QCD at non-zero isospin density:6144 pions in a box

Will Detmold (MIT)

Quark Confinement and the Hadron Spectrum Aug 19th 2024

Based on 2307.15014, 2406.09273 with Ryan Abbott, Fernando Romero-López, Zohreh Davoudi, Marc Illa, Assumpta Parreño, Phiala Shanahan, Mike Wagman [NPLQCD collaboration]

Dense QCD Matter

- QCD phase diagram is an important challenge for nuclear theory: μ_B, μ_I, T
- Probed in heavy-ion experiments
- Observed in supernovae, neutron star interiors, neutron star mergers
- Much is unknown!

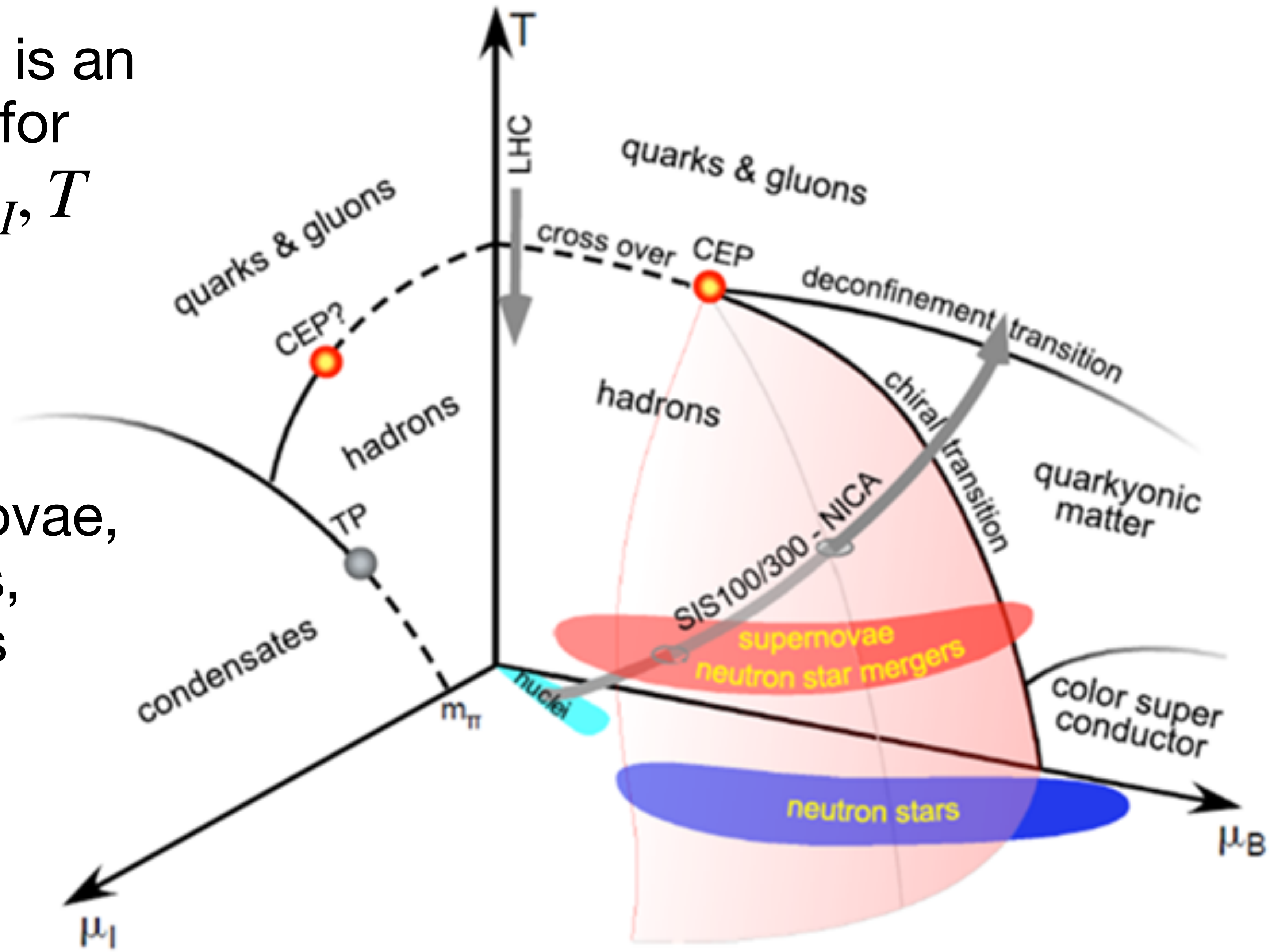
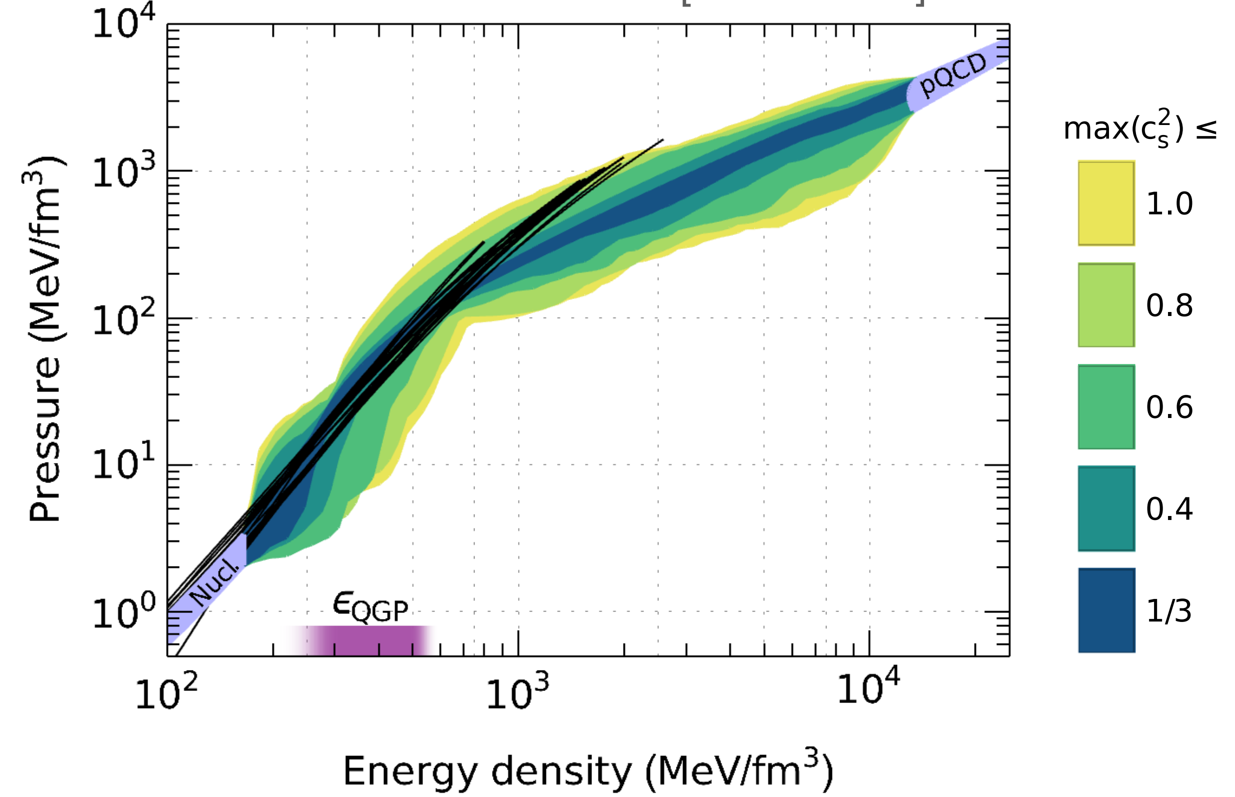


Image: NUPECC Long Range Plan 2017

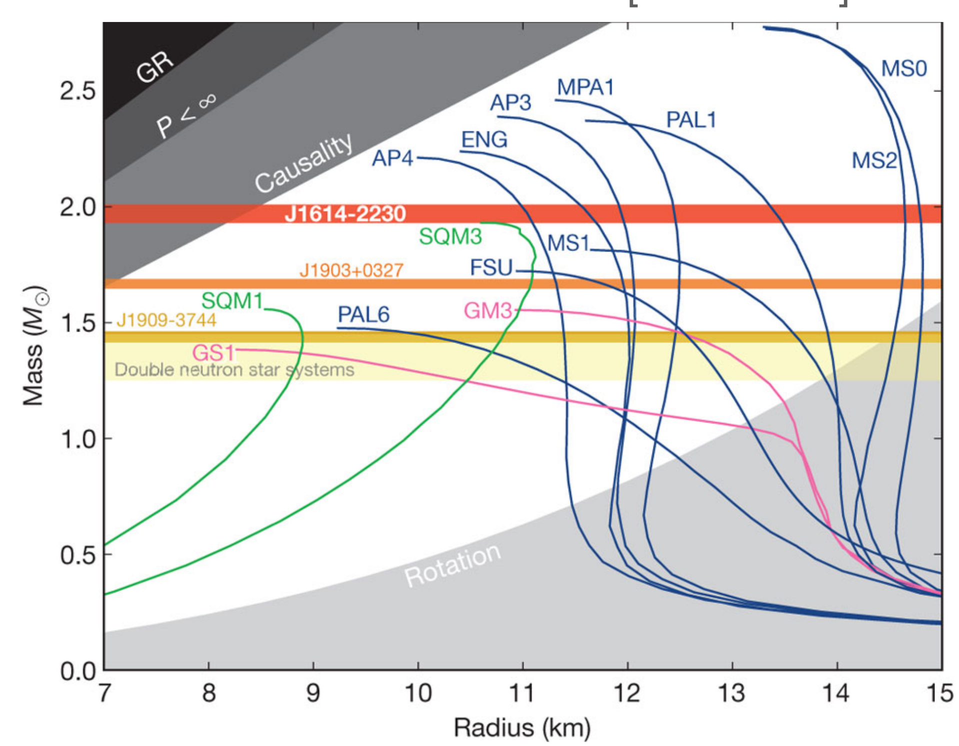
Dense QCD Matter

- Nuclear Equation of State
  - Controls neutron star mass-radius relation
  - Constraints from
    - Nuclear structure
    - pQCD at asymptotic densities
    - (Heavy ion collisions)
    - (Neutron star merger observations)
  - Interpolations very model dependent
- Any rigorous constraints are very valuable
  - LQCD? ✘ Sign problem

Annala et al Nature 2019 [1903.09121]



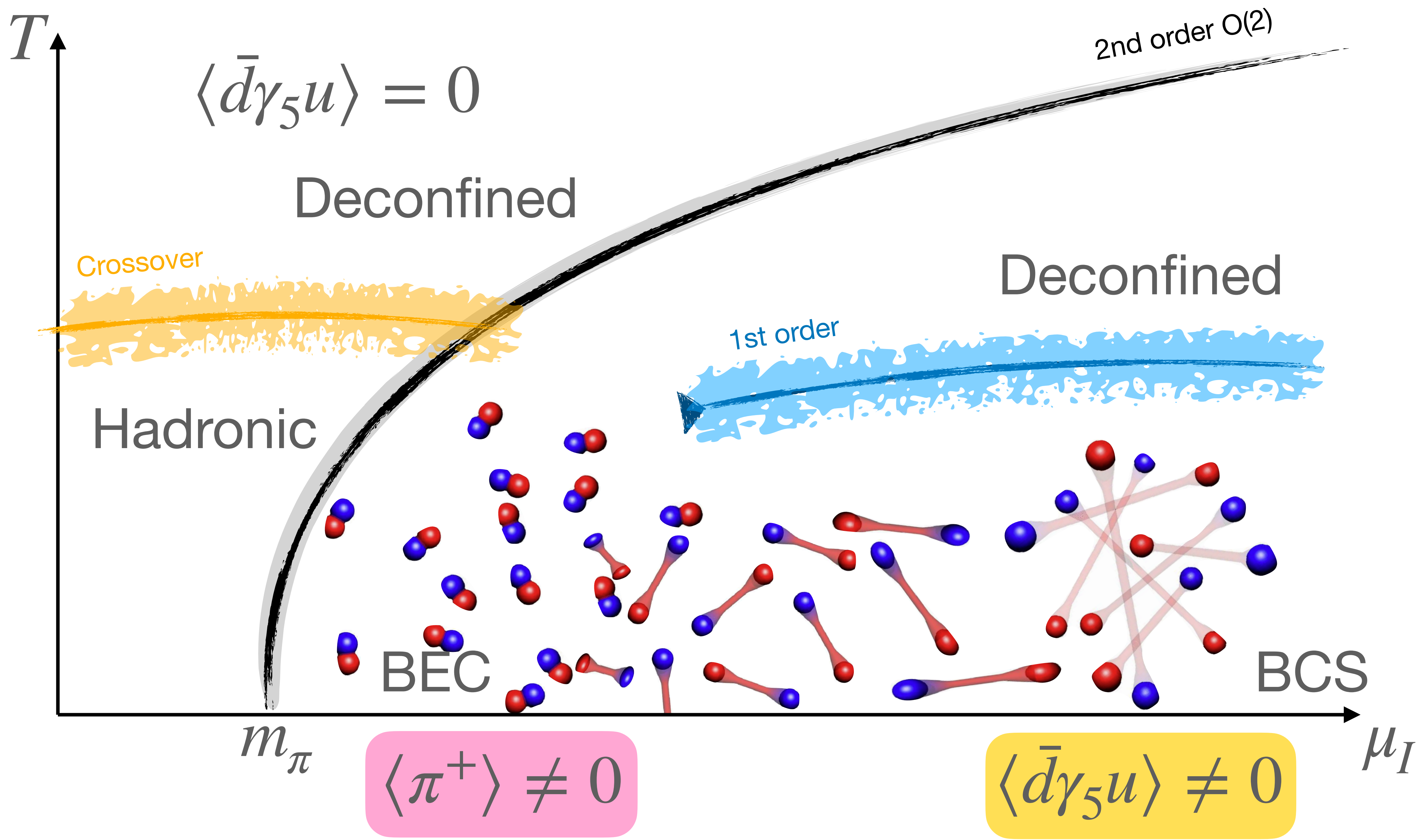
Demorest et al Nature 2010 [1010.5788]



