Polyakov loop, random matrix, and color confinement

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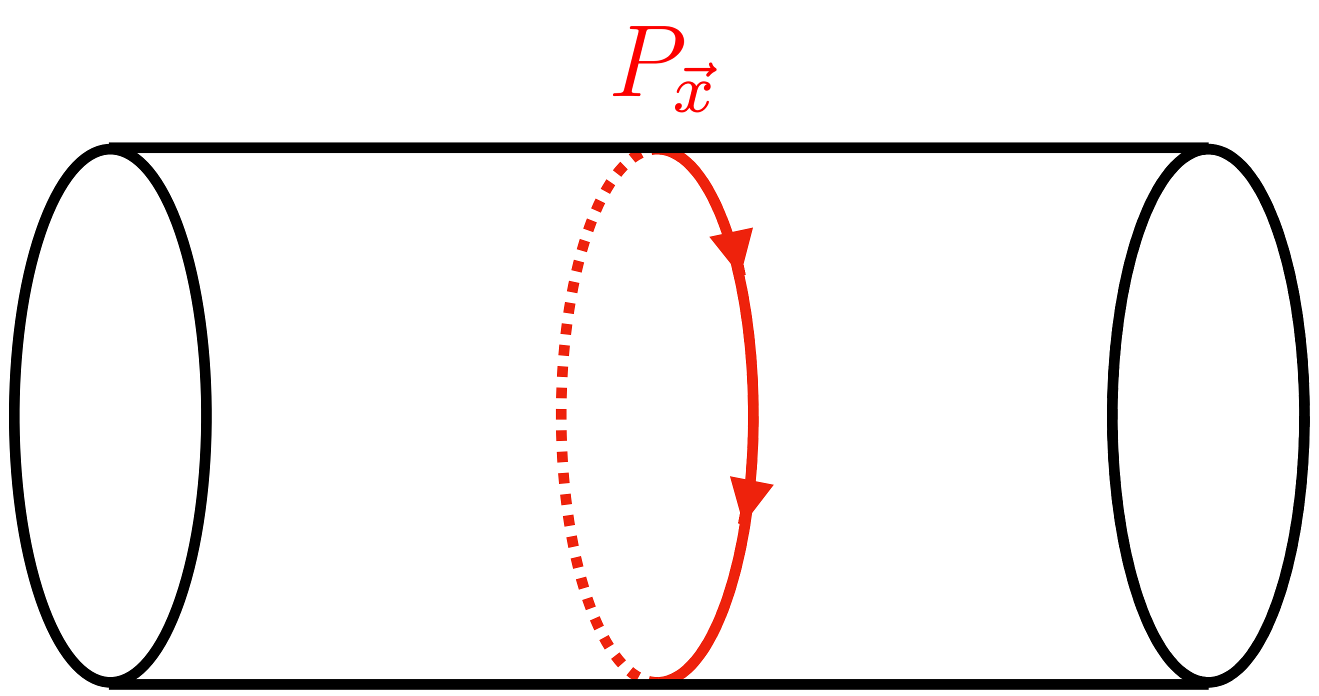
— Polyakov loop has a meaning not related to center symmetry *for any gauge theory* including QCD.

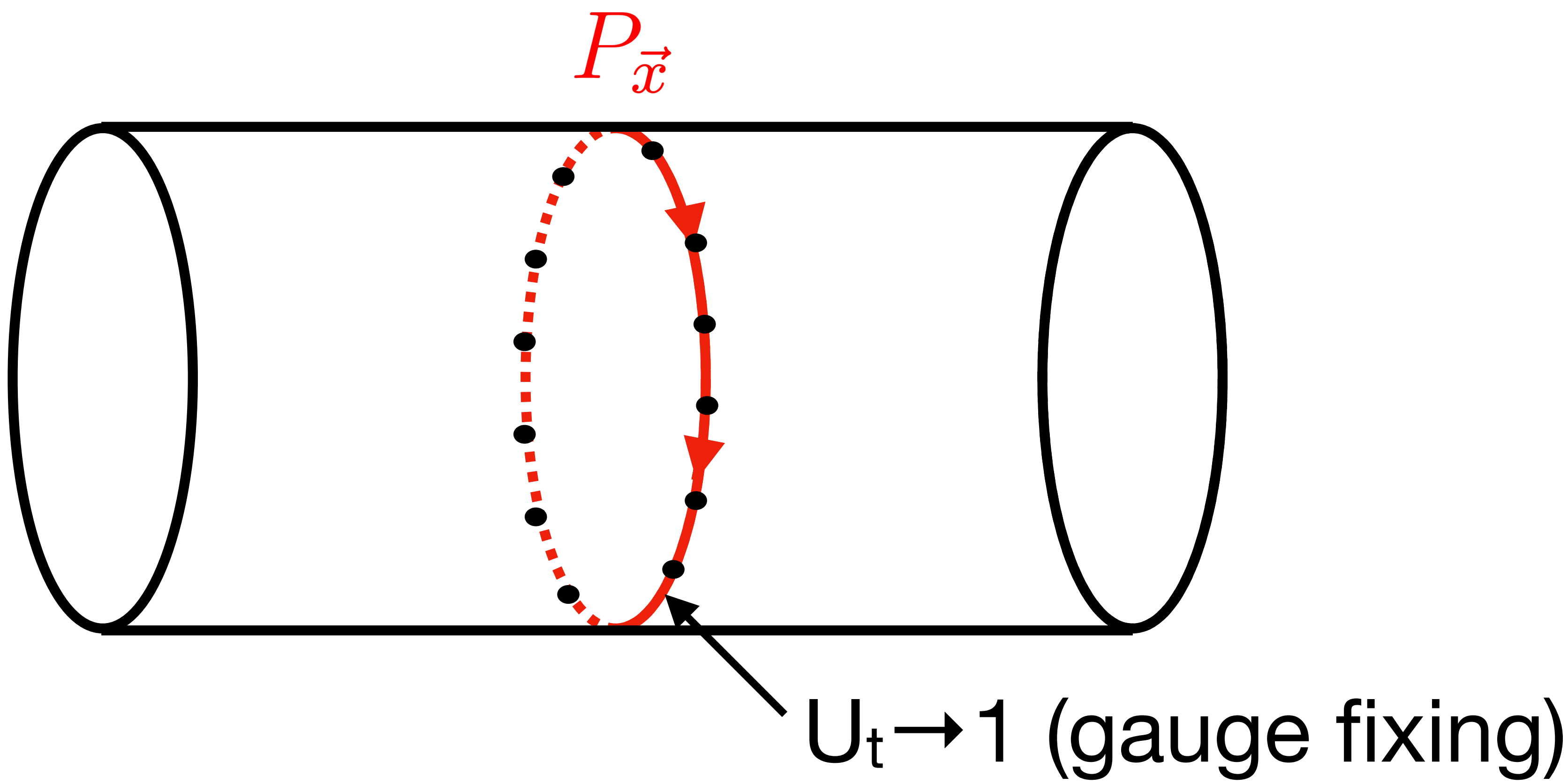
— This is a *mathematical* statement.

Forget about center symmetry.

— Polyakov loop has a meaning in Hamilton formulation as well.



Product of link variables U_t



$$P = U_t \cdot U_t \cdot \ldots \cdot U_t \cdot U_t \xrightarrow[\textbf{fixing}]{\textbf{Gauge}} 1 \cdot 1 \cdot \ldots \cdot 1 \cdot P$$

All but one link can be set 1.
Remaining link = Polyakov loop.

Integrating Pol = Integrating U_t → singlet constraint

$$Z(T) = \int [dA_t][d\phi]e^{-S[A_t,\phi]}$$

$$= \frac{1}{\mathrm{Vol}G}\int dg \mathrm{Tr}_{\mathcal{H}_{\mathrm{ext}}}\left(\hat{g}e^{-\hat{H}(\hat{\phi})/T}\right)$$

$$= \mathrm{Tr}_{\mathcal{H}_{\mathrm{inv}}}\left(e^{-\hat{H}(\hat{\phi})/T}\right)$$

Polyakov loop

Hamiltonian in A_t=0 gauge

Feynman's Ph.D. Thesis

Singlet + non-singlet

Kogut and Susskind didn't explain this step…

Singlet only

Polyakov loop ~ twisted boundary condition

Summation over all twisted boundary conditions → Gauging

$$\int dg \, \mathrm{Tr}_{\mathcal{H}_{\mathrm{ext}}} \left(\hat{g} e^{-\hat{H}(\hat{\phi})/T} \right)$$

$$\sim \int dg \, \langle \mathrm{typical} | \, \hat{g} \, | \mathrm{typical} \rangle \; e^{-E_{\mathrm{typical}}/T}$$

⇒ Dominant contributions come from such g that satisfy

$$\hat{g} \, | \mathrm{typical} \rangle \sim | \mathrm{typical} \rangle$$

Polyakov loop is a stabiliser of typical states

- Polyakov loop is connected to gauge symmetry.
(*Mathematical statement!*)

Common "objection"

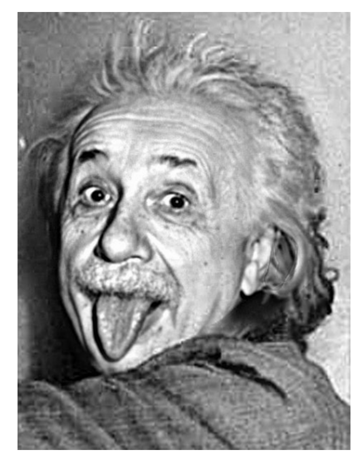
- "Gauge symmetry" is not a symmetry, it's redundancy!

