NEW RESULTS ON α_s FROM HADRONIC τ DECAY

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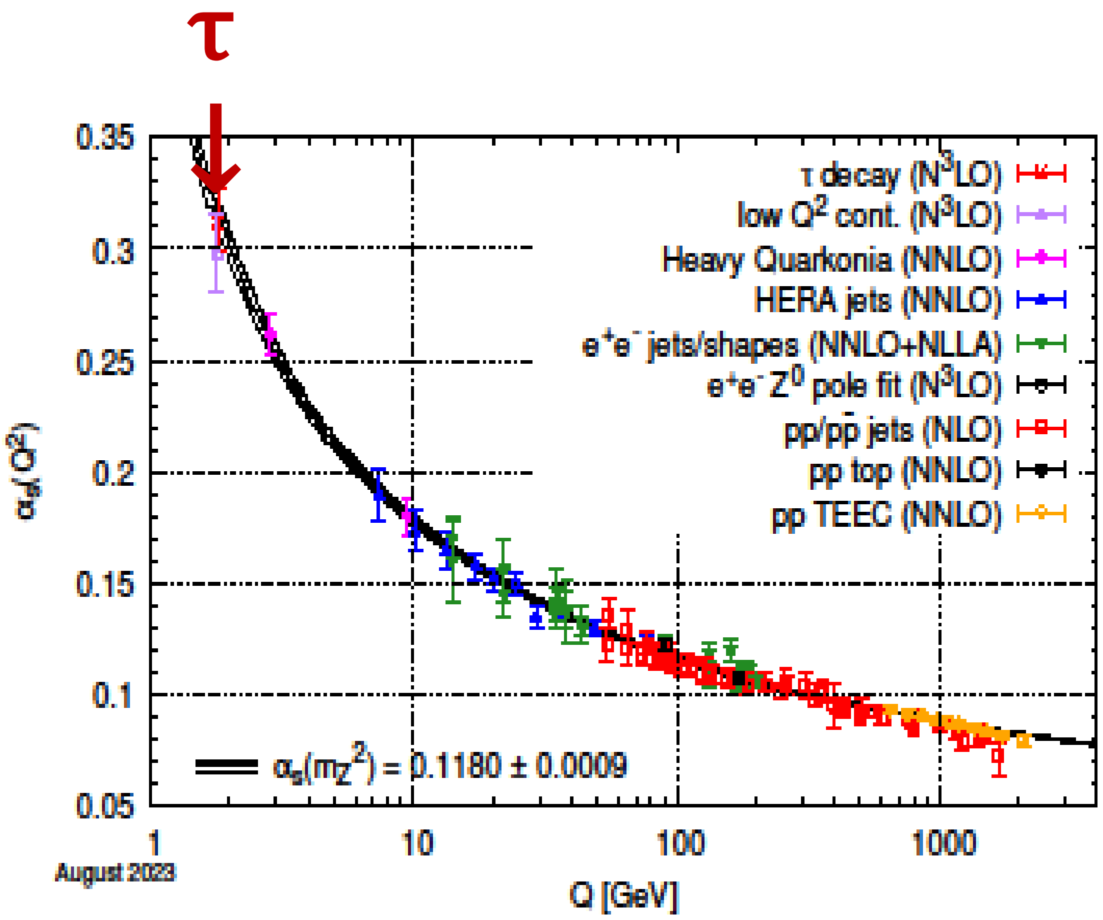
Based on

(1) *"Quark-hadron duality and the determination of α_s from hadronic τ decay: facts vs. myths"* [arXiv: 2402.00588 [hep-ph]]

(2) PRD103(2021) 034028 [arXiv:2012.10440]

CONTEXT: PDG NON-LATTICE α_s DETERMINATIONS



- Increase in precision at M_Z with decreasing μ (for fixed precision at μ):
  $$[\delta\alpha_s(M_Z^2)/\alpha_s(M_Z^2)] \simeq [\alpha_s(M_Z^2)/\alpha_s(\mu^2)]\ [\delta\alpha_s(\mu^2)/\alpha_s(\mu^2)]$$
- $[\alpha_s(M_Z^2)/\alpha_s(\mu^2)] \simeq 1/3$ for $\mu \simeq m_\tau \Rightarrow$ advantage for low-scale τ analysis
- This talk: previously unrecognized issues with one of the two main approaches to the τ determination

INGREDIENTS OF THE τ DETERMINATION (1)

- **V and A vector two-point functions, scalar polarizations and spectral functions**

$$\begin{aligned}\Pi_{\mu\nu}^{V/A}(q) &= i\int d^4x\, e^{iq\cdot x}\langle 0|T\left\{J_\mu^{(V/A)}(x)J_\nu^{(V/A)\dagger}(0)\right\}|0\rangle \\ &= \left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\Pi^{(1)}(q^2) + q_\mu q_\nu \Pi^{(0)}(q^2) \\ &= \left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\Pi^{(1+0)}(q^2) + q^2 g_{\mu\nu}\Pi^{(0)}(q^2)\end{aligned}$$

$$\rho^{(J)}(s) = \frac{1}{\pi}\,\mathrm{Im}\Pi^{(J)}(s)$$

- **Hadronic τ decay in the SM in terms of V and A current spectral functions:**

$$R_{V/A;ud} = \frac{\Gamma[\tau \to (\mathrm{hadrons})_{V/A;ud}\nu_\tau(\gamma)]}{\Gamma[\tau \to e\bar{\nu}_e\nu_\tau(\gamma)]}$$

$$\frac{dR_{V/A;ud}(s)}{ds} = 12\pi^2|V_{ud}|^2 S_{\mathrm{EW}}\,\frac{1}{m_\tau^2}\left[w_T(s;m_\tau^2)\rho_{V/A;ud}^{(1+0)}(s) - w_L(s;m_\tau^2)\rho_{V/A;ud}^{(0)}(s)\right]$$

