function of one real variable.

General structure of self-dual electrodynamics

- Omitting the requirement of $\Lambda(\omega,\bar{\omega})$ being real analytic at $\omega = 0$, new solutions of the self-duality equation become possible.
- ModMax theory

$$L_{\rm MM}(\omega,\bar{\omega}) = -\tfrac{1}{2}(\omega+\bar{\omega})\cosh\gamma + \sqrt{\omega\bar{\omega}}\sinh\gamma\ ,$$
$$\omega = \alpha + {\rm i}\,\beta\ , \qquad \alpha = \tfrac{1}{4}\,F^{ab}F_{ab}\ , \quad \beta = \tfrac{1}{4}\,F^{ab}\widetilde{F}_{ab}\ ,$$

with $\gamma \geq 0$ a parameter.

Bandos, Lechner, Sorokin & Townsend arXiv:2007.09092

- The corresponding $\Lambda(\omega,\bar{\omega})$ is

$$\Lambda_{\rm MM}(\omega,\bar{\omega}) = \frac{\sinh\gamma}{\sqrt{\omega\bar{\omega}}} - \frac{1}{2}\Big(\frac{1}{\omega}+\frac{1}{\bar{\omega}}\Big)(\cosh\gamma - 1)\ ,$$

- Unique conformal solution of the self-duality equation.

Formulation with manifestly U(1) invariant interaction

- Self-duality equation $G \cdot \widetilde{G} + F \cdot \widetilde{F} = 0$ is a nonlinear equation on the Lagrangian $L(F)$, and U(1) duality-invariant deformations of $L(F)$ are difficult to control.
- In 2001, Ivanov & Zupnik proposed a reformulation of nonlinear electrodynamics with the property that U(1) duality invariance becomes equivalent to manifest U(1) invariance of the interaction.
- Nonlinear twisted self-duality constraint, which was put forward by Bossard & Nicolai (2011) and by Carrasco, Kallosh & Roiban (2012), proves to be a variant of the Ivanov-Zupnik formulation.

Formulation with manifestly U(1) invariant interaction

- The Ivanov-Zupnik formulation involves an auxiliary (unconstrained) antisymmetric tensor $V_{ab} = -V_{ba}$, which is equivalently described by a symmetric rank-2 spinor $V_{\alpha\beta} = V_{\beta\alpha}$ and its conjugate $\bar{V}_{\dot\alpha\dot\beta}$, where $\alpha, \beta = 1, 2$.
- $L(F_{ab})$ is replaced with a new Lagrangian

$$L(F_{ab}, V_{ab}) = \frac{1}{4}F^{ab}F_{ab} + \frac{1}{2}V^{ab}V_{ab} - V^{ab}F_{ab} + L_{\rm int}(V_{ab}) \ .$$

  The original Lagrangian $L(F_{ab})$ is obtained from $L(F_{ab}, V_{ab})$ by integrating out the auxiliary variables.
- The condition of U(1) duality invariance proves to be equivalent to the requirement that the self-interaction

$$L_{\rm int}(V_{ab}) = L_{\rm int}(\nu, \bar\nu) \ , \qquad \nu := V_+^{ab}V_{+ab}$$

$$V_\pm^{ab} = \frac{1}{2}\big(V^{ab} \pm {\rm i}\widetilde{V}^{ab}\big) \ , \quad \widetilde{V}_\pm = \mp{\rm i}V_\pm \ , \quad V = V_+ + V_-$$

  is invariant under linear U(1) transformations $\nu \to {\rm e}^{{\rm i}\varphi}\nu$, with $\varphi \in \mathbb{R}$,

$$L_{\rm int}(\nu, \bar\nu) = L_{\rm int}({\rm e}^{{\rm i}\varphi}\nu, {\rm e}^{-{\rm i}\varphi}\bar\nu) \quad \Longrightarrow \quad L_{\rm int}(\nu, \bar\nu) = h(\nu\bar\nu) \ ,$$

  h an arbitrary real function of one real variable (functional freedom).

