Restoration of residual gauge symmetries due to topological defects and color confinement in the Lorenz, Maximal Abelian gauges

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Introduction

- Quark confinement is well understood based on the dual super conductor picture where condensation of magnetic monopoles and antimonopoles occurs.
- Gluon confinement is less understood with the dual superconductor picture.

$\rightarrow$ We recall the color confinement criterion derived by Kugo and Ojima (1979).

- If the Kugo and Ojima (KO) criterion is satisfied, all colored objects cannot be observed.
- The KO criterion was derived only in the Lorenz gauge $\partial^\mu \mathscr{A}_\mu = 0$, and clearly gauge dependent.

$\rightarrow$ The KO criterion is not directly applied to the other gauge fixing conditions.

- In the Lorenz gauge, KO criterion is also derived as the condition of the restoration of the residual gauge symmetry with the special choice of the gauge transformation function $\omega(x)$. (Hata(1982))
- What is the residual gauge symmetry?

Introduction

- The residual local gauge symmetry is the local gauge symmetry remaining even after imposing the gauge fixing condition.
- This symmetry is “spontaneously broken” in the perturbative vacuum.
- We consider the color confinement phase to be a disordered phase where all of symmetries are unbroken.
- The residual gauge symmetry is also restored in the true confining vacuum of QCD and we can consider the condition of restoration as the condition of the disappearance of the massless NG pole associated with sponteneously breaking.
- We can generalize the condition of the restoration of the residual gauge symmetry into more general gauge transformation including topological configurations. (Kondo, Fukushima(2022))
- In the previous work, we have used the infinitesimal gauge transformation.
- To correctly take into account topological configurations, we must consider a finite gauge transformation.

$\rightarrow$ In this talk, we reconsider the condition of the restoration of the residual local gauge symmetry by using finite gauge transformation.

Plan

- Introduction
- The restoration condition of the residual local gauge symmetry in the Lorenz gauge
- The restoration condition of the residual local gauge symmetry in the Maximal Abelian gauge
- Summary

The restoration condition of the residual local gauge symmetry in the Lorenz gauge

- In the operator formalism on the indefinite metric state space $\mathcal{V}$, we suppose that the (nilpotent) BRST symmetry exists.
- Let $\mathcal{V}_{\text{phys}}$ be the physical state space with a semi-positive definite metric $\langle \text{phys}|\text{phys}\rangle \geq 0$. Using BRST charge Q_{B},

$$\mathcal{V}_{\text{phys}} = \{|\text{phys}\rangle \in \mathcal{V}; Q_{\text{B}}\,|\text{phys}\rangle = 0\} \subset \mathcal{V}. \tag{1}$$

- We choose the Lorenz gauge fixing:

$$\partial^\mu \mathscr{A}_\mu^A(x) = 0. \tag{2}$$

- The Lagrangian density in the Lorenz gauge is given by

$$\begin{aligned} \mathscr{L} &= \mathscr{L}_{\text{inv}} + \mathscr{L}_{\text{GF+FP}}, \\ \mathscr{L}_{\text{inv}} &= -\frac{1}{4}\mathscr{F}_{\mu\nu}^A \mathscr{F}^{\mu\nu A} + \mathscr{L}_{\text{matter}}(\psi, D_\mu\psi), \\ \mathscr{L}_{\text{GF+FP}} &= -i\delta_{\text{B}}\left\{\bar{\mathscr{C}}^A\left(\partial^\mu \mathscr{A}_\mu^A + \frac{\alpha}{2}\mathscr{B}^A\right)\right\}. \end{aligned} \tag{3}$$

The restoration condition of the residual local gauge symmetry in the Lorenz gauge

- We consider the finite local gauge transformation $U(x)$

$$
\begin{aligned}
\delta_U \mathscr{A}_\mu(x) =& \Omega_\mu(x) + U(x)\mathscr{A}_\mu(x)U^\dagger(x) - \mathscr{A}_\mu(x), \quad \boxed{\Omega_\mu(x) := ig^{-1}U(x)\partial_\mu U^\dagger(x)}, \\
\delta_U \mathscr{B}(x) =& U(x)\mathscr{B}(x)U^\dagger(x) - \mathscr{B}(x), \ \delta_U \mathscr{C}(x) = U(x)\mathscr{C}(x)U^\dagger(x) - \mathscr{C}(x), \ \delta_U \bar{\mathscr{C}}(x) = U(x)\bar{\mathscr{C}}(x)U^\dagger(x) - \bar{\mathscr{C}}(x).
\end{aligned} \tag{4}
$$

For $U(x)$ to be a residual symmetry for the Lorenz gauge, $U(x)$ should satisfy

$$
\mathscr{D}^\mu \tilde{\Omega}_\mu = \partial^\mu \tilde{\Omega}_\mu(x) - igU[\mathscr{A}^\mu, \tilde{\Omega}_\mu]U^\dagger = 0, \ (\tilde{\Omega}_\mu(x) := -ig^{-1}U^\dagger(x)\partial_\mu U(x)). \tag{5}
$$

- We can calculate the Noether current $\mathscr{J}_U^\mu$ associated with (4) and the divergence of this current is equal to the transformation of the Lagrangian density under the transformation (4)

$$
\begin{aligned}
\partial_\mu \mathscr{J}_U^\mu = \delta_U \mathscr{L} = \delta_U \mathscr{L}_{\text{GF+FP}} =& -i\delta_U\delta_{\text{B}}\left[\text{tr}\left\{\bar{\mathscr{C}}(\partial^\mu \mathscr{A}_\mu + \frac{\alpha}{2}\mathscr{B})\right\}\right] = -i\delta_{\text{B}}\delta_U\left[\text{tr}\left\{\bar{\mathscr{C}}(\partial^\mu \mathscr{A}_\mu + \frac{\alpha}{2}\mathscr{B})\right\}\right] \\
=& -i\delta_{\text{B}}[\text{tr}\{\mathscr{D}_\mu\bar{\mathscr{C}}ig^{-1}U^\dagger\partial^\mu U\}] = i\delta_{\text{B}}[(\mathscr{D}_\mu\bar{\mathscr{C}})^A]\tilde{\Omega}^{\mu A}.
\end{aligned} \tag{6}
$$