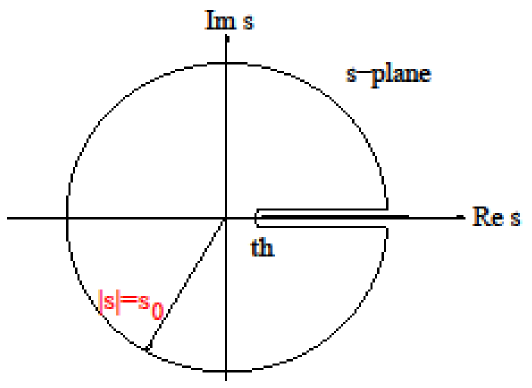
$$w_T(s;s_0) = \left(1-\frac{s}{s_0}\right)^2\left(1+\frac{2s}{s_0}\right),\quad w_L(s,s_0) = \frac{2s}{s_0}\left(1-\frac{s}{s_0}\right)^2$$

INGREDIENTS OF THE τ DETERMINATION (2)

- **Polynomially weighted finite-energy sum rules (FESRs)**

Polynomial w(s), kinematic-singularity-free $\Pi(Q^2)$ ⇒ Cauchy Theorem (FESR) relation

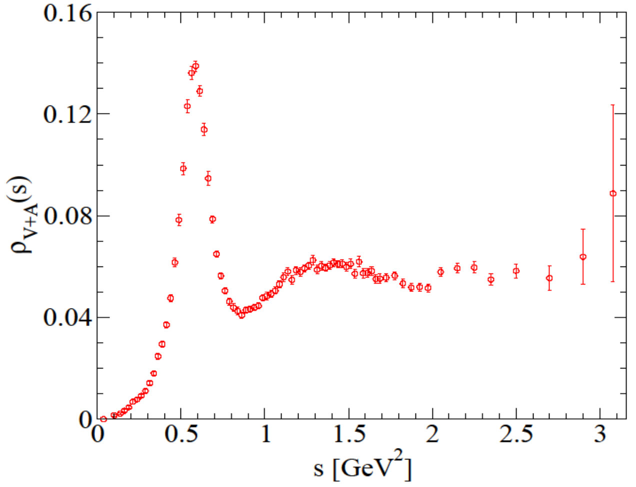
$$\int_{s_{th}}^{s_0} ds\, w(s)\rho(s) = \frac{-1}{2\pi i}\oint_{|s|=s_0} ds\, w(s)\Pi(s)$$



- τ decay α_s determinations: experimental V, A (dR/ds) on LHS, theory (QCD) on RHS
- Theory side: approximate $\Pi(Q^2)\equiv \Pi(Q^2\,)^{OPE}(+\Pi_{DV}(Q^2))$ (α_s in perturbative part of OPE)
- **Two common approaches: tOPE (ALEPH, OPAL, Pich et al), and DV-model (Boito et al)**

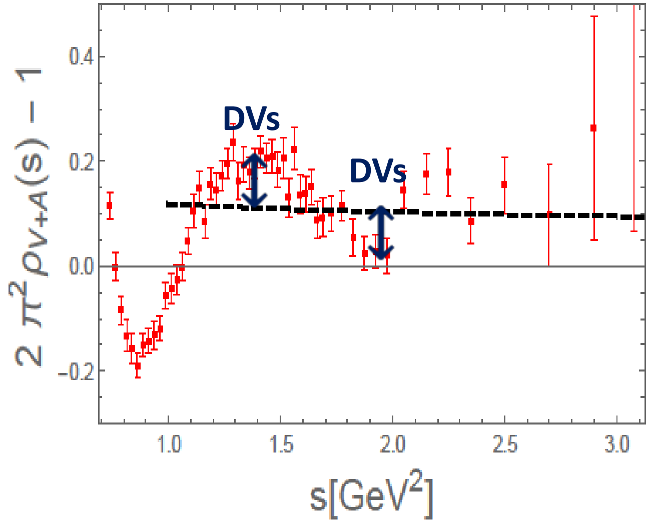
τ DETERMINATION INGREDIENTS (3): I=1, J=0+1 V+A SPECTRAL DATA

- ALEPH 2013 τ, I=1 V+A spectral function, showing "reduced" DVs above s ~1.5-2 GeV ^{2} (reduced c.f. those for V or A alone)



- In the literature: often used to argue for neglect of DVs in this region/claim that PT works "well" for V+A as low as s≃1 GeV^2

- **C.f. the τ, I=1 V+A figure, now with the non-dynamical, α_s-independent parton model contribution removed**

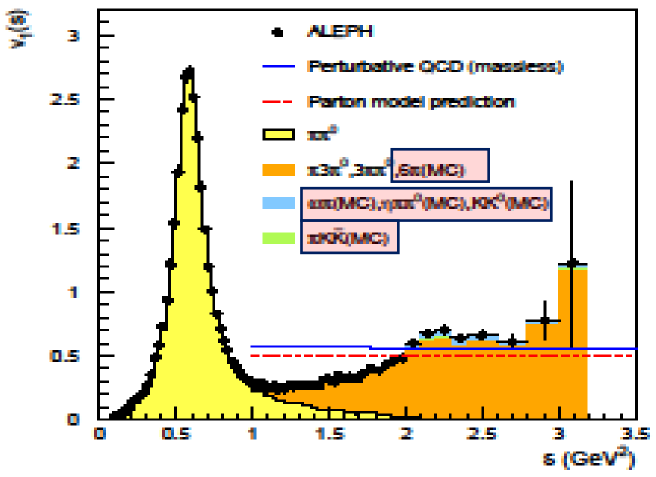


(e.g. same figure with different (larger) α_s-independent contribution)