Well, that's true, but that's not the end of the story.

- **The amount redundancy** has important physical consequences.

Historically the first example of
non-Abelian gauge theory
in the large-N limit

Bose



Einstein

N indistinguishable bosons

N bosons in 3d harmonic trap

$$\hat{H} = \sum_{i=1}^{N} \left(\frac{\hat{\vec{p}}_i^2}{2m} + \frac{m\omega^2}{2} \hat{\vec{x}}_i^2 \right)$$

$$\hat{\vec{x}}_i = (\hat{x}_i, \hat{y}_i, \hat{z}_i)$$

$$\hat{\vec{p}}_i = (\hat{p}_{x,i}, \hat{p}_{y,i}, \hat{p}_{z,i})$$

Fock states $|\vec{n}_1, \vec{n}_2, \cdots, \vec{n}_N\rangle \equiv \prod_{i=1}^{3} \frac{\hat{a}_{i1}^{\dagger n_{i1}}}{\sqrt{n_{i1}!}} \frac{\hat{a}_{i2}^{\dagger n_{i2}}}{\sqrt{n_{i2}!}} \cdots \frac{\hat{a}_{iN}^{\dagger n_{iN}}}{\sqrt{n_{iN}!}} |0\rangle$

States related by S_N permutation are identical.

S_N permutation is gauged.

Summation over singlet states $Z(T) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{inv}}}(e^{-\hat{H}/T})$

Summation over all states & projection to singlet states

$$Z(T) = \frac{1}{\mathrm{vol}(G)} \int_G dg \mathrm{Tr}_{\mathcal{H}_{\mathrm{ext}}}(\hat{g} e^{-\hat{H}/T})$$

$G = \mathrm{SU}(N)$ + adjoint fields → Yang-Mills, Matrix Model

$G = \mathrm{S}_N$ + fundamental fields → N indistinguishable bosons

Non-interacting bosons * N

$$\hat{H} = \sum_{i=1}^{N} \left(\frac{\hat{\vec{p}}_i^2}{2m} + \frac{m\omega^2}{2} \hat{\vec{x}}_i^2 \right)$$

S_N gauge symmetry

Non-interacting bosons * N^2

$$\hat{H}_{\text{Gaussian}} = \sum_I \mathrm{Tr} \left(\frac{1}{2} \hat{P}_I^2 + \frac{1}{2} \hat{X}_I^2 \right)$$

SU(N) gauge symmetry

Non-interacting bosons * N

$$\hat{H} = \sum_{i=1}^{N} \left(\frac{\hat{\vec{p}}_i^2}{2m} + \frac{m\omega^2}{2} \hat{\vec{x}}_i^2 \right)$$

S_N gauge symmetry

Bose-Einstein Condensation

$$|\vec{n}_1, \cdots, \vec{n}_M, \vec{0}, \cdots, \vec{0}\rangle$$

Non-interacting bosons * N^2

$$\hat{H}_{\text{Gaussian}} = \sum_I \mathrm{Tr} \left(\frac{1}{2} \hat{P}_I^2 + \frac{1}{2} \hat{X}_I^2 \right)$$

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$$Z(T) = \frac{1}{N!} \sum_{\sigma \in \mathrm{S}_N} \sum_{\vec{n}_1, \cdots, \vec{n}_N} \langle \vec{n}_1, \cdots, \vec{n}_N | \hat{\sigma} e^{-\hat{H}/T} | \vec{n}_1, \cdots, \vec{n}_N \rangle$$

$$= \frac{1}{N!} \sum_{\vec{n}_1, \cdots, \vec{n}_N} e^{-(E_{\vec{n}_1} + \cdots + E_{\vec{n}_N})/T} \left(\sum_{\sigma \in \mathrm{S}_N} \langle \vec{n}_1, \cdots, \vec{n}_N | \vec{n}_{\sigma(1)}, \cdots, \vec{n}_{\sigma(N)} \rangle \right)$$

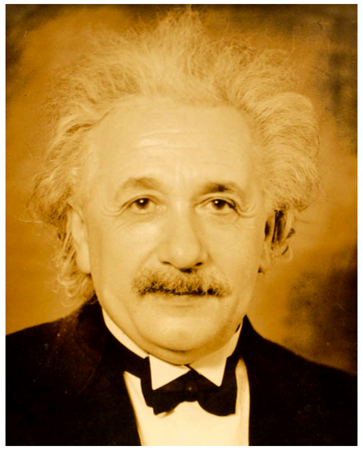
measures the amount of redundancy

Permutation σ = Polyakov loop

京都　三十三間堂

Sanjusangendo, Kyoto

N=1001



Einstein’s trip to Kyoto: 1922

$$Z(T) = \frac{1}{N!} \sum_{\sigma \in \mathrm{S}_N} \sum_{\vec{n}_1, \cdots, \vec{n}_N} \langle \vec{n}_1, \cdots, \vec{n}_N | \hat{\sigma} e^{-\hat{H}/T} | \vec{n}_1, \cdots, \vec{n}_N \rangle$$

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$|\vec{0}, \vec{0}, \cdots, \vec{0}\rangle$ $\quad N!$

$|\vec{n}_1, \cdots, \vec{n}_N\rangle$ $\quad 1$

(all of them are different)

$|\vec{n}_1, \cdots, \vec{n}_M, \vec{0}, \cdots, \vec{0}\rangle$ $\quad (N-M)!$

This enhancement triggers BEC.

Einstein, 1924

$$Z(T) = \frac{1}{\mathrm{vol}(G)} \int_G dg \mathrm{Tr}_{\mathcal{H}_{\mathrm{ext}}}(\hat{g} e^{-\hat{H}/T})$$

$G = \mathrm{S}_N$ + fundamental fields → N indistinguishable bosons

$$N! = \mathrm{vol}(\mathrm{S}_N)$$

This enhancement triggers BEC.

(Einstein, 1924)

$G = \mathrm{SU}(N)$ + adjoint fields → Yang-Mills, Matrix Model

$$\mathrm{vol}(\mathrm{SU}(N)) \sim e^{N^2}$$

This enhancement triggers color confinement.

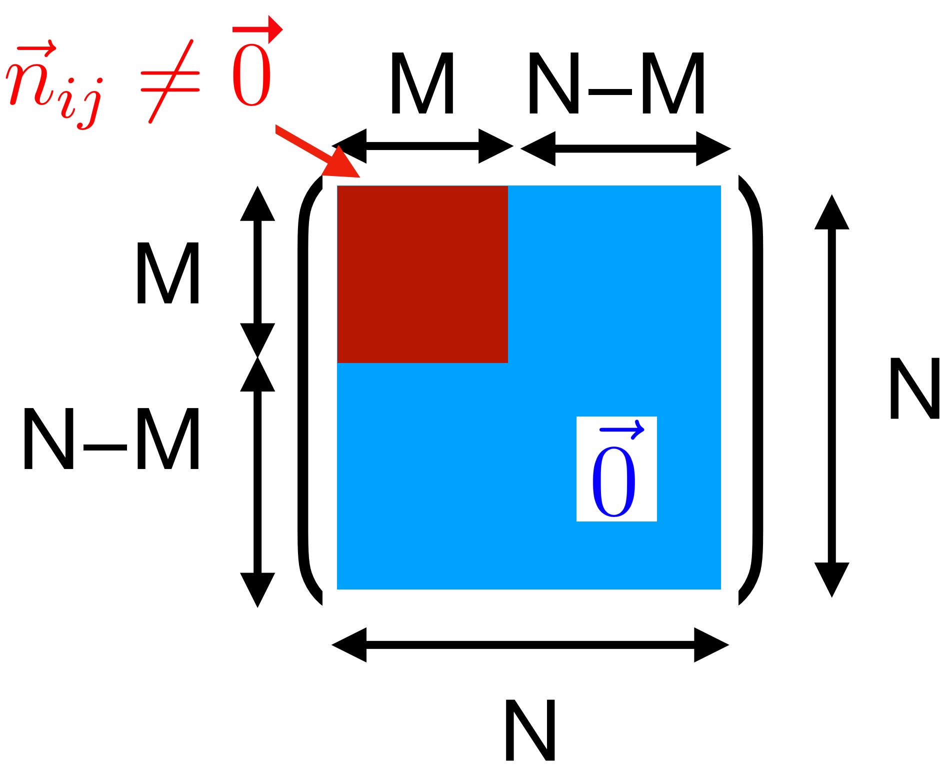
(MH-Shimada-Wintergerst, 2020)

Partially-BEC state

$$|\vec{n}_1, \cdots, \vec{n}_M, \vec{0}, \cdots, \vec{0}\rangle \qquad (N-M)!$$

Partially-confined state

(MH-Maltz, 2016; Berenstein, 2018;
MH-Ishiki-Watanabe, 2018;
MH-Jevicki-Peng-Wintergerst, 2019;
Watanabe et al, 2020)



$$\mathrm{vol}(\mathrm{SU}(N-M)) \sim e^{(N-M)^2}$$

Non-interacting bosons * N

$$\hat{H} = \sum_{i=1}^{N} \left(\frac{\hat{\vec{p}}_i^2}{2m} + \frac{m\omega^2}{2} \hat{\vec{x}}_i^2 \right)$$

