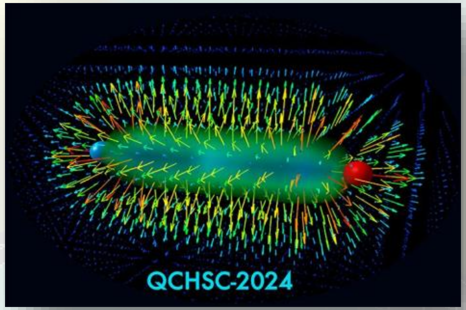
The QCD confining string and the world-sheet axion



18th – 24th of August 2024

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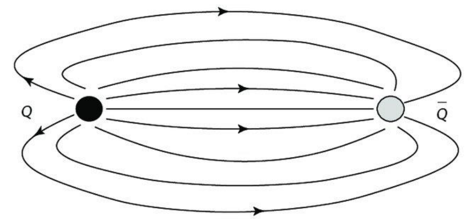
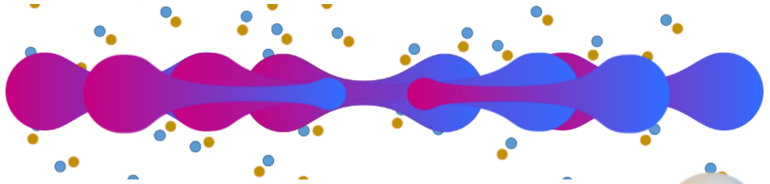
Computation Phenomenology Theory

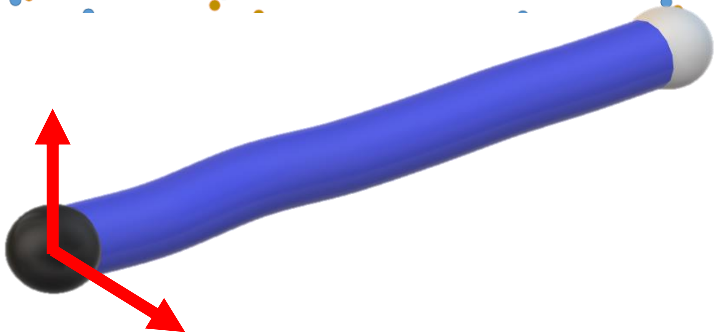
In collaboration with Pedro Bicudo, Sergei Dubovksy, Conghuan Luo, Alireiza Sharifian and Michael Teper

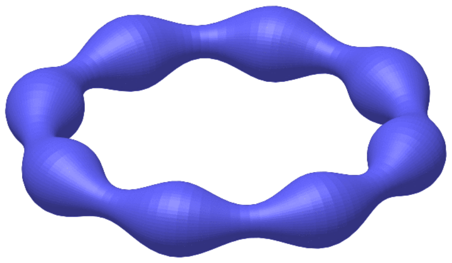
Are flux tubes strings?

- In QCD quarks are confined in bound states called flux-tubes
- Long flux tubes behave pretty much like thin strings
  - Energy increases with separation: $V \approx \sigma r, \quad \sqrt{\sigma} \approx 440$ MeV
  - At some point the string breaks – String breaking –
  - We need dynamical fermions to observe string breaking
  - We work in pure gauge theory
- There are $\boldsymbol{D} - \mathbf{2}$ massless Goldstone modes from broken translation invariance in the $\boldsymbol{D}-\mathbf{2}$ directions
- There should be a Low Energy Effective String Theory model describing the energy spectrum of the flux tube
- Questions to be addressed
  - What is this effective string theory?
  - How good an approximation such an effective string theory is?
  - Are there additional massive excitations along the flux-tube?









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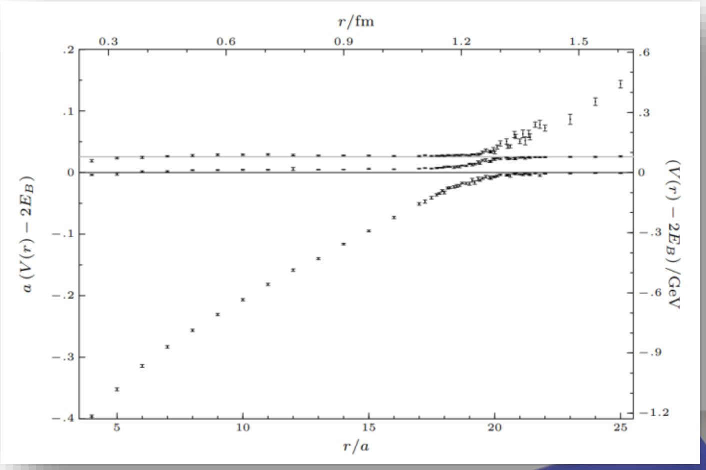
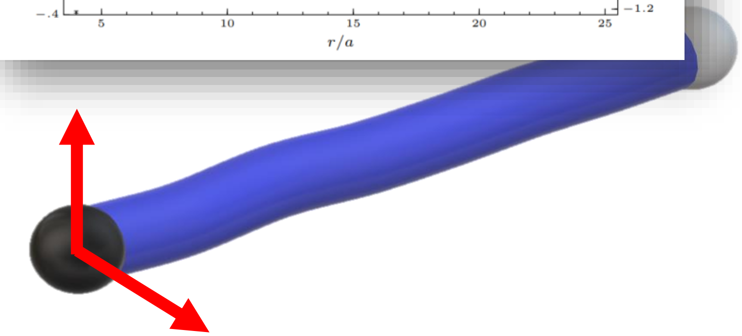
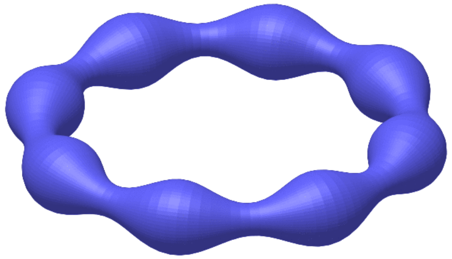
J. Bulava et al, Phys.Lett.B 793 (2019), 493-498







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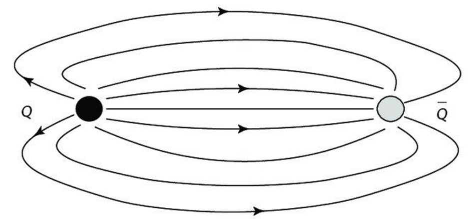

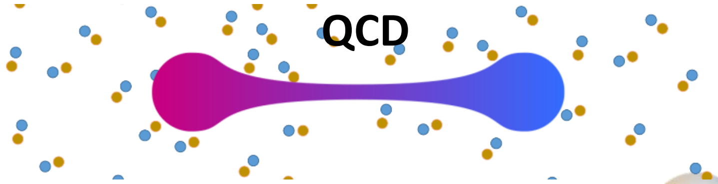

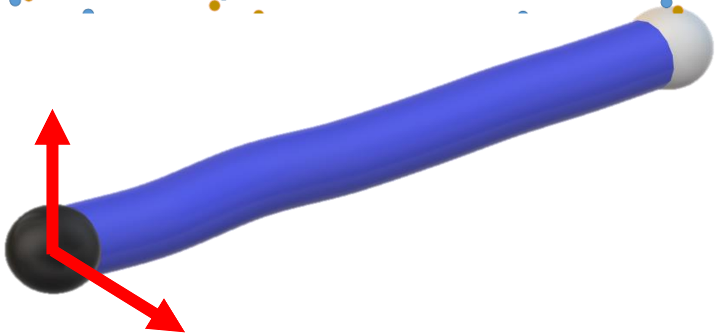

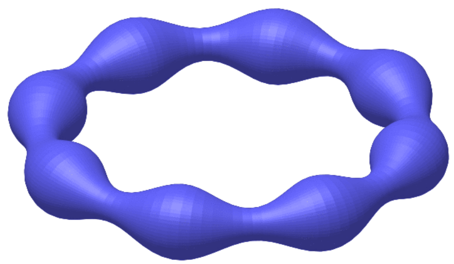

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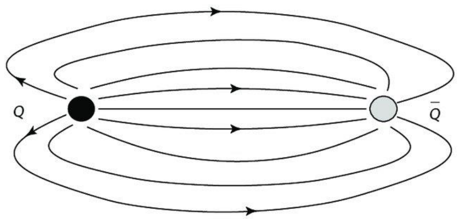
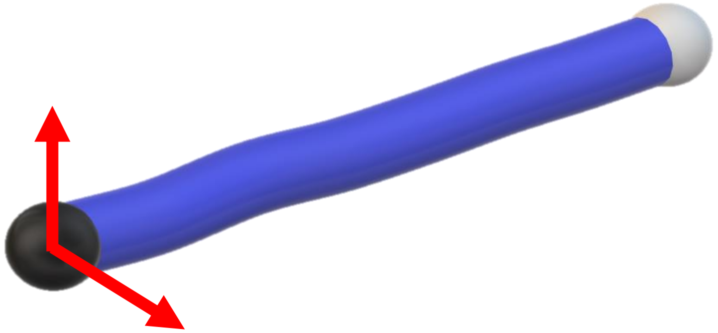
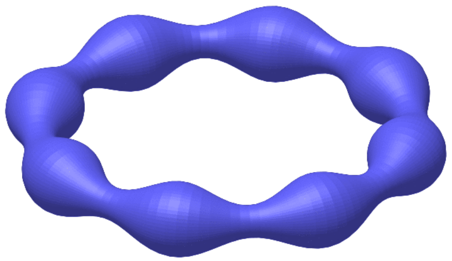


SU(3)





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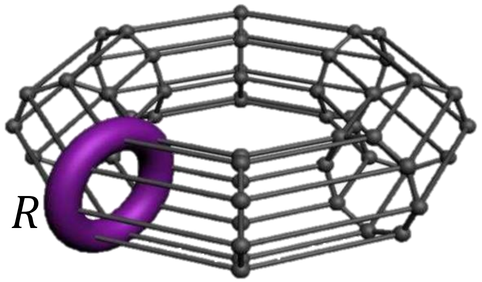
How to investigate

- Strategy:
  - Is there a theoretical description in agreement with Lattice data for the flux-tube?
  - Is there a group of lattice data in striking disagreement with the theory?
  - What does this disagreement teach about the theory? Can the theory be extended?
- Two flux-tube set ups:

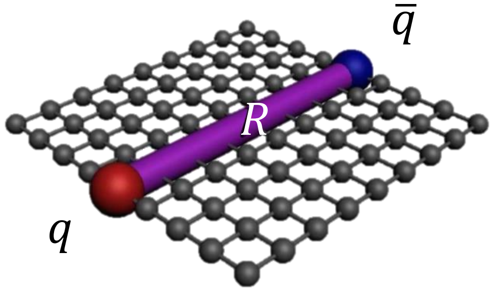
Closed Flux-Tube (Torelon)



Periodic boundaries

Compactification

Open Flux-Tube



[Teper, Barak, AA]

- No boundary terms
- Computationally expensive (length=lattice extent)
- Richer spectrum due to flux-compactification ($N_L - N_R = q$)

[Caselle, Bicudo, Brandt, Kuti, AA]

- One needs to deal with the boundary terms
- Computationally not so expensive

Climbing the Large–N steep

- Pure gauge $SU(3)$ phenomena are also present on flux-tube investigations
  - Glueball – Flux-tube mixing
- A low-energy effective string theoretical description:
  - Cannot capture pure gauge phenomena…
  - How about moving to the large$-N$ limit?
- Hence, we investigate the spectrum of closed flux-tubes in the Large$-N$ limit
- What do we know so far:

Closed Flux-tube	Gauge Group	Open Flux-tube	Gauge Group
arXiv:1007.4720	$SU(3), SU(5)$	arXiv:hep-lat/9608019	$SU(3)$
arXiv:1702.03717	$SU(2-12)$	arXiv:2105.12159	$SU(3)$
arXiv:2112.1121	$SU(3), SU(4), SU(6)$	arXiv:2303.15152	$SU(3)$
arXiv:2205.03642	$SU(3), SU(4), SU(6)$	arXiv:xxxx.xxxxx	$SU(N > 3)$

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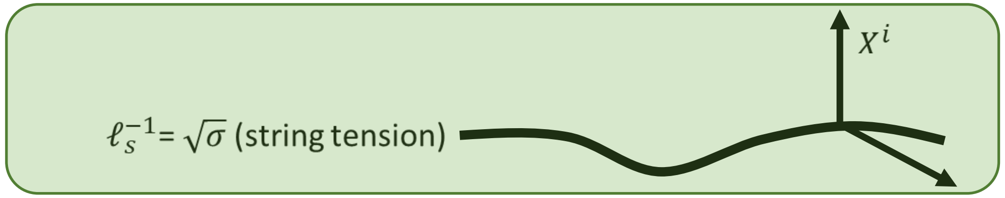
The effective string theory of long strings

- Universal properties of the QCD string studied extensively [Dubovsky, Gorbenko, Aharony]
- Re-parametrisation invariance and D-dimensional target space Poincaré symmetry

$$S = -\int d^2\sigma \sqrt{-\det h_{\alpha\beta}}\,\left[\ell_s^{-2} + \mathcal{R} + a_2 K^2 + b_2 K^{\mu}_{\alpha\beta} K^{\alpha\beta}_{\mu} + O(\ell_s^2)\right]$$



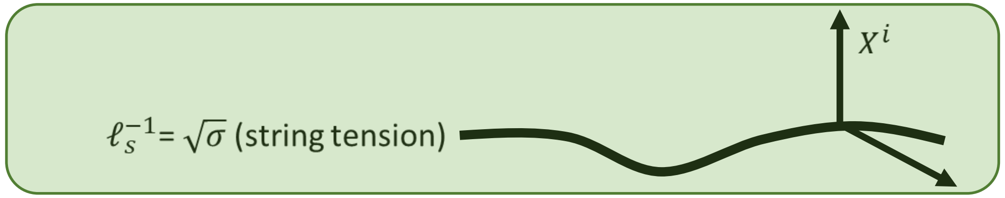
- First non-trivial subleading correction starts at ℓ_s^2 level
- Hence, the perturbative perspectives of the theory are universally determined by Nambu-Goto action

$$S_{\mathrm{NG}} = -\ell_s^{-2} \int d^2\sigma \sqrt{-\det h_{\alpha\beta}}\,, \qquad h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu$$

The Goddard-Goldstone-Rebbi-Thorn spectrum for closed string

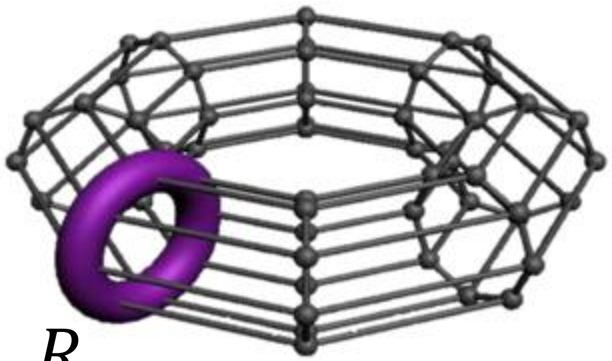
- We quantize the Closed bosonic string (Nambu-Goto)

$$S_{\mathrm{NG}} = -\ell_s^{-2} \int d^2\sigma \sqrt{-\det h_{\alpha\beta}}$$



- The spectrum of a closed bosonic string compactified around a torus is given by:

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$



R

- The spectrum is described by

1. The winding momentum $p_{||} = 2\pi q/R$ with $q = 0, \pm1, \pm2, \ldots$
2. The total contribution of $N_L = \sum_{k>0} n_L(k)k$ and $N_R = \sum_{k'>0} n_R(k')k'$
3. Level matching Constrain: $N_L - N_R = q$