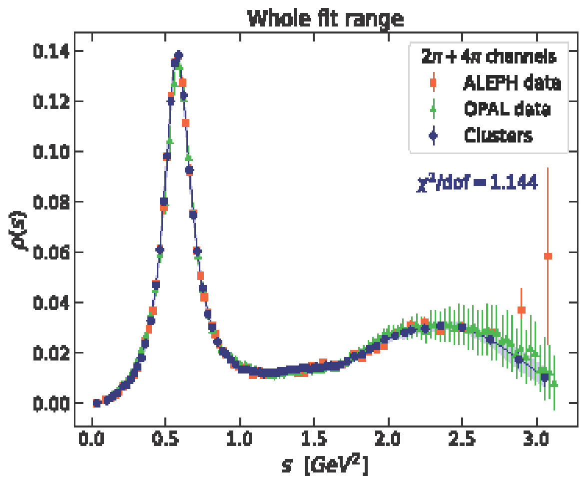
τ DETERMINATION INGREDIENTS (4): I=1, J=0+1 V SPECTRAL DATA

- **Improved I=1, V channel spectral distribution** [Boito et al PRD103(2021) 034028]
- **ALEPH $K\bar{K}$, higher-multiplicity-mode Monte Carlo input replaced with BaBar $\tau \to K\bar{K}\upsilon_\tau$, e^+e^-+ CVC input for higher-multiplicity modes**

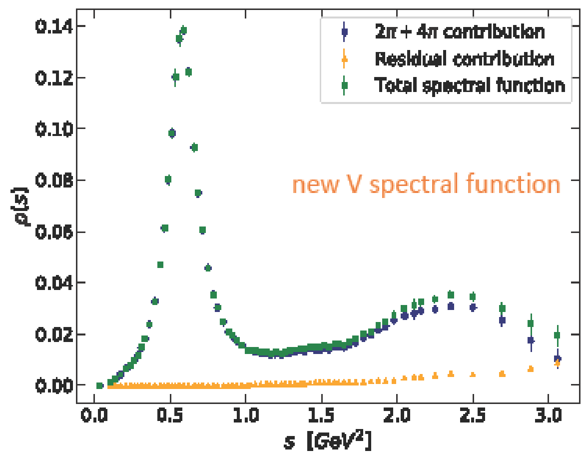
ALEPH 2013



Experimental data (non-strange vector spectral function): [PRD103(2021) 034028]



OPAL: Ackerstaff *et al.* '98
ALEPH: Schael *et al.* '05, Davier *et al.* '14
Combination: Boito *et al.* '20



Residual modes from (mostly) electroproduction (instead of Monte-Carlo) Boito *et al.* '20

τ DETERMINATION INGREDIENTS (5): FESR THEORY-SIDE INPUT

- D=0 (perturbative) series known to O(α_s^4) (Baikov et al '08; Herzog et al '17)
- D=0 OPE integrals ~$1+\alpha_s/\pi+\ldots$

  $\alpha_s(m_\tau^2)$~0.3, hence α_s-dependent contributions numerically significant

- higher D: $[\Pi(Q^2)]^{OPE}_{D\geq 4} \equiv \Sigma_{D\geq 4}\,[C_D/Q^D]$ with effective condensates C_D
  (D=4: chiral and gluon condensates, D=6: 4-quark condensates,…)

  Expansion in powers of 1/s; known to be asymptotic (at best)

- (up to α_s-suppressed log corrections) for polynomial $w(y) = w(s/s_0) = \Sigma_{k\geq 0}\, b_k y^k$

  $$\frac{-1}{2\pi i}\oint_{|s|=s_0} (ds/s_0)\, w(y)\, [\Pi(Q^2)]^{OPE}_{D\geq 4} = \Sigma_{k\geq 1}\, (-1)^k\, b_k\, C_{2(k+1)}/s_0^{k+1}$$

⇒ dim D scales as $1/s_0^{D/2}$; degree N w(y) ↔ OPE contributions to D=2N+2

- DVs: Resonance oscillations in experimental $\rho_{V,A}(s)$ not captured by perturbation theory/the OPE (believed localized to vicinity of timelike point on RHS contour)

- **tOPE vs DV-model-strategy analysis option choice (more on this below)**

tOPE vs. DV-MODEL ANALYSIS STRATEGY COMPARISONS

FESR: $\int_0^{s_0} ds\, w(s\, \rho(s) = -\frac{1}{2\pi i}\oint_{|z|=s_0} dz\, w(z)\, (\Pi_{\mathrm{pert.th.}}(z;\alpha_s) + \Pi_{\mathrm{OPE}}(z)) + \int_{s_0}^{\infty} ds\, w(s)\, \rho_{\mathrm{DV}}(s)$

tOPE:
- set DV part equal to zero (this is a model for duality violations!)
- include high-degree polynomials (with DVs suppressed via zeros at $z = s_0$) ("pinched" weights)
- use a single s_0 value, as close as possible to m_τ^2 , dropping OPE parameters until # fit parameters < # FESRs; OPE treated as if convergent to very high order (up to $1/z^8$)

DV: Since OPE is asymptotic, use only to low orders (max $1/z^5$), don't drop OPE parameters
≥ 1 FESR with unsuppressed DVs, model with QCD-motivated *ansatz* (Regge theory and $1/N_c$)

$$\rho_{\mathrm{DV}}(s) = e^{-\delta-\gamma s}\sin(\alpha + \beta s + \mathcal{O}(\log s))\left(1 + \mathcal{O}\left(\frac{1}{s}, \frac{1}{N_c}, \frac{1}{\log s}\right)\right)$$

use, and test consistency of approach by varying, s_0 between $\sim 1.5\ \mathrm{GeV}^2$ and m_τ^2