S_N gauge symmetry

Bose-Einstein Condensation

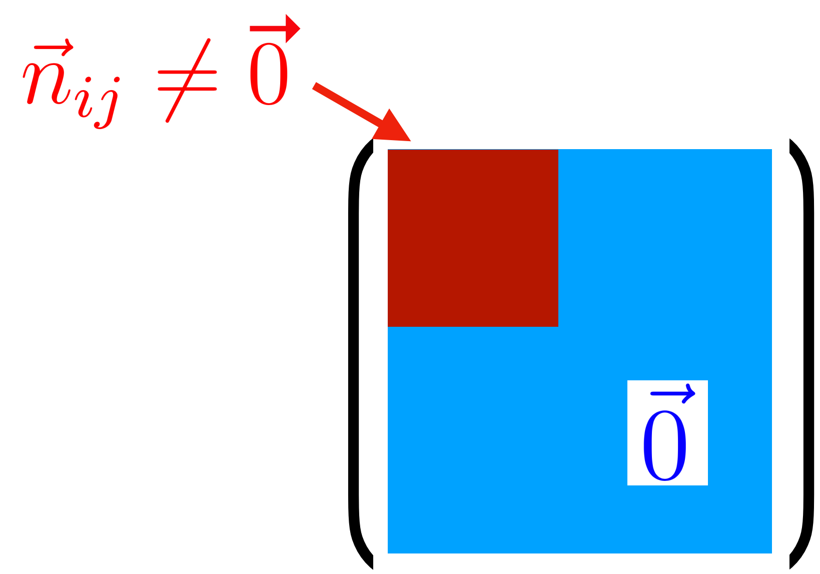
$$\left| \vec{n}_1, \cdots, \vec{n}_M, \vec{0}, \cdots, \vec{0} \right\rangle$$

Non-interacting bosons * N^2

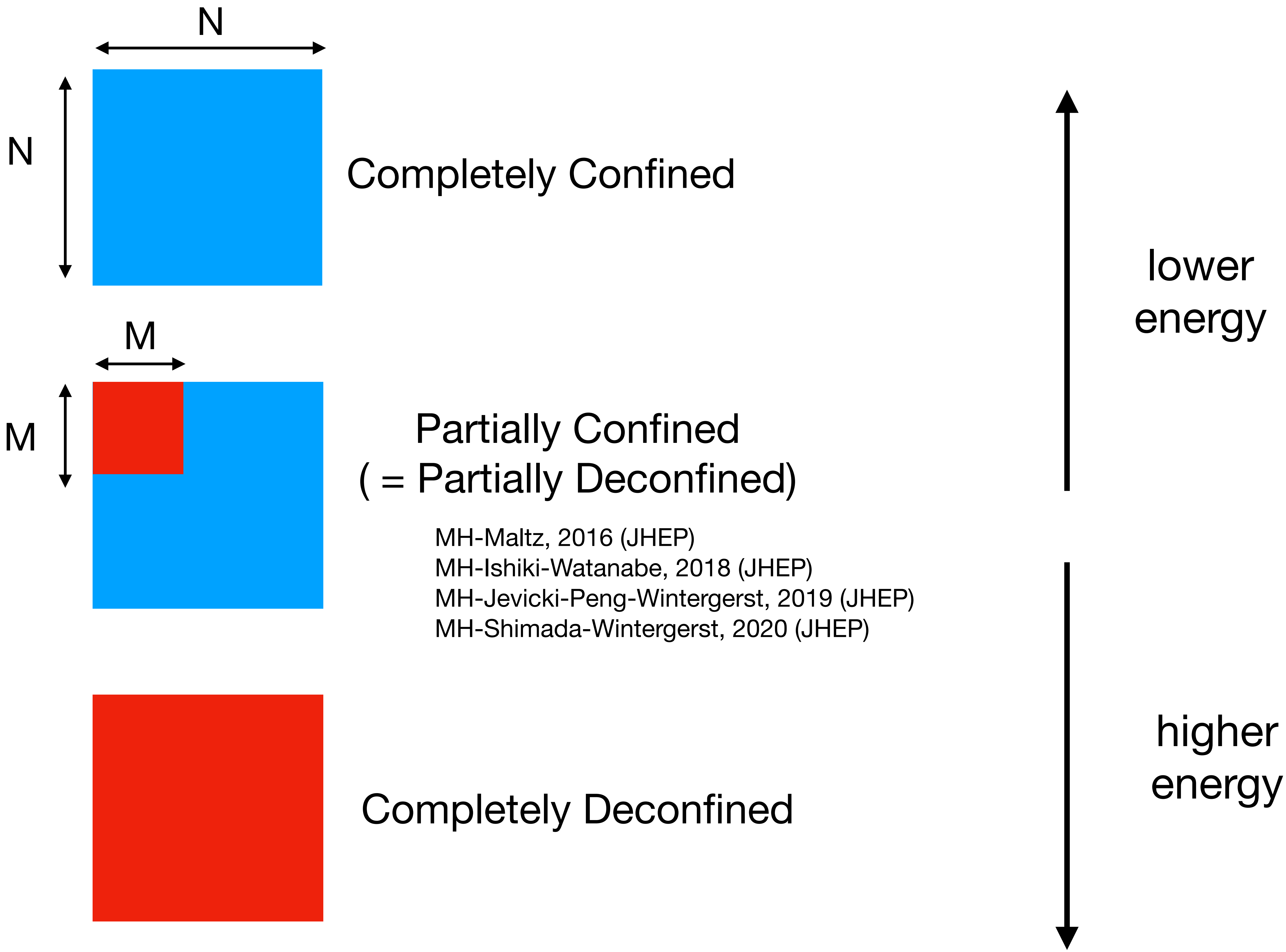
$$\hat{H}_{\text{Gaussian}} = \sum_I \text{Tr} \left(\frac{1}{2} \hat{P}_I^2 + \frac{1}{2} \hat{X}_I^2 \right)$$

SU(N) gauge symmetry

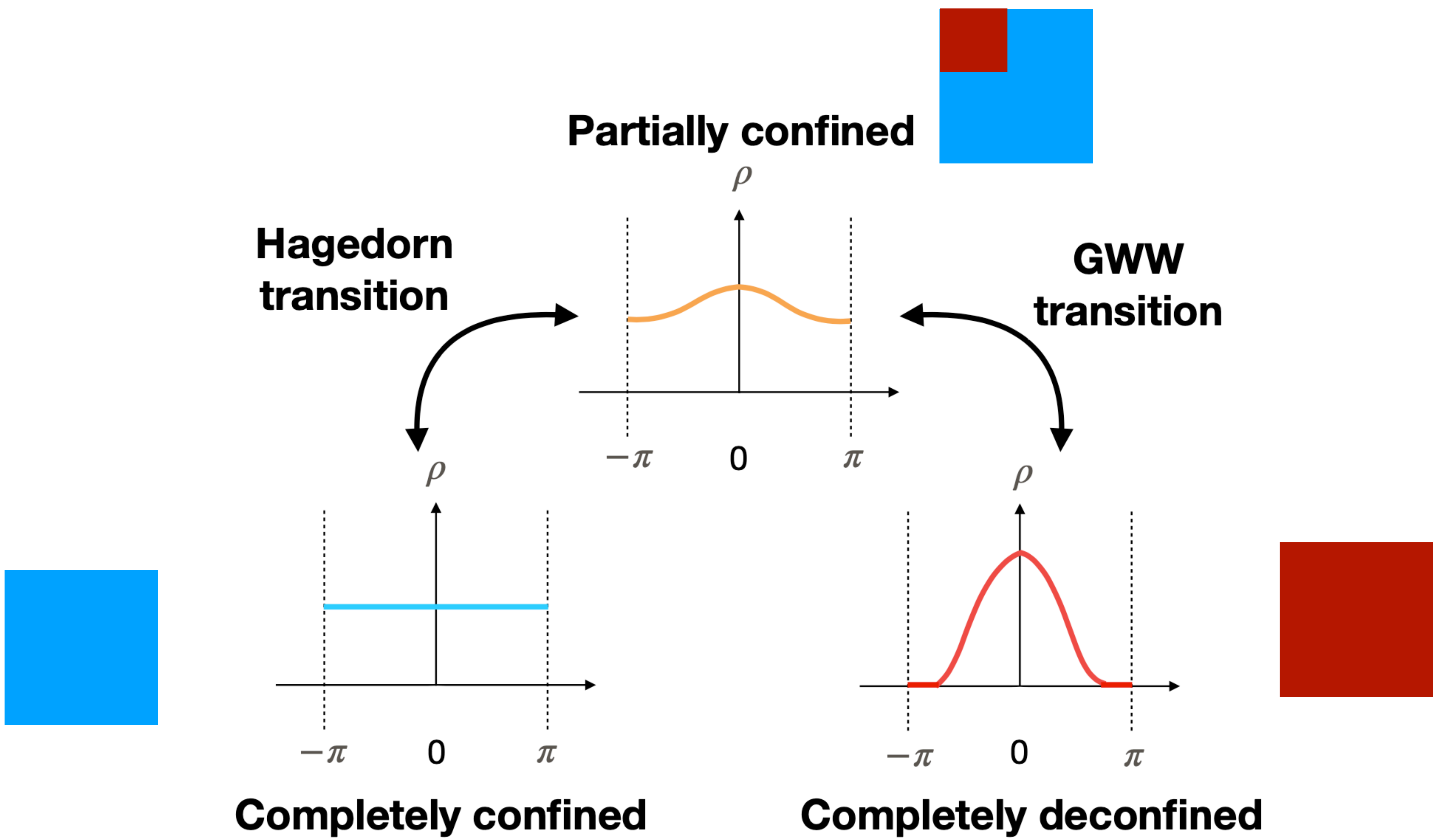
Partial confinement



Generalization to finite coupling:
MH, 2021


N
N
Completely Confined
lower
energy
M
M
Partially Confined
(= Partially Deconfined)
MH-Maltz, 2016 (JHEP)
MH-Ishiki-Watanabe, 2018 (JHEP)
MH-Jevicki-Peng-Wintergerst, 2019 (JHEP)
MH-Shimada-Wintergerst, 2020 (JHEP)
higher
energy
Completely Deconfined