QCD at $\mu_I \neq 0$ & $\mu_B = 0$

Conjectured phase diagram

- Structure conjectured by Son& Stephanov PRL 2001



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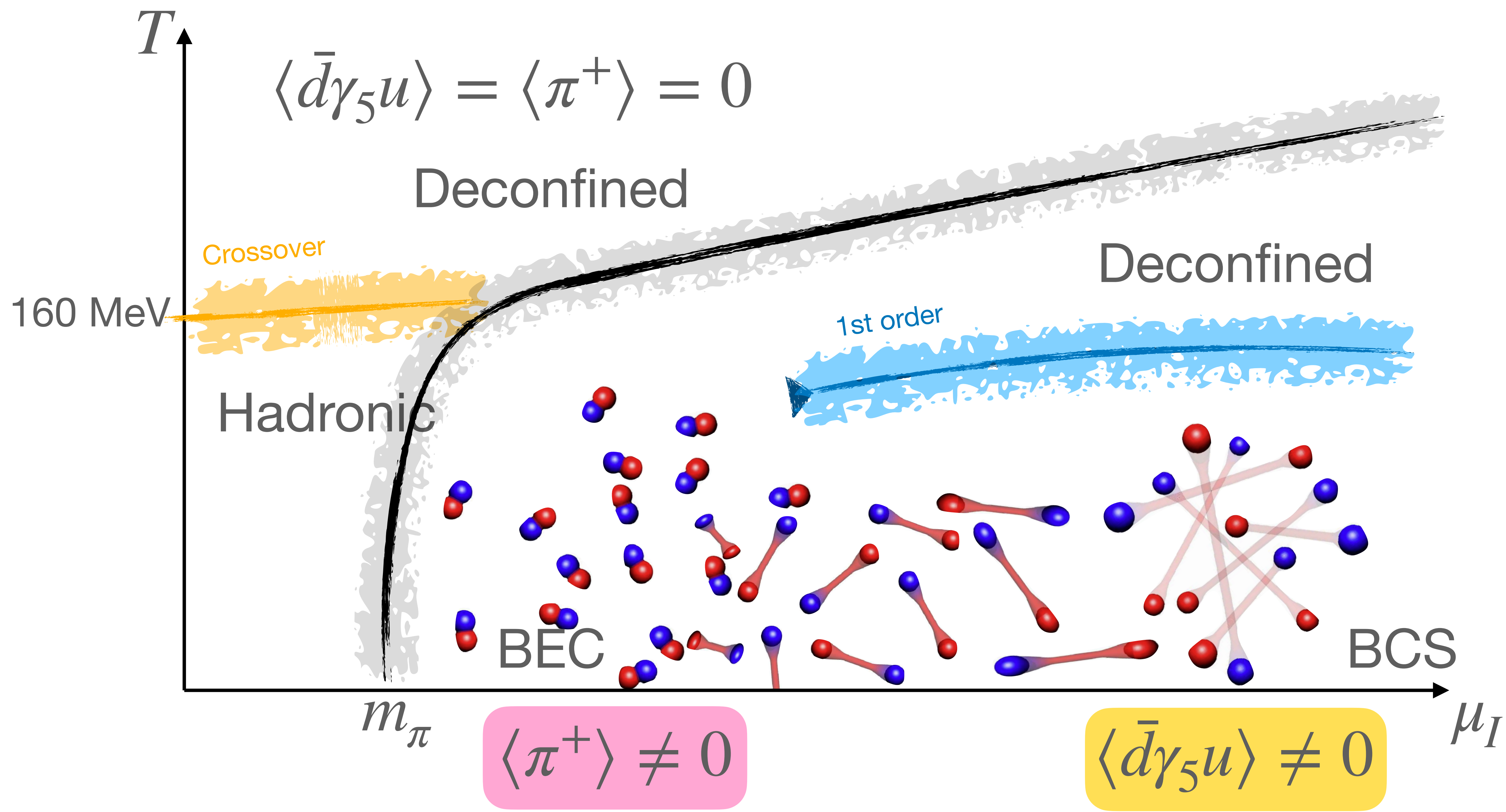
- Status in 2024 - [Kogut-Sinclair], [Brandt, Cuteri & Endrödi],…



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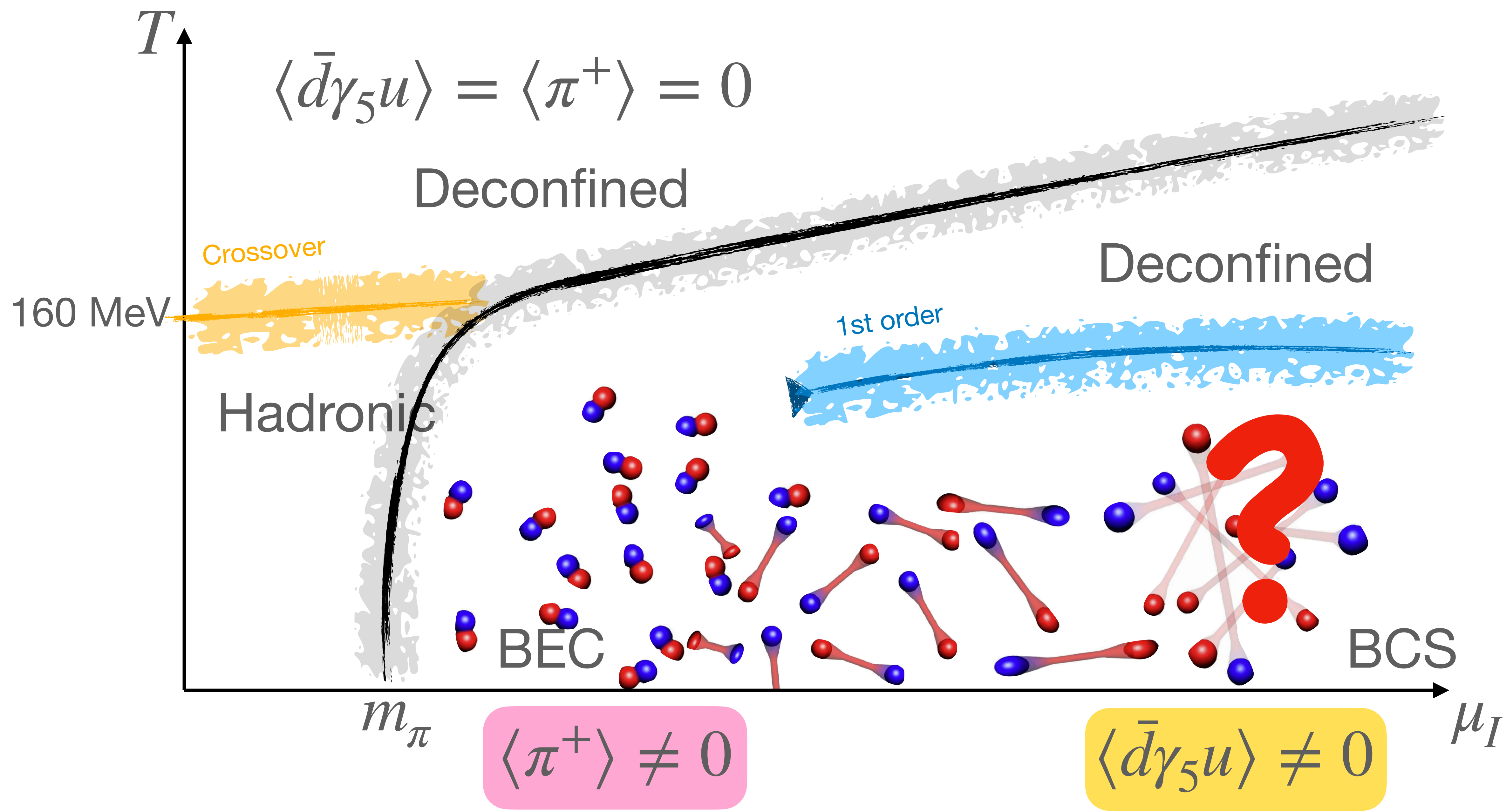

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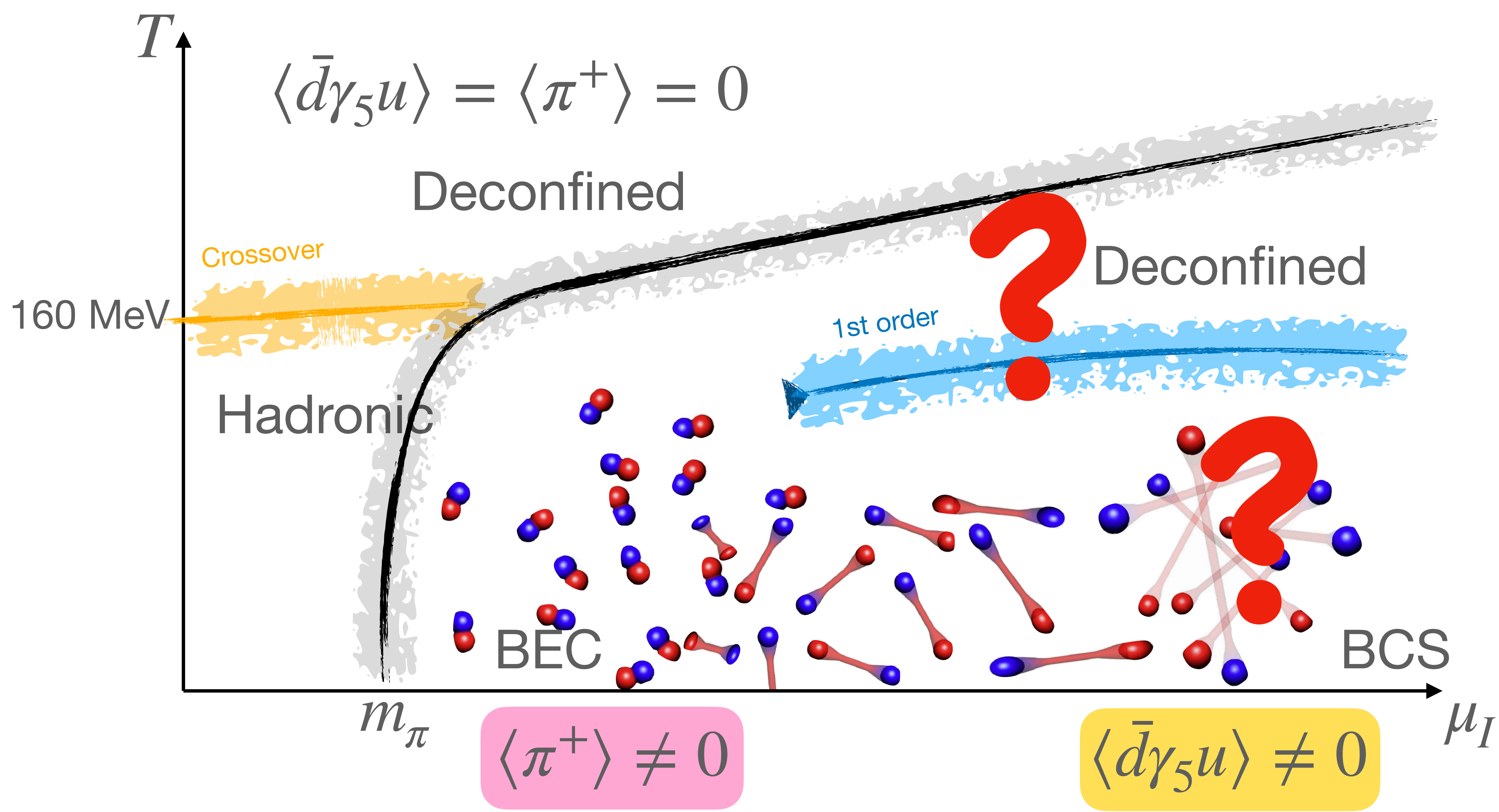


Isospin chemical potential

Three approaches

1. Path integral formulation (new MC calculation for each μ_I)

$$Z(\beta, \mu_I) = \int_\beta [d\phi] e^{-(S[\phi] + \mu_I N_I[\phi])}$$

2. Grand canonical partition function

$$Z(\beta, \mu_I) = \sum_s e^{-\beta(E(s) - \mu_I I_z(s))}$$

- Low temperature limit dominated by $I = I_z = n$ ground states

$$Z(\beta \to \infty, \mu_I) \sim \sum_n e^{-\beta(E_n^{(0)} - \mu_I n)}$$

3. Canonical partition function

$$Z_n(\beta) = \sum_s \delta_{N_s, n} e^{-\beta E(s)} \qquad \Longrightarrow \qquad \mu_I = \left.\frac{dE}{dn}\right|_V$$

Isospin chemical potential

(Grand) canonical approach

- Need ground state energies of system as isospin charge changes
- Correlation functions with quantum numbers of many charged pions

$$C_n(t) = \left\langle \left(\sum_x \pi^-(\mathbf{x}, 0) \right)^n \prod_{i=1}^{n} \pi^+(\mathbf{y}_i, t) \right\rangle$$

  Late time behaviour

$$C_n(t \to \infty) \to Z e^{-E_n^{(0)} t}$$

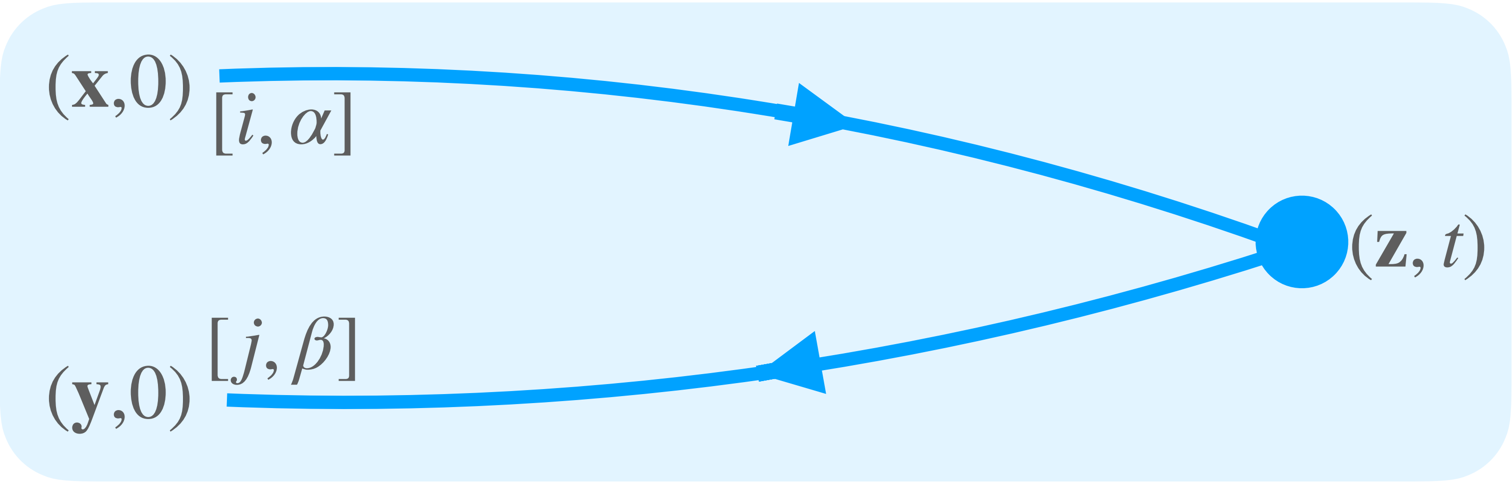
- Large number of Wick contractions
  - E.g. $\sim 10^{40,000}$ for $n = 6144$

Many pion correlation functions

Pion blocks

- Previous studies used:
    - Traces, Recursion relations, Vandermonde matrices & FFTs
    - Limited in n by cost (best algorithm $\sim \mathcal{O}(n^4)$) and numerical precision demands
- Made use of zero-momentum pion block ($12L^3 \times 12L^3$ matrix)

$$\Pi_{(i,\alpha)(j,\beta)}(\mathbf{x},\mathbf{y};t) = \sum_{k,\gamma,\mathbf{z}} S_{(i,\alpha)(k,\gamma)}(\mathbf{x},0;\mathbf{z},t) S^{\dagger}_{(k,\gamma)(j,\beta)}(\mathbf{y},0;\mathbf{z},t)$$



Many pion correlation functions

Symmetric polynomial algorithm

- New algorithm based on symmetric polynomials over eigenvalues of Π (denoted $\vec{x} = \{x_1, \ldots x_N\}$ with $N = 12L^3$)

$$C_n(t) = n!\, E_n(\vec{x})$$

  where for $1 \leq n \leq N$

$$E_n(\vec{x}) \equiv E_n(\{x_1, \ldots, x_N\}) \equiv \sum_{i_1 < \cdots < i_n}^{N} x_{i_1} \ldots x_{i_n}$$

- Recurrence relation for

$$E_k(\{x_1, \ldots, x_M\}) = x_M E_{k-1}(\{x_1, \ldots x_{M-1}\}) + E_k(\{x_1, \ldots, x_{M-1}\}),$$

  (numerically stable and cost is $\mathcal{O}(N^2)$ for all $n \in \{1, \ldots, N\}$)