(Catà *et al.* '05, Boito *et al.* '17)

Advantage of such multi-s_0 analysis approaches: variable s_0 and D-dependent OPE, oscillatory DV scalings with s_0 ↔ non-trivial internal self-consistency tests

NECESSITY OF OPE TRUNCATION IN SINGLE-s_0 tOPE ANALYSES

- OPE sides of doubly (or higher) pinched-weight FESRs needed to suppress DV contributions involve not just α_s but higher D non-perturbative condensates C_D

  E.g.,the J=0+1 kinematic weight $w_T(y) = 1 - 3y^2 + 2y^3$ ⇒ theory representation of non-strange inclusive τ decay width depends on $D = 6$ and 8 condensates as well as α_s

  ⇒ fit of α_s impossible using only a single FESR (needs C_D input)

- Classic tOPE analysis “solution”: add higher-degree-weight FESRs to fit needed C_D

  E.g. classic “(km) spectral weights” $w_{km}(x) = (1-x)^2(1+2x)(1-x)^k x^m$, km=00, 10, 11, 12, 13 (ALEPH, OPAL, Pich et al.): 5 FESRs to fit 4 OPE parameters α_s, C_4, C_6, C_8

- **Basic problem: new higher degree weights add new unknown C_D ⇒ must drop OPE terms in principle present to keep # fit parameters< # spectral integral inputs**

  E.g. classic “(km) spectral weight” analyses truncate OPE at D=8, dropping $C_{10}, C_{12}, C_{14}, C_{16}$ counting on assumed suppression by additional powers of $1/s_0$ to make this safe

- **Basic truncation assumption issue: with only single s_0, impossible to use D-dependent scaling with s_0 to test self-consistency of assumed truncation**

"REDUNDANCY" AND THE tOPE AND DV STRATEGY APPROACHES (1)

- Theory-side s_0-dependence self-consistency tests need multi-weight, multi-s_0 analyses

- If all $s_0 > s_0^{min}$ for given experimental binning used, only one of a 2nd-weight spectral integral set {I(w_2,s_0)} is independent of the corresponding 1st-weight set {I(w_1,s_0)}

- $\Rightarrow$ In fit to data $\{d_k\}$ with theory representations $\{t_k(\eta_m)\}$ involving parameters $\{\eta_m\}$, either give up s_0-dependent multi-weight, multi-s_0 self-consistency tests to use standard χ^2 fit (as in single-s_0 tOPE analyses), or keep multi-weight, multi-s_0 set and use non-χ^2 fit (propagating full set of correlations separately). Generally

$$Q^2(\vec{\eta}) = [\vec{d}-\vec{t}(\vec{\eta})]^T \tilde{C}^{-1}[\vec{d}-\vec{t}(\vec{\eta})]$$

  - **If data covariance matrix C non-singular, can set $\tilde{C}$=C, $Q^2 = \chi^2$**
  - **If C singular, alternate choice for $\tilde{C}$, $Q^2 \neq \chi^2$ and must propagate full covariances separately**

- E.g. Boito et al. V+A, V channel DV-strategy multi-weight, multi-s_0 spectral integral set fits: block-diagonal Q^2 with single-weight, multi-s_0 covariance matrices on the diagonal

"REDUNDANCY" AND THE tOPE AND DV STRATEGY APPROACHES (2)

Redundancy Theorem: Consider a data set $\{d_k$, k=1…N$\}$ with non-singular covariance matrix D, and associated theory representations $\{t_k(\eta_m)$, k=1…N$\}$ involving parameters $\{\eta_m$, m=1…M, M<N$\}$. Now add a single new data point d_{N+1} such that (i) the extended (N+1)-point data set covariance matrix C is also non-singular and (ii) only one additional theory parameter, η_{M+1}, enters the theory representation, t_{N+1}, of d_{N+1}.

In this situation

- **the parameters $\eta_1,\ldots,\eta_M$ obtained from the extended (N+1)-point χ^2 fit are identical to those obtained from the unextended N-point χ^2 fit,**
- **the minimum χ^2 of the extended (N+1)-point fit is identical to that of the original N-point fit and**
- **the extended-fit result for η_{M+1} serves only to make the theory representation t_{N+1}exactly reproduce $d_{N+!}$, regardless of the form chosen for t_{N+1}**

The extended fit is entirely "redundant", producing no new information on the parameters of the original fit, and no physically meaningful constraint on the new parameter η_{M+1}

"REDUNDANCY" AND THE tOPE AND DV STRATEGY APPROACHES (3)

- Single-s_0 tOPE spectral integrals involving a set of linearly independent weights are linearly independent, and hence have a non-singular covariance matrix.
  Results obtained from the associated standard χ^2 tOPE fits in the literature are thus subject to the results of the Redundancy Theorem. (More on this below.)