Conformal duality-invariant electrodynamics

- ModMax theory

$$L_{\rm MM}(\omega,\bar\omega) = -\tfrac{1}{2}\cosh\gamma\Big(\omega+\bar\omega\Big) + \sinh\gamma\sqrt{\omega\bar\omega}~,$$

$$\omega = \alpha + {\rm i}\,\beta~, \qquad \alpha = \tfrac{1}{4}\,F^{ab}F_{ab}~, \quad \beta = \tfrac{1}{4}\,F^{ab}\widetilde{F}_{ab}~,$$

with $\gamma \geq 0$ a parameter.

Bandos, Lechner, Sorokin & Townsend arXiv:2007.09092

- Derivation of ModMax using the Ivanov-Zupnik approach

SMK arXiv:2106.07173

This unique conformal duality-invariant model corresponds to

$$L_{\rm int}(\nu,\bar\nu) = \kappa\sqrt{\nu\bar\nu}~,$$

with κ a coupling constant. Integrating out the auxiliary variables $V_{\alpha\beta}$ and $\bar V_{\dot\alpha\dot\beta}$ leads to $L_{\rm MM}(\omega,\bar\omega)$ with

$$\sinh\gamma = \frac{\kappa}{1-(\kappa/2)^2}~.$$

$T\bar{T}$-like flows in four dimensions

$T\bar{T}$ deformations of QFTs in two dimensions:

Zamolodchikov (2004)
Smirnov & Zamolodchikov, arXiv:1608.05499
Cavaglià, Negro, Szécsényi &Tateo, arXiv:1608.05534 ......

Remarkably active research direction

In four dimensions, we are forced to work with effective field theories, hence $T\bar{T}$-like flows

$T\bar{T}$-like deformations in four dimensions

Two examples of $T\bar{T}$-like flows

- Born-Infeld theory ($\lambda = g^2$)

$$L_{\rm BI}(F) = \frac{1}{\lambda}\Big\{1 - \sqrt{-\det(\eta_{ab} + \sqrt{\lambda}F_{ab})}\Big\}$$
$$= \frac{1}{\lambda}\Big\{1 - \sqrt{1 + \frac{\lambda}{2}F^2 - \frac{\lambda^2}{16}(F\widetilde{F})^2}\Big\}$$

It holds that

$$\frac{\partial L_{\rm BI}}{\partial \lambda} = \frac{1}{8}\Big(T^{ab}T_{ab} - \frac{1}{2}(T^a{}_a)^2\Big)$$

- ModMax theory

$$L_{\rm MM} = -\frac{1}{4}F^2\cosh(\gamma) + \frac{1}{4}\sqrt{(F^2)^2 + \Big(F\widetilde{F}\Big)^2}\,\sinh(\gamma)$$

It holds that (root $T\bar{T}$)

$$\frac{\partial L_{\rm MM}}{\partial \gamma} = \frac{1}{2}\sqrt{T^{ab}T_{ab} - \frac{1}{4}(T^a{}_a)^2} = \frac{1}{2}\sqrt{T^{ab}T_{ab}}$$

- Can the above results be manifestations of a general pattern?

$T\bar{T}$-like deformations

Ferko, SMK, Smith & Tartaglino-Mazzucchelli (2023)

Consider a U(1) duality-invariant theory with Lagrangian $L(F)$. An observable $\mathcal{O}(F)$ is duality invariant if it obeys the equation

$$\frac{\partial \mathcal{O}}{\partial F_{ab}} G_{ab} = 0 \ , \qquad \delta_\phi F_{ab} = \varphi G_{ab}(F)$$