The closed string expansion in $1/R\ell_s^{-1}$

- Topic received contributions since the early 80s

[M. Lüscher ’81, J. Polchinski & A. Strominger ‘90, M. Lüscher & P. Weisz ’04, O. Aharony et al ’07 – 11, S. Dubovsky et al ’12 – 19]

$$
\begin{aligned}
E_n(R) &= l_s^{-1}(Rl_s^{-1}) \\
&+ \frac{4\pi l_s^{-1}}{(Rl_s^{-1})}\left(n - \frac{D-2}{24}\right) \\
&- \frac{8\pi^2 l_s^{-1}}{(Rl_s^{-1})^3}\left(n - \frac{D-2}{24}\right)^2 \\
&+ \frac{32\pi^3 l_s^{-1}}{(Rl_s^{-1})^5}\left(n - \frac{D-2}{24}\right)^3 + \left(\frac{1}{(Rl_s^{-1})^7}\right)
\end{aligned}
$$

Linear Confinement

Lüscher 1980, Polchinski&Strominger 1991

Lüscher&Weisz 2004, Drummond 2004

Aharony&Karzbrun 2009

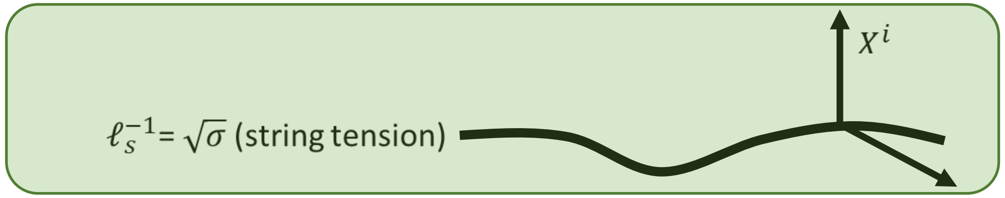
- Relation to the GGRT spectrum:

$$
\begin{aligned}
E_n(R) &= l_s^{-1}(Rl_s^{-1}) + \frac{l_s^{-1}c_1^{\mathrm{GGRT}}}{(Rl_s^{-1})} + \frac{l_s^{-1}c_2^{\mathrm{GGRT}}}{(Rl_s^{-1})^3} + \frac{l_s^{-1}c_3^{\mathrm{GGRT}}}{(Rl_s^{-1})^5} + \left(\frac{1}{(Rl_s^{-1})^7}\right) \\
&= E_{\mathrm{GGRT}}(N_L = 0, N_R = 0, R) + \left(\frac{1}{(Rl_s^{-1})^7}\right)
\end{aligned}
$$

The Nambu-Goto spectrum for open strings

- Quantize the Open bosonic string (Nambu-Goto)

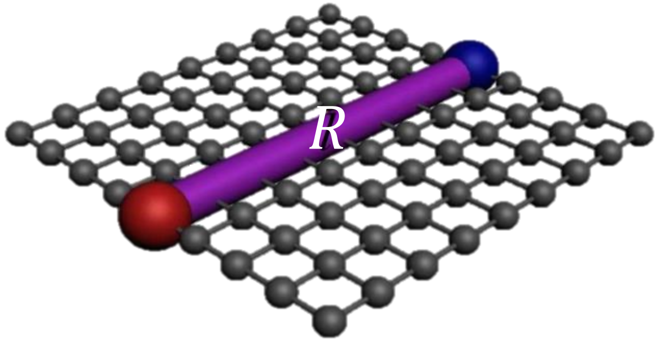
$$S_{\rm NG} = -\ell_s^{-2} \int d^2\sigma \sqrt{-\det h_{\alpha\beta}}$$



- The spectrum of an open bosonic string is given by:

$$E_{\rm NG}(N) = \sqrt{\frac{R^2}{\ell_s^4} + \frac{2\pi}{\ell_s^2}\left(N - \frac{D-2}{24}\right)}$$

[Arvis '83]



- The spectrum is described by

1. The total contribution of $N = \sum_{m=1} m(n_{m+} + n_{m-})$

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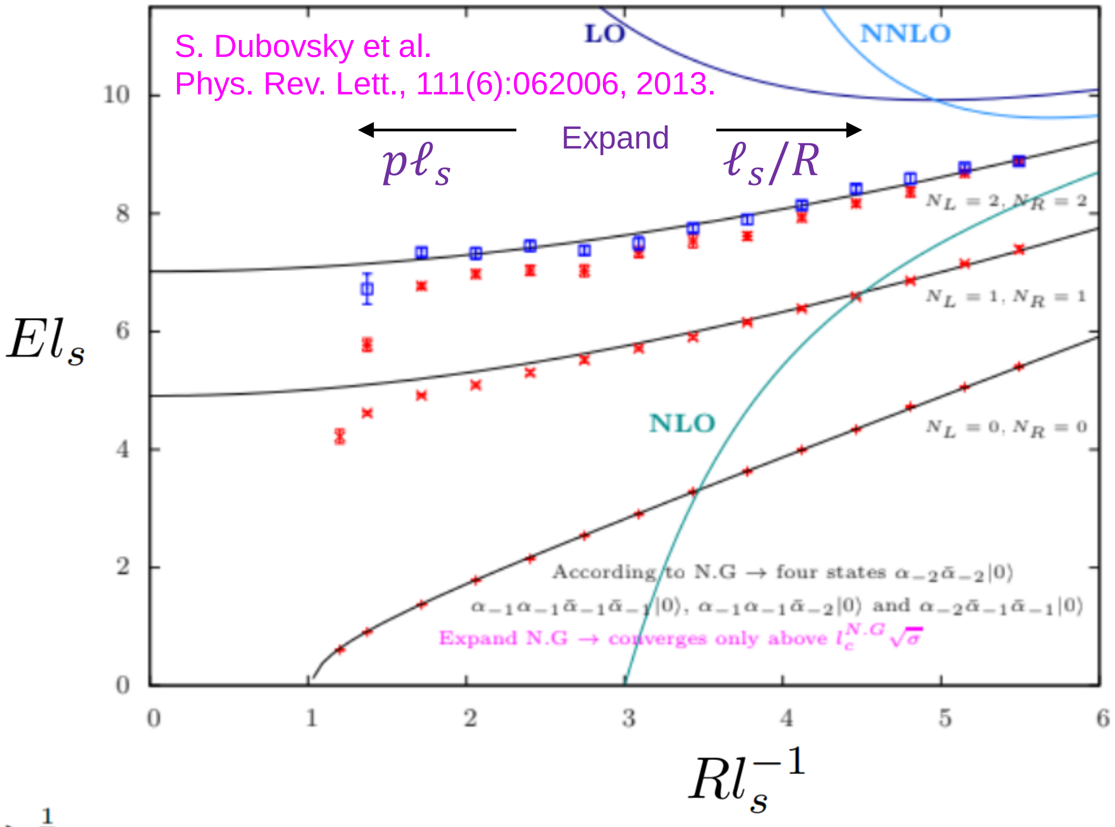
$$\begin{aligned} E_n(R) &= \ell_s^{-1}(R\ell_s^{-1}) && \text{Linear Confinement} \\ &+ \frac{\pi\ell_s^{-1}}{(R\ell_s^{-1})}\left(n - \frac{D-2}{24}\right) && \text{Lüscher 1980, Polchinski\&Strominger 1991} \\ &- \frac{\pi^2\ell_s^{-1}}{2(R\ell_s^{-1})^3}\left(n - \frac{D-2}{24}\right)^2 && \text{Lüscher\&Weisz 2004, Drummond 2004} \\ &+ \frac{\bar{b}_2\pi^3\ell_s^{-1}}{(R\ell_s^{-1})^4}\left(B_n^l - \frac{D-2}{60}\right) && \text{Aharony et al 2010, 2012} \\ &+ \frac{\pi^3\ell_s^{-1}}{16(R\ell_s^{-1})^5}\left(n - \frac{D-2}{24}\right)^3 + \frac{\pi^3\ell_s^{-1}(D-26)}{48(R\ell_s^{-1})^5}C_n^l + \left(\frac{1}{(R\ell_s^{-1})^6}\right) && \text{Aharony et al 2009, 2010} \end{aligned}$$

- Relation to the NG spectrum:

$$E_n(R) = E_{\mathrm{NG}}(n, R) + \frac{\bar{b}_2\pi^3\ell_s^{-1}}{(R\ell_s^{-1})^4}\left(B_n^l - \frac{D-2}{60}\right) + \frac{\pi^3\ell_s^{-1}(D-26)}{48(R\ell_s^{-1})^5}C_n^l + \left(\frac{1}{(R\ell_s^{-1})^6}\right)$$

The expansion in $1/R\ell_s^{-1}$ - the $D = \mathbf{2+1}$ closed flux tube case

N_L, N_R	q	$P_\perp$	String State
$N_L = N_R = 0$	0	$+$	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	$-$	$a_1\lvert 0\rangle$
$N_L = N_R = 1$	0	$+$	$a_1 a_{-1}\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$+$ $-$	$a_1 a_1\lvert 0\rangle$ $a_2\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$+$ $-$	$a_2 a_{-1}\lvert 0\rangle$ $a_1 a_1 a_{-1}\lvert 0\rangle$
$N_L = 3, N_R = 0$	3	$+$ $-$ $-$	$a_2 a_1\lvert 0\rangle$ $a_3\lvert 0\rangle$ $a_1 a_1 a_1\lvert 0\rangle$
$N_L = N_R = 2$	0	$+$ $+$ $-$ $-$	$a_2 a_{-2}\lvert 0\rangle$ $a_1 a_1 a_{-1} a_{-1}\lvert 0\rangle$ $a_2 a_{-1} a_{-1}\lvert 0\rangle$ $a_1 a_1 a_{-2}\lvert 0\rangle$



Expansion valid for $Rl_s^{-1} >> R_C^{\mathrm{GGRT}} l_s^{-1} = \left\{8\pi\left(n - \frac{1}{12}\right)\right\}^{\frac{1}{2}}$

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$

- In $D = 2 + 1$ only string like states have been observed so far
- $D = 3 + 1$ case is more complex and interesting to look for such states

Closed flux tube spectrum from world-sheet scattering

- Thermodynamic Bethe Ansatz (TBA)[Zamolodchikov, 1990]

$\longrightarrow$ finite volume spectrum of a $(1+1)-D$ integrable theory from 2 → 2 scattering

- Leading spectrum is given by integrable theory of $D-2$ scalars with phase shift:

$$e^{2i\delta(s)} = e^{is\ell_s^2/4}$$

- Using TBA leads to the GGRT spectrum
- Phonon scattering amplitudes can be calculated with perturbation theory
- Diagonalization of the S −matrix:

$$2\delta_{sym} = \frac{\ell_s^2 s}{4} - \frac{11\ell_s^4 s^2}{192\pi} + \frac{A-B}{256\pi^2}\ell_s^6 s^3 + O(s^4) \quad \longleftarrow \quad 2^{\pm}$$

$$2\delta_{anti} = \frac{\ell_s^2 s}{4} + \frac{11\ell_s^4 s^2}{192\pi} - \frac{A+B}{256\pi^2}\ell_s^6 s^3 + O(s^4) \quad \longleftarrow \quad 0^{-}$$

$$2\delta_{sing} = \frac{\ell_s^2 s}{4} + \frac{11\ell_s^4 s^2}{192\pi} + \frac{3A-B}{256\pi^2}\ell_s^6 s^3 + O(s^4) \quad \longleftarrow \quad 0^{+}$$

Closed flux tube spectrum from world-sheet scattering

- TBA formulates with the generalized quantization condition around the circle:

ABA

$$p_{li}R + \sum_j 2\delta_{a_i a_j}(p_{li}, p_{rj}) - i\sum_b \int_0^\infty \frac{dq}{2\pi}\frac{d2\delta_{a_i b}(ip_{li}, q)}{dq}\ln\left(1 - e^{-R\epsilon_r^b(q)}\right) = 2\pi N_{li}\,,$$

$$p_{ri}R + \sum_j 2\delta_{a_j a_i}(p_{ri}, p_{lj}) + i\sum_b \int_0^\infty \frac{dq}{2\pi}\frac{d2\delta_{b a_i}(-ip_{ri}, q)}{dq}\ln\left(1 - e^{-R\epsilon_l^b(q)}\right) = 2\pi N_{ri}\,,$$

- The pseudo-energies $\epsilon^a_{l(r)}$ satisfy:

$$\epsilon_l^a(q) = q + \frac{i}{R}\sum_i 2\delta_{ab_i}(q, -ip_{ri}) + \frac{1}{2\pi R}\sum_b \int_0^\infty dq' \frac{d2\delta_{ab}(q,q')}{dq'}\ln\left(1 - e^{-R\epsilon_r^b(q')}\right)$$

$$\epsilon_r^a(q) = q - \frac{i}{R}\sum_i 2\delta_{b_i a}(q, ip_{li}) + \frac{1}{2\pi R}\sum_b \int_0^\infty dq' \frac{d2\delta_{ba}(q,q')}{dq'}\ln\left(1 - e^{-R\epsilon_l^b(q')}\right)$$

- The energy of a state is written as

$$\Delta E = \sum_i p_{li} + \sum_i p_{ri} + \frac{1}{2\pi}\sum_a \int_0^\infty dq \ln\left(1 - e^{-R\epsilon_l^a(q)}\right) + \frac{1}{2\pi}\sum_a \int_0^\infty dq \ln\left(1 - e^{-R\epsilon_r^a(q)}\right)$$

- + Approximations

Closed flux tube spectrum from world-sheet scattering

- $T\bar{T}$ simplifies TBA because the undressed theory is free in the leading order

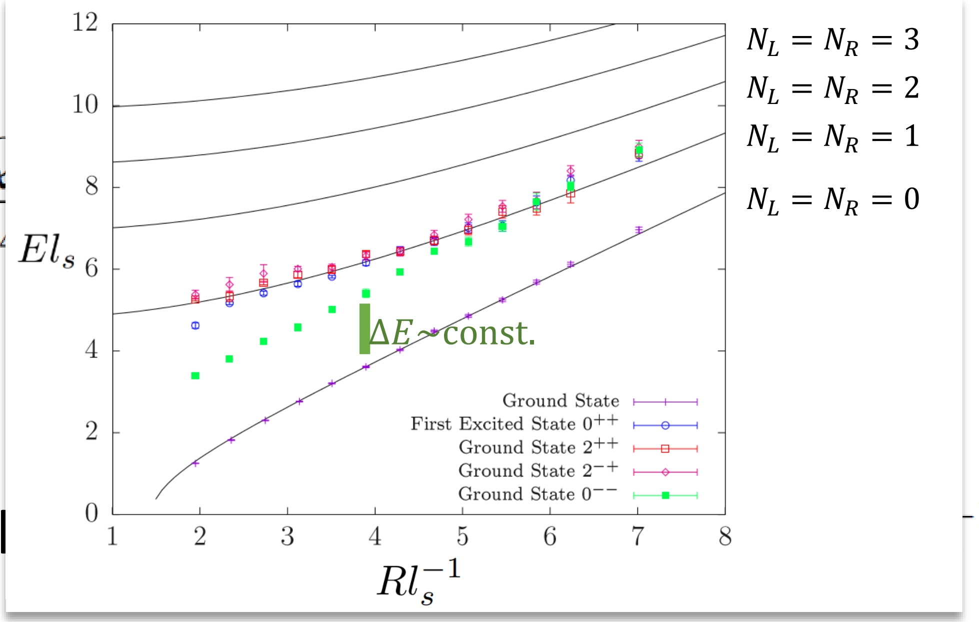
$$E(R,\ell_s) = \frac{1}{\mathcal{R}_0}(R + \frac{\ell_s^2}{2}E(R,\ell_s))E\left(\mathcal{R}_0, 0\right) + \frac{\ell_s^2}{2\mathcal{R}_0}P(R)P\left(\mathcal{R}_0\right)$$

- We neglect all the winding corrections in the undressed spectrum
- Momentum Quantization Condition becomes the Asymptotic Bethe Ansatz (ABA)
- We can investigate the spectrum of phonons – massive excitations using ABA + $T\bar{T}$ deformations. [Chen, Conkey, Dubovsky, Hernández-Chifflet 2018]

- Recipe
  - Start with a world-sheet theory of free phonons (phonons can interact at subleading order in low energy limit) and massive particles.
  - Compute the finite volume spectrum of this theory using ABA.
  - Deform the theory by $T\bar{T}$ operator to the string scale, which will automatically incorporate the axion-interaction at leading order.