Polyakov Loop $P = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$



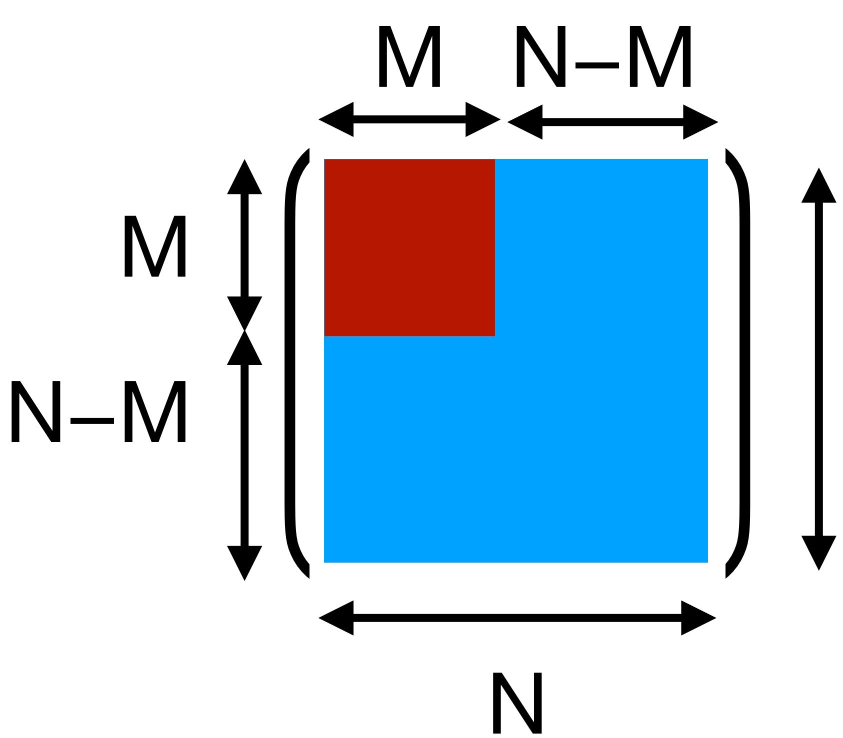
MH-Maltz, 2016 (JHEP)
MH-Ishiki-Watanabe, 2018 (JHEP)
MH-Shimada-Wintergerst, 2020 (JHEP)

$$Z(T) = \frac{1}{\mathrm{vol}G}\int_G dg\,\mathrm{Tr}_{\mathcal{H}_{\mathrm{ext}}}\left(\hat{g}e^{-\hat{H}/T}\right)$$

$$\sim \frac{1}{\mathrm{vol}G}e^{-E_{\mathrm{typical}}/T}\int_G dg\langle\mathrm{typical}|\hat{g}|\mathrm{typical}\rangle$$

Polyakov loop

Typical $\hat{g}$'s which leave $|\mathrm{typical}\rangle$ unchanged dominate the phase distribution

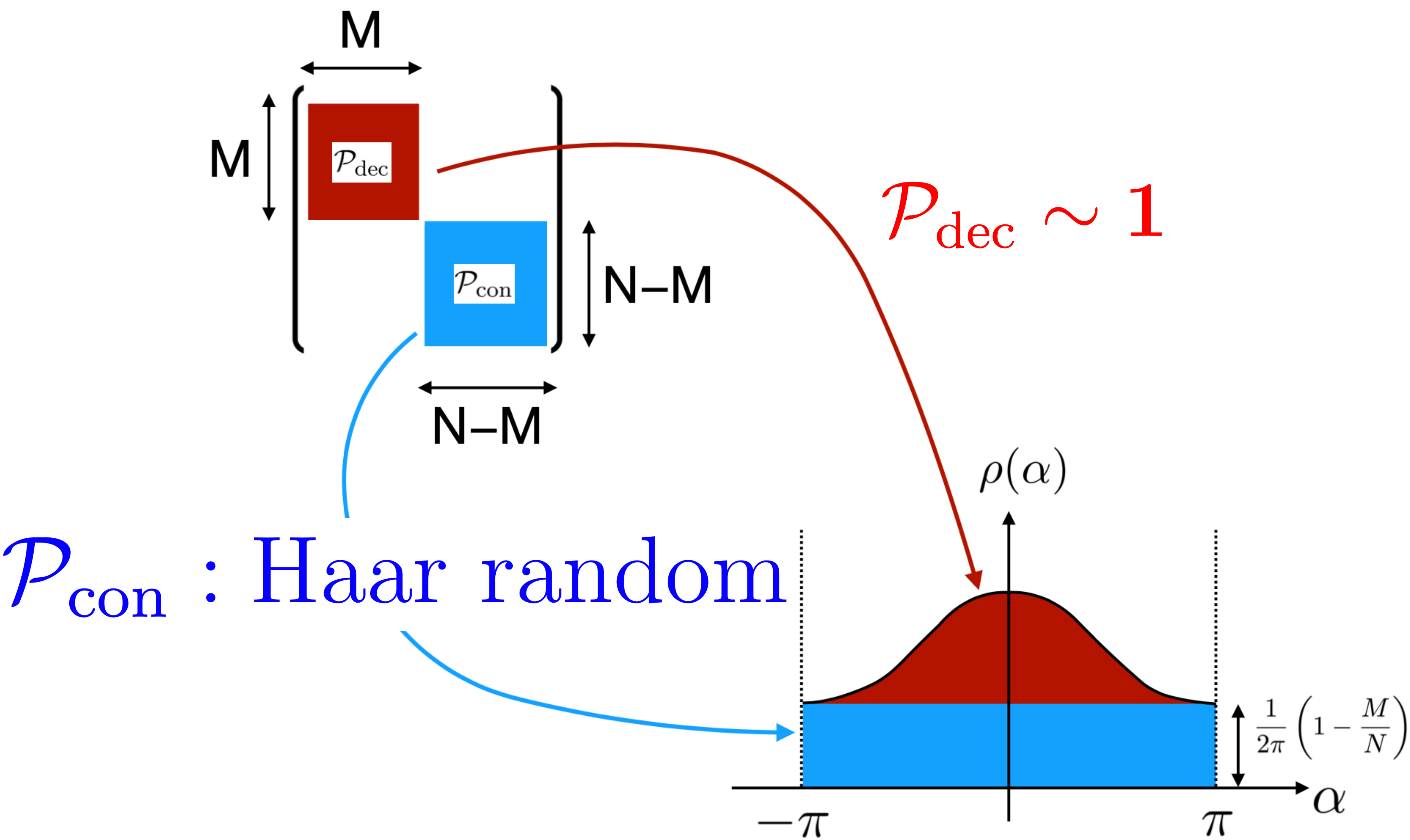


$$g \sim \begin{pmatrix} 1 & \\ & \Omega \end{pmatrix}$$

Haar random

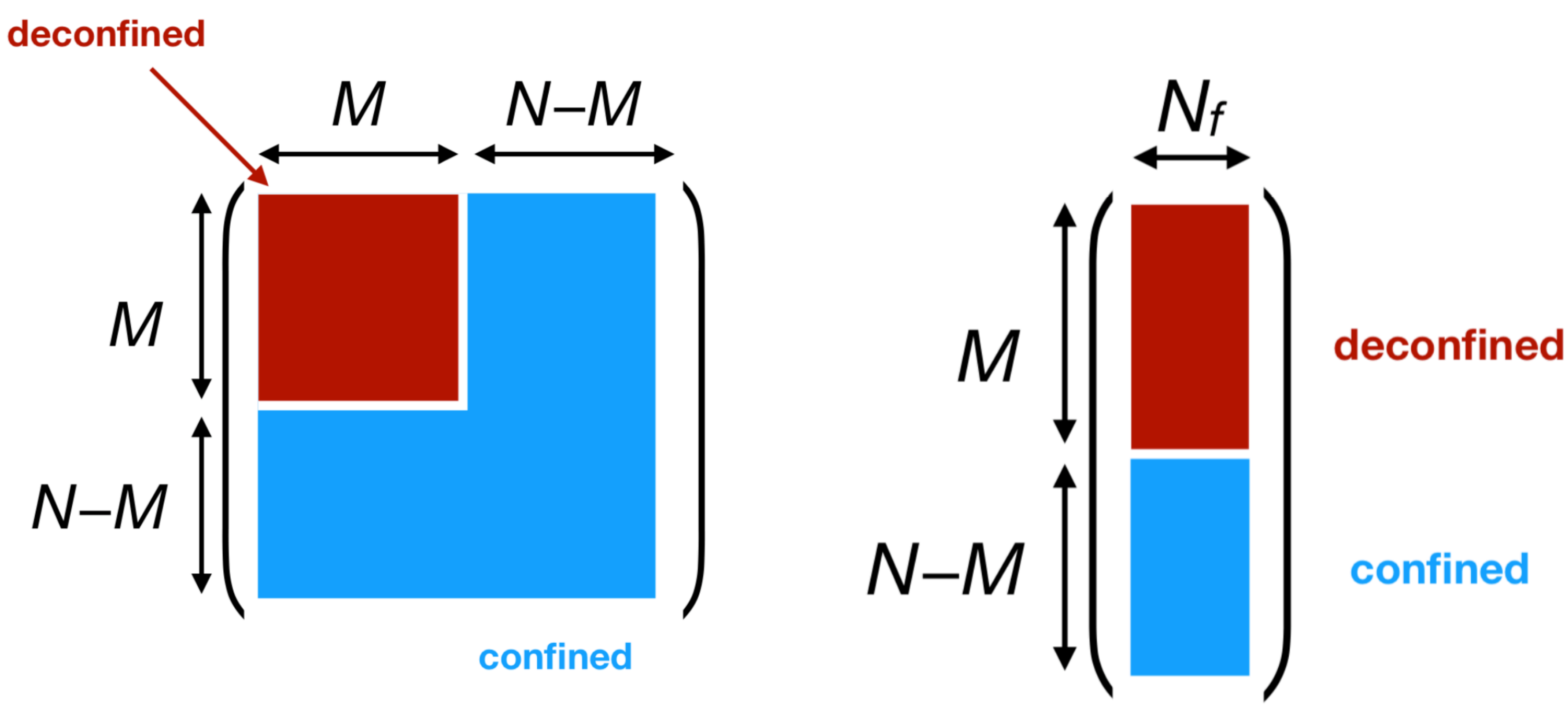
$$|\vec{n}_1,\cdots,\vec{n}_M,\vec{0},\cdots,\vec{0}\rangle$$