- Theorem 1: Any two duality-invariant observables $\mathcal{O}_1(F)$ and $\mathcal{O}_2(F)$ prove to be functionally dependent,

$$\Upsilon(\mathcal{O}_1, \mathcal{O}_2) = 0$$

- Theorem 2: Every duality-invariant observable $\mathcal{O}(F)$ is as a function of the energy-momentum tensor, $\mathcal{O} = f(T_{ab})$.
- Corollary: Given a one-parameter family of U(1) duality-invariant theories, $L(F, g)$, Lagrangian obeys $T\bar{T}$-like flow equation

$$\frac{\partial}{\partial g} L = \mathfrak{S}(T_{ab}) \ .$$

$T\bar{T}$-like deformations in four dimensions

Consider a one-parameter family of theories $L^{(\lambda)}(F)$ satisfying the differential equation and boundary condition

$$\frac{\partial L^{(\lambda)}(F)}{\partial \lambda} := \mathcal{O}^{(\lambda)}(F) = \mathcal{O}(F;\lambda) \ , \qquad L^{(0)}(F) = L(F) \ ,$$

with $\mathcal{O}^{(\lambda)}(F)$ being a duality-invariant function,

$$\frac{\partial \mathcal{O}^{(\lambda)}}{\partial F_{ab}} G^{(\lambda)}_{ab}(F) = 0$$

If the Lagrangian $L(F)$ describes a U(1) duality-invariant theory satisfying

$$G^{ab}\,\widetilde{G}_{ab} + F^{ab}\,\widetilde{F}_{ab} = 0 \ ,$$

then all theories associated with the Lagrangians $L^{(\lambda)}(F)$ are duality invariant.

$T\bar{T}$-like flows for gauge $(2n-1)$-forms in $4n$ dimensions

- The Gaillard-Zumino-Gibbons-Rasheed formalism for nonlinear electrodynamics in $d = 4$ was extended to $d = 4n$ dimensions, $n > 1$, in the late 1990s.

  Gibbons & Rasheed (1995), Araki & Tanii (1999)

- In a curved space $\mathcal{M}^{4n}$, a self-interacting theory of a gauge p-form $A_{\mu_1 \dots \mu_p}$ (for $p = 2n-1$) such that its Lagrangian, $L = L(F)$, is a function of the field strength $F_{\mu_1 \dots \mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]}$.
- In order for this theory to possess U(1) duality invariance, its Lagrangian must satisfy the self-duality equation

$$G^{\mu_1 \dots \mu_{p+1}} \widetilde{G}_{\mu_1 \dots \mu_{p+1}} + F^{\mu_1 \dots \mu_{p+1}} \widetilde{F}_{\mu_1 \dots \mu_{p+1}} = 0 ,$$

  with $\widetilde{G}^{\mu_1 \dots \mu_{p+1}}(F) = (p+1)! \partial L(F) / \partial F_{\mu_1 \dots \mu_{p+1}}$

- Every solution of the self-duality equation defines a U(1) duality-invariant theory. Infinitesimal U(1) duality transformation is

$$\delta \begin{pmatrix} G \\ F \end{pmatrix} = \begin{pmatrix} 0 & -\varphi \\ \varphi & 0 \end{pmatrix} \begin{pmatrix} G \\ F \end{pmatrix} , \qquad \varphi \in \mathbb{R}$$

$T\bar{T}$-like flows for gauge $(2n-1)$-forms in $4n$ dimensions

- Duality-invariant observable $\mathcal{O}(F)$

$$\frac{\partial \mathcal{O}(F)}{\partial F_{\mu_1\dots\mu_{p+1}}} G_{\mu_1\dots\mu_{p+1}} = 0 \ .$$

- Such observables generate consistent flows in the space of field theories describing the dynamics of self-interacting gauge p-forms.
- Let $L^{(\gamma)}(F)$ and $\mathcal{O}^{(\gamma)}(F)$ be two scalar functions that depend on a real parameter γ and satisfy the following conditions:
  1. $L^{(\gamma)}$ and $\mathcal{O}^{(\gamma)}$ obey the equations

$$\frac{\partial}{\partial \gamma} L^{(\gamma)} = \mathcal{O}^{(\gamma)} , \qquad \frac{\partial \mathcal{O}^{(\gamma)}(F)}{\partial F_{\mu_1\dots\mu_{p+1}}} G^{(\gamma)}_{\mu_1\dots\mu_{p+1}} = 0 \ .$$

  2. $L(F) \equiv L^{(0)}(F)$ is a solution of the self-duality equation.