$\rightarrow$ The local gauge current $\mathscr{J}_U^\mu$ is conserved in the physical state space.((6) is BRST transformation δ_{B} exact)

The restoration condition of the residual local gauge symmetry in the Lorenz gauge

- The restoration condition of the residual symmetry is written as the condition of the disappearance of the massless pole. In the one-gluon sector $\Phi = \mathscr{A}_\nu^B(y)$, we focus on the following Ward-Takahashi (WT) identity

$$i\partial_\mu^x \langle \mathscr{J}_U^\mu(x)\Phi\rangle = \delta^D(x-y)\left\langle \delta^U \mathscr{A}_\nu^A(x)\right\rangle + i\left\langle \partial_\mu \mathscr{J}_U^\mu(x)\mathscr{A}_\nu^A(y)\right\rangle . \tag{7}$$

- Performing Fourier transformation, we obtain

$$\int d^Dx\ e^{ip(x-y)}\partial_\mu^x \langle \mathscr{J}_U^\mu(x)\mathscr{A}_\nu^A(y)\rangle = \langle \delta^U \mathscr{A}_\nu^A(y)\rangle + i\int d^Dx\ e^{ip(x-y)}\langle \delta^U \mathscr{L}(x)\mathscr{A}_\nu^A(y)\rangle . \tag{8}$$

- From (6), the second term of (8) is reduced to

$$\begin{aligned}
&i\int d^Dx\ e^{ip(x-y)}\langle \delta^U \mathscr{L}(x)\mathscr{A}_\nu^B(y)\rangle \\
&= -\int d^Dx\ e^{ip(x-y)}\tilde{\Omega}^{\mu A}(x)\langle \boldsymbol{\delta}(\mathscr{D}_\mu\bar{\mathscr{C}})^A(x)]\mathscr{A}_\nu^B(y)\rangle = -\int d^Dx\ e^{ip(x-y)}\tilde{\Omega}^{\mu A}(x)\langle (\mathscr{D}_\mu\bar{\mathscr{C}})^A(x)\boldsymbol{\delta}\mathscr{A}_\nu^B(y)\rangle \\
&= -\int d^Dx\ e^{ip(x-y)}\tilde{\Omega}^{\mu A}(x)\left[\frac{\partial_\mu^x\partial_\nu^x}{\partial_x^2}\delta^D(x-y) + \left(g_{\mu\rho} - \frac{\partial_\mu^x\partial_\rho^x}{\partial_x^2}\right)\langle g(\mathscr{A}^\rho \times \bar{\mathscr{C}})^A(x)(\mathscr{D}_\nu\mathscr{C})^B(y)\rangle\right],
\end{aligned} \tag{9}$$

where we have used $\langle \delta_{\mathrm{B}}[\cdots]\rangle = 0$ and Schwinger-Dyson equation $\int d\mu\ \frac{\delta}{\delta\Phi(x)}e^{iS}F = 0$.

- We obtain the restoration condition of the residual local gauge symmetry in the one-gluon sector $\mathscr{A}_\nu^B(y)$

$$I_\nu^B = \lim_{p\to 0}\int d^Dx\, e^{ip(x-y)}\left\{\left(\Omega_\nu^B(x) - \tilde{\Omega}^{\mu B}(x)\frac{\partial_\mu^x\partial_\nu^x}{\partial_x^2}\right)\delta^D(x-y) + \tilde{\Omega}^{\mu A}(x)\left(g_{\mu\nu} - \frac{\partial_\mu^x\partial_\nu^x}{\partial_x^2}\right)u^{AB}(x-y)\right\}$$
$$\begin{cases} = 0 & \text{restoration} \\ \neq 0 & \text{no restoration} \end{cases}. \qquad (10)$$

where we have defined the Kugo-Ojima (KO) function u^{AB} in the configuration space

$$\left(g_{\mu\nu} - \frac{\partial_\mu^x\partial_\nu^x}{\partial_x^2}\right)\langle g(\mathscr{A}^\nu \times \bar{\mathscr{C}})^A(x)(\mathscr{D}_\nu\mathscr{C})^B(y)\rangle = -\left(g_{\mu\nu} - \frac{\partial_\mu^x\partial_\nu^x}{\partial_x^2}\right)u^{AB}(x-y). \qquad (11)$$

- In the non-compact gauge theory, it is enough to condider the infinitesimal gauge transformation and residual symmetry $\Omega^{\mu A}(x) = \tilde{\Omega}^{\mu A}(x) = \delta^{AC}b^\mu$ (C is arbitrary component of the Lie algebra).

→ (10) is reduced to Kugo-Ojima criterion as shown by Hata(1982)

$$0 = \lim_{p^2\to 0}[\delta^{AB} + u^{AB}(p^2)]. \qquad (12)$$

The restoration condition of the residual local gauge symmetry in the Lorenz gauge

- We restrict the topological configurations satisfying

$$0 = \lim_{p\to 0} \int d^D p \; e^{ipx} [\Omega^{\mu B}(x) - \tilde{\Omega}^{\mu B}(x)] \frac{\partial^x_\mu \partial^x_\nu}{\partial^2_x} \delta^D(x-y)$$
$$= \lim_{p\to 0} \int \frac{d^D k}{(2\pi)^2} e^{-i(p-k)y} [\Omega^{\mu B}(p-k) - \tilde{\Omega}^{\mu B}(p-k)] \frac{k_\mu k_\nu}{k^2}. \quad (13)$$

- If the condition (13) is satisfied, the restoration condition of the residual gauge symmetry (10) is reduced to

$$I^B_\nu = \lim_{p\to 0} \int \frac{d^D k}{(2\pi)^2} e^{-i(p-k)y} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) [\Omega^{\mu B}(p-k)\delta^{AB} + \tilde{\Omega}^{\mu B}(p-k) u^{AB}(k^2)]. \quad (14)$$

- Unlike the KO criterion (12), δ and u have different factors, Ω^μ and $\tilde{\Omega}^\mu$.
- In order to examine the effect of the topological configuration to the restoration of the residual symmetry, in $SU(2)$, we consider following topological configuration with one defect:

$$\Omega_\mu(x) = \sigma^A \eta^A_{\mu\nu} \frac{x^\nu}{x^2} h(x^2), \; \tilde{\Omega}_\mu(x) = -\sigma^A \bar{\eta}^A_{\mu\nu} \frac{x^\nu}{x^2} h(x^2) \; (\mu, \nu = 1, \cdots, D). \quad (15)$$