- Overall cost dominated by finding the eigenvalues: $\mathcal{O}(N^3)$

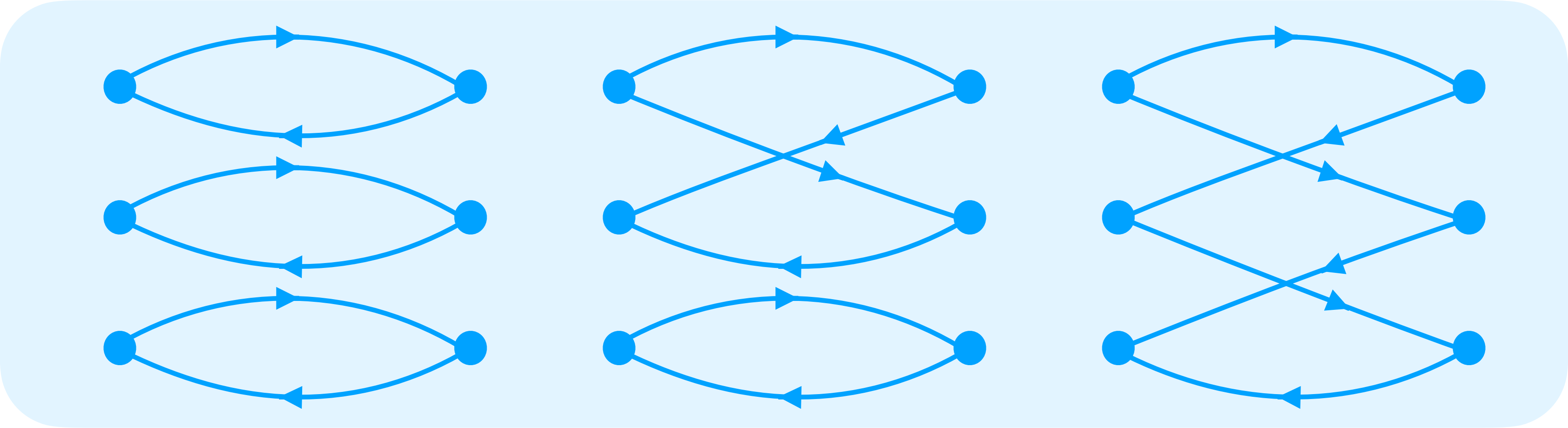
- See 2307.15014 for proof

Many pion correlation functions

Simple example (n=3 for N=4)

- $C_3(t)$ given by

$$C_3 = \text{Tr}(\Pi)^3 - 3\text{Tr}(\Pi^2)\text{Tr}(\Pi) + 2\text{Tr}(\Pi^3)$$



- Expand using trace as sum of powers of eigenvalues

$$\begin{aligned}
&= (x_1 + x_2 + x_3 + x_4)^3 \\
&\qquad -3(x_1^2 + x_2^2 + x_3^2 + x_4^2)(x_1 + x_2 + x_3 + x_4) \\
&\qquad +2(x_1^3 + x_2^3 + x_3^3 + x_4^3) \\
&= 6(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4)
\end{aligned}$$

Many pion correlation functions

Lattice QCD calculations

- Study on four ensembles of $N_f = 2 + 1$ clover gauge configurations with (close-to/below) physical quark masses

Label	$N_{\rm conf}$	β_g	C_{SW}	am_{ud}	am_s	$(L/a)^3 \times (L_4/a)$	a (fm)	M_π (MeV)	L (fm)	$M_\pi L$	T (MeV)
A	665	6.3	1.20537	-0.2416	-0.2050	$48^3 \times 96$	0.091(1)	166(2)	4.37	3.75	22.8
B	1262	6.3	1.20537	-0.2416	-0.2050	$64^3 \times 128$	0.091(1)	167(2)	5.82	5.08	17.1
C	846	6.5	1.17008	-0.2091	-0.1778	$72^3 \times 192$	0.070(1)	166(2)	5.04	4.33	14.7
D	246	6.5	1.17008	-0.2095	-0.1793	$96^3 \times 192$	0.070(1)	128(2)	6.72	4.40	14.7

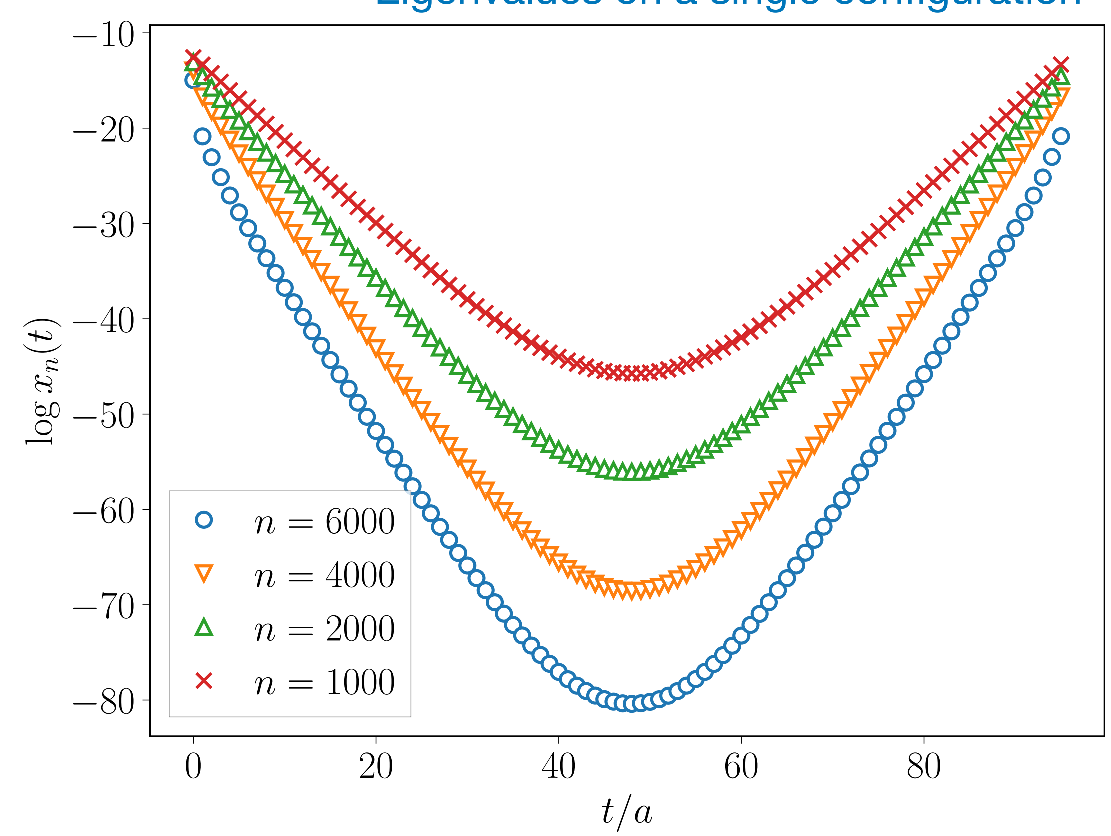
New in 2406.09273

- Sparsened quark propagators computed from grid of 8^3 sites on one timeslice: $N = 12 \times L^3 = 12 \times 8^3 = 6144$

- Eigenvalues computed by SVD of time sliced quark-propagator (since $\Pi = S^\dagger S$)

- Calculations performed in double, 2-double and 3-double

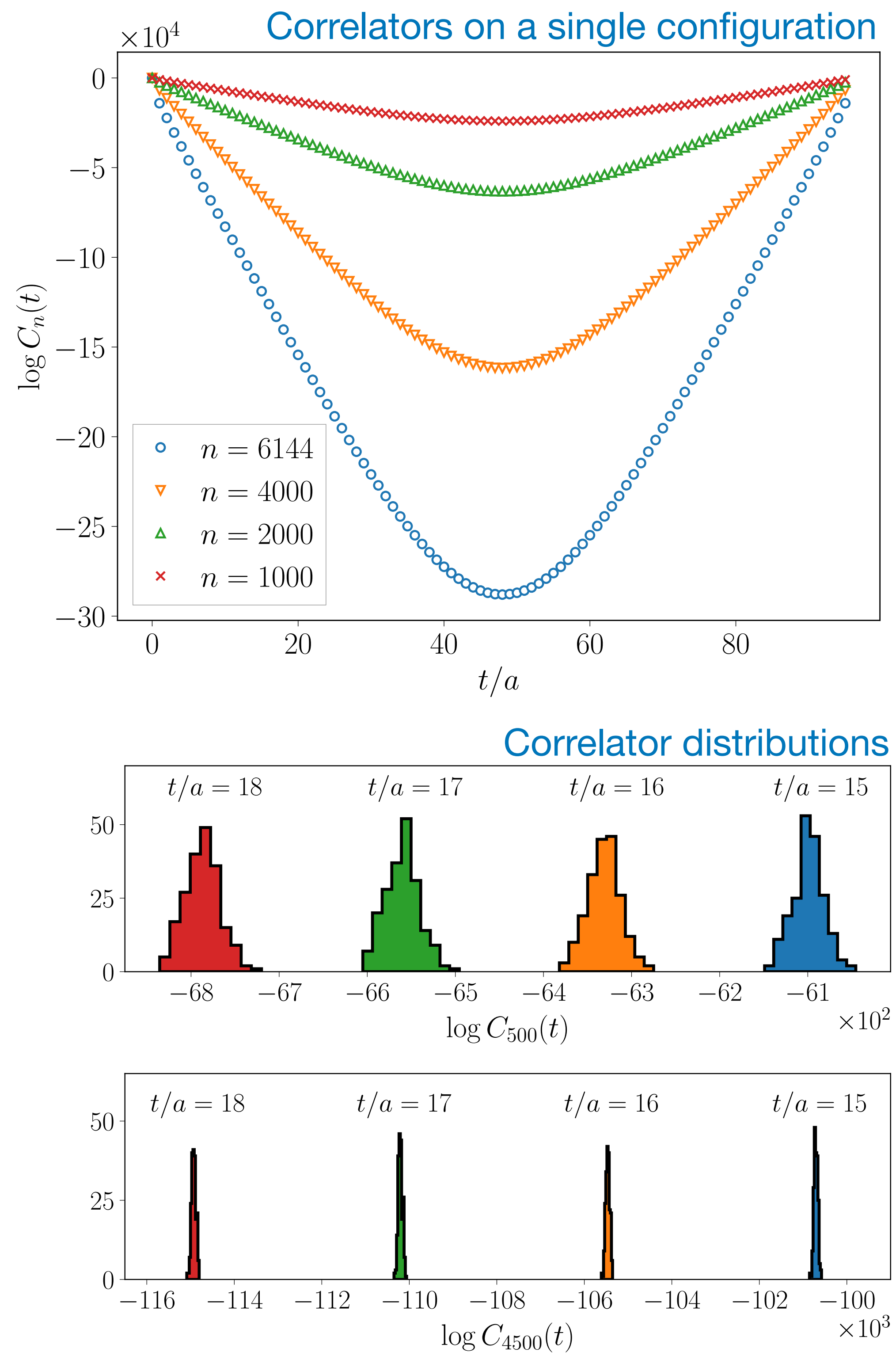
Eigenvalues on a single configuration



Many pion correlation functions

Lattice QCD calculations

- Correlation functions vary rapidly in Euclidean time
  - $C_{6144}(t)$ varies by $> 10^5$ orders of magnitude
- Correlation functions vary between samples by many orders of magnitude
  - Central Limit Theorem only valid at unachievable sample size 😒
  - Correlation function distributions are approximately log-normal 😌



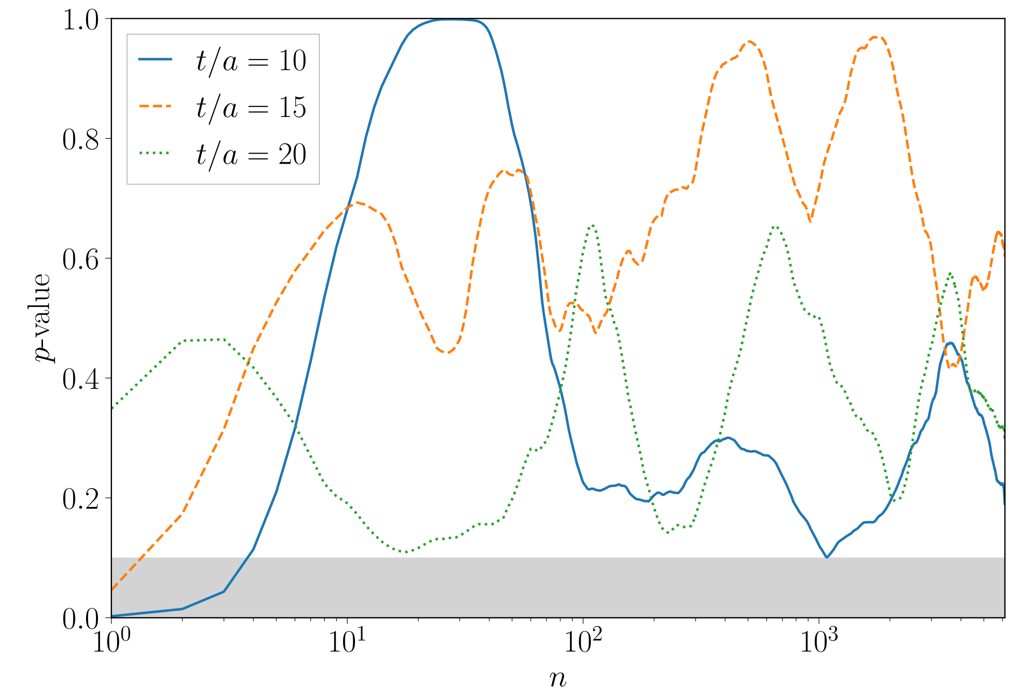
Many pion correlation functions

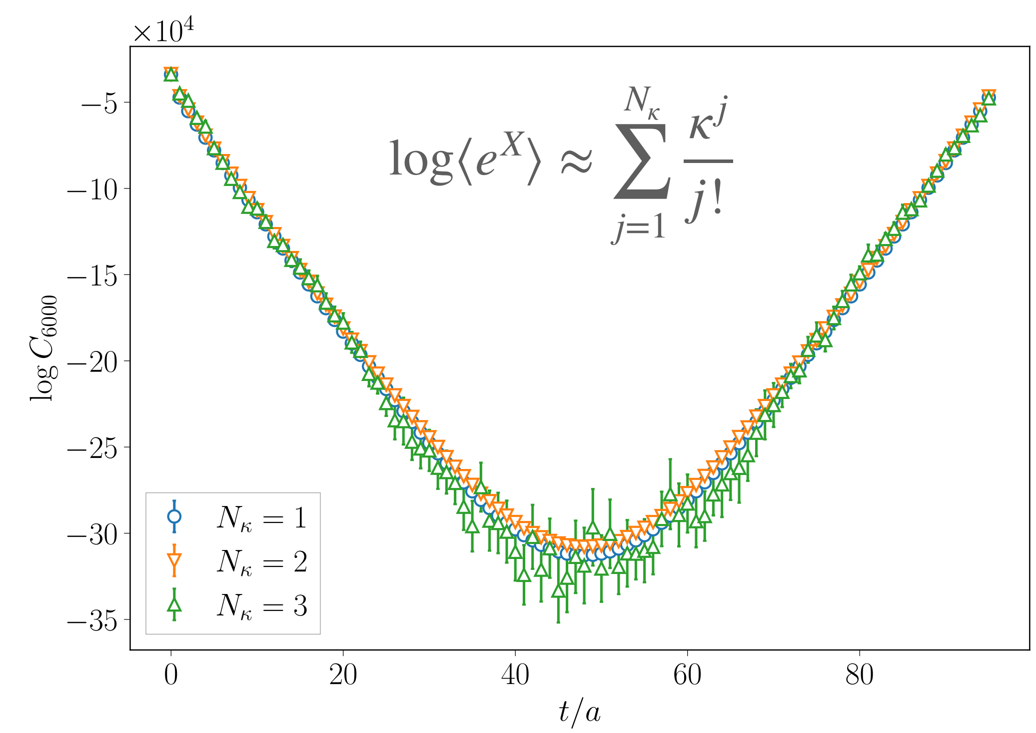
Log-normality tests and cumulants

- No statistically significant deviations from log-normal for $n > 4$
    - Shapiro-Wilk test $p > 0.1$
- Deviations can be incorporated through cumulants but only contribute noise
- Henceforth assume data are log-normal
    - i.e. $\log C_n(t)^{[U]} \sim \mathcal{N}(\mu_n(t), \sigma_n(t))$ where

$$\mu_n = \frac{1}{N_{\text{conf}}} \sum_{i=1}^{N_{\text{conf}}} \log C_n^{[U_i]}(t)$$

$$\sigma_n^2 = \frac{1}{N_{\text{conf}} - 1} \sum_{i=1}^{N_{\text{conf}}} \left(\log C_n^{[U_i]}(t) - \mu_n \right)^2$$