  Then $L^{(\gamma)}(F)$ is a solution of the self-duality equation $\forall\gamma$.

In the $n>1$ case, we do not yet know whether all flows of self-dual theories are generated by the energy-momentum tensor.

Deformations of chiral two-form gauge theories in $d = 6$

- $d = 6$ counterparts of U(1) duality-invariant models for nonlinear electrodynamics are interacting chiral two-form gauge theories.
- PST formulation for chiral two-forms in six dimensions
  Pasti, Sorokin & Tonin (1996, 1997)
- Every deformation of interacting chiral two-form gauge theory is generated by the energy-momentum tensor.
  Ferko, SMK, Lechner, Sorokin & Tartaglino-Mazzucchelli (2024)
  Some technical details are provided below.

$T\bar{T}$-like flows for chiral $2n$-forms in $4n+2$ dimensions

- PST formulation for chiral p-forms in $4n+2 \equiv 2p+2$ dimensions

  Pasti, Sorokin & Tonin (1996, 1997)

  Buratti, Lechner and Melotti (2019)

- $A_{\mu(p)} = A_{\mu_1 \dots \mu_p}$ is a gauge p-form potential on a time orientable spacetime $\mathcal{M}^d$ with metric $g_{\mu\nu}$, and

$$F_{\mu(p+1)} = (p+1)\partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]}$$

  its gauge-invariant field strength.

- Introduce a normalized timelike vector field v^μ,

$$v^\mu v_\mu = -1\,.$$

  Its existence is guaranteed on $(\mathcal{M}^d, g_{\mu\nu})$.

- Associate with $F_{\mu(p+1)}$ the electric field

$$E_{\mu(p)} = F_{\mu_1 \dots \mu_p \nu} v^\nu\,, \qquad E_{\mu_1 \dots \mu_{p-1}\sigma} v^\sigma = 0\,,$$

  and the magnetic field

$$B_{\mu(p)} = \widetilde{F}_{\mu_1 \dots \mu_p \nu} v^\nu\,, \qquad B_{\mu_1 \dots \mu_{p-1}\sigma} v^\sigma = 0\,, \qquad \widetilde{\widetilde{F}} = F$$

$T\bar{T}$-like flows for chiral $2n$-forms in $4n+2$ dimensions

- Action functional

$$S[A,a] = \int \mathrm{d}^d x \sqrt{-g}\left[\frac{1}{2p!} E\cdot B - \mathcal{H}(B_{\mu(p)}, g_{\mu\nu})\right] , \quad v_\mu = \frac{\partial_\mu a}{\sqrt{-\partial a \cdot \partial a}}$$

Existence of a scalar field $a(x)$, such that v^μ is past directed and timelike, is guaranteed on every globally hyperbolic spacetime.

- The scalar function $\mathcal{H}(B_{\mu(d)}, g_{\mu\nu})$ must satisfy a differential condition in order for $S[A,a]$ to be invariant under PST gauge transformations (see below). Defining the derivative of $\mathcal{H}$ by

$$\delta_B \mathcal{H}(B,g) = \frac{1}{p!}\delta B^{\mu_1 \dots \mu_p} H_{\mu_1 \dots \mu_p} , \qquad H_{\mu_1 \dots \mu_{p-1}\nu} v^\nu = 0 ,$$

the master equation on $\mathcal{H}$ is

$$B_{[\mu_1 \dots \mu_p} B_{\mu_{p+1} \dots \mu_{2p}]} = H_{[\mu_1 \dots \mu_p} H_{\mu_{p+1} \dots \mu_{2p}]} ,$$