The Axionic String Ansatz (ASA)

- Lattice calculations demonstrate that there is one massive resonance
- We add a massive resonance [Dubovsky et al 2013]



$$S_a = \int d^2\sigma \sqrt{-h}\left(-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{Q}{\cdots}\right.$$

- ϕ is a pseudoscalar – the world-sheet axion $t^{\mu\nu}$

- From Monte-Carlo data of $4D$ $SU(3)$ Yang-Mill ... $^{-1}$

- Integrable coupling: $Q_{\text{integrable}} = \sqrt{7/(16\pi)} \approx 0.373$

- Can we describe all states in $D = 3 + 1$ with the worldsheet fields only?

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- We add a massive resonance [Dubovsky et al 2013]

$$S_a = \int d^2\sigma\sqrt{-h}\left(-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{Q_\phi}{4}h^{\alpha\beta}\epsilon_{\mu\nu\lambda\rho}\partial_\alpha t^{\mu\nu}\partial_\beta t^{\lambda\rho}\phi\right)$$

- ϕ is a pseudoscalar – the world-sheet axion $t^{\mu\nu} = \frac{\epsilon^{\alpha\beta}}{\sqrt{-h}}\partial_\alpha X^\mu \partial_\beta X^\nu$

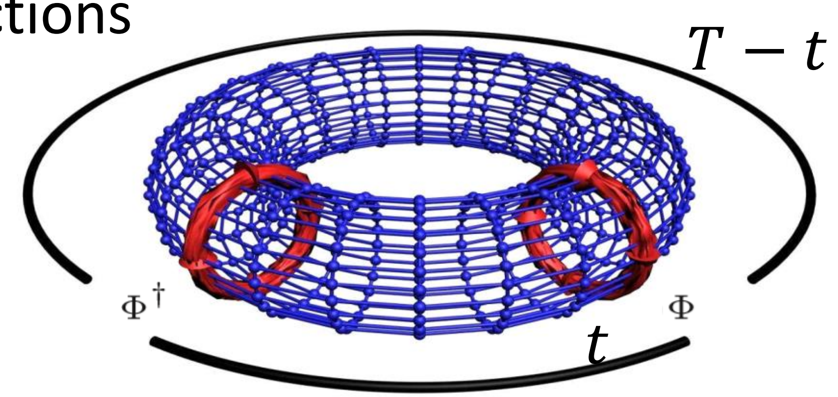
- From Monte-Carlo data of $4D$ $SU(3)$ Yang-Mills: $Q_\phi \approx 0.38 \pm 0.04, \quad m \approx 1.85^{+0.02}_{-0.03}\ell_s^{-1}$

- Integrable coupling: $Q_{\text{integrable}} = \sqrt{7/(16\pi)} \approx 0.373$

- Can we describe all states in $D = 3 + 1$ with the worldsheet fields only?

Extracting the spectrum of close flux tubes

- The spectrum is extracted using torelon correlation functions



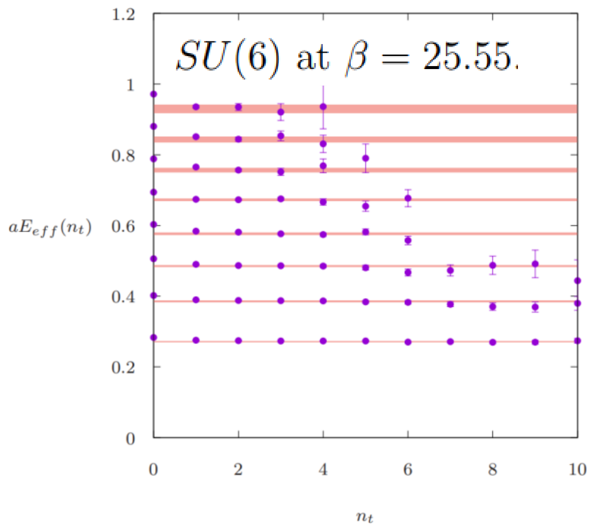
$$\langle \phi^\dagger(t=an_t)\phi(0)\rangle = \langle \phi^\dagger e^{-Han_t}\phi\rangle = \sum_i |c_i|^2 e^{-aE_i n_t}$$
$$\overset{t\to\infty}{=} |c_0|^2 e^{-aE_0 n_t}$$

- Using the effective mass

Provides the ground state

$$\lim_{t\to\infty}\left[-\ln\left(\frac{C(t)}{C(t-a)}\right)\right] = aE_0$$

Example of effective masses



- For excitation spectrum we use the GEVP

[Presentation by Francesco Giacomo Knechtli]

Extracting the spectrum of open flux tubes

- The spectrum is extracted using Wilson loops $W(r,t)$

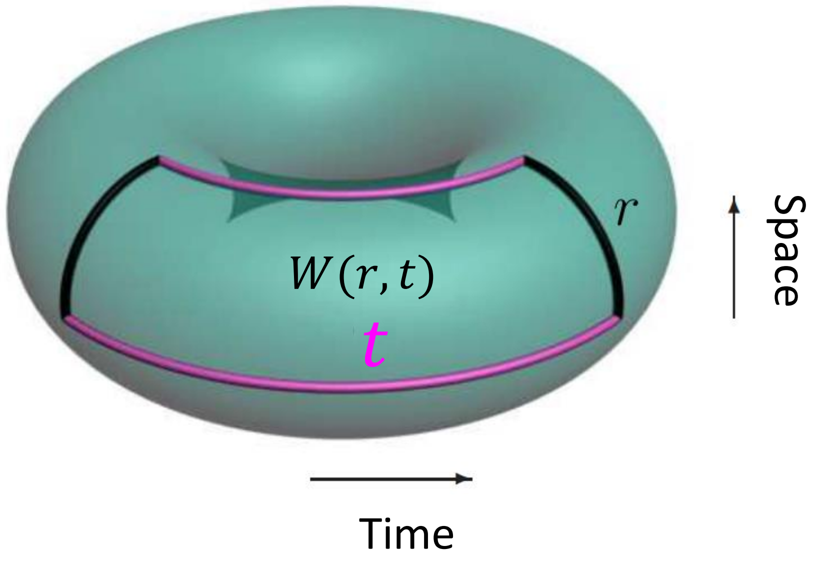
$$\langle W(r,t)\rangle = \sum_{i=0}^{\infty} A_n e^{-E_n(r)t}$$

$$\frac{W(r,t)}{W(r,t+1)} = e^{E_0}\frac{1 + A_1 e^{-(E_1-E_0)t} + \ldots}{1 + A_1 e^{-(E_1-E_0)(t+1)} + \ldots}.$$

- We use the effective energy:

$$E_0(r,t) = \lim_{t\to\infty} \ln \frac{W(r,t)}{W(r,t+1)}$$

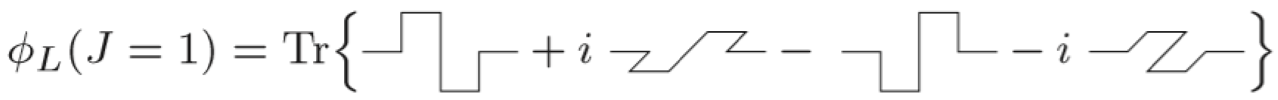
- For excitation spectrum we use once again the GEVP



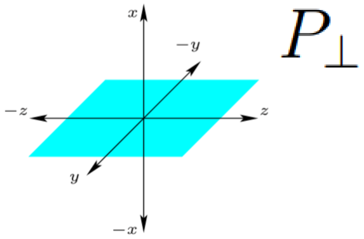
The Quantum Numbers of closed flux tubes

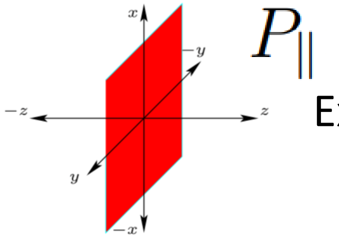
Quantum numbers for the closed string

- Spin J: $\phi_L(J) = \sum_n e^{iJn\frac{\pi}{2}} \phi_{n\frac{\pi}{2}}$ Example

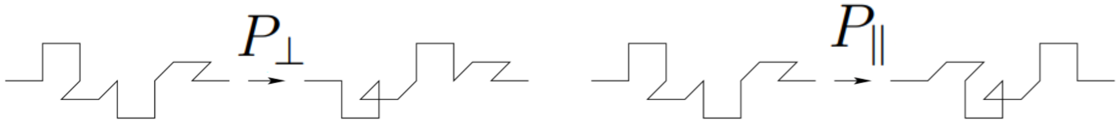


- Parity:



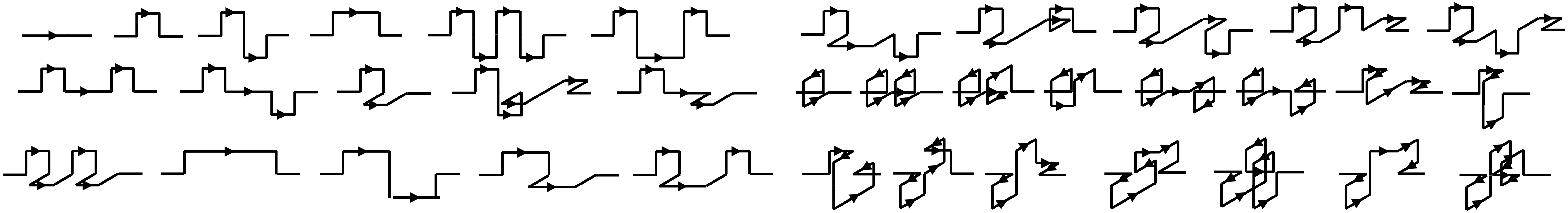


Example



We build operators described by the quantum numbers of J, $P_\perp$, $P_\parallel$

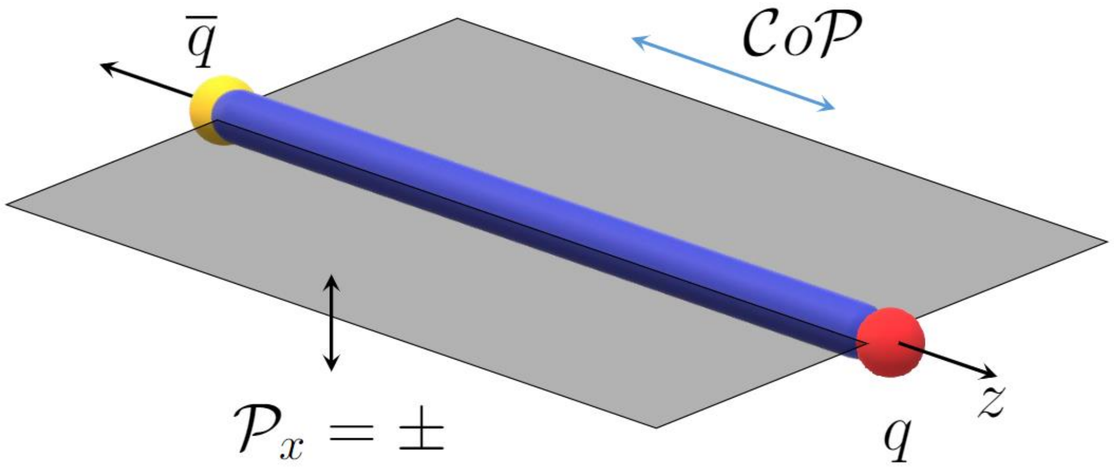
- We use a large basis of operators with transverse deformations:



The Quantum Numbers of open flux tubes

Quantum numbers for the open flux tube

- Spin Λ: Projection of angular momentum J onto the charge axis $J \cdot \hat{R}$
- Parity and Charge Conjugation : $\mathcal{C}\circ\mathcal{P}$, with eigenvalues $g(u) \equiv 1(-1)$
- For $\Lambda = 0$ we have one more quantum number $\mathcal{P}_x = \pm$



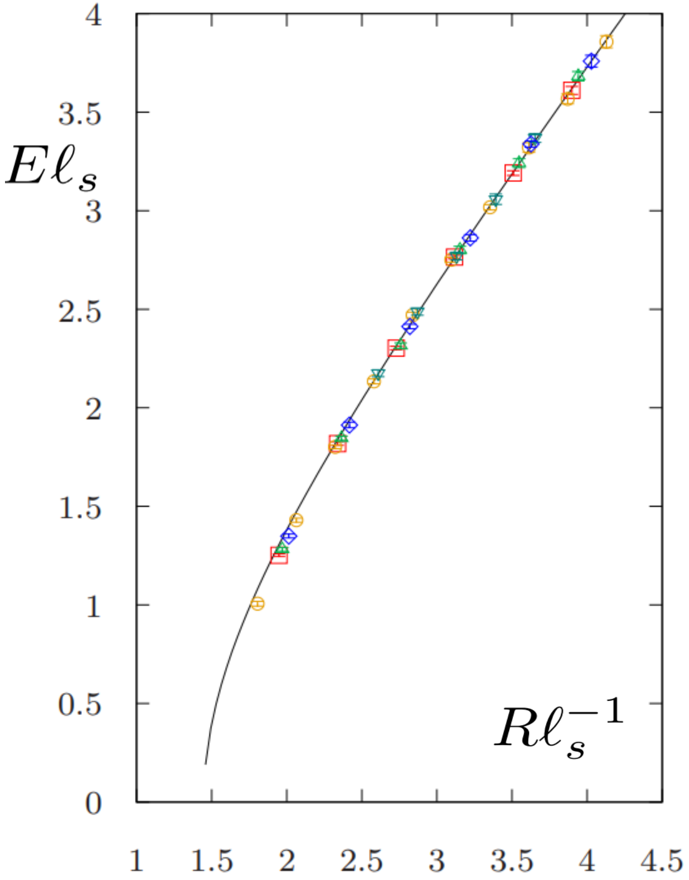
We project onto the irreducible representations

- Σ_g^+, Σ_g^-, Σ_u^+ and Σ_u^-, where $\Sigma \to \Lambda = 0$
- Π_g and Π_u, where $\Pi \to \Lambda = 1$
- Δ_g and Δ_u, where $\Delta \to \Lambda = 2$