For QFT, Haar random slowly changing in space



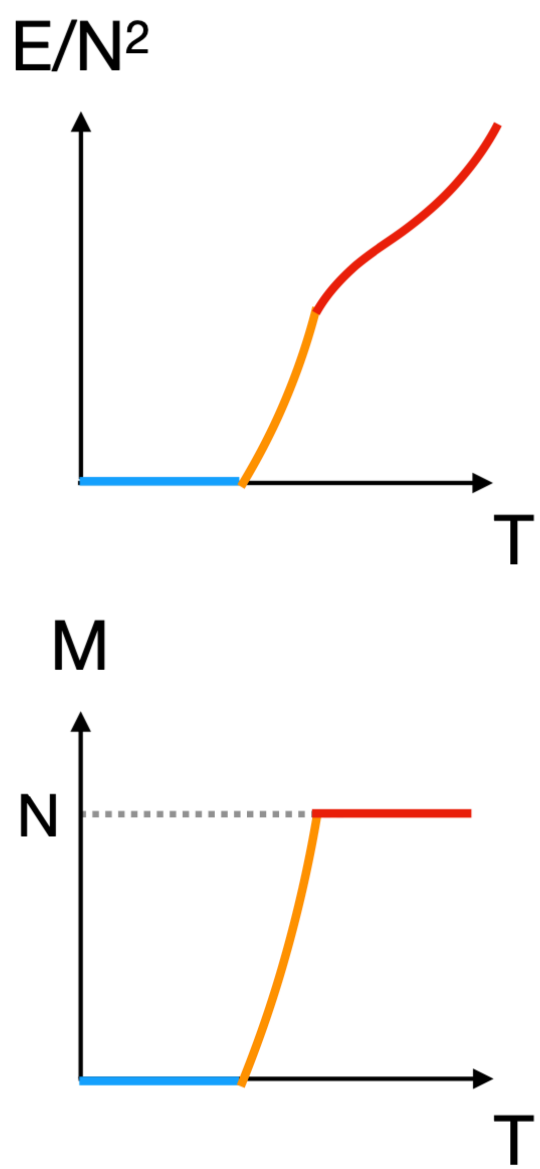
MH-Shimada-Wintergerst, 2020
(Essentially: Feynman, 1953)