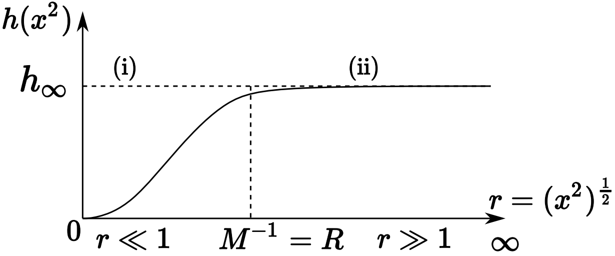
- This topological configuration is the residual symmetry.
- If $h(x^2) \equiv 1$, (15) includes $D=2$: Abrikosov-Nielsen-Olesen vortex, $D=3$: Wu-Yang monopole, $D=4$:Alfaro-Fubini-Furlan meron but the Euclidean action is divergent.

The restoration condition of the residual local gauge symmetry in the Lorenz gauge

$\rightarrow$ In order for such configurations to give a finite Euclidean action and to contribute to the path integral, $h(x^2) \rightarrow 0$ moderately as $x \rightarrow 0$

$$e^{-S_{\mathrm{E}}} > 0 \Leftarrow S_{\mathrm{E}} < \infty. \tag{16}$$

- To obtain the finite S_E, $h(x^2)$ must approaches 0 of order $r^{2\delta}$ ($r = (x^2)^{\frac{1}{2}}$, $\delta > 1/4$) in $r \ll 1$ and it must approach h_∞ (a finite value) rapidly in $r \gg 1$. ($D = 2$: Kondo(2018), $D = 3$: Nishino et al. (2018))



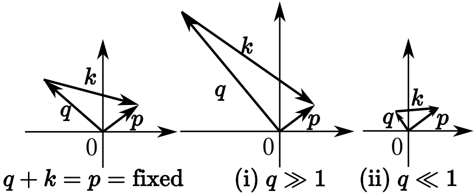
The restoration condition of the residual local gauge symmetry in the Lorenz gauge

- We put $q := p - k$

$$\Omega^B_\mu(q) := \int d^D x \; e^{iqx} \Omega^B_\mu(x), \; \tilde{\Omega}^B_\mu(q) := \int d^D x \; e^{iqx} \tilde{\Omega}^B_\mu(x)$$

$$\lim_{p \to 0} \int \frac{d^D k}{(2\pi)^D} \left(\Omega^{\mu B}(q) g_{\mu\nu} - \tilde{\Omega}^{\mu B}(q) \frac{k_\mu k_\nu}{k^2} \right). \quad (17)$$

- We fix $p \ll 1$.



- $r \ll 1 \Rightarrow \; q \gg 1 \Rightarrow \; k \gg 1$.

$\to$ UV region (of k).

- $r \gg 1 \Rightarrow \; q \ll 1 \Rightarrow \; k = p \to \; 0$.

$\to$ IR region (of k).

The restoration condition of the residual local gauge symmetry in the Lorenzn gauge

- When $h(x^2) \equiv 1$, Fourier transformation of the topological configuration (15) is given by

$$\Omega^A_\mu(q) := iC_D \eta^A_{\mu\nu} \frac{2q_\nu}{|q|^D}, \; \tilde{\Omega}^A_\mu(q) := -iC_D \bar{\eta}^A_{\mu\nu} \frac{2q_\nu}{|q|^D}, \; C_D := 2^{D-1}\pi^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right). \tag{18}$$

- In $r \ll 1$, we take $h(r^2) \sim r^{2\delta}$ $(\delta > 1/4)$.

→ In $q \gg 1$, it corresponds to dividing $\Omega(q), \tilde{\Omega}(q)$ of (18) by $q^{2\delta}$

$$\Omega^A_\mu(q) \sim iC_D \eta^A_{\mu\nu} \frac{2q_\nu}{|q|^{D+2\delta}}, \; \tilde{\Omega}^A_\mu(q) \sim -iC_D \bar{\eta}^A_{\mu\nu} \frac{q_\nu}{|q|^{D+2\delta}}. \tag{19}$$

- We discuss the condition (13) by using these $\Omega_\mu, \tilde{\Omega}_\mu$.

$$\begin{aligned}(13) =& i\frac{2C_D}{D}\left(\frac{D}{2}+\delta\right)\lim_{p\to 0} 2\varepsilon_{4A\nu\rho}p^\lambda \int_0^1 d\alpha \int \frac{d^D\ell}{(2\pi)^D} \frac{\alpha^{\frac{D}{2}+\delta-1}\ell^2}{\{\ell^2+\alpha(1-\alpha)p^2\}^{\frac{D}{2}+\delta+1}} \\ =& i\left(\frac{D}{4}+\frac{\delta}{2}\right)\frac{\Gamma(\delta)\Gamma(1-\delta)\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{D}{2}\right)}{\Gamma\left(\frac{D}{2}+\delta+1\right)\Gamma\left(\frac{D}{2}-\delta+1\right)}\lim_{p\to 0} 2\varepsilon_{4A\nu\rho}p^\rho(p^2)^{-\delta}, \; (\ell = k-\alpha p).\end{aligned} \tag{20}$$

where we take $y = 0$.

- In this case, we can take the limit $p \to 0$ in the integrand of the k when we estimate the contribution from the UV region $k \gg k_R$ i.e. $p \ll k$

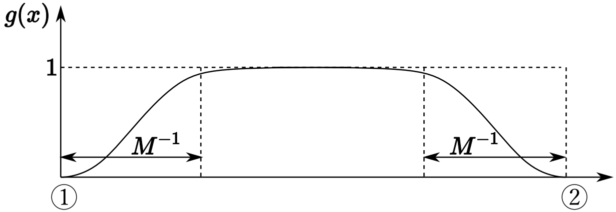
$$(13) = \boxed{i\frac{C_D}{\delta(4\pi)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}+1\right)}\lim_{p\to 0} 2\varepsilon_{4A\nu\rho}p^\rho \frac{1}{k_R^{2\delta}}}. \tag{21}$$

→ δ play the role of avoiding the UV divergence $(\delta > 1/4)$.