Closed flux tube results: The absolute ground state $N_L = N_R = 0$

N_L, N_R	$\lvert J \rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$(a_1^+ a_{-1}^- + a_1^- a_{-1}^+)\lvert 0\rangle$
	0	−	−	$(a_1^+ a_{-1}^- - a_1^- a_{-1}^+)\lvert 0\rangle$
	2	+	+	$(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-)\lvert 0\rangle$
	2	−	+	$(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$(a_2^+ a_{-2}^- + a_2^- a_{-2}^+)\lvert 0\rangle$
	0	−	−	$(a_2^+ a_{-2}^- - a_2^- a_{-2}^+)\lvert 0\rangle$
	0	+	+	$(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+)\lvert 0\rangle$
	0	−	−	$(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+)\lvert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^-\lvert 0\rangle$
	1	±	+	$\left[(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+) \pm (a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+)\right]\lvert 0\rangle$
	1	±	−	$\left[(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+) \pm (a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+)\right]\lvert 0\rangle$
	1	±	+	$\left[(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+) \pm (a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-)\right]\lvert 0\rangle$
	1	±	−	$\left[(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+) \pm (a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-)\right]\lvert 0\rangle$
	2	+	+	$(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-)\lvert 0\rangle$
	2	−	+	$(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-)\lvert 0\rangle$
	2	+	+	$\left[(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+) + (a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+)\right]\lvert 0\rangle$
	2	+	−	$\left[(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+) - (a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+)\right]\lvert 0\rangle$
	2	−	+	$\left[(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+) + (a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+)\right]\lvert 0\rangle$
	2	−	−	$\left[(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+) - (a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+)\right]\lvert 0\rangle$
	3	±	+	$\left[(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+) \pm (a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-)\right]\lvert 0\rangle$
	3	±	−	$\left[(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+) \pm (a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-)\right]\lvert 0\rangle$
	4	+	+	$(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-)\lvert 0\rangle$
	4	−	+	$(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-)\lvert 0\rangle$

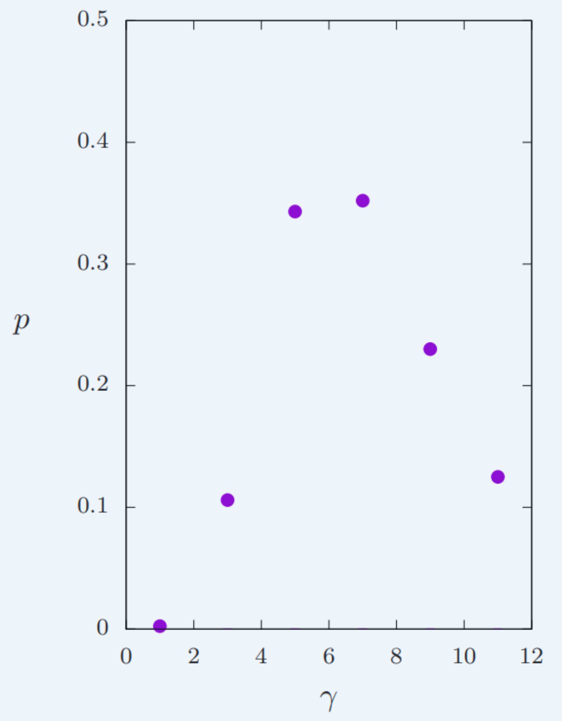
$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$



$SU(3), \beta = 6.0625$ | $SU(5), \beta = 17.63$ | $SU(6), \beta = 25.55$

$SU(3), \beta = 6.338$ | $SU(5), \beta = 18.375$

$$E(R) = E_{\mathrm{GGRT}}(R) - \frac{\tilde{c}\, l_s^{-1}}{(R l_s^{-1})^{\gamma}}$$

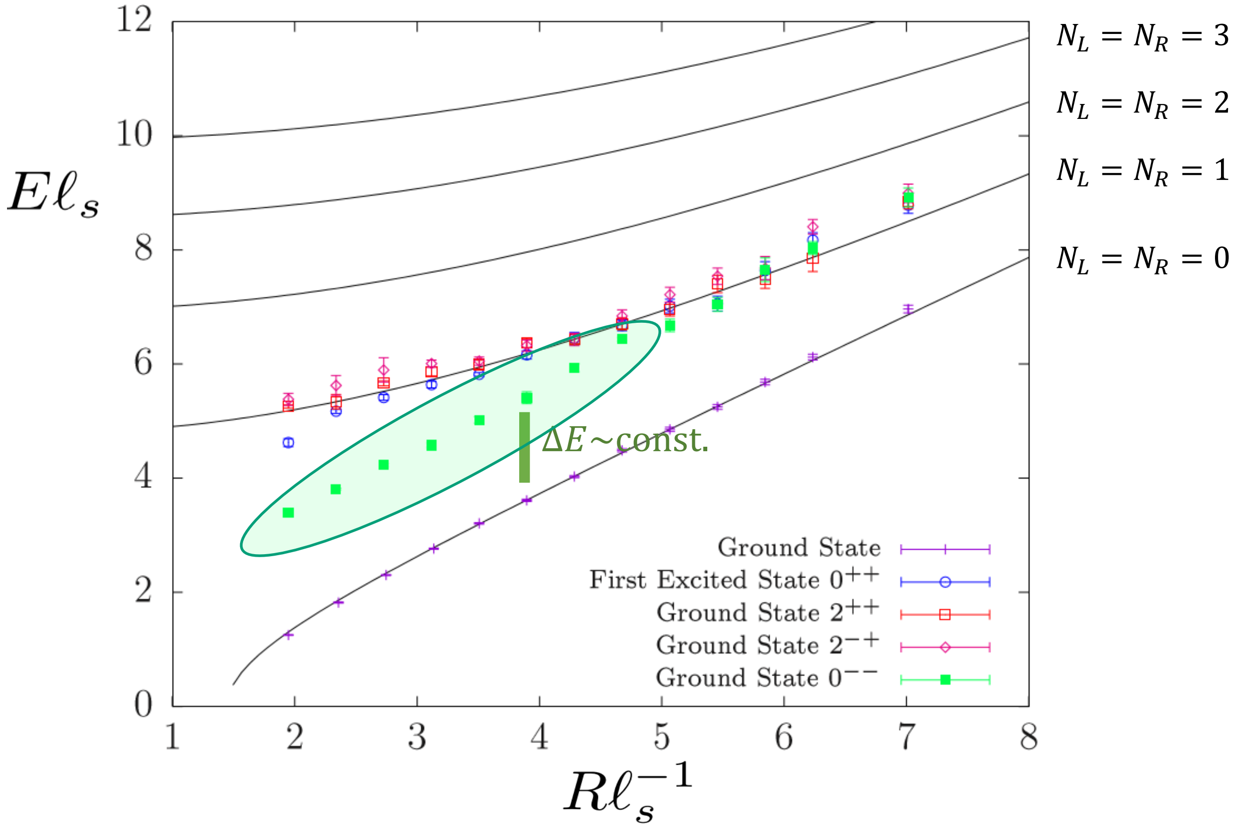


Closed flux tube results: the GGRT state with $N_L = N_R = 1$

N_L, N_R	$\lvert J \rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	+	+	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^-\lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	2	+	+	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	−	+	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	+	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	+	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	3	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	3	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	4	+	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$
	4	−	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$

Results for $SU(3)$ and $\beta = 6.0625$

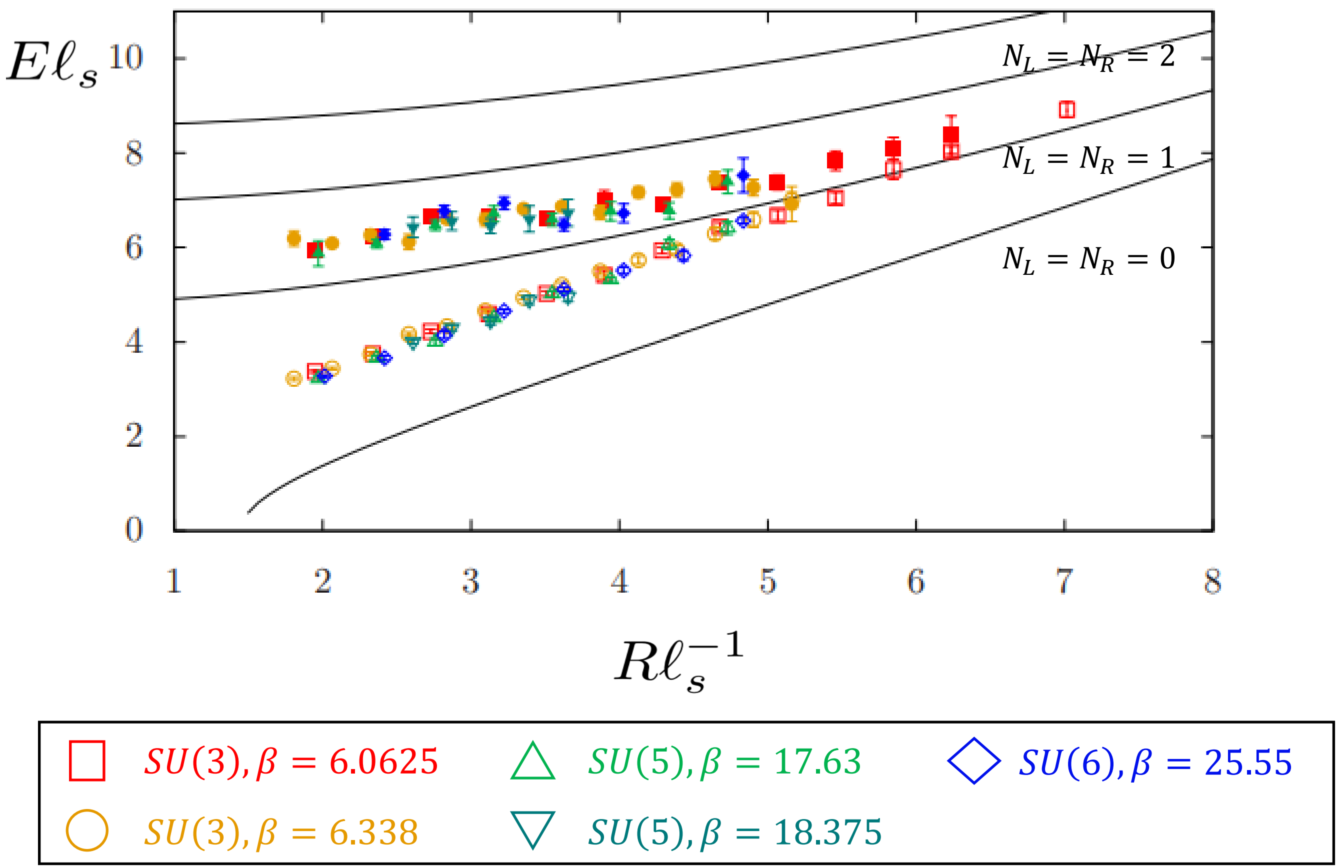


Closed flux tube results: the GGRT state with $N_L = N_R = 1$

N_L, N_R	$\lvert J \rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	$+$	$+$	$\lvert 0\rangle$
$N_L = N_R = 1$	0	$+$	$+$	$(a_1^+ a_{-1}^- + a_1^- a_{-1}^+)\lvert 0\rangle$
	0	$-$	$-$	$(a_1^+ a_{-1}^- - a_1^- a_{-1}^+)\lvert 0\rangle$
	2	$+$	$+$	$(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-)\lvert 0\rangle$
	2	$-$	$+$	$(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	$+$	$+$	$(a_2^+ a_{-2}^- + a_2^- a_{-2}^+)\lvert 0\rangle$
	0	$-$	$-$	$(a_2^+ a_{-2}^- - a_2^- a_{-2}^+)\lvert 0\rangle$
	0	$+$	$+$	$(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+)\lvert 0\rangle$
	0	$-$	$-$	$(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+)\lvert 0\rangle$
	0	$+$	$+$	$a_1^+ a_1^- a_{-1}^+ a_{-1}^-\lvert 0\rangle$
	1	$\pm$	$+$	$\left[(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+) \pm (a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+)\right]\lvert 0\rangle$
	1	$\pm$	$-$	$\left[(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+) \pm (a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+)\right]\lvert 0\rangle$
	1	$\pm$	$+$	$\left[(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+) \pm (a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-)\right]\lvert 0\rangle$
	1	$\pm$	$-$	$\left[(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+) \pm (a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-)\right]\lvert 0\rangle$
	2	$+$	$+$	$(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-)\lvert 0\rangle$
	2	$-$	$+$	$(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-)\lvert 0\rangle$
	2	$+$	$+$	$\left[(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+) + (a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+)\right]\lvert 0\rangle$
	2	$+$	$-$	$\left[(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+) - (a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+)\right]\lvert 0\rangle$
	2	$-$	$+$	$\left[(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+) + (a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+)\right]\lvert 0\rangle$
	2	$-$	$-$	$\left[(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+) - (a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+)\right]\lvert 0\rangle$
	3	$\pm$	$+$	$\left[(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+) \pm (a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-)\right]\lvert 0\rangle$
	3	$\pm$	$-$	$\left[(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+) \pm (a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-)\right]\lvert 0\rangle$
	4	$+$	$+$	$(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-)\lvert 0\rangle$
	4	$-$	$+$	$(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-)\lvert 0\rangle$

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$

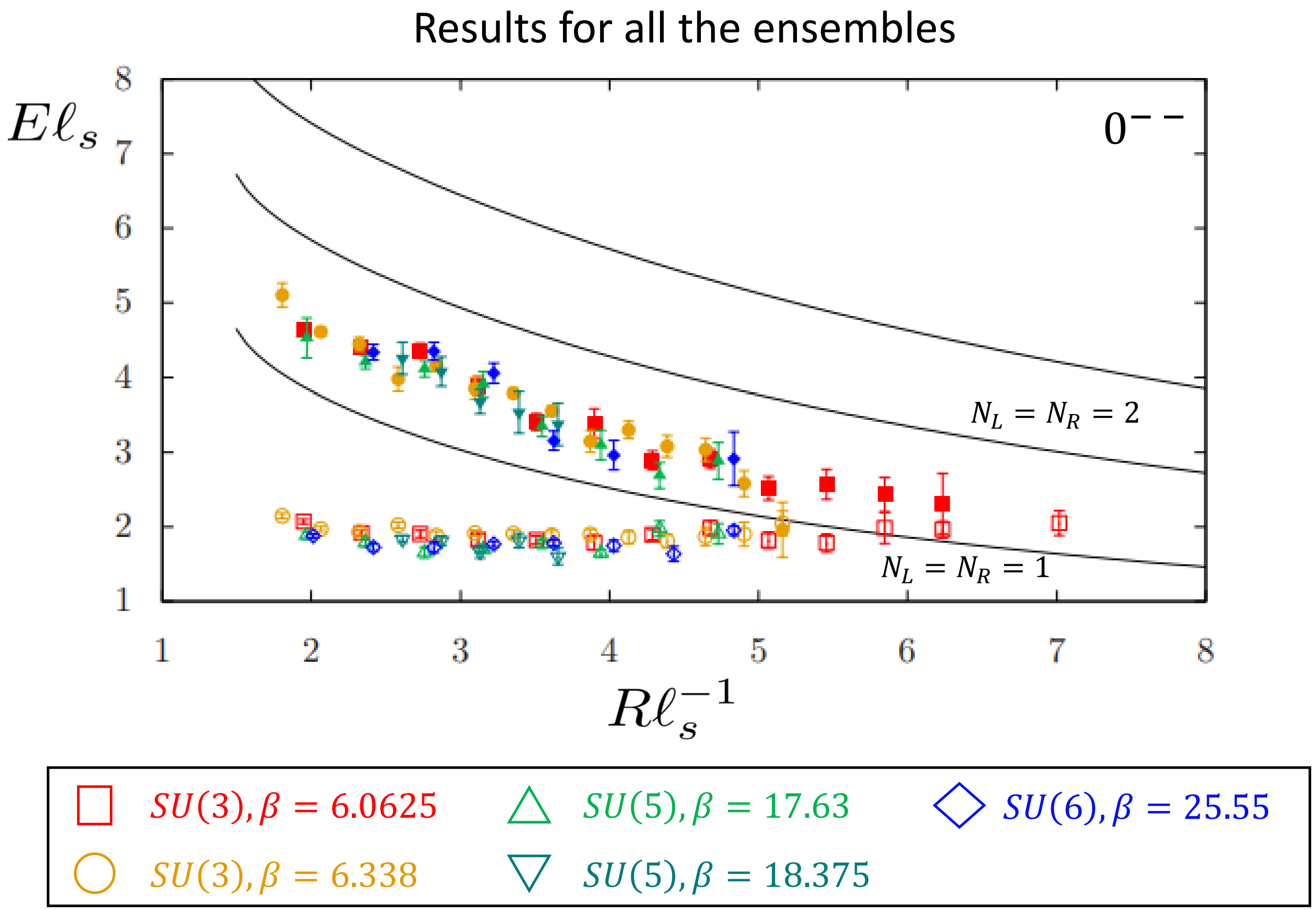
Results for all the ensembles



Closed flux tube results: the GGRT state with $N_L = N_R = 1$

N_L, N_R	$\lvert J\rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	$+$	$+$	$\lvert 0\rangle$
$N_L = N_R = 1$	0	$+$	$+$	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
	0	$-$	$-$	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
	2	$+$	$+$	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
	2	$-$	$+$	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	$+$	$+$	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	$-$	$-$	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	$+$	$+$	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	$-$	$-$	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	$+$	$+$	$a_1^+ a_1^- a_{-1}^+ a_{-1}^-\lvert 0\rangle$
	1	$\pm$	$+$	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	$\pm$	$-$	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	$\pm$	$+$	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	1	$\pm$	$-$	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	2	$+$	$+$	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	$-$	$+$	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	$+$	$+$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	$+$	$-$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	$-$	$+$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	$-$	$-$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	3	$\pm$	$+$	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	3	$\pm$	$-$	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	4	$+$	$+$	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$
	4	$-$	$+$	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$