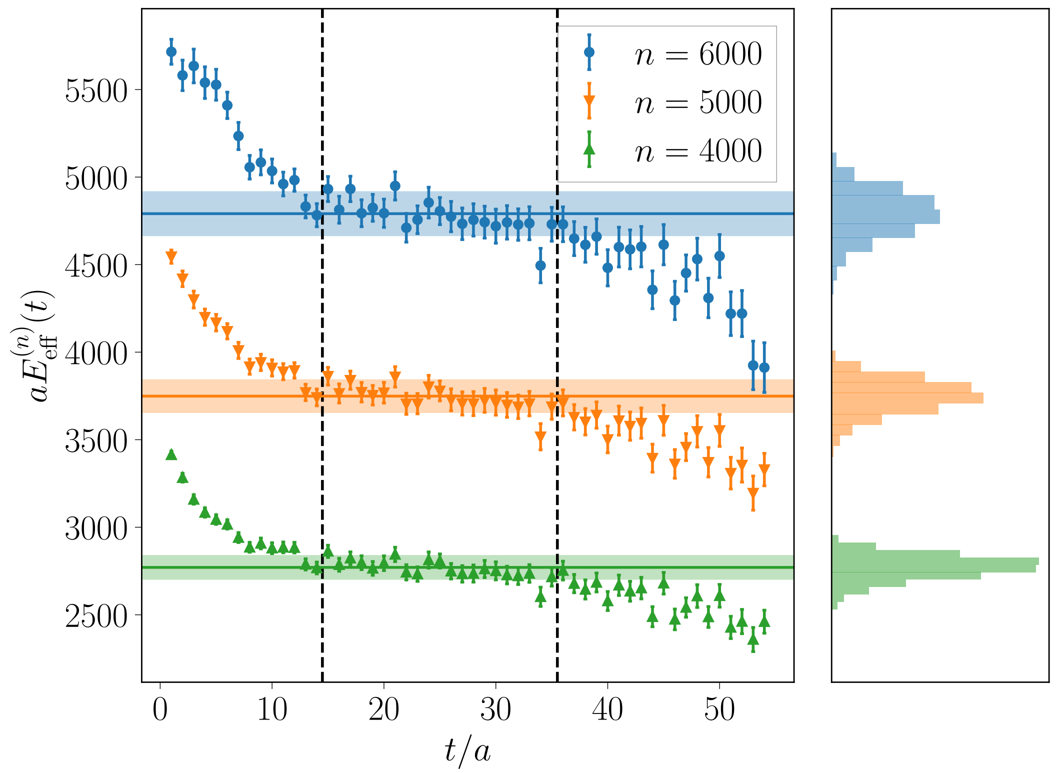
Many pion correlation functions

Many pion energies

- Effective energy from log-normality

$$E_{\text{eff}}^{(n)}(t) = \mu_n(t) - \mu_n(t-1) + \frac{\sigma_n^2(t)}{2} - \frac{\sigma_n^2(t-1)}{2}$$

- ~~CLT~~: χ^2-fitting makes no sense
- Bootstrap analysis takes value of $E_{\text{eff}}^{(n)}$ for random timeslice in plateau region
- Entire bootstrap histogram propagated into subsequent analysis
- Energy significantly larger than that of n free pions



Many pion correlation functions

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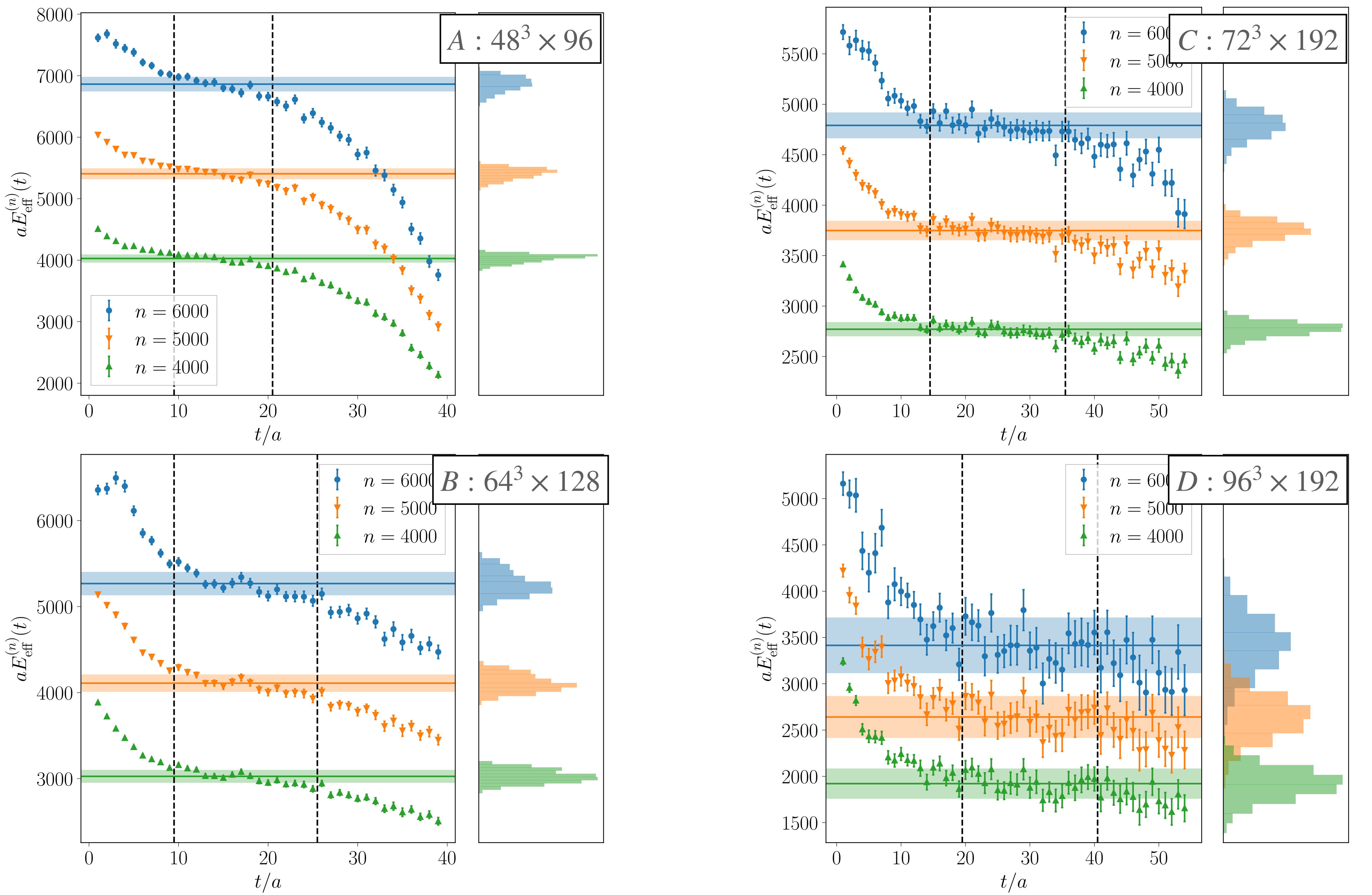

Many pion correlation functions

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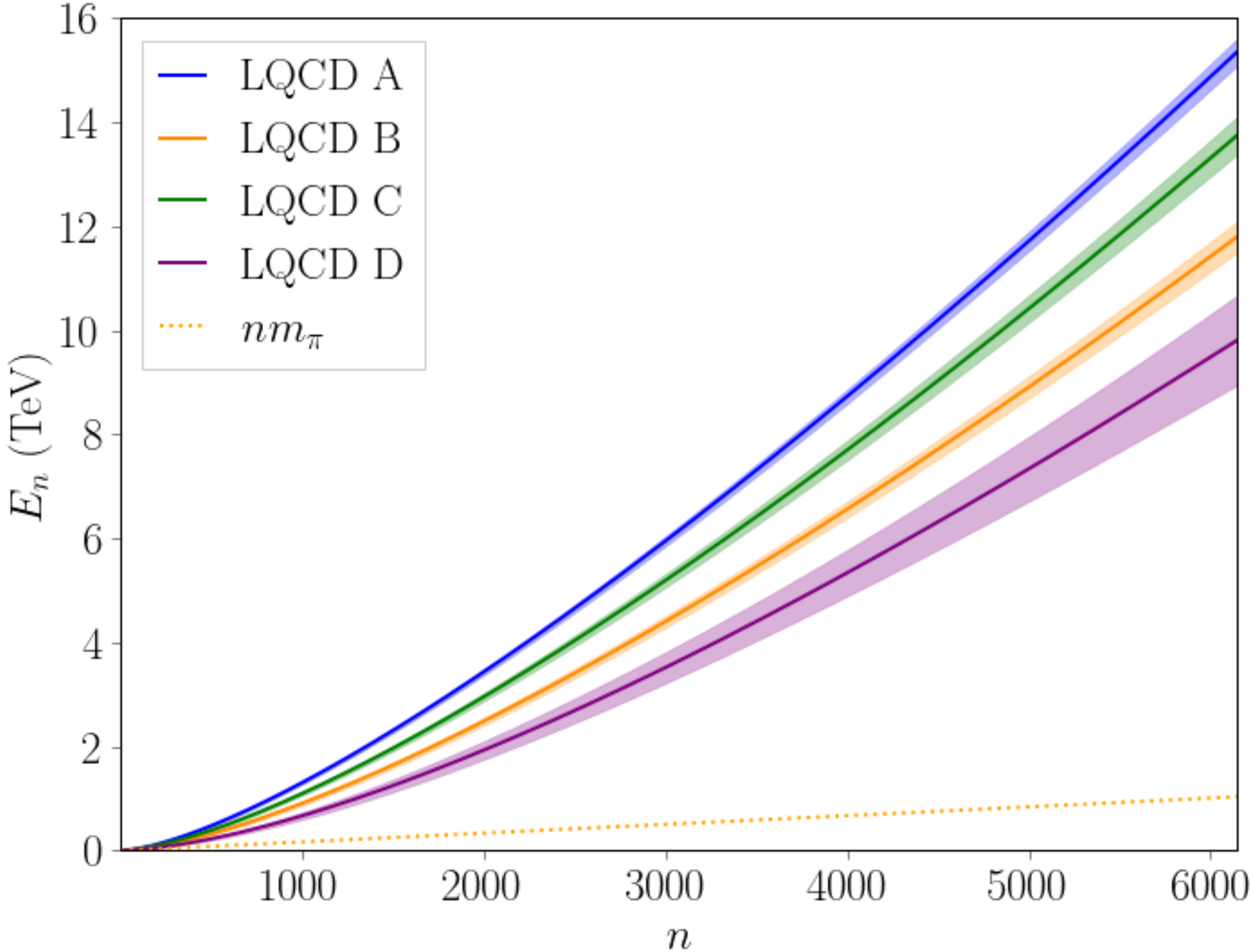


Many pion correlation functions

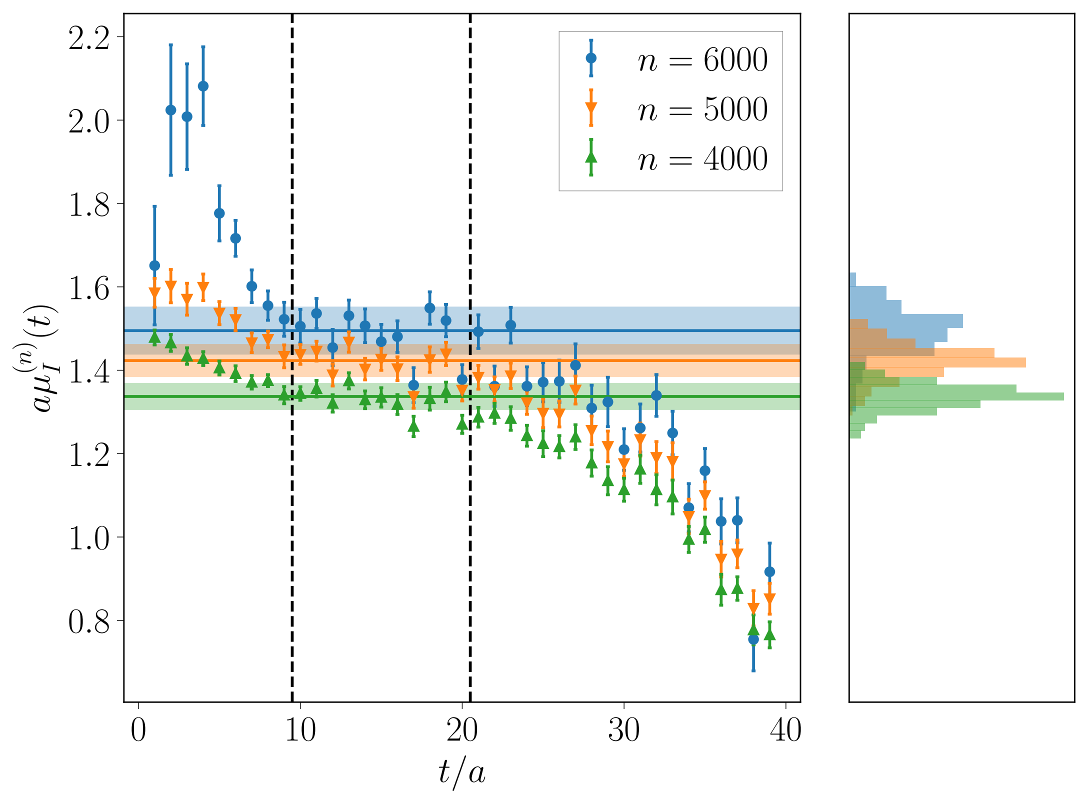
Isospin chemical potential

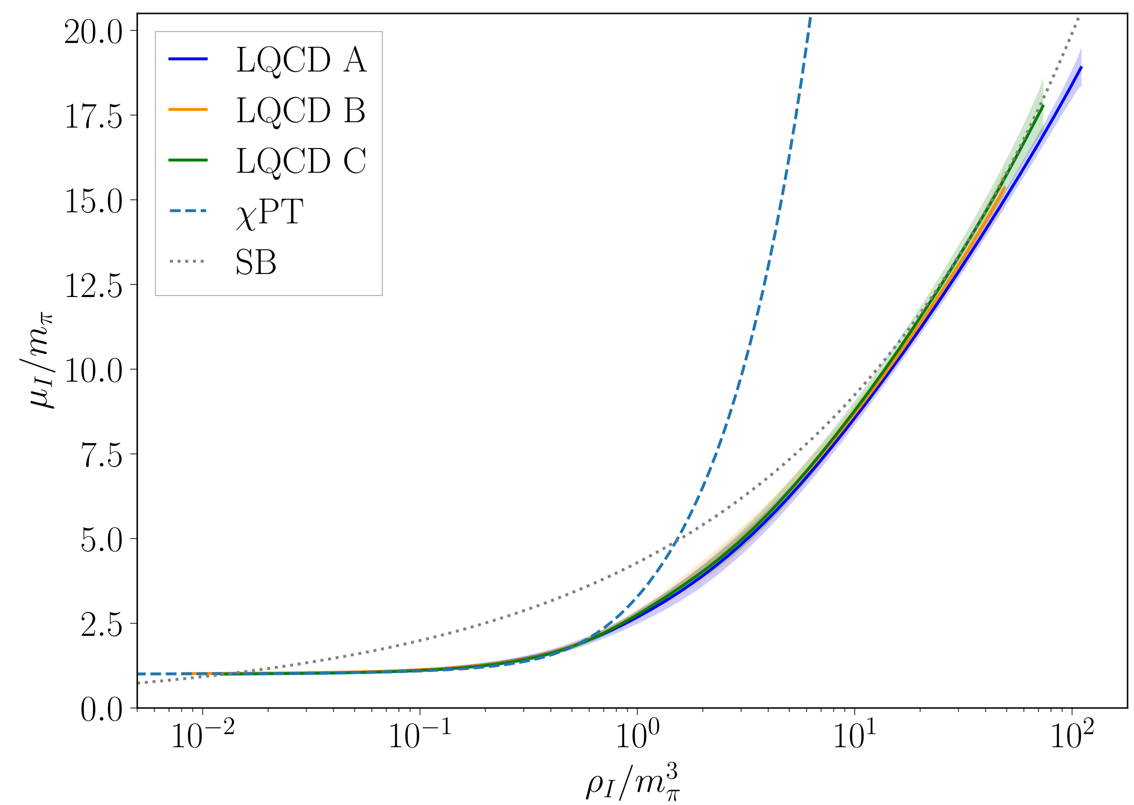
- Isospin chemical potential

$$\mu_I(n) = \left.\frac{dE_n}{dn}\right|_{V\text{ const}} \approx \frac{E_{n+1} - E_{n-1}}{2}$$