- Since $q \ll 1$ corresponds to $r \gg 1$, $\delta \to 0$.
- We cannot avoid IR divergence when the configuration has only one defect.

$\rightarrow$ We introduce the topological configuration with the multiple pairs of defect.

$$\Omega_\mu^A(x) = \sum_{s=1}^{n} \eta_{\mu\nu}^A \frac{(x-x_s)^\nu}{(x-x_s)^2} g((x-x_s)^2) \ (\mu,\nu = 1,\cdots,D). \tag{22}$$



- When topological defects are sufficiently separated, the IR region of one defect ① corresponds to the UV region of another defect ②.

$\rightarrow$ IR divergence can be eliminated.

The restoration condition of the residual local gauge symmetry in the Lorenz gauge

- In $D = 3$, we consider the topological configuration consisting of the multiple monopole.

$\rightarrow$ Magnetic monopole plasma cause the Debye screening. (Polyakov(1977))

$\rightarrow$ The gauge field becomes massive.

- This corresponds to the origin of the appearance of the dimensional scale $M^{-1} = R$ of $h(x^2)$. ($D = 4$:Callan-Dashen-Gross(1978))
- If we introduce a topological configuration with multiple defects, these results are consistent with the confinement picture where the vacuum condensation of topological defects leads to confinement.

The restoration condition of the residual local gauge symmetry in the Maximal Abelian gauge

- We decompose the Lie-algebra valued quantity to the diagonal Cartan part and remaining off-diagonal part.
- The gauge field $\mathscr{A}_\mu(x) = \mathscr{A}_\mu^A T_A$ with the generators T_A $(A = 1, \cdots, N^2 - 1)$ of the Lie algebra $su(N)$ has the decomposition:

$$\mathscr{A}_\mu(x) = \mathscr{A}_\mu^A(x) T_A = a_\mu^j(x) H_j + A_\mu^a(x) T_a, \tag{23}$$

where H_j are the Cartan generators and T_a are the remaining generators of the Lie algebra.
- We choose the Maximal Abelian (MA) gauge

$$(\mathscr{D}^\mu[a] A_\mu(x))^a := \partial^\mu A_\mu^a + g f^{ajb} a^{\mu j} A_\mu^b(x) = 0. \tag{24}$$

- Since MA gauge does not fix the diagonal components, we further impose the Lorenz gauge for the diagonal components

$$\partial^\mu a_\mu^j = 0. \tag{25}$$

- The GF+FP term of the Lagrangian in the MA gauge is given as below

$$\mathscr{L}_{\text{GF+FP}} = -i\delta_{\text{B}} \left\{ \bar{C}^a \left((\mathscr{D}^\mu[a] A_\mu)^a + \frac{\alpha}{2} B^a \right) + \bar{c}^j (\partial^\mu a_\mu^j + \frac{\alpha}{2} b^j) \right\}. \tag{26}$$

- Assuming Abelian dominance, we can calculate the divergence of the Noether current $\mathscr{J}_U^\mu$ as in the Lorenz gauge:

$$\partial_\mu \mathscr{J}_U^\mu = \delta_U \mathscr{L} = \delta_U \mathscr{L}_{\text{GF+FP}} = -i\delta_U \delta_{\text{B}} \left[\text{tr} \left\{ \bar{\mathscr{C}} \left(\partial^\mu \mathscr{A}_\mu + \frac{\alpha}{2} \mathscr{B} \right) + g\bar{C}^a (a^\mu \times A_\mu)^a \right\} \right] \to i\delta_{\text{B}} \partial_\mu \bar{c}^j \tilde{\Omega}^{\mu j}. \tag{27}$$

- U is residual local gauge symmetry, it must satisfy

$$\partial^\mu \Omega_\mu^j(x) = 0. \tag{28}$$

The restoration condition of the residual local gauge symmetry in the Maximal Abelian gauge

- The restoration condition of the residual local gauge symmetry in the one diagonal gluon sector $a^j_\nu(y)$ is obtained as in the Lorenz gauge:

$$I^j_\nu = \lim_{p\to 0}\int d^Dx\, e^{ip(x-y)}\left(\Omega^{\mu j}(x)g_{\mu\nu} - \tilde{\Omega}^{\mu j}(x)\frac{\partial^x_\mu\partial^x_\nu}{\partial^2_x}\right)\delta^D(x-y) \tag{29}$$

$$= \lim_{p\to 0}\int \frac{d^Dk}{(2\pi)^D}\, e^{-i(p-k)y}\left(\Omega^{\mu j}(p-k)g_{\mu\nu} - \tilde{\Omega}^{\mu j}(p-k)\frac{k_\mu k_\nu}{k^2}\right)\begin{cases} = 0 & \text{restoration} \\ \neq 0 & \text{no restoration}\end{cases}. \tag{30}$$

- If the condition:

$$\lim_{p\to 0}\int d^Dp\, e^{ipx}[\Omega^{\mu j}(x) - \tilde{\Omega}^{\mu j}(x)]\frac{\partial^x_\mu\partial^x_\nu}{\partial^2_x}\delta^D(x-y) = 0 \tag{31}$$

is satisfied, (30) is reduced to

$$I^j_\nu = \lim_{p\to 0}\int \frac{d^Dk}{(2\pi)^D}\, e^{-i(p-k)y}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Omega^{\mu j}(p-k). \tag{32}$$

- Similar to the discussion of the condition (13), this restoration condition is satisfied for the same $\Omega_\mu, \tilde{\Omega}_\mu$ (indeed, it satisfy the condition of residual gauge symmetry (28)) by correcting the topological configuration to contribute to path integrals and introducing multiple pairs of topological defects.

Summary

- We gave the generalization of the color confinement criterion in the Lorenz gauge, the MA gauge as the residual symmetry restoration conditions for the one gauge field sector with the finite gauge transformation.
- We discussed the restoration of the residual local gauge symmetry by correcting the topological configuration to contribute to path integrals.
- The axial-like gauge is under way.