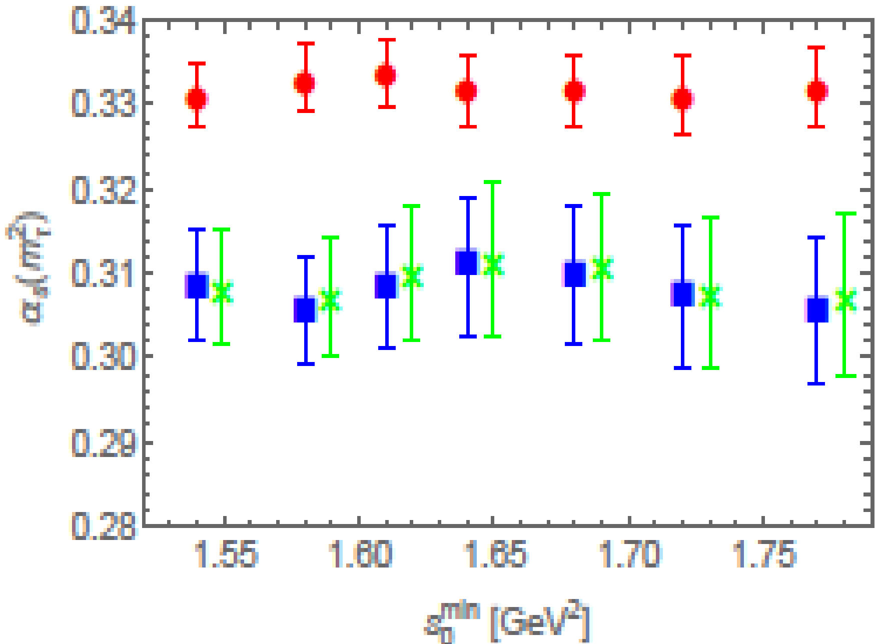
- In contrast, for the block-diagonal, multi-weight, multi-s_0 DV-strategy fits in the literature, ***which cannot, even in principle, be of the standard χ^2 form***,
  - **the conclusions of the Redundancy Theorem do not hold***
  - **the multi-weight, multi-s_0 nature of the fit and differing s_0- and weight-dependences of the different theory contributions lead to highly non-trivial self-consistency checks on the form chosen for the theory representations** (More on this below.)

*A claim to the contrary by Pich and Rodrigues-Sanchez rests on the (unexamined) assumption that the proof for the standard χ^2 fit case (which is valid) carries over to the case of non-χ^2 block-diagonal fits, which do not satisfy the conditions on which that proof is based, and for which it turns out the theorem does not hold

NON-REDUNDANCY OF MULTI-WEIGHT, MULTI-s_0 BLOCK-DIAGONAL DV-STRATEGY FITS

A two-weight, $w_0(x) = 1, w_2(x) = 1 - x^2$, V-channel block-diagonal fit example

- First weight fit: $\alpha_s, \alpha_V, \beta_V, \gamma_V, \delta_V$ from a multi-s_0, single-weight w_0 standard χ^2 fit
- In QCD, the w_2 FESR adds one further NP theory parameter, C_6, in the form C_6/s_0^3
- Consider also an alternate, non-QCD NP form, C'/s_0^5, on the w_2 theory side
- Adding the w_2 FESR at a single s_0, the two-weight w_0&w_2 χ^2 fit returns unchanged α_s, $\alpha_V, \beta_V, \gamma_V, \delta_V$, regardless of the w_2 form used [as per the Redundancy Theorem]
- In contrast: w_0 and w_0&w_2 fits with w_2 FESR *at the same multi-s_0 >s_0^{min} set as w_0:*



- **$\alpha_s(m_\tau^2)$ as a function of s_0^{min}**
  - **Blue:** from the single-weight w_0 fit
  - **Green:** from the w_0&w_2 fit with QCD w_2 form
  - **Red:** from the w_0&w_2 fit with non-QCD w_2 form
- **Bue-red** differences: non-applicability of the Redundancy Theorem for block-diagonal non-χ^2 fits
- Close (but not exact) **blue-green** agreement: (i) non-redundancy and (ii) non-trivial self-consistency tests of the use of the QCD NP form from adding the w_2 FESR also a multiple s_0

REDUNDANCY OF MULTI-WEIGHT, SINGLE-s_0 tOPE STRATEGY FITS (1)

- OPAL, ALEPH, Baikov et al., Pich et al.: classic *km=00, 10, 11, 12, 13* spectral weights, V and V+A channel fits with s_0=m_τ^2, $C_{D>8}$=0 tOPE truncation
- Pich and Rodrigues-Sanchez '16/'22 (PRS), three 5-weight tOPE fits, ALEPH 2013 V+A data, omitting last two large-error bins, hence s_0 = 2.8 GeV^2:
  - *km=00, 10, 11, 12, 13* spectral weights, $C_{D>8}$=0 tOPE truncation
  - Modified *km=00, 10, 11, 12, 13* spectral weights, $\hat{w}_{km}(x)=(1-x)^{k+2}x^m$, $C_{D>8}$=0 tOPE truncation
  - *m=1,…,5 "optimal weights"*, $w^{(2m)}(x) = 1-(m+2)x^{m+1}$+(*m*+*1*)$x^{m+2}$, $C_{D>10}$=0 tOPE truncation

- Technical note: basis transformations: A multi-weight $\{W_k\}$ fit, and fit with alternate weight basis $\{W_k'\}$, $W_k(x) = \sum_m A_{km}\, W_m'(x)$ and equivalently transformed minimizer

  $(Q')^2(\vec{\eta})=[\vec{d}\,'-\vec{t}\,'(\vec{\eta})']^T(\tilde{C}^{-1})'\,[\vec{d}'-\vec{t}(\vec{\eta})'],\ (\tilde{C}^{-1})' = A^T\,\tilde{C}^{-1}A$

  yield identical results for the fit parameters $\{\eta_m\}$

REDUNDANCY OF MULTI-WEIGHT, SINGLE-s_0 tOPE STRATEGY FITS (2)

Post-redundancy-theorem revisions of the conventional understanding of tOPE output (for definiteness, starting from the classic km spectral weight example)