- Curve collapse $\implies$ thermodynamic limit ($T \sim 20$ MeV)

- Agreement with
  - Chiral perturbation theory for $\mu_I \to 0$
  - Stefan-Boltzmann limit for $\mu_I \to \infty$





QCD at $\mu_I \neq 0$ & $\mu_B = 0$

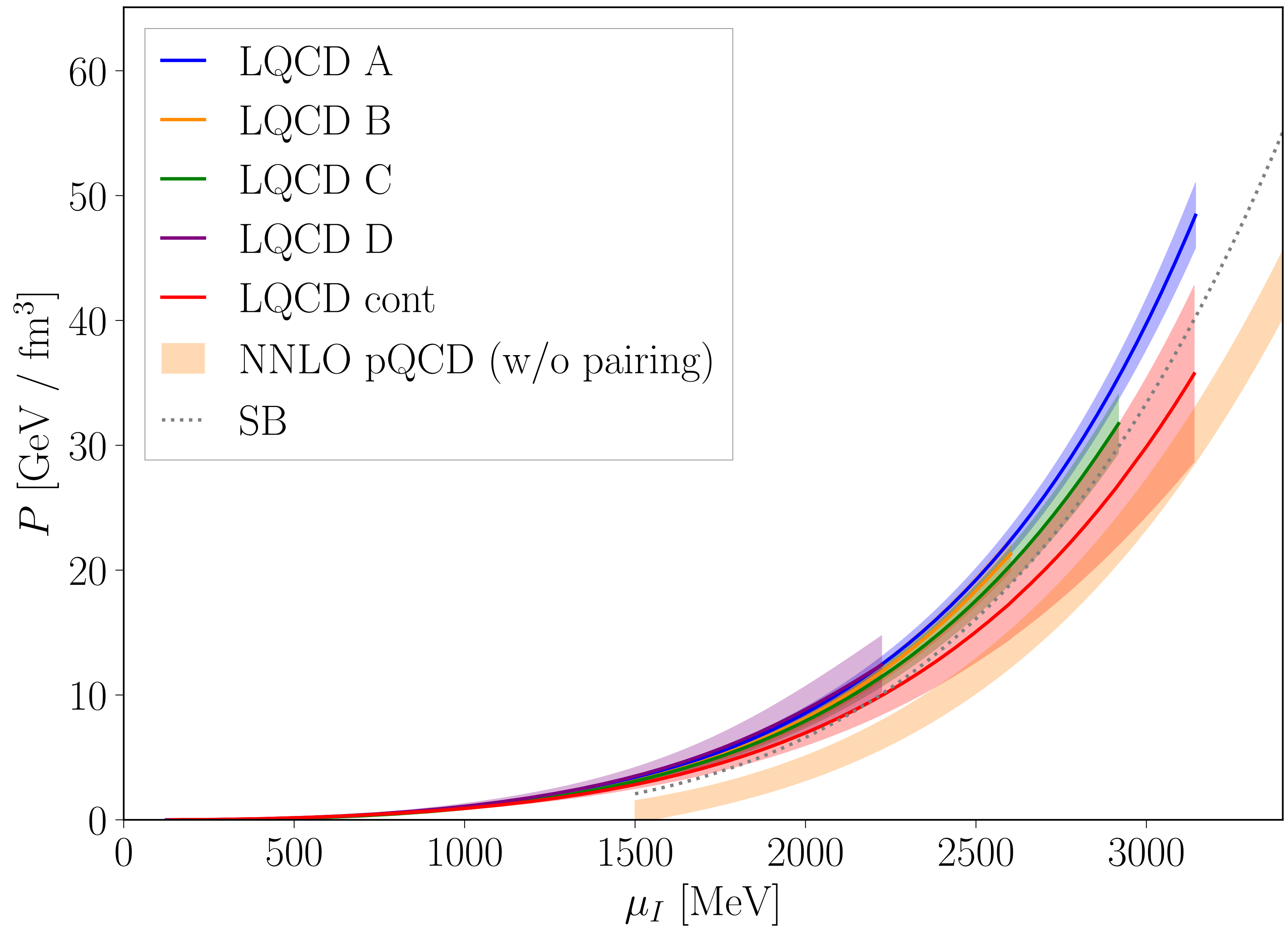
Thermodynamic observables

- Observables defined as

$$\langle \mathcal{O}(E,n) \rangle_{\beta,\mu_I} = \frac{1}{Z(\beta,\mu_I)} \sum_n \mathcal{O}(E_n, n) e^{-\beta(E_n - \mu_I n)}$$

- Pressure

$$P(\beta,\mu_I) = \int_0^{\mu_I} \frac{\langle n \rangle_{\beta,\mu_I}}{V} d\mu$$



QCD at $\mu_I \neq 0$ & $\mu_B = 0$

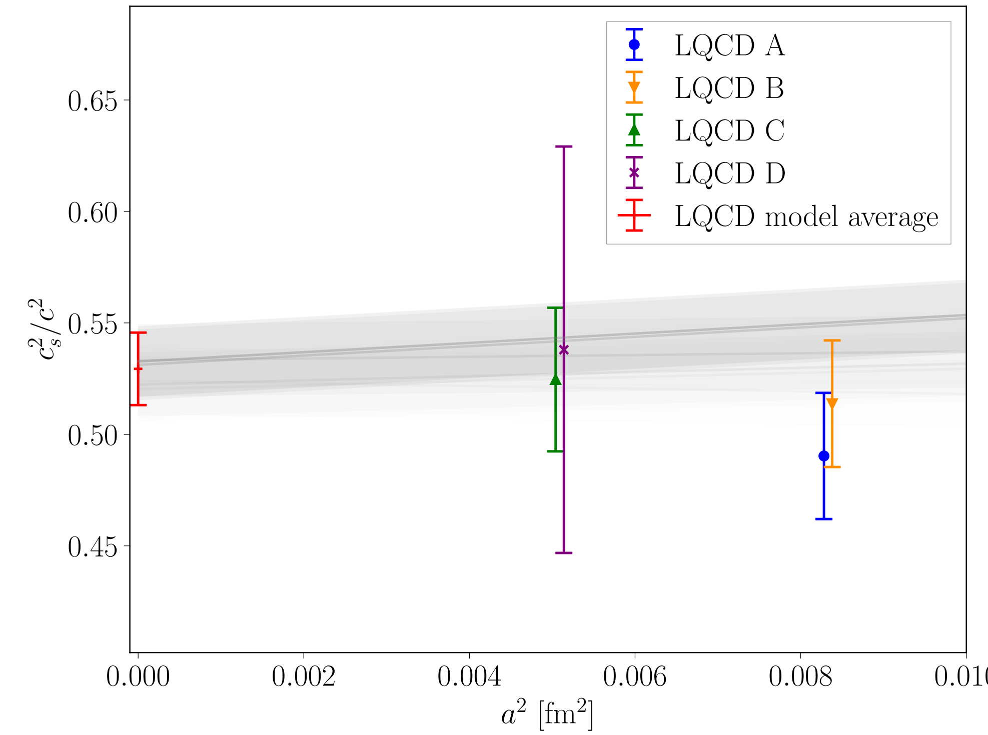
Chiral interpolation & continuum extrapolation

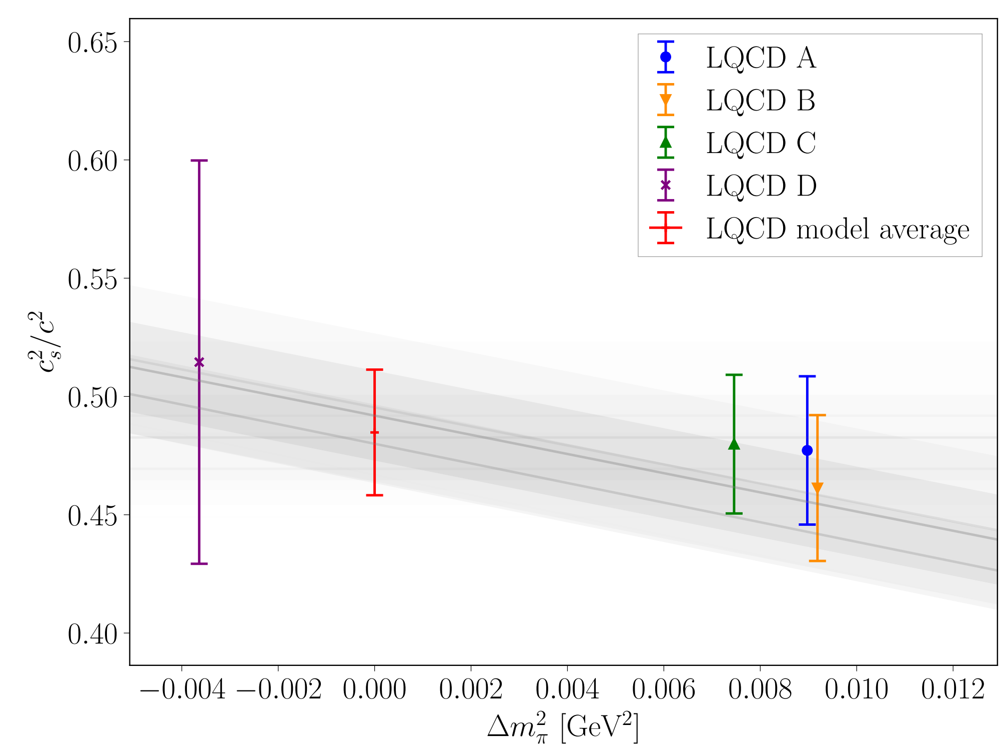
- Action is perturbatively improved: $\mathcal{O}(a^2, g^2(a^{-1})a)$
- Masses range over $m_\pi \in [125, 165]$ MeV
- Extrapolation/interpolation form

$$X^{(j)}(a, \mu_I, m_\pi) = X_0^{(j)}(\mu_I) + X_1^{(j)}a^2 + X_2^{(j)}a^2\mu_I + X_3^{(j)}a^2\mu_I^2 + X_4^{(j)}(m_\pi^2 - \overline{m}_\pi^2)$$

for $X \in \{P, \epsilon, c_s^2, \dots\}$

- Systematic uncertainty from model-averaging all sub-models





QCD at $\mu_I \neq 0$ & $\mu_B = 0$

Superconducting gap

- Asymptotic freedom for $\mu_I \to \infty$: attractive interaction $\Longrightarrow$ superconductivity

- Superconducting gap

$$\langle \bar{u}_a \gamma_5 d_b \rangle = \delta_{ab} \Delta$$

- Solve gap equation [Son&Stephanov 2001, Cohen&Sen 2015, Fujimoto 2023]

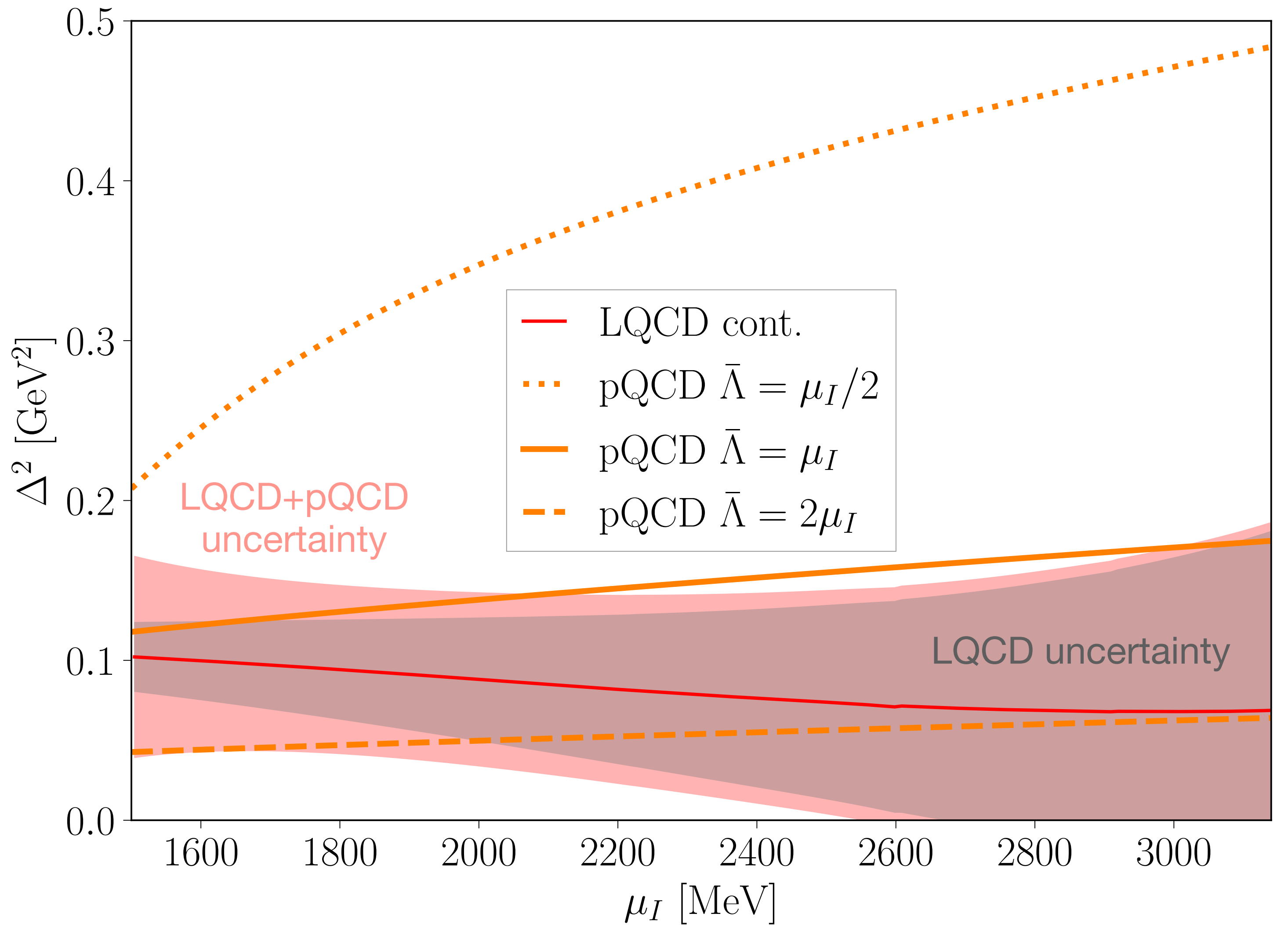
$$\Delta(\mu_I) = b' \exp\left(-\frac{3\pi^2}{2g(\mu_I)} \right)$$