Future work

- To give the complete discussion for confinement, it is necessary to consider interactions among various fields, including different kinds of fields such as the ghost and gauge field.
- A physical picture of symmetry restoration due to topological configurations contributing to the path integral is provided. Therefore, based on this picture, it is necessary to discuss the condensation of topological defects in arbitrary dimensions.
- The concrete calculation in the axial-like gauge.

We calculate the expected value contained in the integrand. If we break it down into a longitudinal part and a transverse part, we get

$$
\begin{aligned}
&\langle (\mathscr{D}_\mu \bar{\mathscr{C}}(x))^B \boldsymbol{\delta} \mathscr{A}_\nu^A(y) \rangle \\
=& g_{\mu\rho} \langle (\mathscr{D}^\rho \bar{\mathscr{C}})^B(x) \boldsymbol{\delta} \mathscr{A}_\nu^A(y) \rangle \\
=& \frac{\partial_\mu^x \partial_\rho^x}{\partial_x^2} \langle (\mathscr{D}^\rho \bar{\mathscr{C}})^B(x) \boldsymbol{\delta} \mathscr{A}_\nu^A(y) \rangle + \left(g_{\mu\rho} - \frac{\partial_\mu^x \partial_\rho^x}{\partial_x^2} \right) \langle (\mathscr{D}^\rho \bar{\mathscr{C}})^B(x) (\mathscr{D}_\nu \mathscr{C})^A(y) \rangle \\
=& \frac{\partial_\mu^x \partial_\rho^x}{\partial_x^2} \langle (\mathscr{D}^\rho \bar{\mathscr{C}})^B(x) (\mathscr{D}_\nu \mathscr{C})^A(y) \rangle + \left(g_{\mu\rho} - \frac{\partial_\mu^x \partial_\rho^x}{\partial_x^2} \right) \langle g(\mathscr{A}^\rho \times \bar{\mathscr{C}})^A(x) (\mathscr{D}_\nu \mathscr{C})^A(y) \rangle \,, \qquad (33)
\end{aligned}
$$

where we have used $\left(g_{\mu\rho} - \frac{\partial_\mu^x \partial_\rho^x}{\partial_x^2} \right) \partial_x^\rho = 0$.

If we note the left derivative $\frac{\delta S}{\delta \mathscr{C}^B(x)} = -i(\mathscr{D}^\rho \partial_\rho \bar{\mathscr{C}})^B(x)$, $\frac{\delta S}{\delta \mathscr{B}^B(x)} = \partial_\rho \mathscr{A}^{\rho B}(x) + \alpha \mathscr{B}^B(x)$, the part of (33) that excludes $\frac{\partial_\mu^x}{\partial_x^2}$ in the first term is

$$\begin{aligned}
&\partial_\rho^x \left\langle (\mathscr{D}^\rho \bar{\mathscr{C}})^B(x) \boldsymbol{\delta} \mathscr{A}_\nu^A(y) \right\rangle \\
&= \int d\mu \; e^{iS} \partial_\rho (\mathscr{D}^\rho \bar{\mathscr{C}})^B(x) \boldsymbol{\delta} \mathscr{A}_\nu^A(y) \\
&= \int d\mu \; e^{iS} [(\mathscr{D}^\rho \partial_\rho \bar{\mathscr{C}})^B(x) \boldsymbol{\delta} \mathscr{A}_\nu^A(y) + (g \partial_\rho \mathscr{A}^\rho \times \bar{\mathscr{C}})^B(x) \boldsymbol{\delta} \mathscr{A}_\nu^A(y)] \\
&= \int d\mu \; e^{iS} \left[i \frac{\delta S}{\delta \mathscr{C}^A(x)} \boldsymbol{\delta} \mathscr{A}_\nu^A(y) + \left(g \left(\frac{\delta S}{\delta \mathscr{B}} - \alpha \mathscr{B} \right) \times \bar{\mathscr{C}} \right)^B (x) \boldsymbol{\delta} \mathscr{A}_\nu^A(y) \right] . \qquad (34)
\end{aligned}$$

From Schwinger-Dyson equation $\int d\mu \frac{\delta}{\delta\Phi^A(x)}(\cdots) = 0$,

$$\int d\mu\ e^{iS} i \frac{\delta S}{\delta \mathscr{C}^B(x)} \boldsymbol{\delta}\mathscr{A}_\nu^A(y) = \int d\mu \frac{\delta}{\delta \mathscr{C}^B(x)} [e^{iS} \boldsymbol{\delta}\mathscr{A}_\nu^A(y)] - \int d\mu\ e^{iS} \frac{\delta}{\delta \mathscr{C}^B(x)} \boldsymbol{\delta}\mathscr{A}_\nu^A(y)$$

$$= -\int d\mu\ e^{iS} \frac{\delta}{\delta \mathscr{C}^B(x)} \boldsymbol{\delta}\mathscr{A}_\nu^A(y) = -\left\langle \frac{\delta}{\delta \mathscr{C}^B(x)} (\mathscr{D}_\nu \mathscr{C})^A(y) \right\rangle = -\delta^{AB} \partial_\nu \delta^D(x-y). \qquad (35)$$

Similarly,

$$\int d\mu\ e^{iS} \left(g \frac{\delta S}{\delta \mathscr{B}} \times \bar{\mathscr{C}} \right)^B (x) \boldsymbol{\delta}\mathscr{A}_\nu^A(y) = \int d\mu\ e^{iS} g f^{BCE} \frac{\delta S}{\delta \mathscr{B}^C(x)} \bar{\mathscr{C}}^E(x) \boldsymbol{\delta}\mathscr{A}_\nu^A(y)$$

$$= \int d\mu \frac{\delta}{\delta i \mathscr{B}^C(x)} [e^{iS} g f^{BCE} \bar{\mathscr{C}}^E(x) \boldsymbol{\delta}\mathscr{A}_\nu^A(y)] + i \int d\mu\ e^{iS} g f^{BCE} \bar{\mathscr{C}}^E(x) \frac{\delta}{\delta \mathscr{B}^C(x)} \boldsymbol{\delta}\mathscr{A}_\nu^A(y)$$

$$= -ig \left\langle \left(\bar{\mathscr{C}} \times \frac{\delta}{\delta \mathscr{B}(x)} \boldsymbol{\delta}\mathscr{A}_\nu^A(y) \right)^B \right\rangle = 0. \qquad (36)$$