Conventional understanding

- α_s largely from lowest degree *km=00* FESR
- $C_{4,6,8}$ from remaining, higher degree FESRs
- Small central condensate values support OPE truncation at D=8
- Similar α_s from modified km spectral weight and (2m) optimal weight analyses represent non-trivial tests "because of their very different dependence on NP condensate contributions"

Post-redundancy-theorem revisions

- α_s from FESRs of two **highest** degree combinations, **with only perturbative contributions on the theory sides**
- α_s of all three 5-weight PRS tOPE fits from $w^{(23)}$, $w^{(24)}$, $w^{(25)}$ FESR combinations
- (Redundantly) determined C_D **from lower-degree**-weight FESRs, **and play no role in the corresponding α_s determinations**
- Generic very large C_D uncertainties from even small NP contaminations in the perturbative-only α_s determinations

REDUNDANCY OF MULTI-WEIGHT, SINGLE-s_0 tOPE STRATEGY FITS (3)

A few details of the classic km spectral weight analysis case (tOPE truncation $C_{D>8}$=0)

- Alternate basis:

$$\begin{aligned}
\hat{w}_1(x) &= 1 - \frac{15}{2}x^4 + 12x^5 - \frac{17}{2}x^6 + 3x^7 = \frac{3}{2}w^{(23)}(x) - w^{(24)}(x) + \frac{1}{2}w^{(25)}(x)\,, \\
\hat{w}_2(s) &= 1 - 9x^4 + 12x^5 - 4x^6 = \frac{9}{5}w^{(23)}(x) - \frac{4}{5}w^{(24)}(x)\,, \\
\hat{w}_3(x) &= 1 + 2x^3 - 9x^4 + 6x^5 = -\frac{1}{2}w^{(22)}(x) + \frac{3}{2}w^{(23)}(x)\,, \\
\hat{w}_4(s) &= 1 - 3x^2 + 2x^3 = w^{(21)}(x)\,, \\
\hat{w}_5(x) &= 1 + \frac{2}{3}x - \frac{23}{3}x^4 + 6x^5 \\
&= -\frac{1}{3}w^{(20)}(x) - \frac{1}{9}w^{(21)}(x) - \frac{1}{18}w^{(22)}(x) + \frac{3}{2}w^{(23)}(x)\,.
\end{aligned}$$

$$\begin{aligned}
w_{00}(x) &= \hat{w}_4(x)\,, \\
w_{10}(x) &= \frac{3}{2}\hat{w}_2(x) + \hat{w}_4(x) - \frac{3}{2}\hat{w}_5(x)\,, \\
w_{11}(x) &= -\frac{11}{6}\hat{w}_2(x) + \frac{1}{3}\hat{w}_4(x) + \frac{3}{2}\hat{w}_5(x)\,, \\
w_{12}(x) &= \frac{1}{2}\hat{w}_2(x) - \frac{1}{6}\hat{w}_3(x) - \frac{1}{3}\hat{w}_4(x)\,, \\
w_{13}(x) &= -\frac{2}{3}\hat{w}_1(x) + \frac{1}{6}\hat{w}_2(x) + \frac{1}{2}\hat{w}_3(x)\,.
\end{aligned}$$

- **With $C_{D>8}$=0 tOPE truncation:**
  - ❖ No theory-side C_D contributions to $\hat{w}_{1,2}$ FESRs ⇒ combined $\hat{w}_1$&$\hat{w}_2$ fit fixes α_s
  - ❖ Add $\hat{w}_3$ FESR (theory side: α_s and C_8): α_s unchanged, redundant determination of C_8
  - ❖ Add $\hat{w}_4$ (theory side: α_s, C_8 and C_6): α_s, C_8 unchanged, redundant determination of C_6
  - ❖ Add $\hat{w}_5$ (theory side: α_s, C_4): α_s unchanged, redundant determination of C_4

REDUNDANCY OF MULTI-WEIGHT, SINGLE-s_0 tOPE STRATEGY FITS (4)

Details of the modified ($\hat{w}_{km}$) spectral weight analysis case (tOPE truncation $C_{D>8}$=0)

- Alternate basis: $\{w^{(2m)}(x), m = 0, \ldots, 4\}$ related to original $\{\hat{w}_{km}(x)\}$ basis by

$\hat{w}_{00}$ (x) =$w^{(20)}$(x)
$\hat{w}_{10}$ (x) =[$3w^{(20)}$(x)-$w^{(21)}$(x)]/2
$\hat{w}_{11}$ (x) =[$-3w^{(20)}$(x)+5$w^{(21)}$(x)-2$w^{(22)}(x)$]/6
$\hat{w}_{12}$ (x) =[$-4w^{(21)}$(x)+7$w^{(22)}$(x)-3$w^{(23)}(x)$]/12
$\hat{w}_{13}$ (x) =[$-5w^{(22)}$(x)+9$w^{(23)}$(x)-4$w^{(25)}(x)$]/20

$w^{(20)}(x) = 1 - 2x + x^2$
$w^{(21)}(x) = 1 - 3x^2 + 2x^3$
$w^{(22)}(x) = 1 - 4x^3 + 3x^4$
$w^{(23)}(x) = 1 - 5x^4 + 4x^5$
$w^{(24)}(x) = 1 - 6x^5 + 5x^6$

- **With $C_{D>8}$=0 tOPE truncation:**
  - ❖ No theory-side $w^{(23)}$, $w^{(24)}$ FESR C_D contributions ⇒ combined 2-weight fit fixes α_s
  - ❖ Add $w^{(22)}$ (theory side: α_s, C_8): α_s unchanged, redundant determination of C_8
  - ❖ Add $w^{(21)}$ (theory side: α_s, C_8, C_6): α_s, C_8 unchanged, redundant determination of C_6
  - ❖ Add $w^{(20)}$ (theory side: α_s, C_4, C_6): α_s, C_6 unchanged, redundant determination of C_4