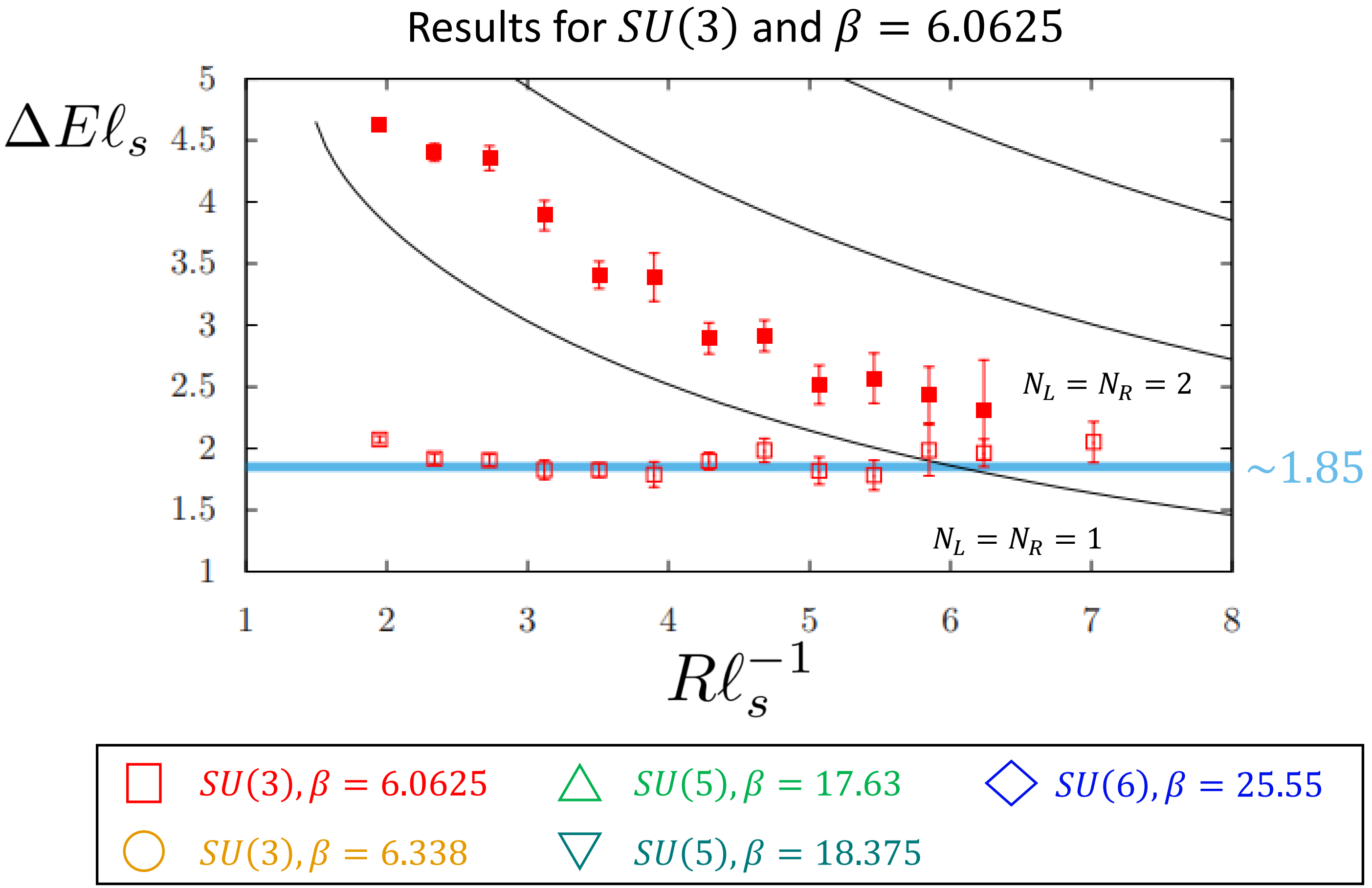
$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2 (N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$



Closed flux tube results: the GGRT state with $N_L = N_R = 1$

N_L, N_R	$\lvert J \rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$(a_1^+ a_{-1}^- + a_1^- a_{-1}^+)\lvert 0\rangle$
	0	−	−	$(a_1^+ a_{-1}^- - a_1^- a_{-1}^+)\lvert 0\rangle$
	2	+	+	$(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-)\lvert 0\rangle$
	2	−	+	$(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$(a_2^+ a_{-2}^- + a_2^- a_{-2}^+)\lvert 0\rangle$
	0	−	−	$(a_2^+ a_{-2}^- - a_2^- a_{-2}^+)\lvert 0\rangle$
	0	+	+	$(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+)\lvert 0\rangle$
	0	−	−	$(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+)\lvert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^-\lvert 0\rangle$
	1	±	+	$\left[(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+) \pm (a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+)\right]\lvert 0\rangle$
	1	±	−	$\left[(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+) \pm (a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+)\right]\lvert 0\rangle$
	1	±	+	$\left[(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+) \pm (a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-)\right]\lvert 0\rangle$
	1	±	−	$\left[(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+) \pm (a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-)\right]\lvert 0\rangle$
	2	+	+	$(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-)\lvert 0\rangle$
	2	−	+	$(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-)\lvert 0\rangle$
	2	+	+	$\left[(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+) + (a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+)\right]\lvert 0\rangle$
	2	+	−	$\left[(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+) - (a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+)\right]\lvert 0\rangle$
	2	−	+	$\left[(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+) + (a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+)\right]\lvert 0\rangle$
	2	−	−	$\left[(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+) - (a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+)\right]\lvert 0\rangle$
	3	±	+	$\left[(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+) \pm (a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-)\right]\lvert 0\rangle$
	3	±	−	$\left[(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+) \pm (a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-)\right]\lvert 0\rangle$
	4	+	+	$(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-)\lvert 0\rangle$
	4	−	+	$(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-)\lvert 0\rangle$

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$

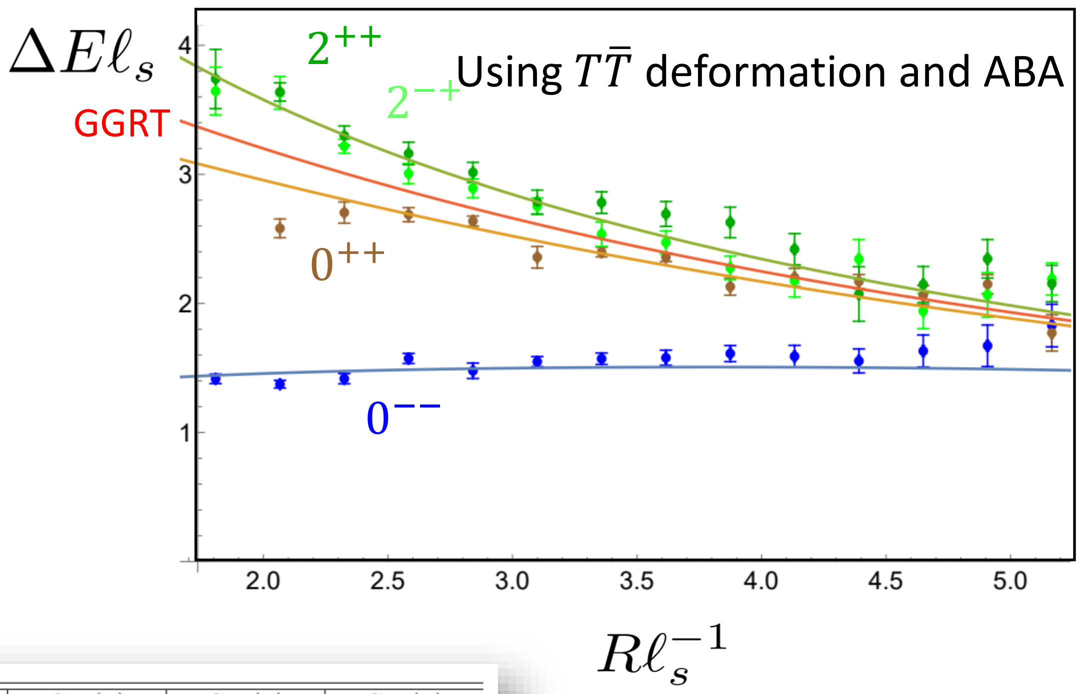


Closed flux tube results: the GGRT state with $N_L = N_R = 1$

N_L, N_R	$\lvert J\rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$(a_1^+a_{-1}^- + a_1^-a_{-1}^+)\lvert 0\rangle$
	0	−	−	$(a_1^+a_{-1}^- - a_1^-a_{-1}^+)\lvert 0\rangle$
	2	+	+	$(a_1^+a_{-1}^+ + a_1^-a_{-1}^-)\lvert 0\rangle$
	2	−	+	$(a_1^+a_{-1}^+ - a_1^-a_{-1}^-)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$(a_2^+a_{-2}^- + a_2^-a_{-2}^+)\lvert 0\rangle$
	0	−	−	$(a_2^+a_{-2}^- - a_2^-a_{-2}^+)\lvert 0\rangle$
	0	+	+	$(a_1^+a_1^+a_{-1}^-a_{-1}^- + a_1^-a_1^-a_{-1}^+a_{-1}^+)\lvert 0\rangle$
	0	−	−	$(a_1^+a_1^+a_{-1}^-a_{-1}^- - a_1^-a_1^-a_{-1}^+a_{-1}^+)\lvert 0\rangle$
	0	+	+	$a_1^+a_1^-a_{-1}^+a_{-1}^-\lvert 0\rangle$
	1	±	+	$[(a_1^+a_1^+a_{-2}^- + a_2^-a_{-1}^+a_{-1}^+) \pm (a_1^-a_1^-a_{-2}^+ + a_2^-a_{-1}^+a_{-1}^+)]\lvert 0\rangle$
	1	±	−	$[(a_1^+a_1^+a_{-2}^- - a_2^-a_{-1}^+a_{-1}^+) \pm (a_1^-a_1^-a_{-2}^+ - a_2^-a_{-1}^+a_{-1}^+)]\lvert 0\rangle$
	1	±	+	$[(a_1^+a_1^-a_{-2}^+ + a_2^+a_{-1}^-a_{-1}^+) \pm (a_1^-a_1^+a_{-2}^- + a_2^-a_{-1}^+a_{-1}^-)]\lvert 0\rangle$
	1	±	−	$[(a_1^+a_1^-a_{-2}^+ - a_2^+a_{-1}^-a_{-1}^+) \pm (a_1^-a_1^+a_{-2}^- - a_2^-a_{-1}^+a_{-1}^-)]\lvert 0\rangle$
	2	+	+	$(a_2^+a_{-2}^+ + a_2^-a_{-2}^-)\lvert 0\rangle$
	2	−	+	$(a_2^+a_{-2}^+ - a_2^-a_{-2}^-)\lvert 0\rangle$
	2	+	+	$[(a_1^+a_1^+a_{-1}^+a_{-1}^- + a_1^-a_1^-a_{-1}^-a_{-1}^+) + (a_1^+a_1^-a_{-1}^-a_{-1}^- + a_1^-a_1^+a_{-1}^+a_{-1}^+)]\lvert 0\rangle$
	2	+	−	$[(a_1^+a_1^+a_{-1}^+a_{-1}^- + a_1^-a_1^-a_{-1}^-a_{-1}^+) - (a_1^+a_1^-a_{-1}^-a_{-1}^- + a_1^-a_1^+a_{-1}^+a_{-1}^+)]\lvert 0\rangle$
	2	−	+	$[(a_1^+a_1^+a_{-1}^+a_{-1}^- - a_1^-a_1^-a_{-1}^-a_{-1}^+) + (a_1^+a_1^-a_{-1}^-a_{-1}^- - a_1^-a_1^+a_{-1}^+a_{-1}^+)]\lvert 0\rangle$
	2	−	−	$[(a_1^+a_1^+a_{-1}^+a_{-1}^- - a_1^-a_1^-a_{-1}^-a_{-1}^+) - (a_1^+a_1^-a_{-1}^-a_{-1}^- - a_1^-a_1^+a_{-1}^+a_{-1}^+)]\lvert 0\rangle$
	3	±	+	$[(a_1^+a_1^+a_{-2}^+ + a_2^+a_{-1}^+a_{-1}^+) \pm (a_1^-a_1^-a_{-2}^- + a_2^-a_{-1}^-a_{-1}^-)]\lvert 0\rangle$
	3	±	−	$[(a_1^+a_1^+a_{-2}^+ - a_2^+a_{-1}^+a_{-1}^+) \pm (a_1^-a_1^-a_{-2}^- - a_2^-a_{-1}^-a_{-1}^-)]\lvert 0\rangle$
	4	+	+	$(a_1^+a_1^+a_{-1}^+a_{-1}^+ + a_1^-a_1^-a_{-1}^-a_{-1}^-)\lvert 0\rangle$
	4	−	+	$(a_1^+a_1^+a_{-1}^+a_{-1}^+ - a_1^-a_1^-a_{-1}^-a_{-1}^-)\lvert 0\rangle$

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L-N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$

Results for $SU(3)$ and $\beta = 6.338$



	$SU(3)$	$SU(5)$	$SU(6)$
2^{++}			
$m\ell_s$	1.812(16)	1.647(23)	1.653(17)
Q_ϕ	0.377(7)	0.387(10)	0.385(10)
2^{+-}			
$m\ell_s$	1.811(16)	1.648(23)	1.656(17)
Q_ϕ	0.354(6)	0.346(7)	0.337(10)

$$2\delta_{res}(p) = 2\sigma_2 \tan^{-1}\left(\frac{8Q_\phi^2\ell_s^4p^6}{m^2-4p^2}\right) + \sigma_1\frac{8Q_\phi^2\ell_s^4p^6}{m^2+4p^2}$$

The Axionic States

N_L, N_R	$\lvert J\rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	+	+	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^-\lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	2	+	+	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	−	+	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	+	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	+	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	3	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	3	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	4	+	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$
	4	−	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$

N_L, N_R	$\lvert J\rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = 1, N_R = 0$	1	±		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	+		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
	0	−		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
	1	±		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
	1	±		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
	2	+		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
	2	−		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
	3	±		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