Moreover

$$-\int d\mu\ e^{iS} g\alpha(\mathscr{B}\times\bar{\mathscr{C}})^A(x)\boldsymbol{\delta}\Phi(y)=\int d\mu\ e^{iS} g\alpha\boldsymbol{\delta}(\mathscr{B}\times\bar{\mathscr{C}})^A(x)\Phi(y)=0 \tag{37}$$

Finally, the first term of (33) is reduced to

$$\partial^x_\nu \langle(\mathscr{D}^\nu\bar{\mathscr{C}})^B(x)\boldsymbol{\delta}\mathscr{A}^A_\nu(y)\rangle=-\delta^{AB}\partial_\nu\delta^D(x-y) \tag{38}$$

Then, (8)

$$\begin{aligned}&-i\int d^Dx\ e^{ip(x-y)}\partial^x_\mu\langle\mathscr{J}^\mu_U(x)\Phi(y)\rangle\\ =&\Omega_\nu(x)+\int d^Dx\ e^{ipx}\tilde{\Omega}^{\mu A}(x)\left[\frac{\partial^x_\mu\partial^x_\nu}{\partial^2_x}\delta^D(x-y)\right.\\ &\left.-\left(g_{\mu\rho}-\frac{\partial^x_\mu\partial^x_\rho}{\partial^2_x}\right)\langle g(\mathscr{A}^\rho\times\bar{\mathscr{C}})^A(x)(\mathscr{D}_\nu\mathscr{C})^A(y)\rangle\right]\end{aligned} \tag{39}$$

In what follows, we demonstrate that the finite gauge transformation δ_U and the BRST transformation δ_{B} commute.
The BRST transformations are given by

$$\begin{aligned} \delta_{\mathrm{B}}\mathscr{A}_\mu &= \mathscr{D}_\mu[\mathscr{A}_\mu]\mathscr{C} = \partial_\mu\mathscr{C} - ig[\mathscr{A}_\mu, \mathscr{C}] \\ \delta_{\mathrm{B}}\mathscr{C} &= ig\mathscr{C}\mathscr{C} \\ \delta_{\mathrm{B}}\bar{\mathscr{C}} &= i\mathscr{B} \\ \delta_{\mathrm{B}}\mathscr{B} &= 0. \end{aligned} \tag{40}$$

The finite gauge transformations are given by

$$\begin{aligned} \delta_U\mathscr{A}_\mu &= U\mathscr{A}_\mu U^\dagger + ig^{-1}U\partial_\mu U^\dagger - \mathscr{A}_\mu \\ \delta_U\mathscr{C} &= U\mathscr{C}U^\dagger - \mathscr{C} \\ \delta_U\bar{\mathscr{C}} &= U\bar{\mathscr{C}}U^\dagger - \bar{\mathscr{C}} \\ \delta_U\mathscr{B} &= U\mathscr{B}U^\dagger - \mathscr{B}. \end{aligned} \tag{41}$$

Then, we obtain

$$\delta_U \delta_{\mathrm{B}} \mathscr{B} = 0, \ \delta_{\mathrm{B}} \delta_U \mathscr{B} = U \delta_{\mathrm{B}} \mathscr{B} U^\dagger - \delta_{\mathrm{B}} \mathscr{B} = 0, \tag{42}$$

$$\delta_U \delta_{\mathrm{B}} \bar{\mathscr{C}} = i \delta_U \mathscr{B} = i(U \mathscr{B} U^\dagger - \mathscr{B}), \ \delta_{\mathrm{B}} \delta_U \bar{\mathscr{C}} = U \delta_{\mathrm{B}} \bar{\mathscr{C}} U^\dagger - \delta_{\mathrm{B}} \bar{\mathscr{C}} = iU \mathscr{B} U^\dagger - i\mathscr{B} \tag{43}$$

$$\begin{aligned} \delta_U \delta_{\mathrm{B}} \mathscr{C} &= ig \delta_U (\mathscr{C}\mathscr{C}) = ig(U \mathscr{C}\mathscr{C} U^\dagger - \mathscr{C}\mathscr{C}), \\ \delta_{\mathrm{B}} \delta_U \mathscr{C} &= U \delta_{\mathrm{B}} \mathscr{C} U^\dagger - \delta_{\mathrm{B}} \mathscr{C} = igU \mathscr{C}\mathscr{C} U^\dagger - ig\mathscr{C}\mathscr{C}, \end{aligned} \tag{44}$$

$$\begin{aligned} \delta_U \delta_{\mathrm{B}} \mathscr{A}_\mu &= \delta_U (\mathscr{D}_\mu[\mathscr{A}]\mathscr{C}) = U \mathscr{D}_\mu[\mathscr{A}]\mathscr{C} U^\dagger - \mathscr{D}_\mu[\mathscr{A}]\mathscr{C}, \\ \delta_{\mathrm{B}} \delta_U \mathscr{A}_\mu &= \delta_{\mathrm{B}}(U \mathscr{A}_\mu U^\dagger + igU \partial_\mu U^\dagger - \mathscr{A}_\mu), \\ &= U \delta_{\mathrm{B}} \mathscr{A}_\mu U^\dagger - \delta_{\mathrm{B}} \mathscr{A}_\mu = U \mathscr{D}_\mu[\mathscr{A}]\mathscr{C} U^\dagger - \mathscr{D}_\mu[\mathscr{A}]\mathscr{C}, \end{aligned} \tag{45}$$