REDUNDANCY OF MULTI-WEIGHT, SINGLE-s_0 tOPE STRATEGY FITS (5)

Details of the $w^{(2m)}$ optimal weight analysis case (with w_{km} tOPE truncation $C_{D>8}$=0)

- The $\{w^{(2m)}(x)$, m=1,…,5$\}$ basis:

$$w^{(21)}(x) = 1 - 3x^2 + 2x^3$$
$$w^{(22)}(x) = 1 - 4x^3 + 3x^4$$
$$w^{(23)}(x) = 1 - 5x^4 + 4x^5$$
$$w^{(24)}(x) = 1 - 6x^5 + 5x^6$$
$$w^{(25)}(x) = 1 - 7x^6 + 6x^7$$

- **With $C_{D>8}$=0 tOPE truncation:**
  - ❖ No theory-side $w^{(23)}$, $w^{(24)}$, $w^{(25)}$ C_D contributions ⇒ combined 3-weight fit fixes α_s
  - ❖ Add $w^{(22)}$ (theory side: α_s, C_8): α_s unchanged, redundant determination of C_8
  - ❖ Add $w^{(21)}$ (theory side: α_s, C_8, C_6): α_s, C_8 unchanged, redundant determination of C_6

REDUNDANCY THEOREM ILLUSTRATION: tOPE OPTIMAL WEIGHT FIT CASE

Redundancy of tOPE strategy

(PRS '22)

irrelevant

- Consider "optimal weights": (PRS, '16)

$$\boxed{\begin{aligned} w_{21}(y) &= 1 - 3y^2 + 2y^3 \\ w_{22}(y) &= 1 - 4y^3 + 3y^4 \\ w_{23}(y) &= 1 - 5y^4 + 4y^5 \end{aligned}}$$

$$w_{24}(y) = 1 - 6y^5 + 5y^6$$
$$w_{25}(y) = 1 - 7y^6 + 6y^7$$

$\alpha_s,\ C_6,\ C_8$ adds C_6

$\alpha_s,\ C_8,\ C_{10}$ adds C_8

$\alpha_s,\ C_{10}$ adds C_{10}

$\left.\begin{array}{l}\alpha_s \\ \alpha_s\end{array}\right\}$ fixes $\alpha_s(m_\tau^2)$

- Results: $w_{24},\ w_{25}:\quad \alpha_s = 0.3168(27),\ \chi^2 = 3.06933$

  $w_{23},\dots w_{25}:\quad \alpha_s = 0.3168(27),\ \chi^2 = 3.06933,\ C_{10} = -0.0041(25)$

  $w_{22},\dots w_{25}:\quad \alpha_s = 0.3168(27),\ \chi^2 = 3.06933,\ C_{10} = -0.0041(25),\ C_8 = 0.0016(14)$

  $w_{21},\dots w_{25}:\quad \alpha_s = 0.3168(27),\ \chi^2 = 3.06933,\ C_{10} = -0.0041(25),\ C_8 = 0.0016(14),\ C_6 = 0.00054(53)$

- $\alpha_s(m_\tau^2)$ purely from perturbation theory, *no* effect from OPE; OPE coefficients not fitted

  Can also get $\alpha_s(m_\tau^2)$ from *only* w_{25} (not a fit!): $\alpha_s = 0.3228(43)$ tests only pert.th., not the OPE!

$\alpha_s(m_\tau^2)$ FROM THE V+A-CHANNEL tOPE OPTIMAL WEIGHT FIT ANALYSIS

From (i) 3-weight $w^{(23)}, w^{(24)}, w^{(25)}$ fit (PT only), redundant C_8, C_6; (ii) 2-weight $w^{(24)}$, $w^{(25)}$ fit (PT only), redundant C_{10}, C_8, C_6; (iii) single-weight $w^{(25)}$ determination (PT only)

- s_0=2.8 GeV^2 (as in PRS 2016/22):
  - 3-weight fit: $0.3125(23)_{ex}$, χ^2/dof =11.6/2 [p-value 0.3%]
  - 2-weight fit: $0.3168(22)_{ex}$, χ^2/dof =3.1 [p-value 7.8%]
  - $w^{(25)}$ determination: $0.3228(43)_{ex}$ [(25) -3-weight difference: $0.0103(37)_{ex}$ $(10)_{th}$]

- **Non-trivial tensions/self-consistency/fit quality issues**
  - **If due to propagating NP contamination of PT-only α_s determination will show up as increasing discrepancy at lower s_0**
  - **$\Rightarrow$ Consider lower s_0 still in range where spectral data consistent with neglect of DVs (for ALEPH data, s_0 = 2.6 GeV^2 or 2.4 GeV^2**

$\alpha_s(m_\tau^2)$ FROM THE V+A-CHANNEL tOPE OPTIMAL WEIGHT FIT ANALYSIS

From (i) 3-weight $w^{(23)}, w^{(24)}, w^{(25)}$ fit (PT only), redundant C_8, C_6; (ii) 2-weight $w^{(24)}$, $w^{(25)}$ fit (PT only), redundant C_{10}, C_8, C_6; (iii) single-weight $w^{(25)}$ determination (PT only)