QCD phase transition

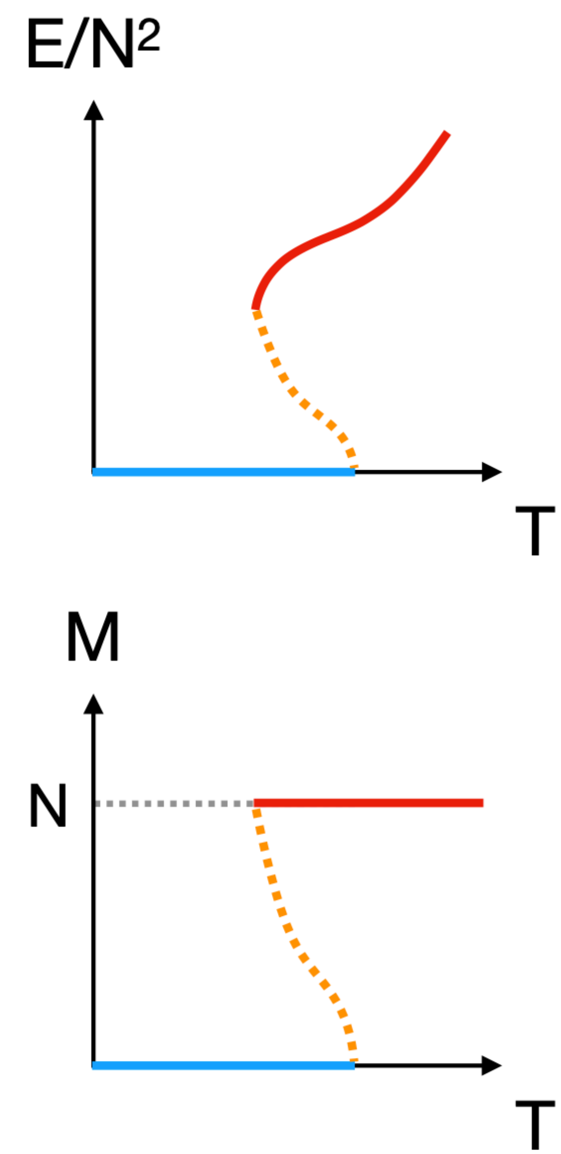


Weak-coupling analysis: MH-Robinson 2019

QCD phase transition



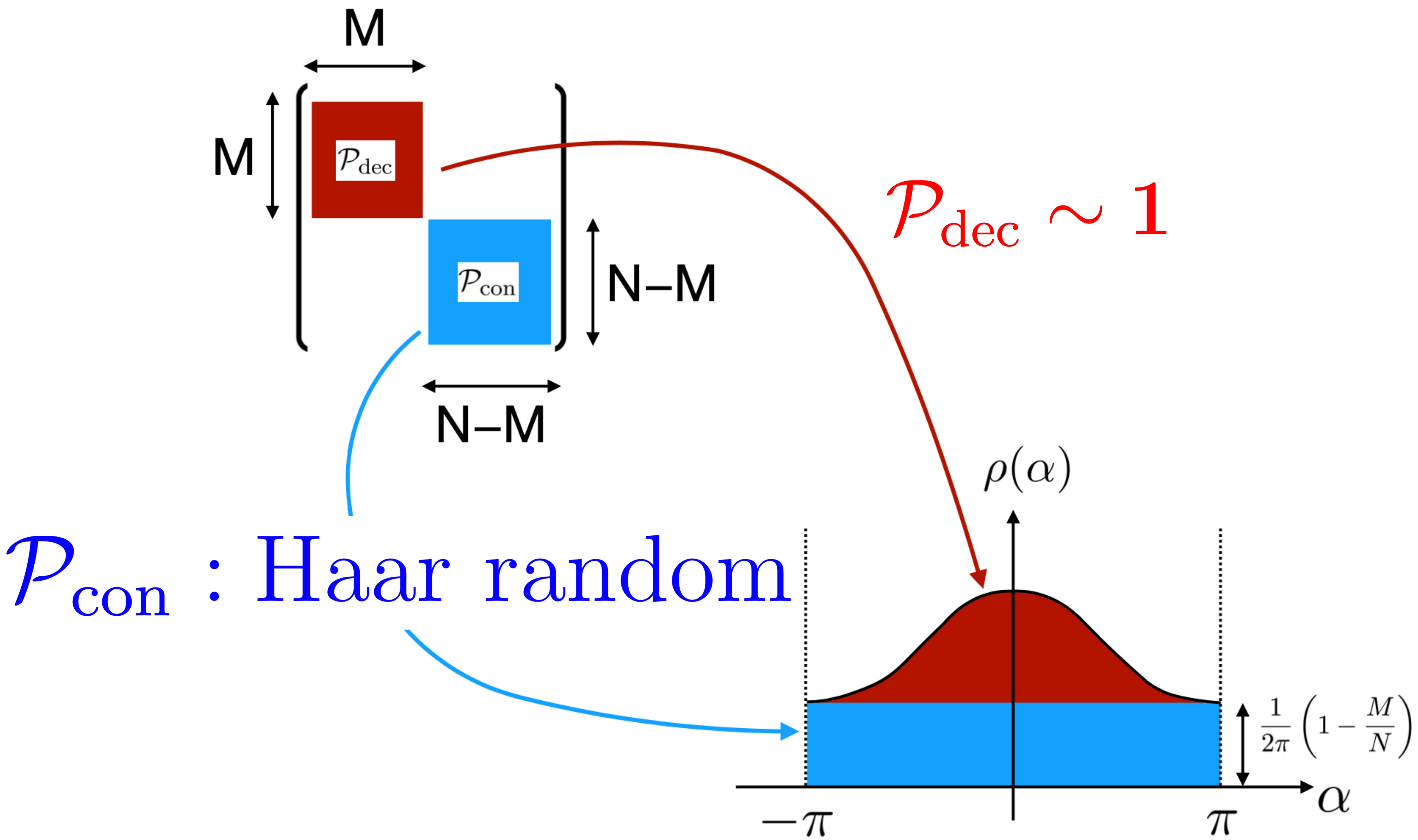
Light quark mass



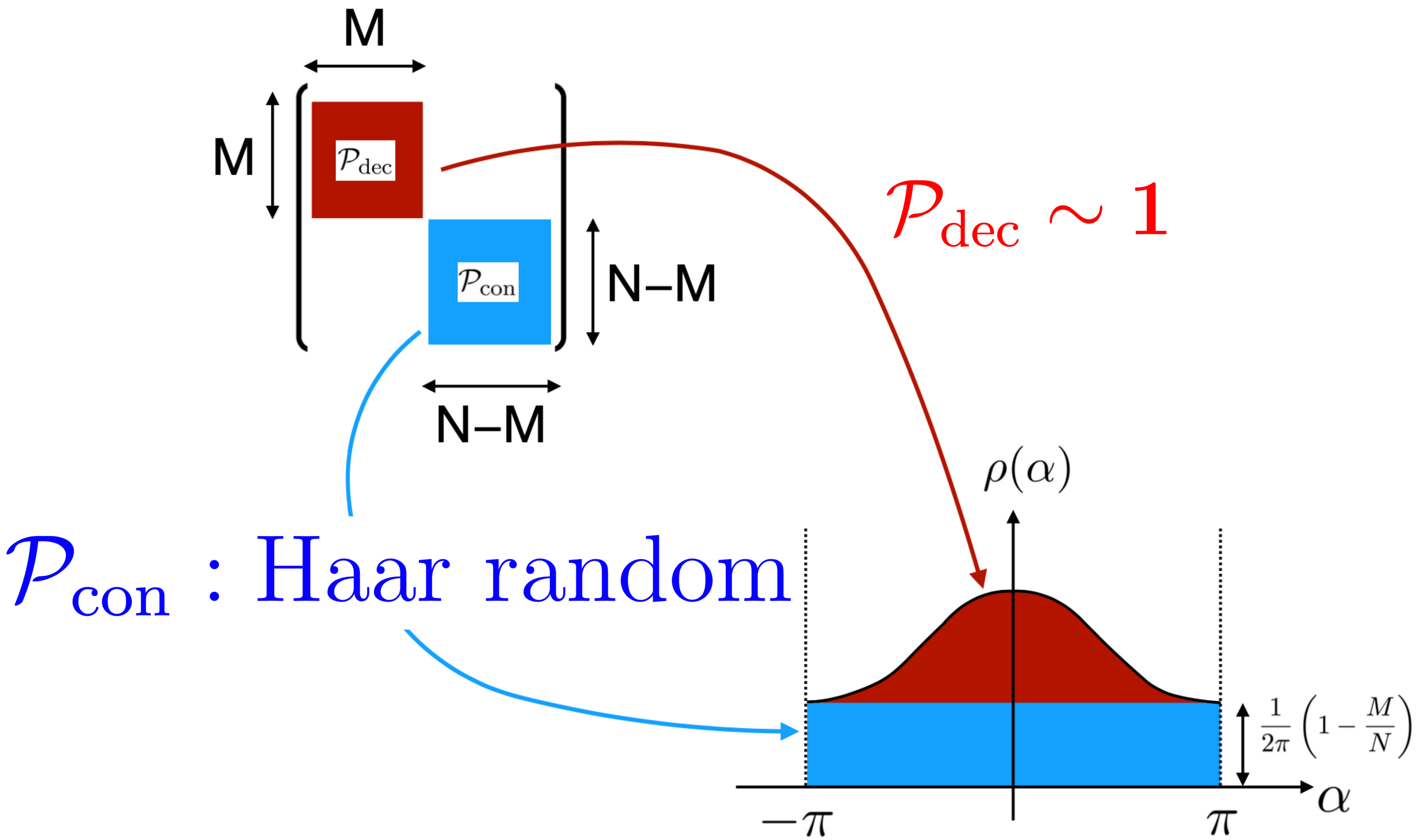
Heavy quark mass

Finite-N theories

MH, Ohata, Shimada, Watanabe, 2023 (PTEP)
MH, Watanabe, 2023 (PTEP)



1/N correction makes "M" ambiguous...



1/N correction makes "M" ambiguous...

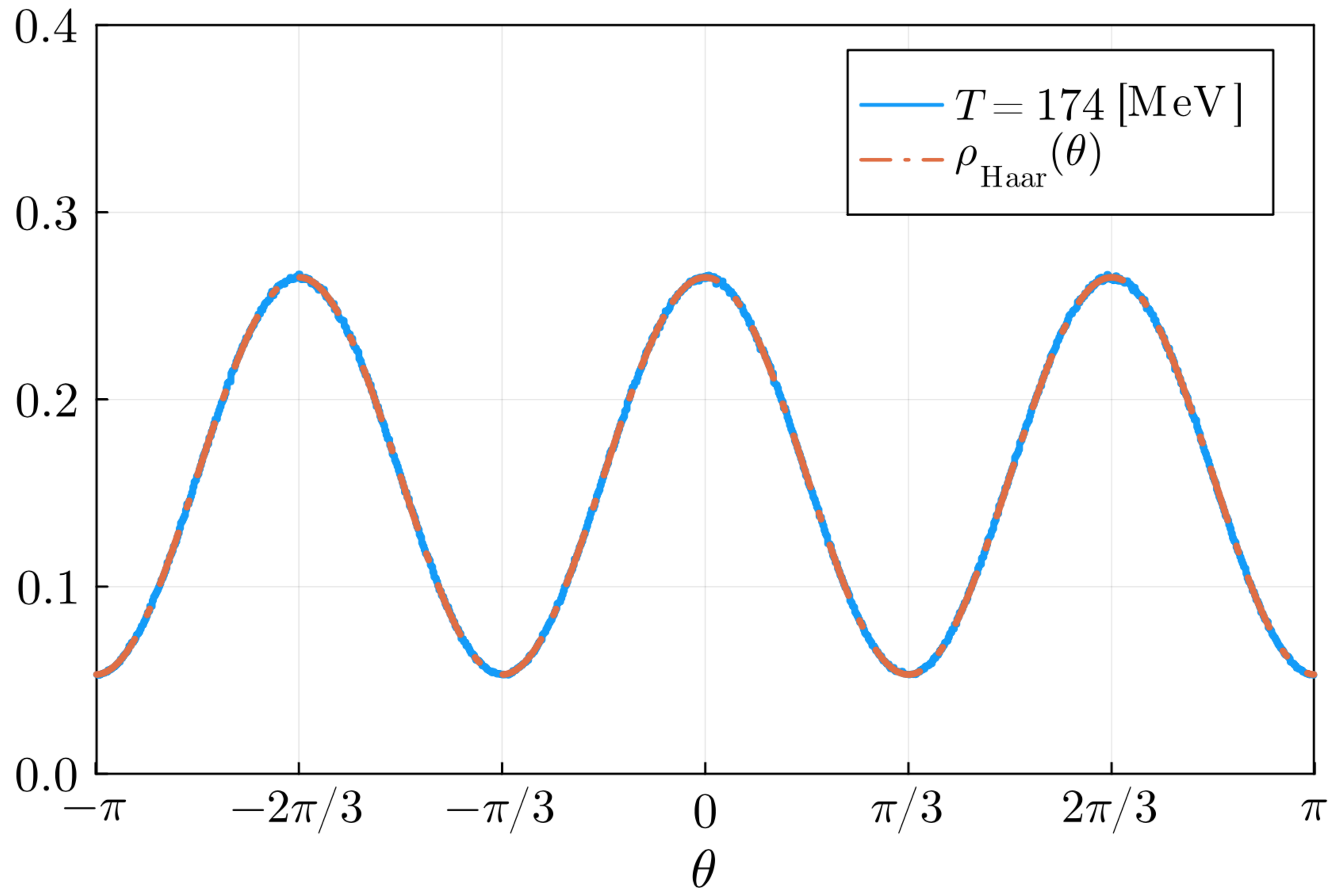
But no ambiguity for M=0

(As for M=N, see Hanada-Watanabe (2023) or ask me after the talk!)