where we have used

$$\begin{aligned} \delta_U (\mathscr{D}_\mu[\mathscr{A}]\mathscr{C}) = \delta_U (\partial_\mu \mathscr{C} - ig[\mathscr{A}_\mu, \mathscr{C}]) & \\ =& \partial_\mu (U \mathscr{C} U^\dagger) - ig[U \mathscr{A}_\mu U^\dagger + ig^{-1} U \partial_\mu U^\dagger, U \mathscr{C} U^\dagger] - (\partial_\mu \mathscr{C} - ig[\mathscr{A}_\mu, \mathscr{C}]) \\ =& U \partial_\mu \mathscr{C} U^\dagger - igU[\mathscr{A}_\mu, \mathscr{C}]U^\dagger + [U \partial_\mu U^\dagger, U \mathscr{C} U^\dagger] + \partial_\mu U \mathscr{C} U^\dagger + U \mathscr{C} \partial_\mu U^\dagger \\ =& U \partial_\mu \mathscr{C} U^\dagger - igU[\mathscr{A}_\mu, \mathscr{C}]U^\dagger - \partial_\mu U \mathscr{C} U^\dagger - U \mathscr{C} \partial_\mu U^\dagger + \partial_\mu U \mathscr{C} U^\dagger + U \mathscr{C} \partial_\mu U^\dagger \\ =& U \partial_\mu \mathscr{C} U^\dagger - igU[\mathscr{A}_\mu, \mathscr{C}]U^\dagger = U \mathscr{D}[\mathscr{A}]\mathscr{C} U^\dagger, \end{aligned} \tag{46}$$

following

$$U \partial_\mu U^\dagger = -\partial_\mu U U^\dagger. \tag{47}$$

Thus, the finite gauge transformation δ_U and the BRST transformation δ_{B} commute.

$$
\begin{aligned}
0 =& U^\dagger[\partial^\mu \Omega_\mu + \partial^\mu (U \mathscr{A}_\mu U^\dagger)]U \\
=& ig^{-1} U^\dagger \partial^\mu U \partial_\mu U^\dagger U + ig^{-1} U^\dagger U \partial^\mu \partial_\mu U^\dagger U + U^\dagger \partial^\mu U \mathscr{A}_\mu + \mathscr{A}_\mu \partial_\mu U^\dagger U \\
=& ig^{-1} \partial^\mu U^\dagger U U^\dagger \partial_\mu U + ig^{-1} \partial^\mu \partial_\mu U^\dagger U + ig \tilde{\Omega}^\mu \mathscr{A}_\mu - \mathscr{A}_\mu U^\dagger \partial^\mu U \\
=& ig^{-1} \partial_\mu (\partial^\mu U^\dagger U) + ig \tilde{\Omega}^\mu \mathscr{A}_\mu - ig \mathscr{A}_\mu \tilde{\Omega}^\mu \\
=& - ig^{-1} \partial_\mu (U^\dagger \partial^\mu U - ig[\mathscr{A}_\mu, \tilde{\Omega}^\mu] \\
=& \partial_\mu \tilde{\Omega}^\mu - ig[\mathscr{A}_\mu, \tilde{\Omega}^\mu] =: \mathscr{D}_\mu \tilde{\Omega}^\mu
\end{aligned}
\tag{48}
$$

$$\mathscr{A}_\mu(x) = \frac{\sigma^A}{2}\bar{\eta}^A_{\mu\nu}x_\nu F(x^2) \tag{49}$$

is a typical example that satisfies the Lorenz condition. As a residual symmetry, we consider

$$\tilde{\Omega}_\mu(x) = \frac{\sigma^A}{2}\bar{\eta}^A_{\mu\nu}x_\nu H(x^2) \tag{50}$$

in this case,

$$\partial_\mu \tilde{\Omega}_\mu = 0, \tag{51}$$

$$\begin{aligned}[\mathscr{A}_\mu, \tilde{\Omega}_\mu] &= \left[\frac{\sigma^A}{2}\bar{\eta}^A_{\mu\nu}x_\nu F(x^2), \frac{\sigma^B}{2}\bar{\eta}^B_{\mu\rho}x_\rho H(x^2)\right] = \left[\frac{\sigma^A}{2}, \frac{\sigma^B}{2}\right]\bar{\eta}^A_{\mu\nu}\bar{\eta}^B_{\mu\rho}x_\nu x_\rho F(x^2)H(x^2) \\ &= i\varepsilon^{ABC}\frac{\sigma^C}{2}\bar{\eta}^A_{\mu\nu}\bar{\eta}^B_{\mu\rho}x_\nu x_\rho F(x^2)H(x^2) = i\frac{\sigma^C}{2}2\bar{\eta}_{C\nu\rho}x_\nu x_\rho F(x^2)H(x^2) = 0,\end{aligned} \tag{52}$$

where we have used

$$\begin{aligned}\varepsilon_{CAB}\bar{\eta}_{A\mu\nu}\bar{\eta}_{B\mu\rho} &= \delta_{\mu\mu}\bar{\eta}_{C\nu\rho} - \delta_{\mu\rho}\bar{\eta}_{C\nu\mu} - \delta_{\nu\mu}\bar{\eta}_{C\mu\rho} + \delta_{\nu\rho}\bar{\eta}_{C\mu\mu} \\ &= 4\bar{\eta}_{C\nu\rho} - \bar{\eta}_{C\nu\rho} - \bar{\eta}_{C\nu\rho} \\ &= 2\bar{\eta}_{C\nu\rho}.\end{aligned} \tag{53}$$

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Now, gauge fixing condition is

$$n^\mu(x)\mathscr{A}_\mu(x) = 0. \tag{54}$$

Lagrangian is

$$\begin{aligned}
\mathscr{L} &= \mathscr{L}_{\mathsf{YM}} + \mathscr{L}_{\mathsf{GF+FP}} + \mathscr{L}_J, \\
\mathscr{L}_{\mathsf{YM}} &= -\frac{1}{4}(\mathscr{F}_{\mu\nu})^2, \\
\mathscr{L}_{\mathsf{GF+FP}} &= -i\delta_{\mathsf{B}}\left(\bar{\mathscr{C}}^A\left(n^\mu\mathscr{A}_\mu^A + \frac{\alpha}{2}\mathscr{B}^A\right)\right) = i\bar{\mathscr{C}}^A n^\mu(\mathscr{D}_\mu\mathscr{C})^A + \mathscr{B}^A n^\mu\mathscr{A}_\mu^A + \frac{\alpha}{2}\mathscr{B}^A\mathscr{B}^A, \\
\mathscr{L}_J &= \mathscr{A}_\mu^A J^{\mu A} + \mathscr{B}^A K^A.
\end{aligned} \tag{55}$$