Operator that creates a massive excitation with momentum $\frac{2\pi k}{R}$

$$A_k \approx a_{k+1}^+ a_{-1}^- - a_{k+1}^- a_{-1}^+ \text{ for } k \geq 0.$$

N_L, N_R	$\lvert J\rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 1$	0	−	−	$A_0\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	−		$A_1\lvert 0\rangle$
	1	±		$A_0\left(a_1^+ \mp a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$A_0 A_0\lvert 0\rangle$
	0	−	−	$A_0\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
	1	±	+	$\left[A_1\left(a_{-1}^+ \mp a_{-1}^-\right) - A_{-1}\left(a_1^+ \mp a_1^-\right)\right]\lvert 0\rangle$
	1	±	−	$\left[A_1\left(a_{-1}^+ \mp a_{-1}^-\right) + A_{-1}\left(a_1^+ \mp a_1^-\right)\right]\lvert 0\rangle$
	2	+	−	$A_0\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
	2	−	−	$A_0\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 3, N_R = 1$	0	−		$A_2\lvert 0\rangle$
	0	−		$A_0 a_1^+ a_1^-\lvert 0\rangle$
	1	±		$A_1\left(a_1^+ \mp a_1^-\right)\lvert 0\rangle$
	1	±		$A_0\left(a_2^+ \mp a_2^-\right)\lvert 0\rangle$
	2	+		$A_0\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\lvert 0\rangle$
	2	−		$A_0\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 3, N_R = 2$	0	+		$A_1 A_0\lvert 0\rangle$
	0	−		$A_{-1} a_1^+ a_1^-\lvert 0\rangle$
	0	−		$A_1\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
	0	−		$A_0\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 3, N_R = 3$	0	+	+	$A_1 A_{-1}\lvert 0\rangle$
	0	−	−	$A_0 A_0 A_0\lvert 0\rangle$
	0	+	+	$A_0\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	−	−	$A_0\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	−	+	$\left[A_{-1}\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right) - A_1\left(a_{-2}^+ a_1^- + a_{-2}^- a_1^+\right)\right]\lvert 0\rangle$
	0	−	−	$\left[A_{-1}\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right) + A_1\left(a_{-2}^+ a_1^- + a_{-2}^- a_1^+\right)\right]\lvert 0\rangle$
	0	−	+	$\left(A_2 a_{-1}^+ a_{-1}^- - A_{-2} a_1^+ a_1^-\right)\lvert 0\rangle$
	0	−	−	$\left(A_2 a_{-1}^+ a_{-1}^- + A_{-2} a_1^+ a_1^-\right)\lvert 0\rangle$

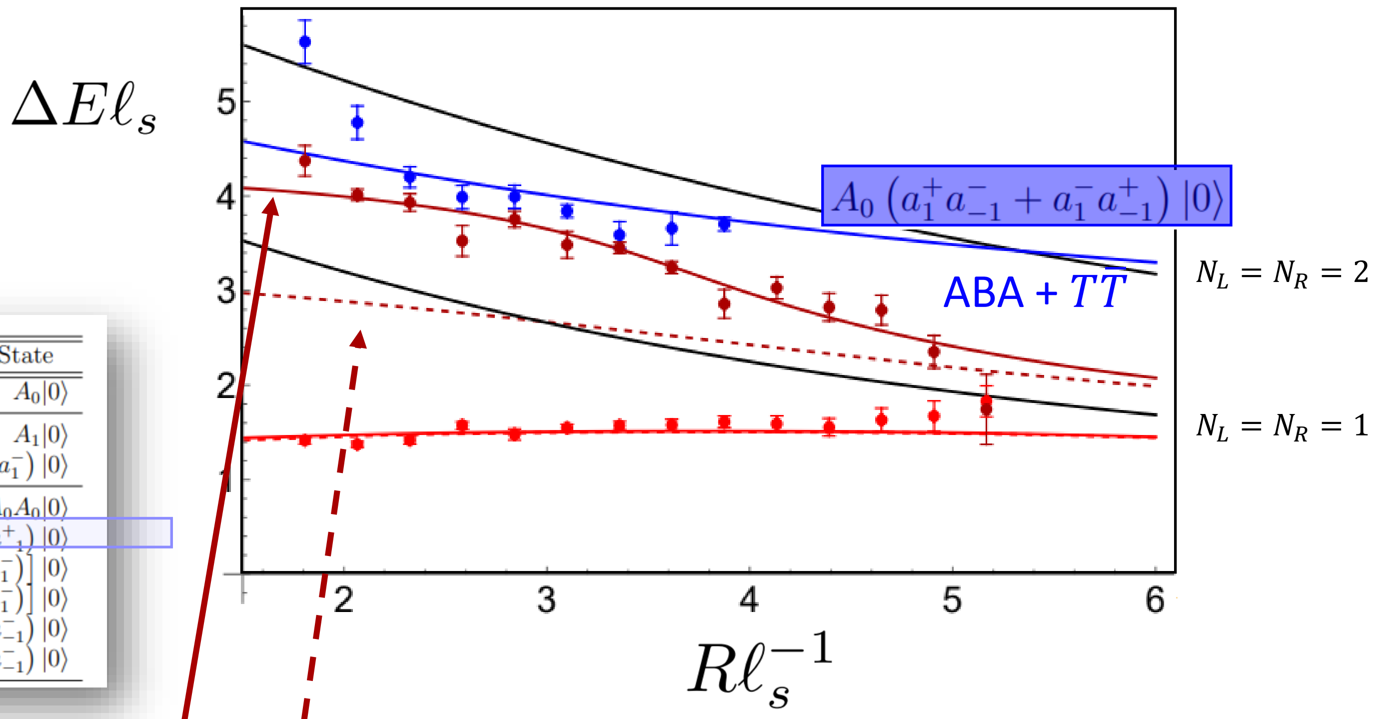
Closed flux tube results: the $\mathbf{0}^{--}$ state and the axion

N_L, N_R	$\lvert J \rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	+	+	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^- \lvert 0\rangle$
	1			
	1			
	1			
	1			
	2			
	2			
	2			
	2			
	2			
	2			
	3			
	3			
	4			
	4	−	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$

N_L, N_R	$\lvert J \rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 1$	0	−	−	$A_0\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	−		$A_1\lvert 0\rangle$
	1	±		$A_0\left(a_1^+ \mp a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$A_0 A_0\lvert 0\rangle$
	0	−	−	$A_0\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
	1	±	+	$\left[A_1\left(a_{-1}^+ \mp a_{-1}^-\right) - A_{-1}\left(a_1^+ \mp a_1^-\right)\right]\lvert 0\rangle$
	1	±	−	$\left[A_1\left(a_{-1}^+ \mp a_{-1}^-\right) + A_{-1}\left(a_1^+ \mp a_1^-\right)\right]\lvert 0\rangle$
	2	+	−	$A_0\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
	2	−	−	$A_0\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2 (N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$

Results for $SU(3)$ and $\beta = 6.338$



TBA with no high orders in s

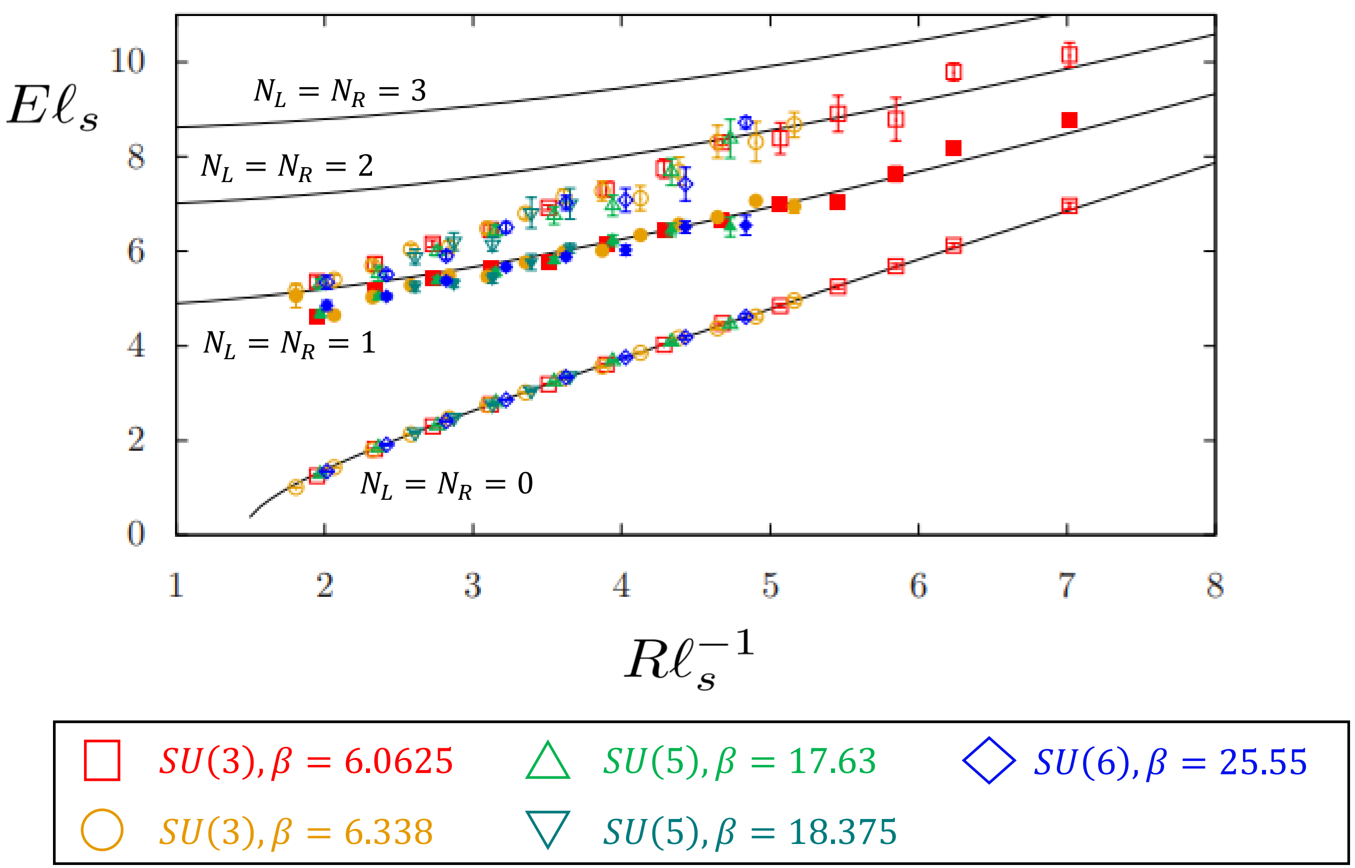
TBA with higher orders $s^4 l_s^{-8}$ $\quad 2\delta_{anti}(s) = \frac{s}{4} + \frac{11 s^2}{192\pi} + a_3 s^3 + a_4 s^4 + \cdots$

Closed flux tube results: the $\mathbf{0}^{++}$ states and the axion

N_L, N_R	$\lvert J\rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	+	+	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^-\lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	2	+	+	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	−	+	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	+	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	+	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	3	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	3	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	4	+	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$
	4	−	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$

Results for all the ensembles

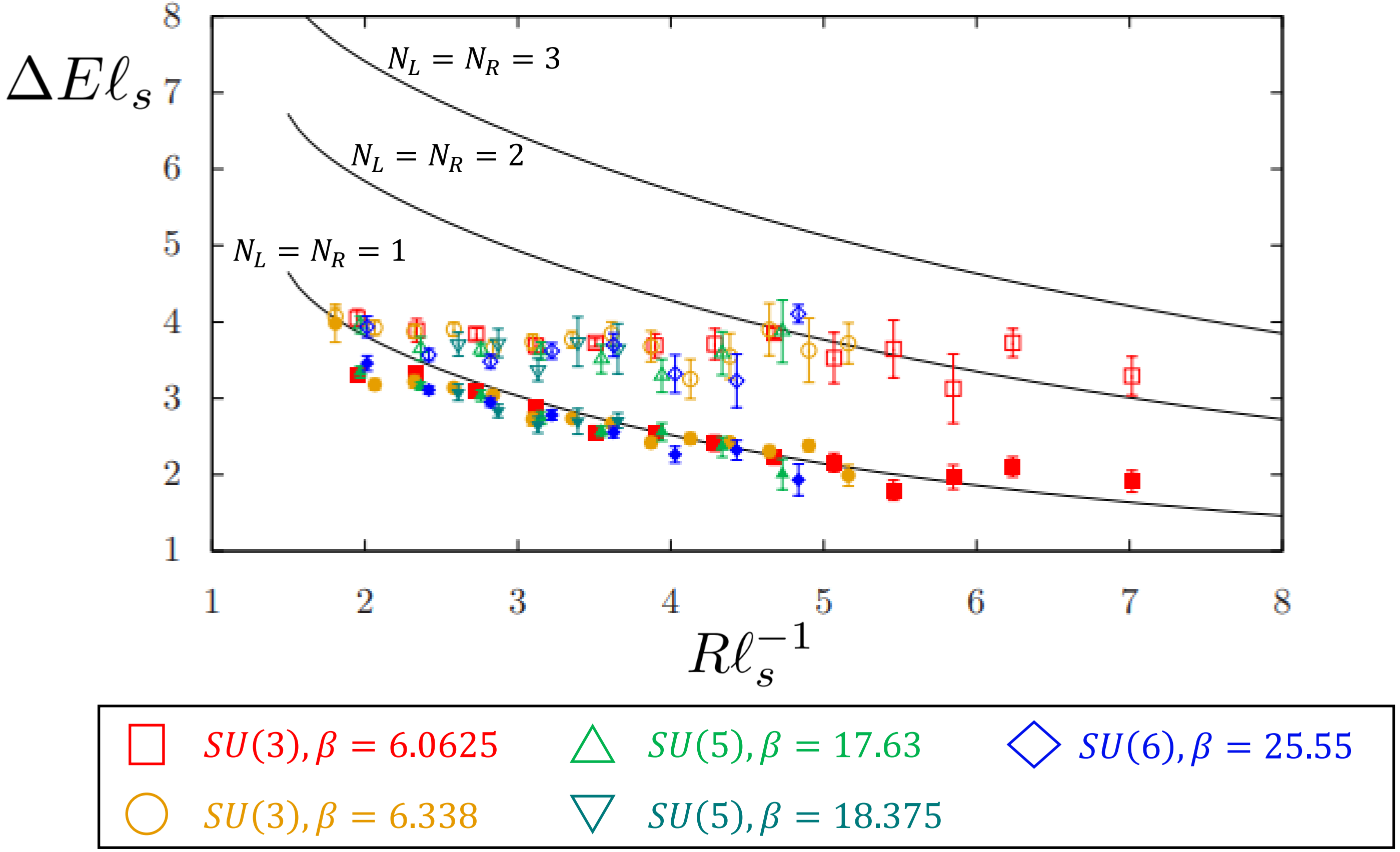


Closed flux tube results: the $\mathbf{0}^{++}$ states and the axion

N_L, N_R	$\lvert J\rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right) \lvert 0\rangle$
$N_L = N_R = 1$	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right) \lvert 0\rangle$
	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right) \lvert 0\rangle$
	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right) \lvert 0\rangle$
	0	+	+	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right) \lvert 0\rangle$
	0	−	−	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right) \lvert 0\rangle$
	0	+	+	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right) \lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right) \lvert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^- \lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right] \lvert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right] \lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right] \lvert 0\rangle$
$N_L = 2, N_R = 2$	1	±	−	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right] \lvert 0\rangle$
	2	+	+	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right) \lvert 0\rangle$
	2	−	+	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right) \lvert 0\rangle$
	2	+	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0\rangle$
	2	+	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0\rangle$
	2	−	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0\rangle$
	2	−	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0\rangle$
	3	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right] \lvert 0\rangle$
	3	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right] \lvert 0\rangle$
	4	+	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right) \lvert 0\rangle$
	4	−	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right) \lvert 0\rangle$