Completely-confined → SU(N) Haar random

(At sufficiently strong coupling)

N=3
WHOT-QCD
configugration



$$\rho_{\text{Haar}}(\theta) = \frac{1}{2\pi}\left(1 - (-1)^N \cdot \frac{2}{N}\cos(N\theta)\right)$$

$$\rho_{\text{Polyakov}}(\theta) = \frac{1}{2\pi} + \frac{1}{2\pi}\sum_{n>0}\left(\tilde{\rho}_n e^{-in\theta} + \tilde{\rho}_{-n} e^{in\theta}\right)$$
$$= \frac{1}{2\pi} + \frac{1}{2\pi}\sum_{n>0} 2\tilde{\rho}_n \cos(n\theta)\,.$$

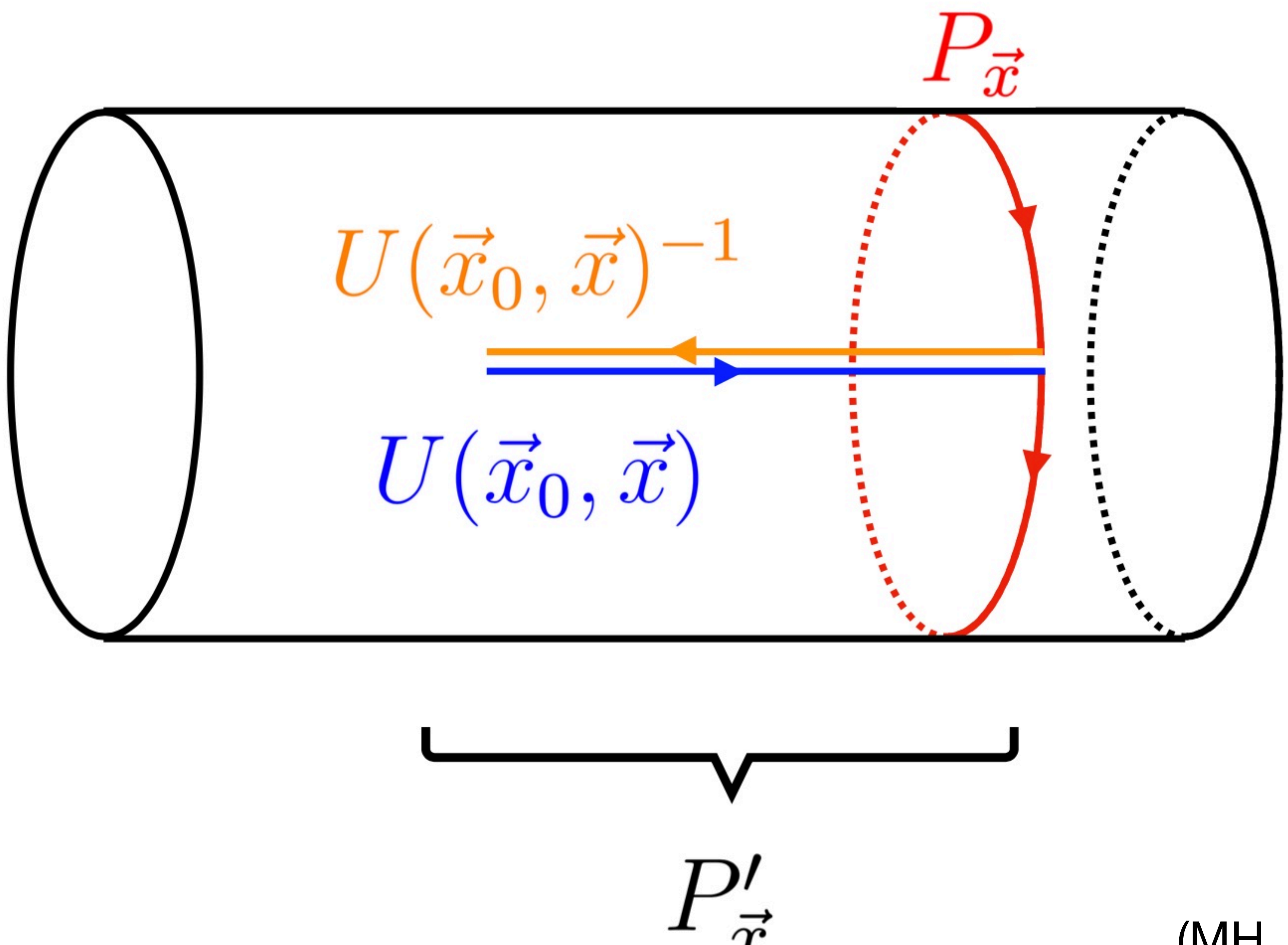
$$\tilde{\rho}_n = \begin{cases} \frac{(-1)^N}{N} & (n = \pm N) \\ 0 & (n \neq \pm N) \end{cases}$$

$$\tilde{\rho}_n = \frac{1}{N}\langle \mathrm{Tr}(\mathcal{P}^n)\rangle$$

baryon!

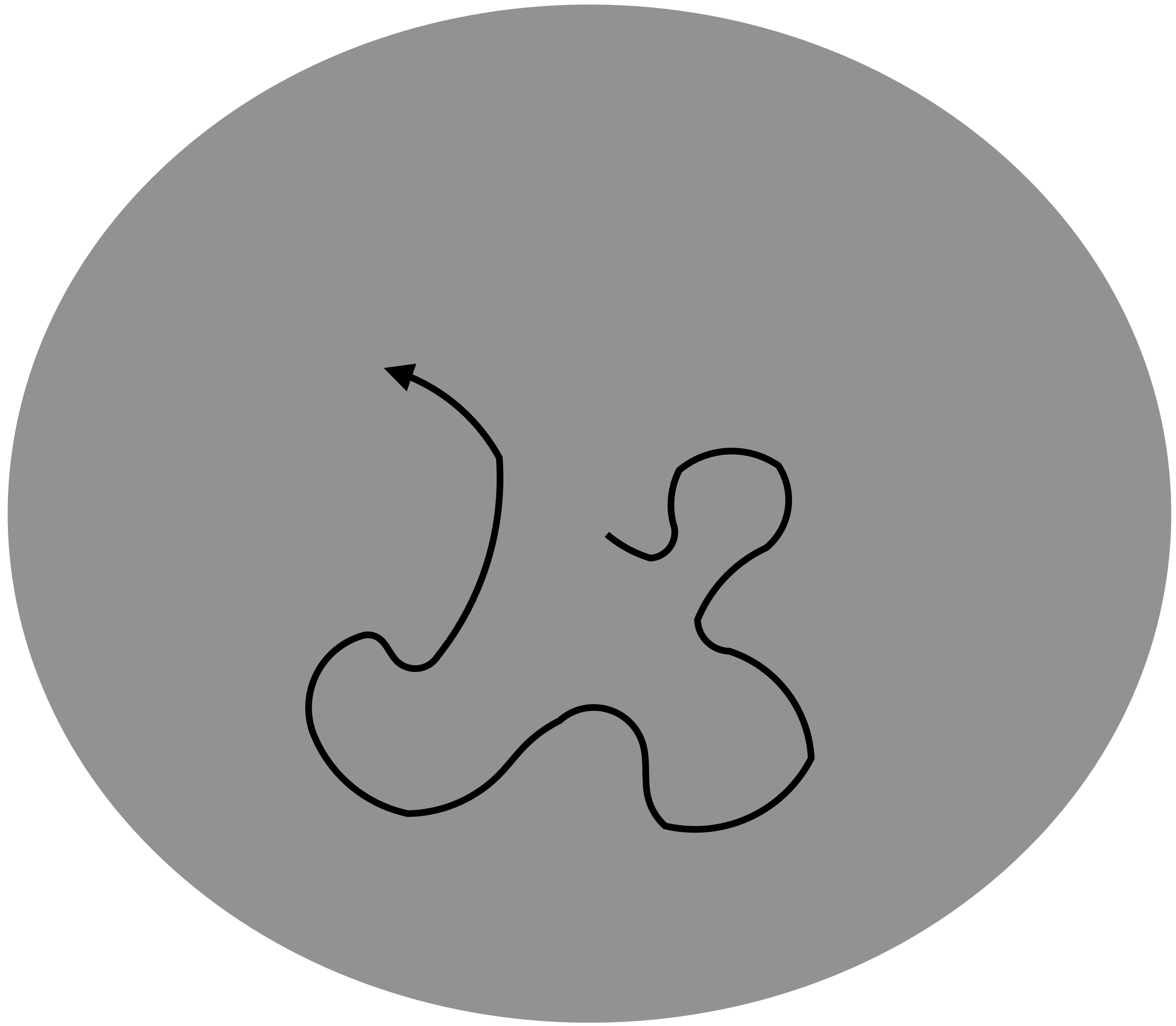
(Strictly speaking, there is a small corrections to Haar-random distribution)

More precise story:

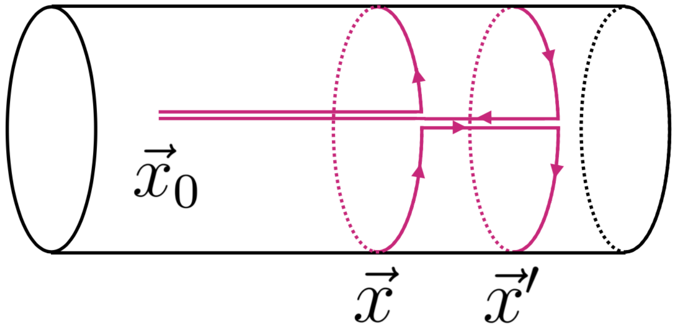


(MH, Watanabe, 2023)

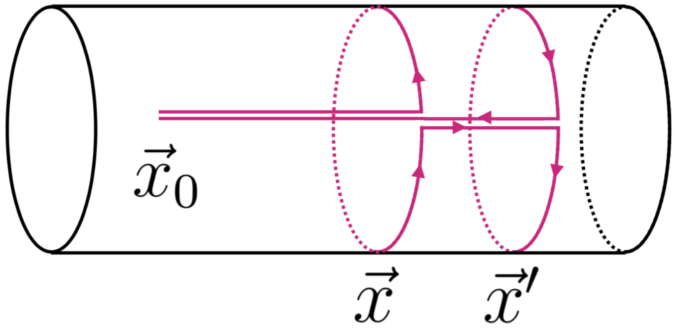
This gentleman random walks slowly on group manifold.