- s_0=2.8 GeV^2 (as in PRS 2016/22):
  - ❖ 3-weight fit: 0.3125(23)$_{ex}$, χ^2/dof =11.6/2 [p-value 0.3%]
  - ❖ 2-weight fit: 0.3168(22)$_{ex}$, χ^2/dof =3.1 [p-value 7.8%]
  - ❖ $w^{(25)}$ determination: 0.3228(43)$_{ex}$ [(25) -3-weight difference: 0.0103(37)$_{ex}$ (10)$_{th}$]
- s_0=2.6 GeV^2 [experimental ρ_{DV}(s) compatible with 0 within errors]
  - ❖ 3-weight fit: 0.3100(22)$_{ex}$, χ^2/dof =18.7/2 [p-value ~0.0001]
  - ❖ 2-weight fit: 0.3153(26)$_{ex}$, χ^2/dof =4.5 [p-value 3.4%]
  - ❖ $w^{(25)}$ determination: 0.3202(34)$_{ex}$ [(25) -3-weight difference: 0.0102(27)$_{ex}$ (10)$_{th}$]
- s_0=2.4 GeV^2 [experimental ρ_{DV}(s) compatible with 0 within errors]
  - ❖ 3-weight fit: 0.3064(22)$_{ex}$, χ^2/dof =31.9/2 [p-value ~10^{-7}]
  - ❖ 2-weight fit: 0.3136(28)$_{ex}$, χ^2/dof =6.3 [p-value 1.2%]
  - ❖ $w^{(25)}$ determination: 0.3178(30)$_{ex}$ [(25) -3-weight difference: 0.0114(22)$_{ex}$ (11)$_{th}$]
- **Deterioration with decreasing s_0 as expected if NP contamination present**

α_s PT-ONLY NP-CONTAMINATION-INDUCED UNCERTAINTY IMPACT ON tOPE C_D

E.g., tOPE optimal weight $C_{D>10}$=0 truncation analysis (α_s from $w^{(24)}$ &$w^{(25)}$ part of fit)

- $\bar{\alpha}_s \equiv$ result for α_s from underling combined $w^{(24)}$ &$w^{(25)}$ fit
- Addition of $w^{(23)}$ FESR yields (redundant) C_{10} determination, $\bar{C}_{10}$:

$$\bar{C}_{10} = -[s_0^5/5]\,[I^{(23)}_{exp}(s_0) - I^{(23)}_{th;D=0}(s_0;\bar{\alpha}_s)]$$

- Strong D=0 dominance of (23) FESR theory side ⇒ strong cancellation on RHS, hence strong sensitivity to any NP contamination in $w^{(24)}$ &$w^{(25)}$ α_s determination
- Similarly: NP contamination of $\bar{\alpha}_s$, $\bar{C}_{10}$ ⇒ strongly enhanced NP contamination of (redundant) $\bar{C}_8$ determination from (22) FESR; NP $\bar{\alpha}_s$, $\bar{C}_{10}$, $\bar{C}_8$ contamination ⇒ strongly enhanced NP contamination of (redundant) $\bar{C}_6$ determination from (21) FESR

$\bar{\alpha}_s$	$\bar{C}_{10}$ $[GeV^{10}]$	$\bar{C}_8$ $[GeV^8]$	$\bar{C}_6$ $[GeV^6]$	
0.3168	-0.0041(41)	0.0016(26)	0.0005(12)	[tOPE fit results]
0.3077	-0.0151(41)	-0.0093(26)	-0.0036(12)	
0.3228	0.0033(41)	-0.0037(26)	0.0033(12)	

tOPE c.f. DV-STRATEGY V CHANNEL OPTIMAL-WEIGHT ANALYSES

s_0=2.882 GeV^2 tOPE analysis

- Sizeable PT-only $\alpha_s(m_\tau^2)$ discrepancies [e.g., $w^{(23)}$, $w^{(25)}$ difference 0.0142(16)]
- Discrepancies so large no combined 3-weight fit possible
- **Even doable 2-weight $w^{(24)}$& $w^{(25)}$ fit yields disastrous χ^2/dof=43.1**

Multi-weight, multi-s_0 DV-strategy fits

- All s_0>s_0^{min}, variable s_0^{min}
- $w_0(x) = 1$, $w_2(x)$=1-x^2, $w_3(x) = 1 - 3x^2 + 2x^3$, $w_3(x) = 1 - 2x^2 + x^4$
- 1-, 2- and 3-weight fits, all including w_0
- α_s, DV parameters in all; C_6 in w_2, w_3 and w_4 FESRs, hence non-trivial self-consistency tests (all successful)
- **$\alpha_s^{(3)}(m_\tau^2)$ from 7-point s_0^{min} stability window: 0.3077(75) $\leftrightarrow$ $\alpha_s^{(5)}(m_Z^2)$=0.1171(10)**

SUMMARY/CONCLUSIONS

- Multi-weight, single-s_0 tOPE determinations suffer from redundancy-induced issues not quantifiable within the tOPE approach
  - ❖ determinations from highest degree weight FESRs with only PT included
  - ❖ limited self-consistency tests showing significant tensions
  - ❖ unconstrained (redundant) C_D determinations with high sensitivity to unidentified NP contamination in the PT-only α_s determination
- Dramatic breakdown (huge χ^2/dof =43.1) in optimal weight V channel tOPE analysis
- V channel DV-strategy analysis with improved $\rho_V(s)$ in upper part of spectrum from electroproduction+CVC input, in contrast,
  - ❖ Passes internal self-consistency tests
  - ❖ Yields current best τ determination $\alpha_s^{(3)}(m_\tau^2)$ = 0.3077(75) ↔ $\alpha_s^{(5)}(m_Z^2)$ = 0.1171(10)
- Multi-weight, multi-s_0 analyses required to test tOPE OPE truncation and DV omission assumptions for self-consistency, even in analyses assuming DVs negligible

BACKUP

DV-STRATEGY w(x)=1-x^2 THEORY COMPONENT SELF-CONSISTENCY CHECK