Analogue of the self-duality equation in $4n$ dimensions

$T\bar{T}$-like flows for gauge $2n$-forms in $4n+2$ dimensions

- PST gauge transformations

$$\delta A_{\mu(p)} = p v_{[\mu_1} \psi_{\mu_2 \ldots \mu_p]}\,, \qquad \delta a = 0\,;$$
$$\delta A_{\mu(p)} = -\frac{\varphi}{\sqrt{-\partial a \partial a}} \Big(E_{\mu(p)} - H_{\mu(p)} \Big)\,, \qquad \delta a = \varphi\,.$$

$a(x)$ is a Stueckelberg field. Useful gauge condition $\partial_\mu a = \delta_\mu{}^0$.

- Gauge freedom associated with the first transformation allows us to write the equation of motion for A in the form

$$E_{\mu(p)} - H_{\mu(p)} = 0$$

Nonlinear self-duality condition

$T\bar{T}$-like flows for chiral $2n$-forms in $4n+2$ dimensions

Ferko, SMK, Lechner, Sorokin & Tartaglino-Mazzucchelli (2024)

- Invariant observable $\mathcal{O}(B_{\mu(p)}, g_{\mu\nu})$ is a scalar function satisfying the first-order differential equation

$$\mathcal{O}_{[\mu_1 \dots \mu_p} H_{\mu_{p+1} \dots \mu_{2p}]} = 0 \,.$$

  On the mass shell such quantities are v^μ-field independent and hence Lorentz (or general coordinate) invariant.

- Suppose the interaction term depends on a parameter γ,

$$S[A, a; \gamma] = \int \mathrm{d}^d x \sqrt{-g} \left[\frac{1}{2p!} E \cdot B - \mathcal{H}(B_{\mu(p)}, g_{\mu\nu}; \gamma) \right] \,,$$

  such that $\mathcal{H}(B_{\mu(p)}, g_{\mu\nu}; \gamma)$ is a solution of the equation

$$B_{[\mu_1 \dots \mu_p} B_{\mu_{p+1} \dots \mu_{2p}]} = H_{[\mu_1 \dots \mu_p} H_{\mu_{p+1} \dots \mu_{2p}]}$$

  for every value of γ. Then

$$\mathcal{O} = \frac{\partial}{\partial \gamma} \mathcal{H}(B, g; \gamma)$$

  is an invariant observable.

$T\bar{T}$-like flows for chiral $2n$-forms in $4n+2$ dimensions

Let $\mathcal{H}^{(\gamma)}(B_{\mu(p)}, g_{\mu\nu})$ and $\mathcal{O}^{(\gamma)}(B_{\mu(p)}, g_{\mu\nu})$ be two scalar functions that depend on a real parameter γ and satisfy the following conditions:

- $\mathcal{H}^{(\gamma)}$ and $\mathcal{O}^{(\gamma)}$ obey the equations

$$\frac{\partial}{\partial\gamma}\mathcal{H}^{(\gamma)} = \mathcal{O}^{(\gamma)}\,, \qquad \mathcal{O}^{(\gamma)}_{[\mu_1\dots\mu_p}H^{(\gamma)}_{\mu_{p+1}\dots\mu_{2p}]} = 0\,;$$

- $\mathcal{H}^{(0)}(B_{\mu(p)}, g_{\mu\nu})$ is a solution of

$$B_{[\mu_1\dots\mu_p}B_{\mu_{p+1}\dots\mu_{2p}]} = H_{[\mu_1\dots\mu_p}H_{\mu_{p+1}\dots\mu_{2p}]}$$

Then $\mathcal{H}^{(\gamma)}(B_{\mu(p)}, g_{\mu\nu})$ is a solution of the master equation at every value of the parameter γ.

Six-dimensional story ($n=1$):

- Any two invariant observables $\mathcal{O}_1$ and $\mathcal{O}_2$ are functionally dependent.
- Every invariant observable $\mathcal{O}$ proves to be a function of the energy-momentum tensor, $\mathcal{O} = f(T_{\mu\nu})$.