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$

Results for all the ensembles

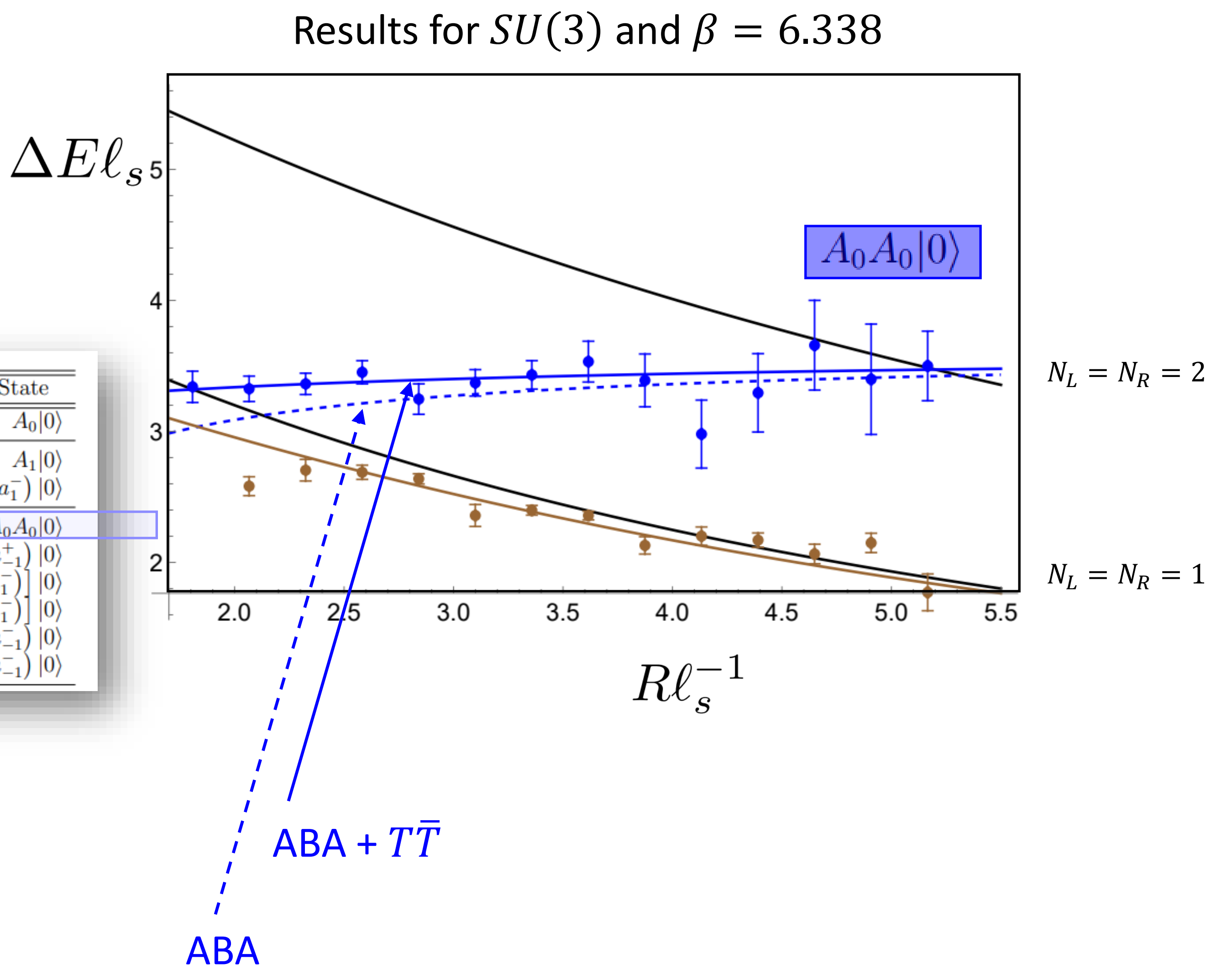


Closed flux tube results: the $\mathbf{0}^{++}$ states and the axion

N_L, N_R	$\lvert J\rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$(a_1^+ a_{-1}^- + a_1^- a_{-1}^+)\,\lvert 0\rangle$
	0	−	−	$(a_1^+ a_{-1}^- - a_1^- a_{-1}^+)\,\lvert 0\rangle$
	2	+	+	$(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-)\,\lvert 0\rangle$
	2	−	+	$(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-)\,\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$(a_2^+ a_{-2}^- + a_2^- a_{-2}^+)\,\lvert 0\rangle$
	0	−	−	$(a_2^+ a_{-2}^- - a_2^- a_{-2}^+)\,\lvert 0\rangle$
	0	+	+	$(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+)\,\lvert 0\rangle$
	0	−	−	$(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+)\,\lvert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^- \lvert 0\rangle$
	1			
	1			
	1			
	1			
	2			
	2			
	2			
	2			
	2			
	2			
	3			
	3			
	4			
	4	−	+	$(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-)\,\lvert 0\rangle$

N_L, N_R	$\lvert J\rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 1$	0	−	−	$A_0\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	−		$A_1\lvert 0\rangle$
	1	±		$A_0\,(a_1^+ \mp a_1^-)\,\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$A_0 A_0\lvert 0\rangle$
	0	−	−	$A_0\,(a_1^+ a_{-1}^- + a_1^- a_{-1}^+)\,\lvert 0\rangle$
	1	±	+	$\left[A_1\,(a_{-1}^+ \mp a_{-1}^-) - A_{-1}\,(a_1^+ \mp a_1^-)\right]\lvert 0\rangle$
	1	±	−	$\left[A_1\,(a_{-1}^+ \mp a_{-1}^-) + A_{-1}\,(a_1^+ \mp a_1^-)\right]\lvert 0\rangle$
	2	+	−	$A_0\,(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-)\,\lvert 0\rangle$
	2	−	−	$A_0\,(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-)\,\lvert 0\rangle$

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$

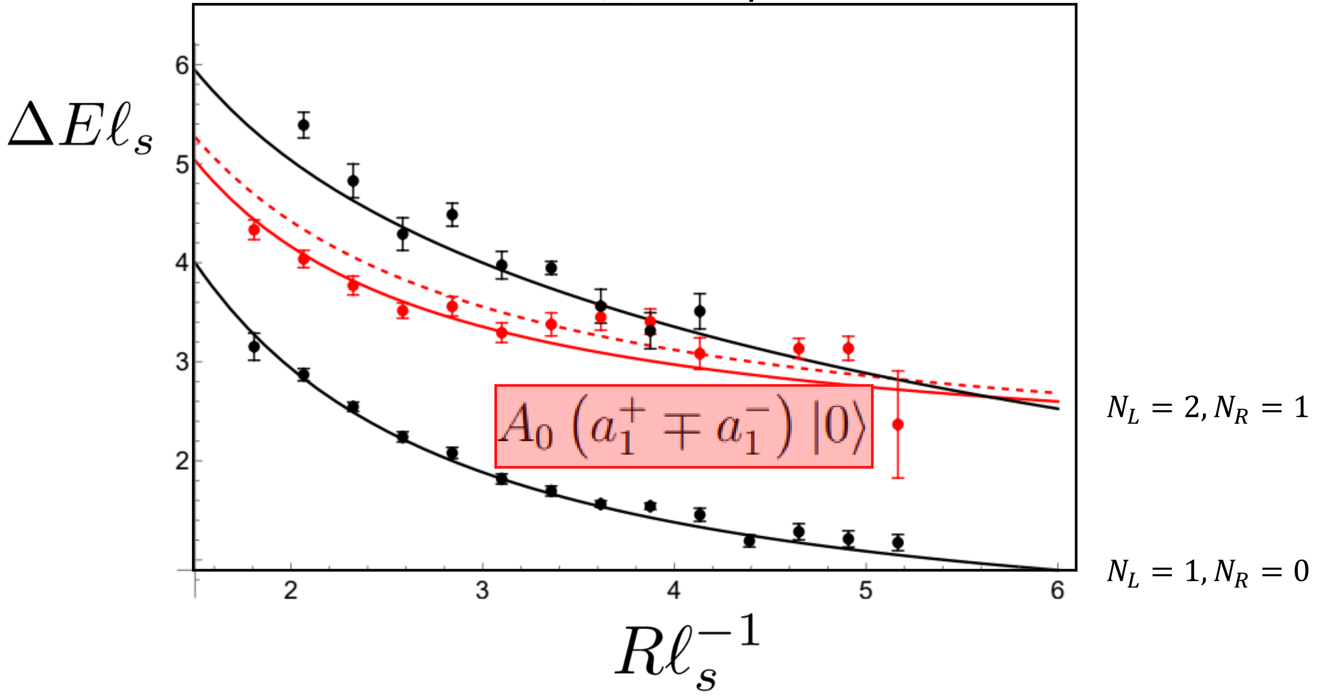


Closed flux tube results: $J = 1, q = 1$ ground, 1^{st} and 2^{nd}

N_L, N_R	$\lvert J \rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = 1, N_R = 0$	1	$\pm$		$(a_1^+ \pm a_1^-)\,\lvert 0\rangle$
	0	$+$		$(a_2^+ a_{-1}^- + a_2^- a_{-1}^+)\,\lvert 0\rangle$
	0	$-$		$(a_2^+ a_{-1}^- - a_2^- a_{-1}^+)\,\lvert 0\rangle$
	1	$\pm$		$(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+)\,\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+)\,\lvert 0\rangle$
	2	$+$		$(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-)\,\lvert 0\rangle$
	2	$-$		$(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-)\,\lvert 0\rangle$
	3	$\pm$		$(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-)\,\lvert 0\rangle$

N_L, N_R	$\lvert J \rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 1$	0	$-$	$-$	$A_0\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$-$		$A_1\lvert 0\rangle$
	1	$\pm$		$A_0\,(a_1^+ \mp a_1^-)\,\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	$+$	$+$	$A_0 A_0\lvert 0\rangle$
	0	$-$	$-$	$A_0\,(a_1^+ a_{-1}^- + a_1^- a_{-1}^+)\,\lvert 0\rangle$
	1	$\pm$	$+$	$[A_1\,(a_{-1}^+ \mp a_{-1}^-) - A_{-1}\,(a_1^+ \mp a_1^-)]\,\lvert 0\rangle$
	1	$\pm$	$-$	$[A_1\,(a_{-1}^+ \mp a_{-1}^-) + A_{-1}\,(a_1^+ \mp a_1^-)]\,\lvert 0\rangle$
	2	$+$	$-$	$A_0\,(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-)\,\lvert 0\rangle$
	2	$-$	$-$	$A_0\,(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-)\,\lvert 0\rangle$

Results for $SU(3)$ and $\beta = 6.338$



$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right) + \left(\frac{2\pi q}{R}\right)^2}$$

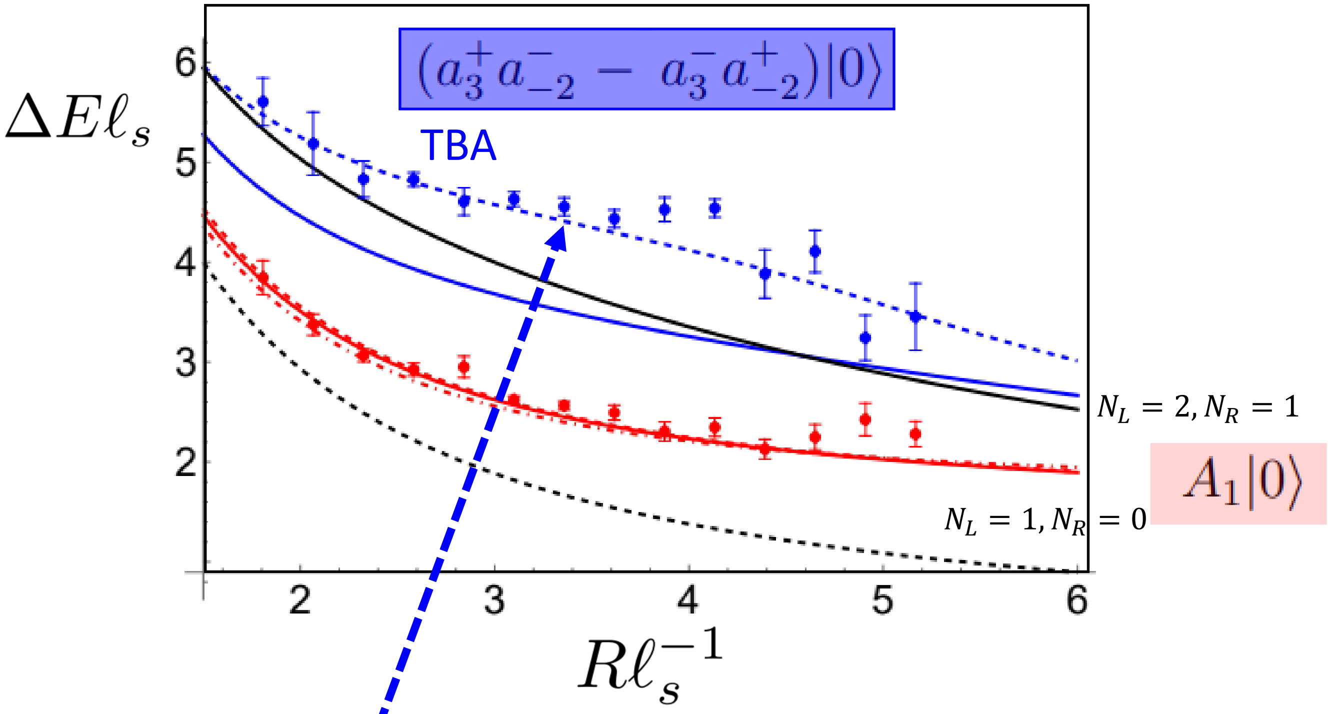
Closed flux tube results: $\mathbf{0}^{-}$, $q = 1$ ground and first excitations

N_L, N_R	$\lvert J\rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
	0	$+$		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
	0	$-$		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
	1	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
	2	$+$		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
	2	$-$		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
	3	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

N_L, N_R	$\lvert J\rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 1$	0	$-$	$-$	$A_0\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$-$		$A_1\lvert 0\rangle$
	1	$\pm$		$A_0\left(a_1^+ \mp a_1^-\right)\lvert 0\rangle$
	0	$+$	$+$	$A_0 A_0\lvert 0\rangle$
	0	$-$	$-$	$A_0\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 2$	1	$\pm$	$+$	$\left[A_1\left(a_{-1}^+ \mp a_{-1}^-\right) - A_{-1}\left(a_1^+ \mp a_1^-\right)\right]\lvert 0\rangle$
	1	$\pm$	$-$	$\left[A_1\left(a_{-1}^+ \mp a_{-1}^-\right) + A_{-1}\left(a_1^+ \mp a_1^-\right)\right]\lvert 0\rangle$
	2	$+$	$-$	$A_0\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
	2	$-$	$-$	$A_0\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right) + \left(\frac{2\pi q}{R}\right)^2}$$