To derive the propagator, we consider up to the second order of the field

$$\frac{\partial\mathscr{L}}{\partial\mathscr{A}_\mu^A} - \partial_\nu\frac{\partial\mathscr{L}}{\partial\partial_\nu\mathscr{A}_\mu^A} = (g^{\mu\nu}\Box - \partial^\mu\partial^\nu)\mathscr{A}_\nu + \mathscr{B}^A n^\mu + J^{\mu A} = 0. \tag{56}$$

If we take the divergence, we get

$$-\partial_\mu J^{\mu A} = n^\mu\partial_\mu\mathscr{B}^A + \partial_\mu n^\mu\mathscr{B}^A \Rightarrow\ B^A = -\frac{1}{\partial_\rho n^\rho + n^\rho\partial_\rho}\partial_\mu J^{\mu A}. \tag{57}$$

Then we obtain

$$\langle\mathscr{B}^A(x)\mathscr{A}_\mu^B(y)\rangle = \frac{\delta\mathscr{B}^A(x)}{\delta J^{\mu B}(y)} = -\frac{1}{\partial_\rho n^\rho + n^\rho\partial_\rho}\partial_\mu\delta^D(x-y)\delta^{AB} \tag{58}$$

$$i\frac{2QC_D}{D}\left(\frac{D}{2}+\delta\right)\lim_{p\to\,0}\bar{\eta}^{j}_{\nu\lambda}p^{\lambda}\int_0^1 d\alpha\int\frac{d^D\ell}{(2\pi)^D}\frac{\alpha^{\frac{D}{2}+\delta-1}\ell^2}{\{\ell^2+\alpha(1-\alpha)p^2\}^{\frac{D}{2}+\delta+1}} \tag{59}$$

If we estimate the contribution of the UV region $k > k_R \gg 1$ to I_ν, we can take the limit $p \to\ 0$ of the integrand, which is equivalent to the replacement $\ell_\mu = k_\mu$:

$$\begin{aligned}
&\to i\frac{2C_D}{D}\left(\frac{D}{2}+\delta\right)\lim_{p\to\,0}2\varepsilon_{4A\nu\lambda}p^{\lambda}\int_0^1 d\alpha\int d\Omega_D\int_{k_R}^{\infty}\frac{dk\ k^{D-1}}{(2\pi)^D}\frac{\alpha^{\frac{D}{2}+\delta-1}}{k^{D+2\delta}}\\
&=i\frac{2QC_D}{(4\pi)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}+1\right)}\left(\frac{D}{2}+\delta\right)\lim_{p\to\,0}2\varepsilon_{4A\nu\lambda}p^{\lambda}\int_0^1 d\alpha\int_{k_R}^{\infty}dk\frac{\alpha^{\frac{D}{2}+\delta-1}}{k^{2\delta+1}}\\
&=i\frac{2C_D}{(4\pi)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}+1\right)}\lim_{p\to\,0}2\varepsilon_{4A\nu\lambda}p^{\lambda}[\alpha^{\frac{D}{2}+\delta}]_0^1\left[-\frac{1}{2\delta}\frac{1}{k^{2\delta}}\right]_{k_R}^{\infty}\\
&=i\frac{2C_D}{2\delta(4\pi)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}+1\right)}\lim_{p\to\,0}2\varepsilon_{4A\nu\lambda}p^{\lambda}\frac{1}{k_R^{2\delta}}=0\ \ (\delta>1/4)\,,
\end{aligned} \tag{60}$$

If we think

$$\mathscr{A}_\mu(x) = \sigma^A \bar{\eta}^A_{\mu\nu} \frac{\tilde{x}_\nu \lambda^2}{\tilde{x}^2(\tilde{x}^2 + \lambda^2)}$$

$$U = U_1 U_0,\ U_0 = \frac{x_4 I + i\sigma_j x_j}{\sqrt{x^2}},\ U_1(x) = \exp\left[i\sigma^A \frac{x_A}{\sqrt{\boldsymbol{x}^2 + \lambda^2}}\left(\arctan\frac{x_4}{\sqrt{\boldsymbol{x}^2 + \lambda^2}} + \frac{\pi}{2}\right)\right] \tag{61}$$

, gauge fixing condition is satisfied.

$$\mathscr{A}''_\mu = \mathscr{A}_\mu + \delta_U \mathscr{A}_\mu = 0\ (|x| \to\ 0) \tag{62}$$

If we introduce the polar coordinate $(r, \theta_1, \theta_2, \theta_3)$ as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} r\sin\theta_1 \sin\theta_2 \sin\theta_3 \\ r\sin\theta_1 \sin\theta_2 \cos\theta_3 \\ r\sin\theta_1 \cos\theta_2 \\ r\cos\theta_1 \end{pmatrix} \tag{63}$$

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Then,

$$U_0 = \cos\theta_1 I + \sin\theta_1 \sin\theta_2 \sin\theta_3 \sigma_1 + \sin\theta_1 \sin\theta_2 \cos\theta_3 \sigma_2 + \sin\theta_1 \cos\theta_2 \sigma_3 \tag{64}$$

$$\begin{aligned} U_1 &= \exp\left[ir\frac{\sin\theta_1 \sin\theta_2 \sin\theta_3 \sigma_1 + \sin\theta_1 \sin\theta_2 \cos\theta_3 \sigma_2 + \sin\theta_1 \cos\theta_2 \sigma_3}{\sqrt{r^2 \sin^2\theta_1 + \lambda^2}}\left(\arctan\frac{r\cos\theta_1}{\sqrt{r^2 \sin^2\theta_1 + \lambda^2}} + \frac{\pi}{2}\right)\right] \\ &= \exp\left[i\frac{\sin\theta_1 \sin\theta_2 \sin\theta_3 \sigma_1 + \sin\theta_1 \sin\theta_2 \cos\theta_3 \sigma_2 + \sin\theta_1 \cos\theta_2 \sigma_3}{\sqrt{\sin^2\theta_1 + (\lambda/r)^2}}\left(\arctan\frac{\cos\theta_1}{\sqrt{\sin^2\theta_1 + (\lambda/r)^2}} + \frac{\pi}{2}\right)\right] \end{aligned} \tag{65}$$

Therefore, if we rescale $r \to rp$, U is 0 order of p, where since λ is unphysical parameter, we can take $\lambda \to 0$ as $p \to 0$