SU(N) group manifold



$$P'^{-1}_{\vec{x}} P'_{\vec{x}'} = e^{i\Delta X}$$: Gaussian random



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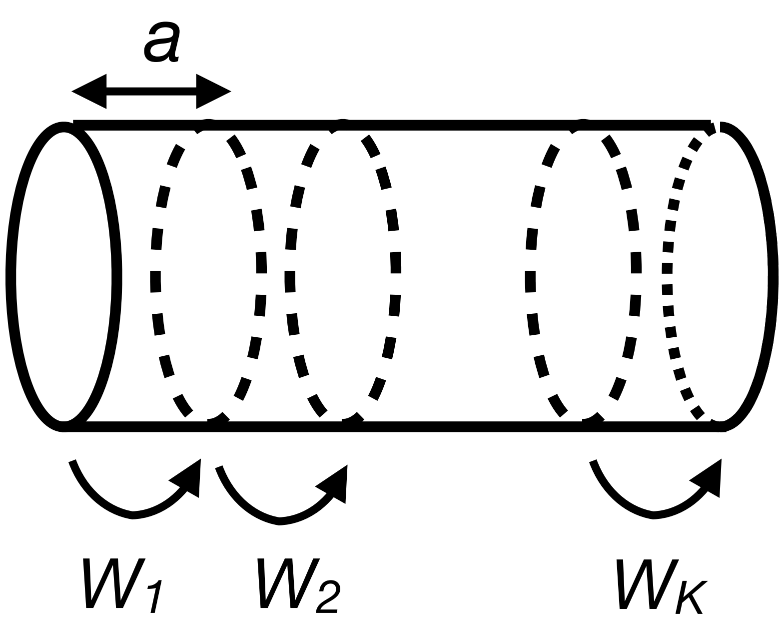
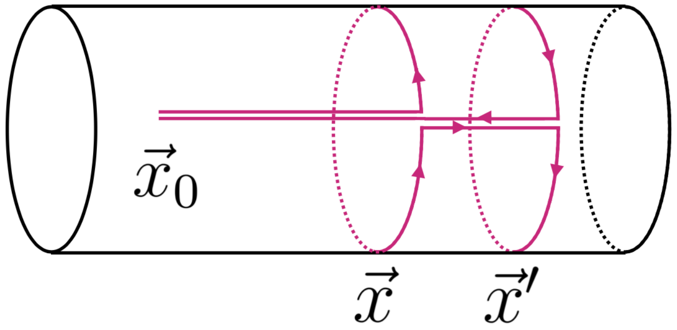


Suppose all W's are independent
(crude approximation)



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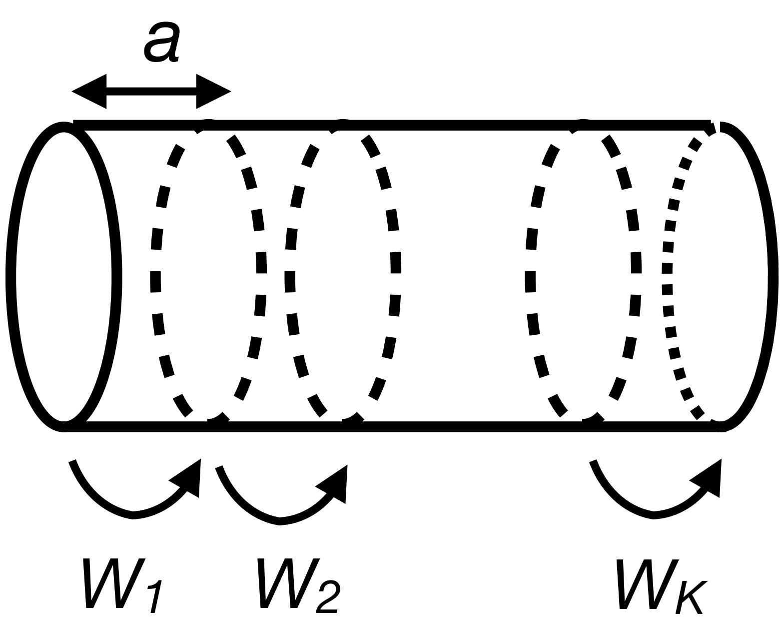


Suppose all W's are independent
(crude approximation)

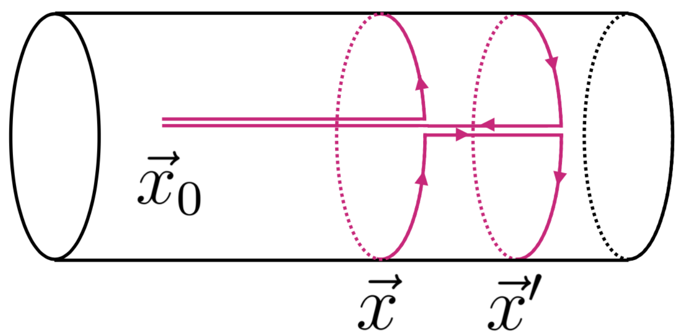
$$\left\langle W_1^{(\mathrm{r})} W_2^{(\mathrm{r})} \cdots W_k^{(\mathrm{r})} \right\rangle = \left\langle W_1^{(\mathrm{r})} \right\rangle \left\langle W_2^{(\mathrm{r})} \right\rangle \cdots \left\langle W_k^{(\mathrm{r})} \right\rangle = \left(\left\langle W^{(\mathrm{r})} \right\rangle \right)^k$$

$$\left\langle W^{(\mathrm{r})} \right\rangle = \left\langle \mathbf{1} + i\Delta x^\alpha T_\alpha^{(\mathrm{r})} - \frac{\Delta^2 x^\alpha x^\beta}{2} T_\alpha^{(\mathrm{r})} T_\beta^{(\mathrm{r})} + \cdots \right\rangle$$
$$= \mathbf{1} - \frac{\Delta^2}{2} (T_\alpha^{(\mathrm{r})})^2 + \cdots$$
$$= \mathbf{1} - 2\Delta^2 C_{\mathrm{r}} \mathbf{1} + \cdots .$$

$$\left\langle W_1^{(\mathrm{r})} \right\rangle \left\langle W_2^{(\mathrm{r})} \right\rangle \cdots \left\langle W_K^{(\mathrm{r})} \right\rangle \simeq e^{-2\Delta^2 C_{\mathrm{r}} K} \mathbf{1}$$

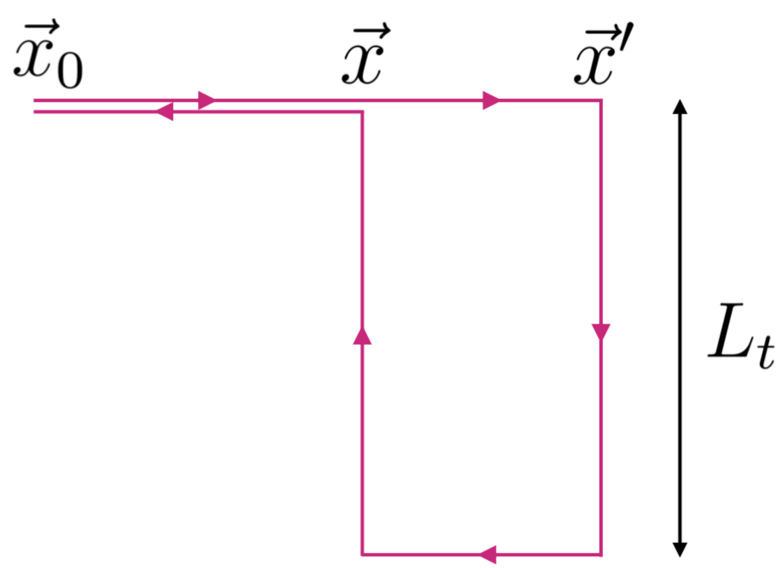
Casimir scaling
up to O(Δ^4)

Polyakov loop



$$P'^{-1}_{\vec{x}} P'_{\vec{x}'} = e^{i\Delta X}$$: Gaussian random

Wilson loop



Natural conjecture:

This is also Gaussian random

$$W = e^{i\Delta X}$$

Numerically confirmed for 3d SU(2) pure YM

Polyakov loop

$$
\begin{aligned}
Z(T) &= \int [dA_t][d\phi]e^{-S[A_t,\phi]} \\
&= \frac{1}{\mathrm{Vol}G}\int dg \, \mathrm{Tr}_{\mathcal{H}_{\mathrm{ext}}}\left(\hat{g}e^{-\hat{H}(\hat{\phi})/T}\right) \\
&= \mathrm{Tr}_{\mathcal{H}_{\mathrm{inv}}}\left(e^{-\hat{H}(\hat{\phi})/T}\right)
\end{aligned}
$$

→ Random walk on group manifold

→ Linear confinement potential with Casimir scaling

(Bergner, Gautam, MH, 2023)

Summary

- Einstein studied large-N limit of non-Abelian gauge theory.
- Color confinement at large N = Bose-Einstein condensation.
- Polyakov loop is connected to gauge symmetry.
  (*Mathematical statement!*)

Forget about center symmetry!

- Polyakov lines random walks on group manifold.
- Linear confinement potential with Casimir scaling follows from random walk.