- Nontrivial background

$$P(\Delta) - P(\Delta = 0) = \frac{N_c \mu_I^2}{8\pi^2} \Delta^2$$

- Estimate of gap from

$$P_{\mathrm{LQCD}} - P_{\mathrm{pQCD}}^{\mathrm{NNLO}}(\Delta = 0)$$



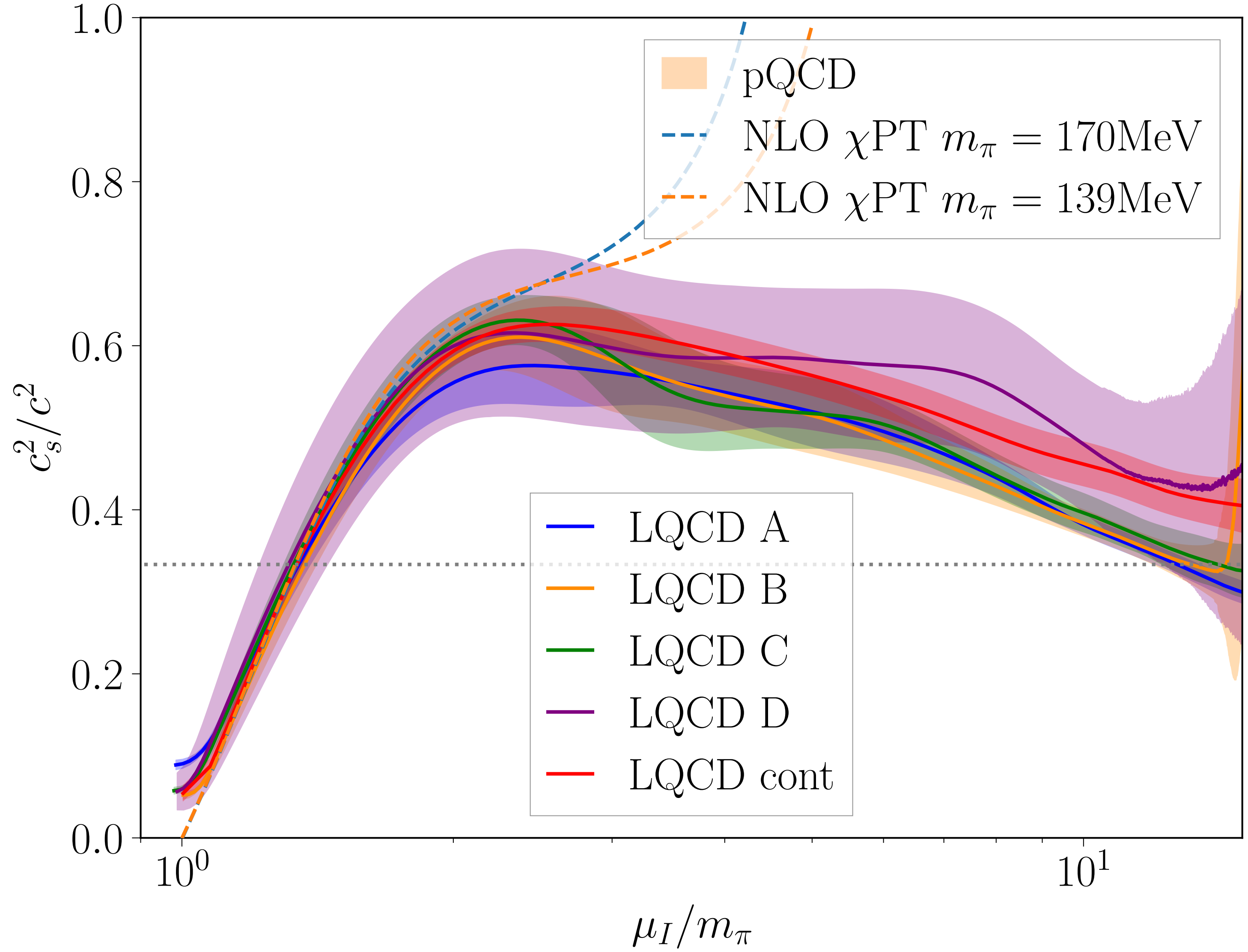
QCD at $\mu_I \neq 0$ & $\mu_B = 0$

Speed of sound

- Since temperature is $0 \sim T \leq 20$ MeV, isentropic speed-of-sound can be determined

$$\frac{1}{c_s^2} = \frac{\partial \epsilon}{\partial P} = \frac{1}{\langle n \rangle_{\beta,\mu_I}} \frac{\partial}{\partial \mu_I} \langle E \rangle_{\beta,\mu_I}$$

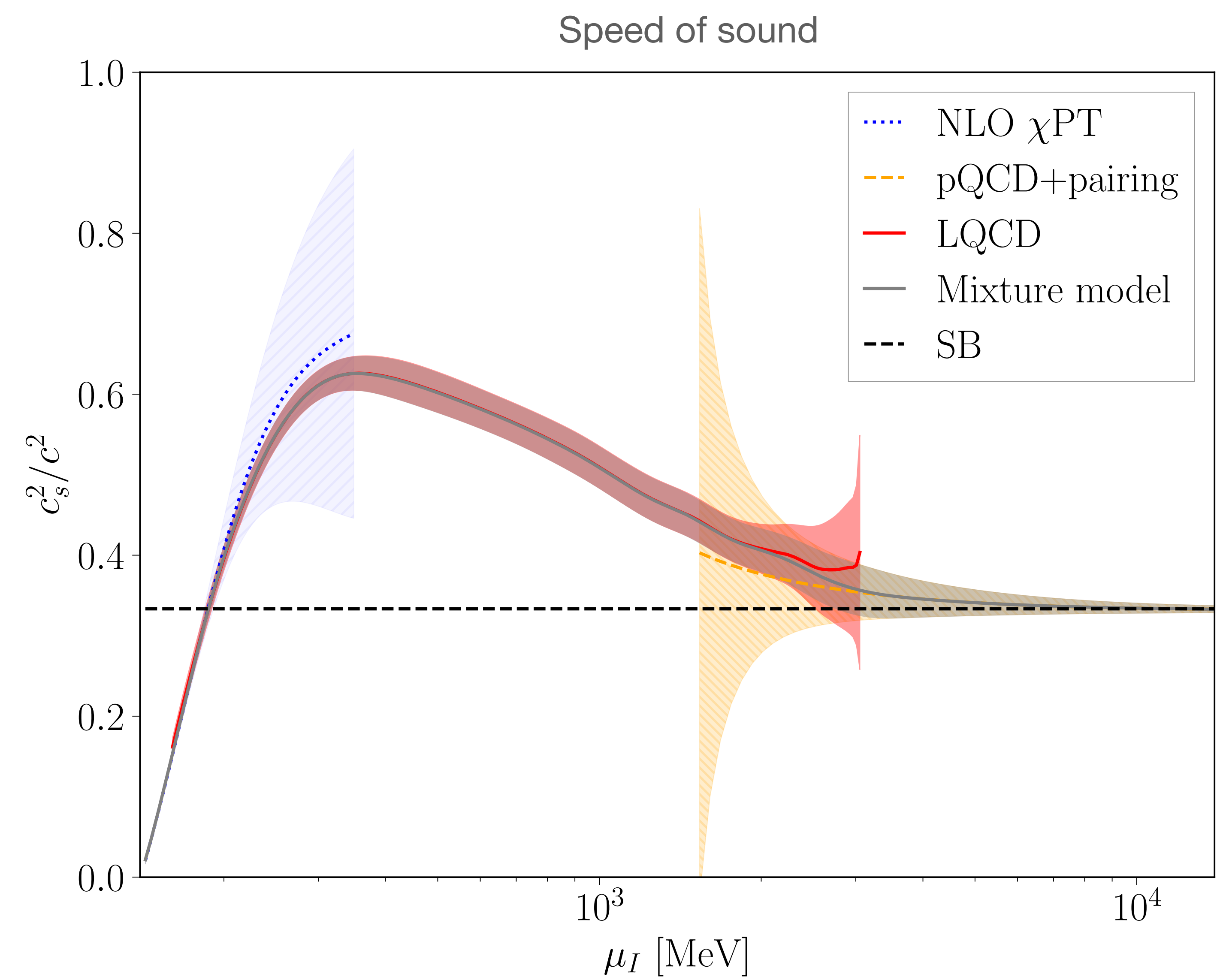
- Exceeds conformal bound $c_s^2 \leq 1/3c^2$ over wide range of μ_I
- Similar behaviour seen for small μ_I [Brandt, Cuteri, Endrodi 2022]
- Similar behaviour seen in $N_c = 2$ QCD [E. Itou et al]
- Eventually relaxes to pQCD/ideal gas



QCD at $\mu_I \neq 0$ & $\mu_B = 0$

Isospin dense equation of state

- Combine information from LQCD, χPT, pQCD
- Bayesian model mixing
  - χPT: NLO, errors from NLO-LO
  - pQCD: NNLO with pairing gap, errors from scale variation
  - LQCD: continuum, physical mass
  - Details follow nuclear EoS methods



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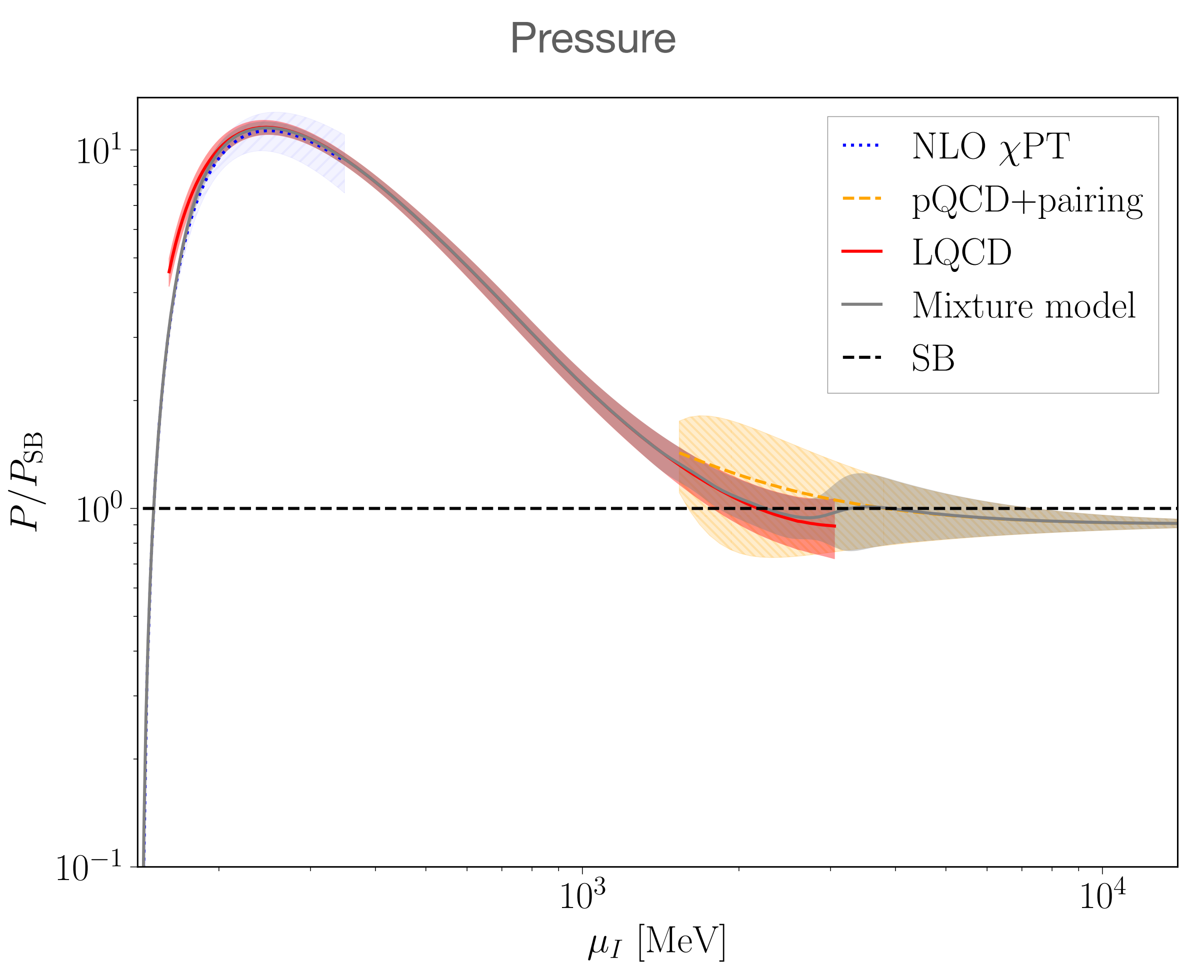

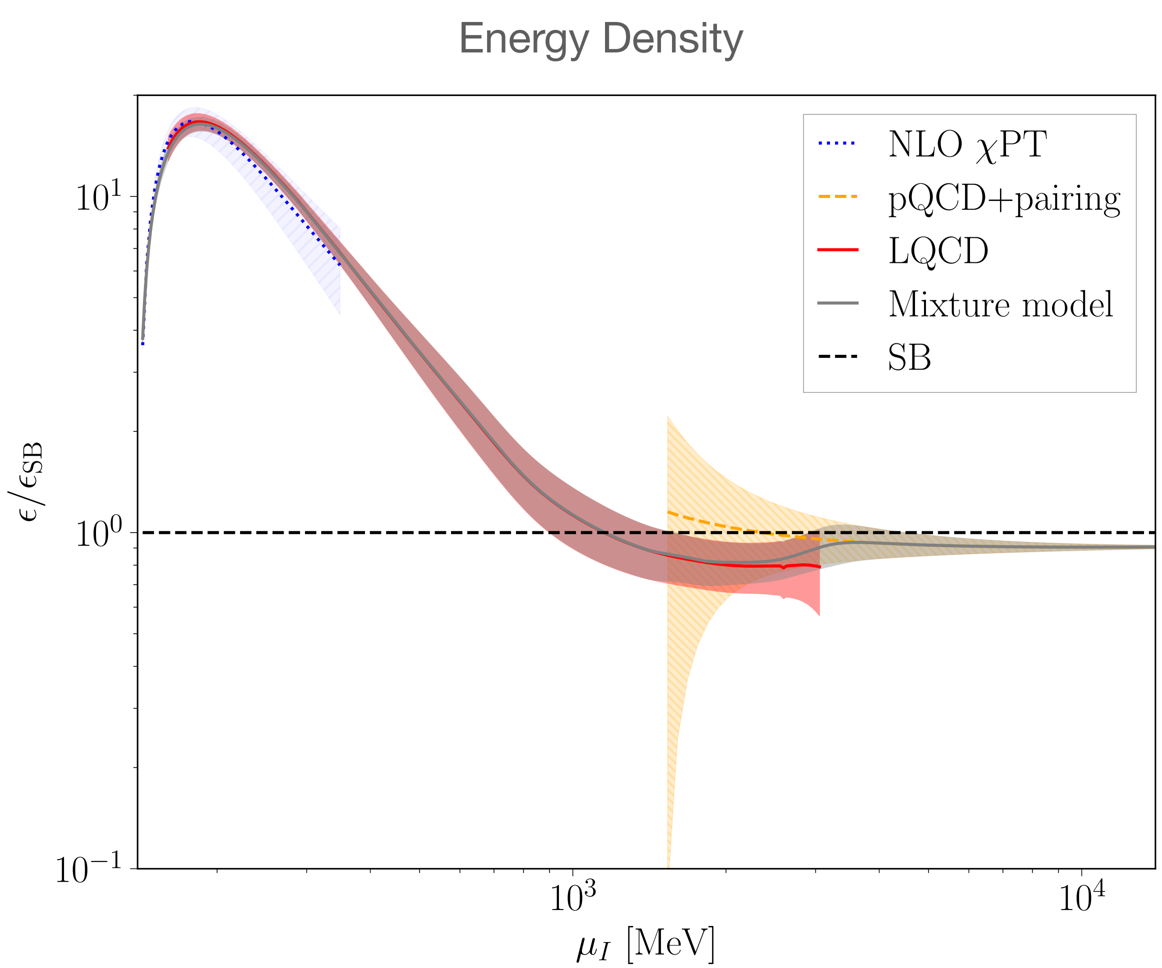

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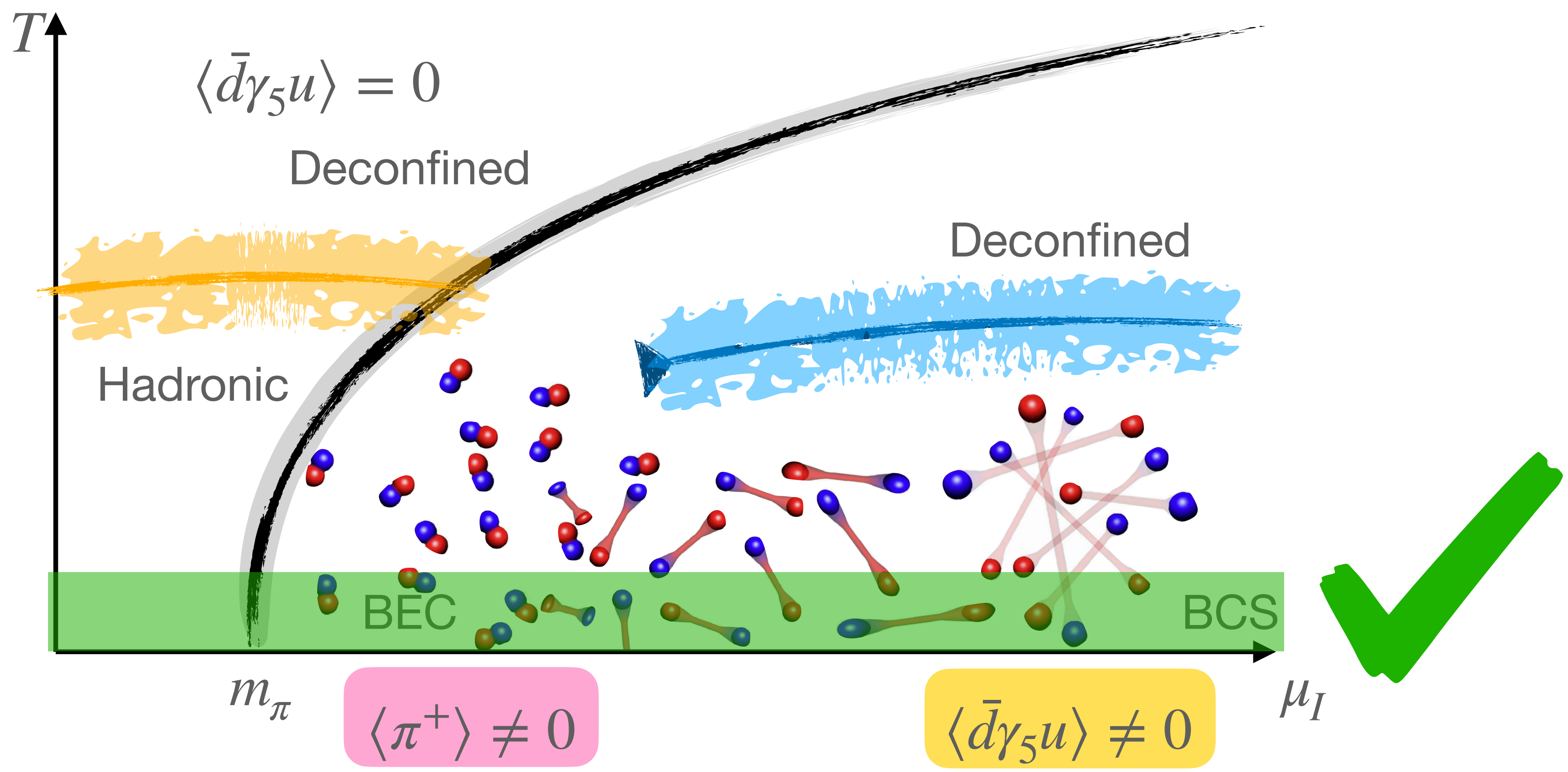


Bounding the nuclear equation of state

QCD inequalities [T Cohen 2004]

- $N_f = 2$ partition function for baryon chemical potential

$$Z_{\mathrm{B}}\left(\beta,\mu_{\mathrm{B}}\right) = \int_\beta [dA] \det \mathscr{D}\left(\frac{\mu_{\mathrm{B}}}{N_{\mathrm{c}}}\right)^2 e^{-S_{\mathrm{G}}}$$

where $\mathscr{D}(\mu) = \not{D} + m - \mu\gamma_0$

  - Fermion determinant complex for $\mu_B \neq 0$

- $N_f = 2$ partition function for isospin chemical potential

$$Z_{\mathrm{I}}\left(\beta,\mu_{\mathrm{I}}\right) = \int_\beta [dA] \det \mathscr{D}\left(-\frac{\mu_{\mathrm{I}}}{2}\right) \det \mathscr{D}\left(\frac{\mu_{\mathrm{I}}}{2}\right) e^{-S_{\mathrm{G}}} \quad = \int_\beta [dA] \left|\det \mathscr{D}\left(\frac{\mu_{\mathrm{I}}}{2}\right)\right|^2 e^{-S_{\mathrm{G}}}$$

- QCD inequality

$$Z_{\mathrm{B}}\left(\beta,\mu_{\mathrm{B}}\right) \leq \int_\beta [dA] \left|\det \mathscr{D}\left(\frac{\mu_{\mathrm{B}}}{N_{\mathrm{c}}}\right)\right|^2 e^{-S_{\mathrm{G}}} = Z_I(\beta, \mu_I = 2\mu_B/N_c)$$

- Translates to pressure bound ($PV = T\log(Z)$)

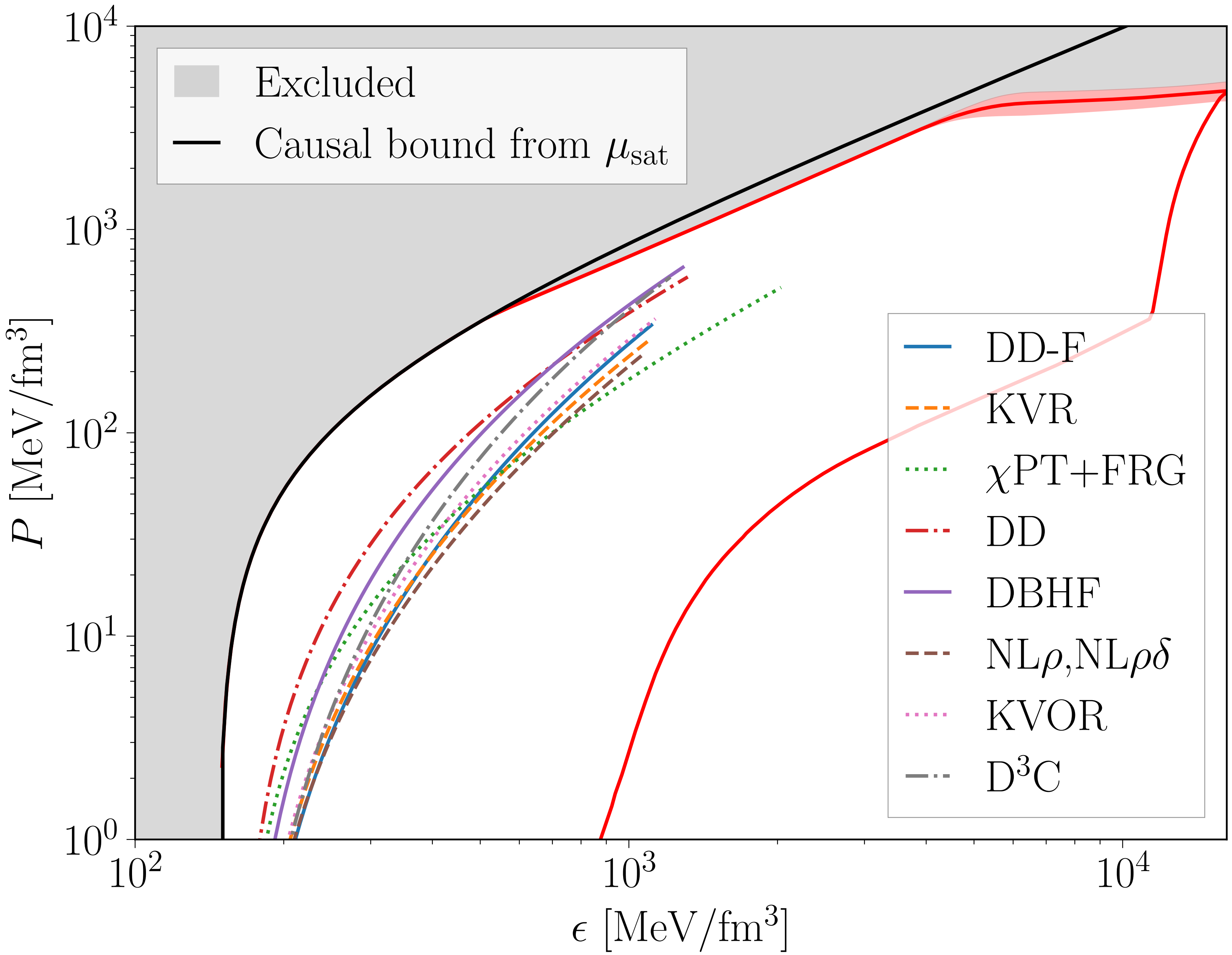
$$\boxed{P_B(\mu_B) \leq P_I(\mu_I = 2\mu_B/N_c)}$$

- Saturated up to in pQCD [Moore&Gorda 2023]

Bounding the nuclear equation of state

QCD inequalities [T Cohen 2004, Fujimoto & Reddy 2023]

- Using GP-model isospin-dense EoS gives bound on symmetric nuclear matter EoS



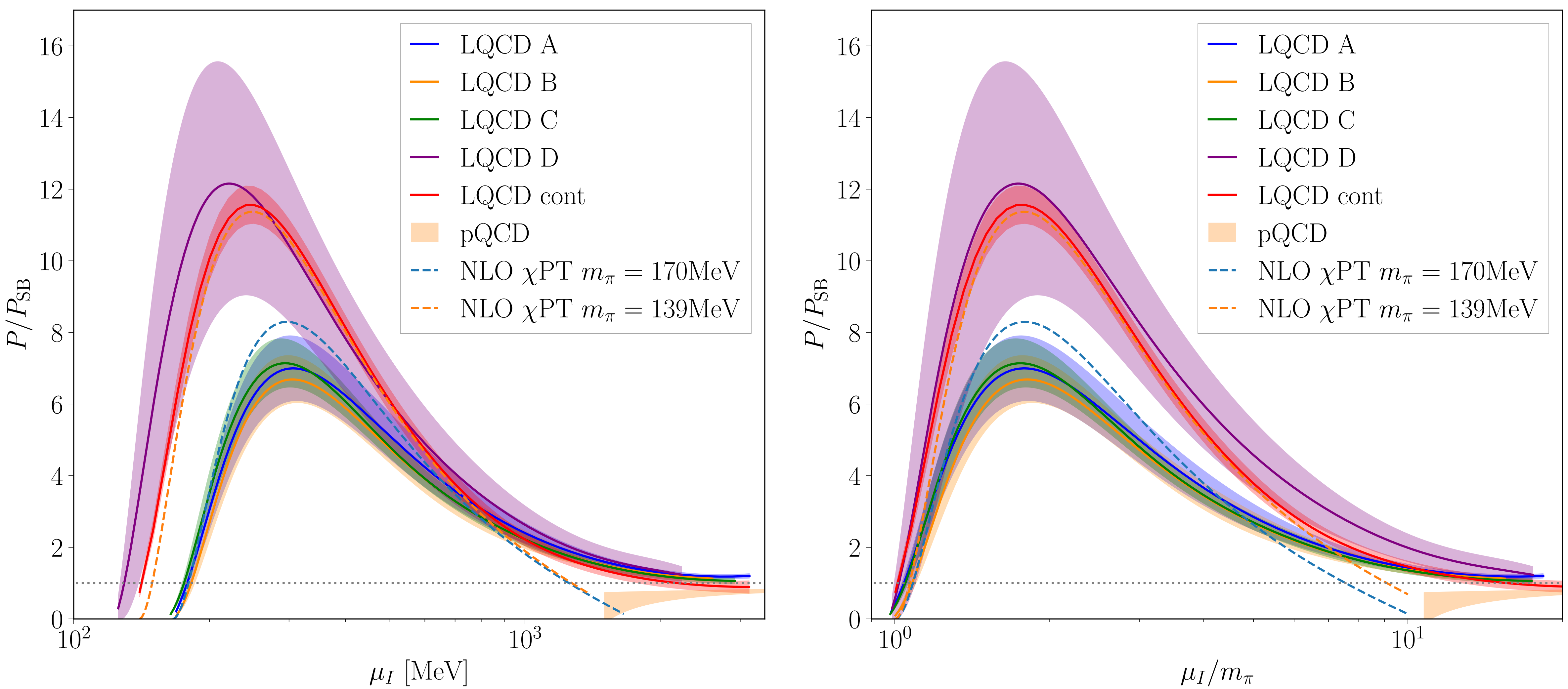
QCD at $\mu_I \neq 0$ & $\mu_B = 0$

A fascinating playground

- Complete determination of isospin-dense matter equation of state
    - LQCD+χPT+pQCD
- Clear signal for transition to pion BEC and eventually to BCS superconducting state
    - Determination of superconducting gap from difference to pQCD
- Conformal bound $c_s^2 < c^2/3$ clearly exceeded as in $N_c = 2$ QCD
- Rigorous bound on EoS of symmetric nuclear matter (but not so practical ☹️)

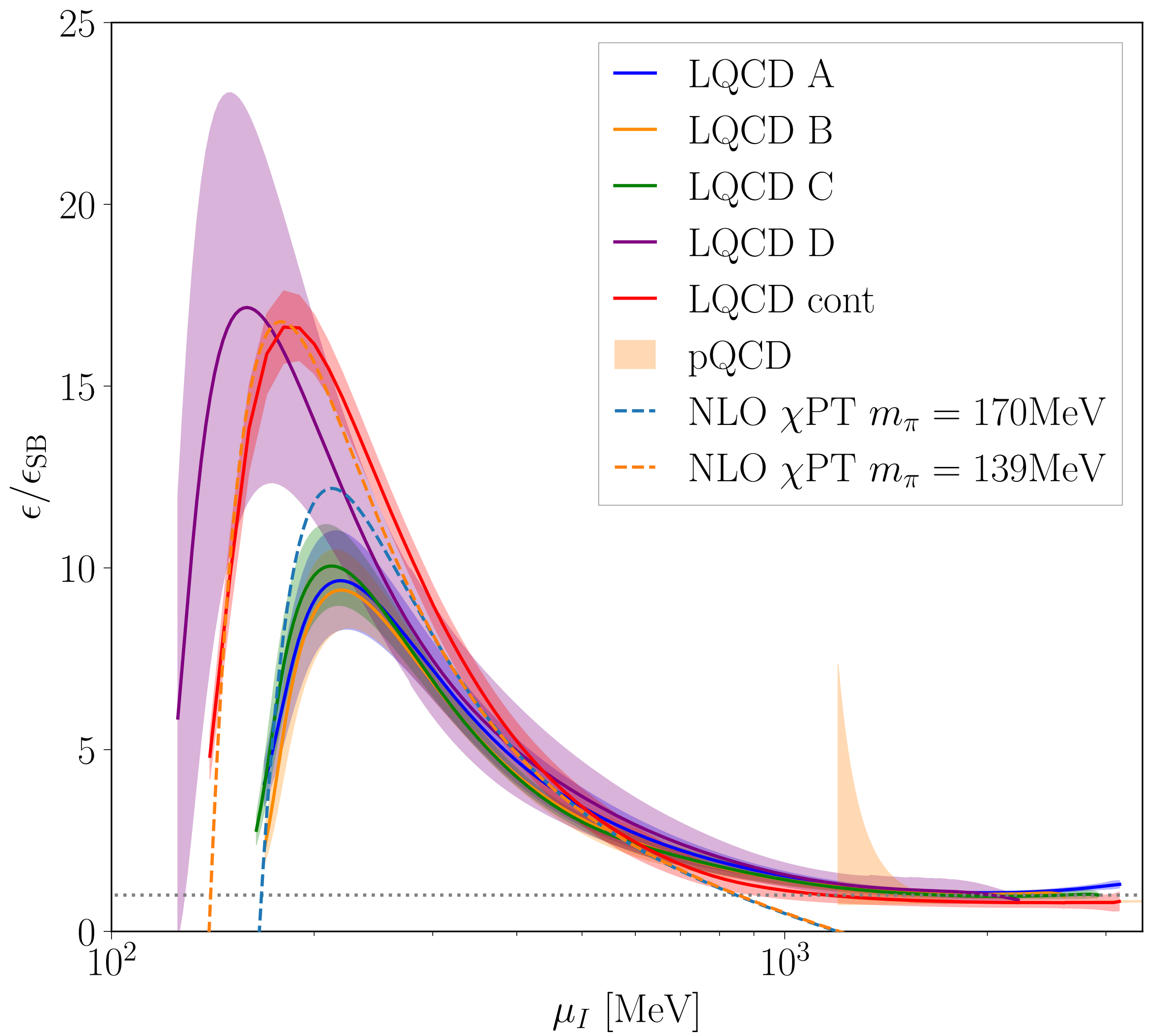
QCD at $\mu_I \neq 0$ & $\mu_B = 0$

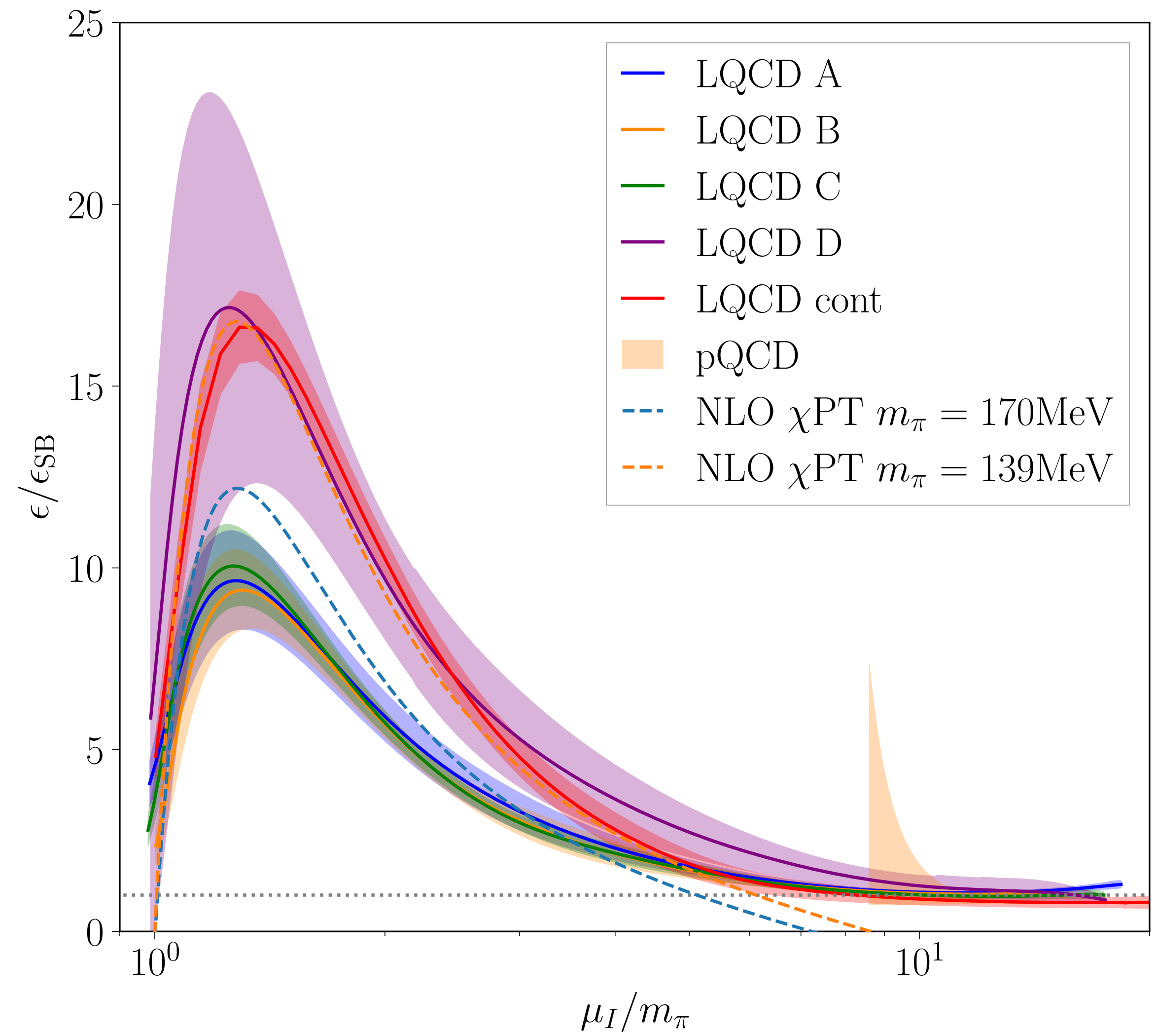
Pressure



QCD at $\mu_I \neq 0$ & $\mu_B = 0$

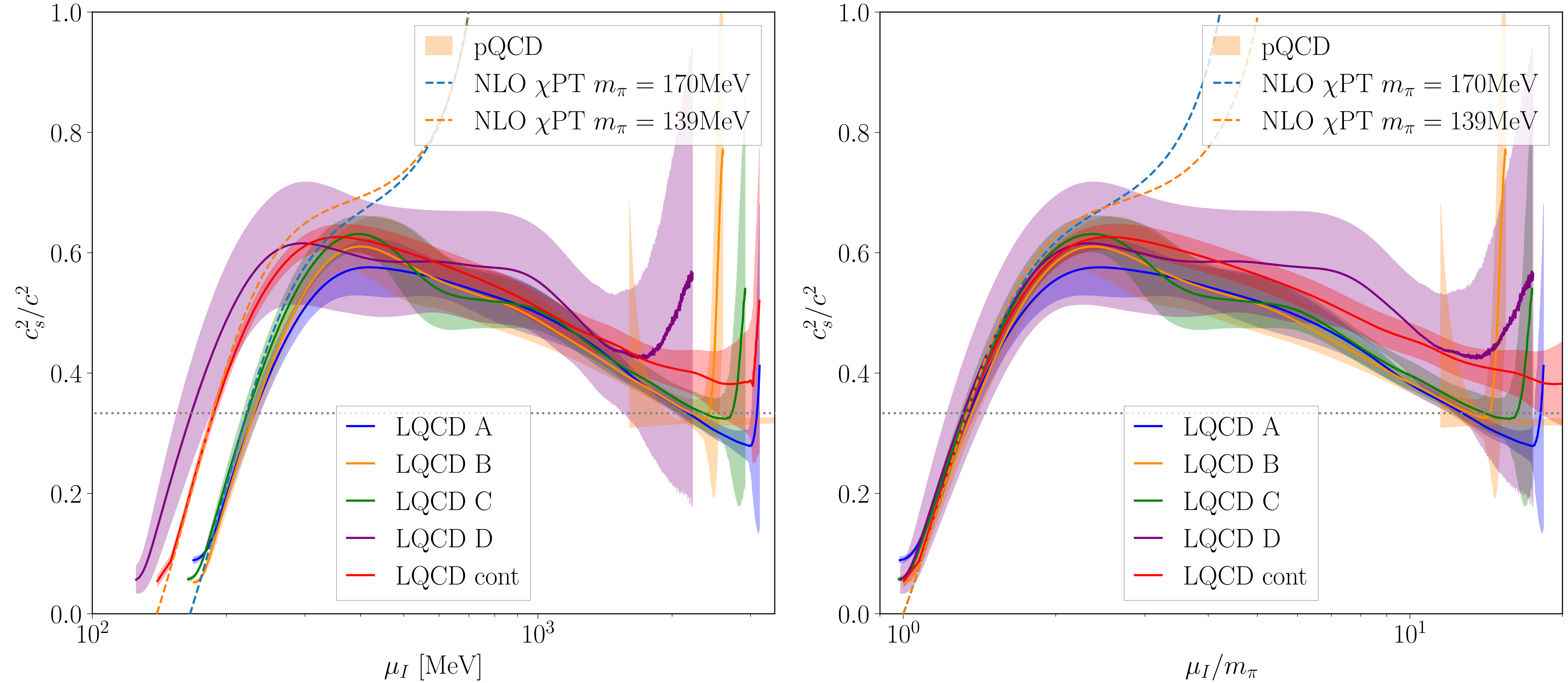
Energy Density





QCD at $\mu_I \neq 0$ & $\mu_B = 0$

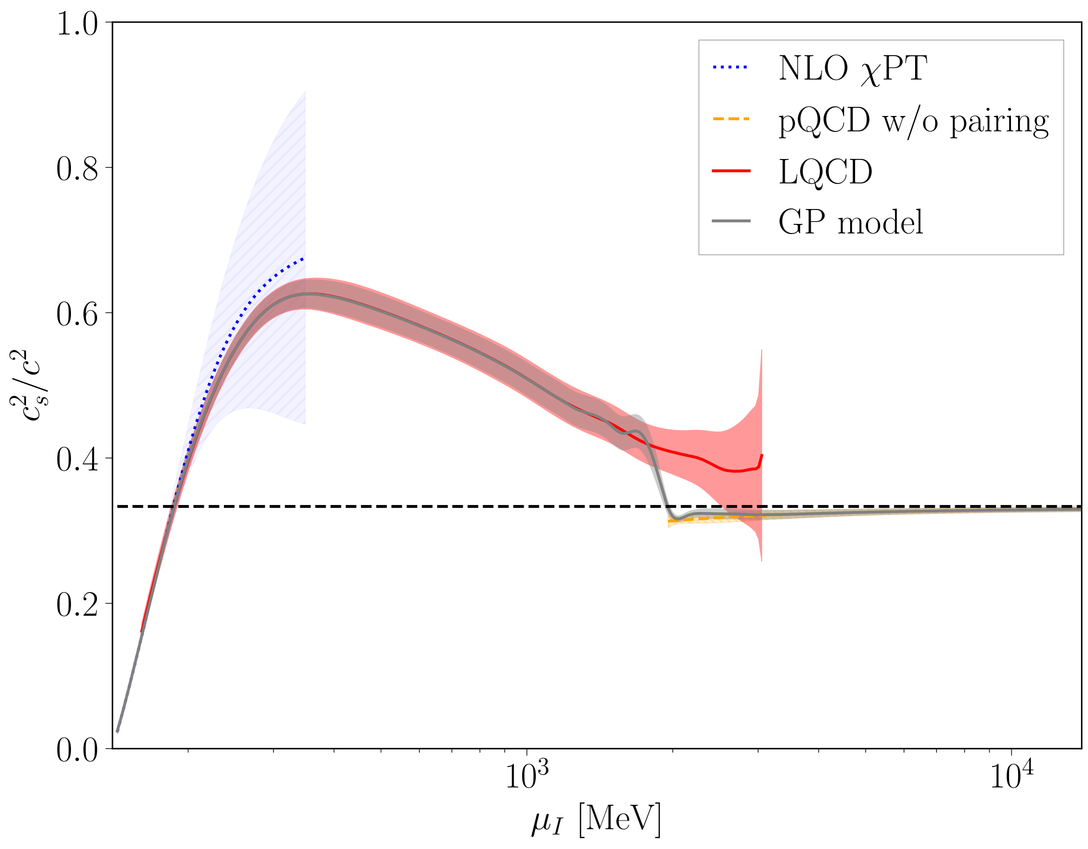
Speed of sound

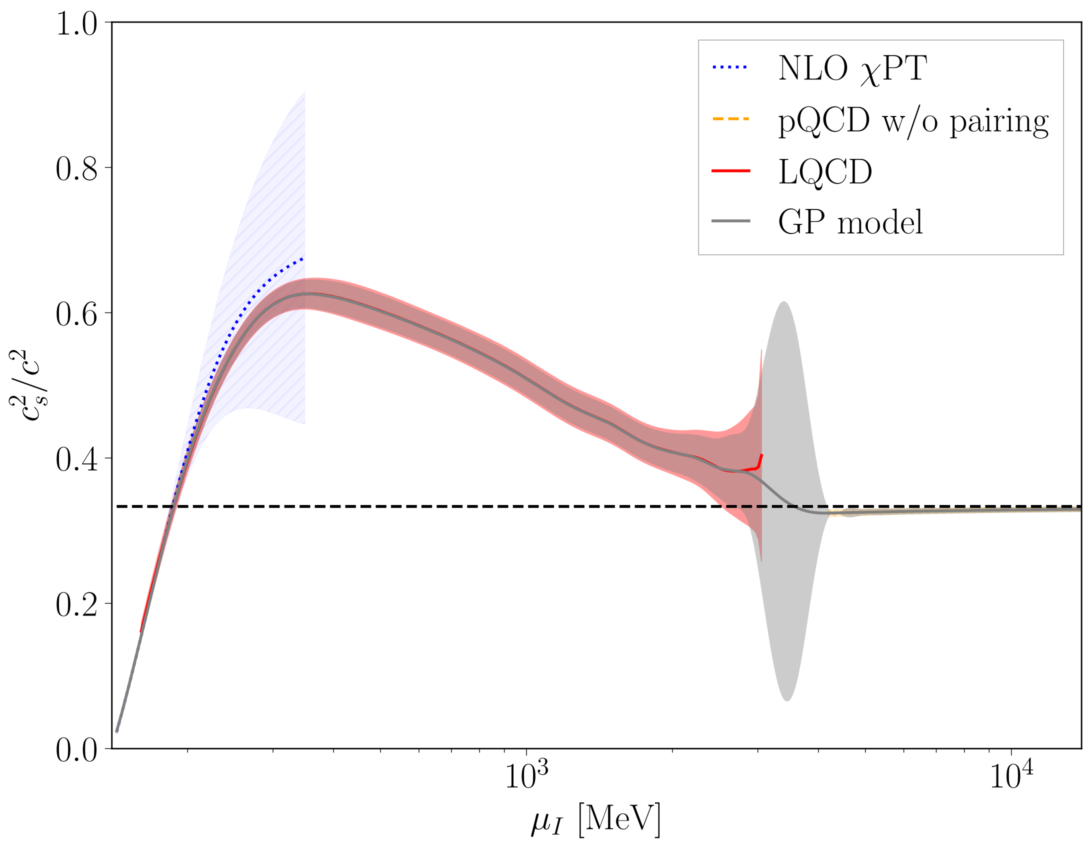


QCD at $\mu_I \neq 0$ & $\mu_B = 0$

Details of mixture model

- Models without pairing





QCD at $\mu_I \neq 0$ & $\mu_B = 0$

Details of mixture model

- Models without LQCD input (similar to baryon density case)