Results for $SU(3)$ and $\beta = 6.338$

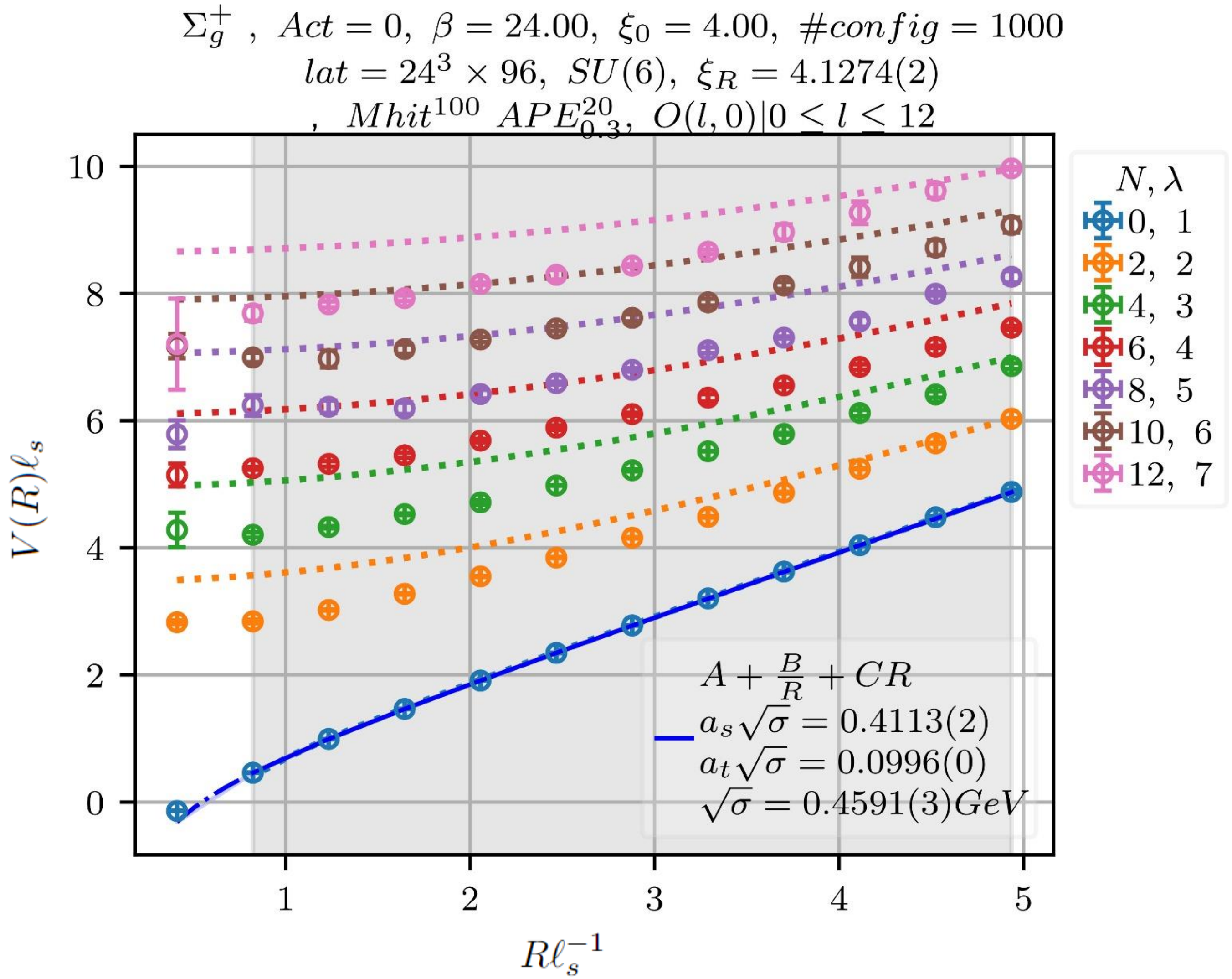


From $\mathbf{0}^{--}$ first excited state a_3 and a_4

$$2\delta_{anti}(s) = \frac{s}{4} + \frac{11 s^2}{192\pi} + a_3 s^3 + a_4 s^4 + \cdots$$

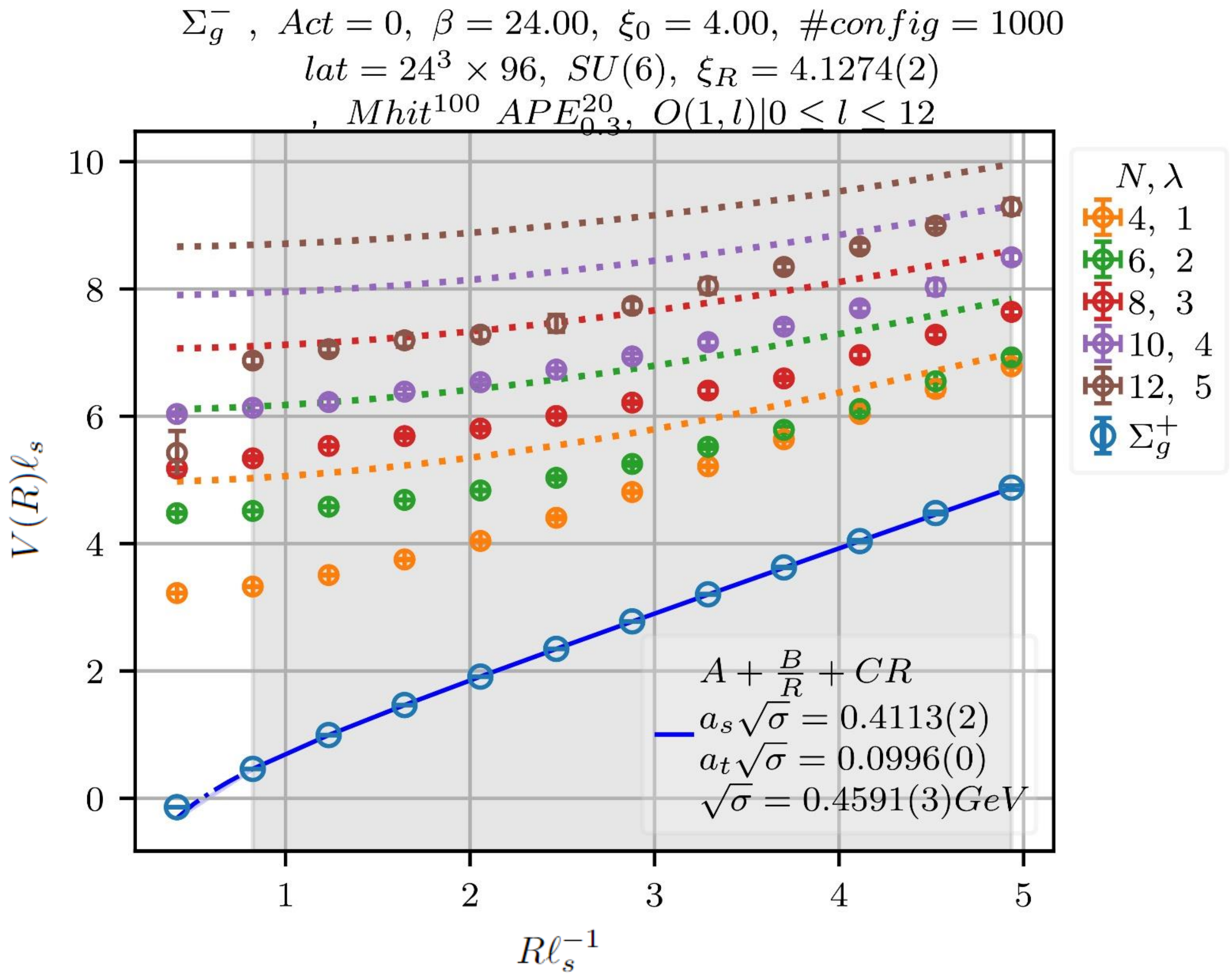
Open flux tube results: Σ_g^+ representation in SU(6)

Excitation	Symmetry	State	
$N=0$	Σ_g^+	$\vert 0\rangle$	
$N=1$	Π_u	$a^\dagger_{1+}\vert 0\rangle$	$a^\dagger_{1-}\vert 0\rangle$
$N=2$	$\Sigma_g^{+\prime}$	$a^\dagger_{1+}a^\dagger_{1-}\vert 0\rangle$	
	Π_g	$a^\dagger_{2+}\vert 0\rangle$	$a^\dagger_{2-}\vert 0\rangle$
	Δ_g	$(a^\dagger_{1+})^2\vert 0\rangle$	$(a^\dagger_{1-})^2\vert 0\rangle$
$N=3$	$\Sigma_u^{+\prime}$	$(a^\dagger_{1+}a^\dagger_{2-}+a^\dagger_{1-}a^\dagger_{2+})\vert 0\rangle$	
	Σ_u^-	$(a^\dagger_{1+}a^\dagger_{2-}-a^\dagger_{1-}a^\dagger_{2+})\vert 0\rangle$	
	$\Pi_u^\prime$	$a^\dagger_{3+}\vert 0\rangle$	$a^\dagger_{3-}\vert 0\rangle$
	$\Pi_u^{\prime\prime}$	$(a^\dagger_{1+})^2a^\dagger_{1-}\vert 0\rangle$	$a^\dagger_{1+}(a^\dagger_{1-})^2\vert 0\rangle$
	Δ_u	$a^\dagger_{1+}a^\dagger_{2+}\vert 0\rangle$	$a^\dagger_{1-}a^\dagger_{2-}\vert 0\rangle$
	ϕ_u	$(a^\dagger_{1+})^3\vert 0\rangle$	$(a^\dagger_{1-})^3\vert 0\rangle$
$N=4$	$\Sigma_g^{+\prime\prime}$	$a^\dagger_{2+}a^\dagger_{2-}\vert 0\rangle$	
	$\Sigma_g^{+\prime\prime\prime}$	$(a^\dagger_{1+})^2(a^\dagger_{1-})^2\vert 0\rangle$	
	$\Sigma_g^{+(\mathrm{iv})}$	$(a^\dagger_{1+}a^\dagger_{3-}+a^\dagger_{1-}a^\dagger_{3+})\vert 0\rangle$	
	Σ_g^-	$(a^\dagger_{1+}a^\dagger_{3-}-a^\dagger_{1-}a^\dagger_{3+})\vert 0\rangle$	
	$\Pi_g^\prime$	$a^\dagger_{4+}\vert 0\rangle$	$a^\dagger_{4-}\vert 0\rangle$
	$\Pi_g^{\prime\prime}$	$(a^\dagger_{1+})^2a^\dagger_{2-}\vert 0\rangle$	$(a^\dagger_{1-})^2a^\dagger_{2+}\vert 0\rangle$
	$\Pi_g^{\prime\prime\prime}$	$a^\dagger_{1+}a^\dagger_{1-}a^\dagger_{2+}\vert 0\rangle$	$a^\dagger_{1+}a^\dagger_{1-}a^\dagger_{2-}\vert 0\rangle$
	$\Delta_g^\prime$	$a^\dagger_{1+}a^\dagger_{3+}\vert 0\rangle$	$a^\dagger_{1-}a^\dagger_{3-}\vert 0\rangle$
	$\Delta_g^{\prime\prime}$	$(a^\dagger_{2+})^2\vert 0\rangle$	$(a^\dagger_{2-})^2\vert 0\rangle$
	$\Delta_g^{\prime\prime\prime}$	$(a^\dagger_{1+})^3a^\dagger_{1-}\vert 0\rangle$	$a^\dagger_{1+}(a^\dagger_{1-})^3\vert 0\rangle$
	Φ_g	$(a^\dagger_{1+})^2a^\dagger_{2+}\vert 0\rangle$	$(a^\dagger_{1-})^2a^\dagger_{2-}\vert 0\rangle$
	Γ_g	$(a^\dagger_{1+})^4\vert 0\rangle$	$(a^\dagger_{1-})^4\vert 0\rangle$



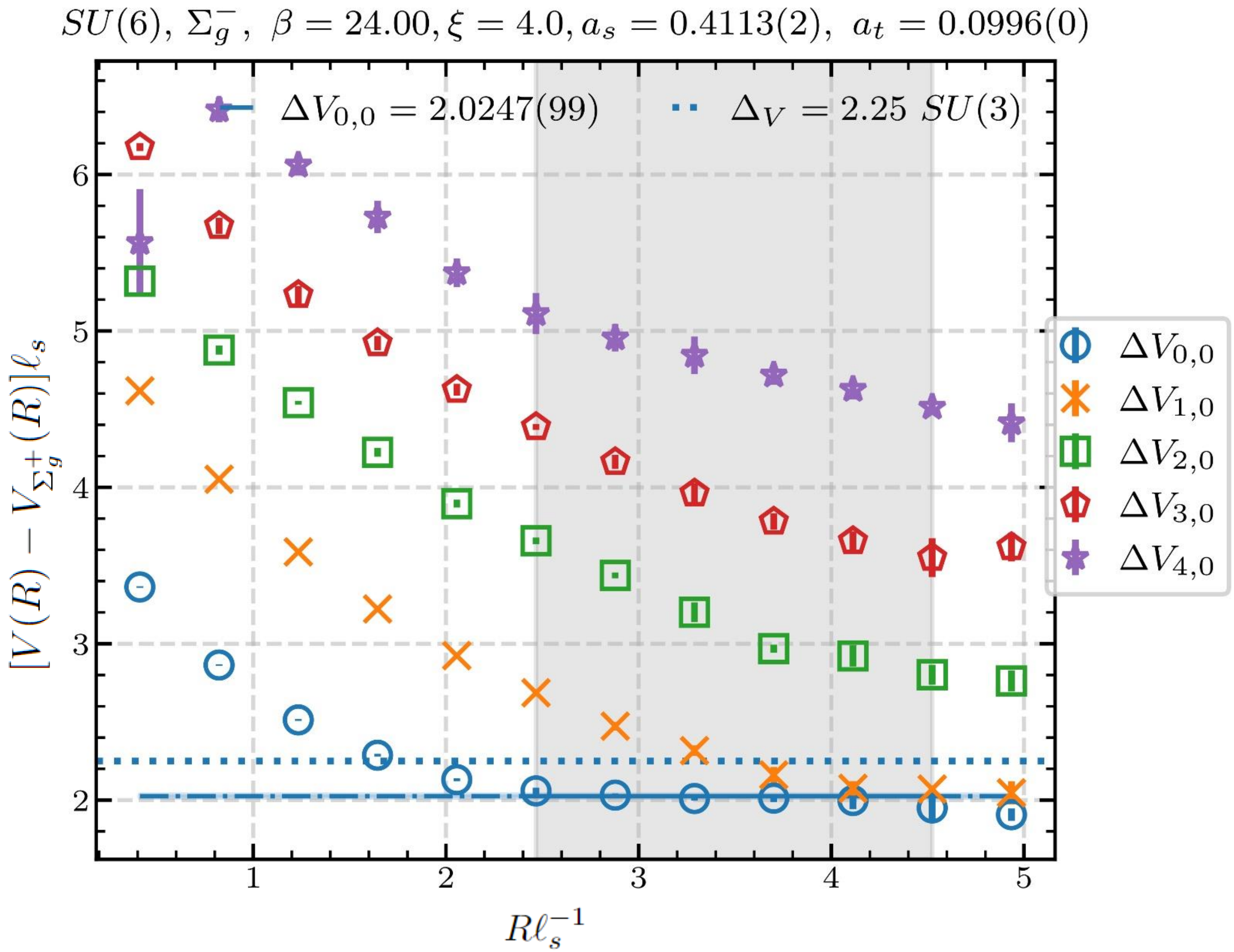
Open flux tube results: Σ_g^- representation in SU(6)

Excitation	Symmetry	State	
$N=0$	Σ_g^+	$\vert 0\rangle$	
$N=1$	Π_u	$a_{1+}^\dagger \vert 0\rangle$	$a_{1-}^\dagger \vert 0\rangle$
$N=2$	$\Sigma_g^{+\prime}$	$a_{1+}^\dagger a_{1-}^\dagger \vert 0\rangle$	
	Π_g	$a_{2+}^\dagger \vert 0\rangle$	$a_{2-}^\dagger \vert 0\rangle$
	Δ_g	$(a_{1+}^\dagger)^2 \vert 0\rangle$	$(a_{1-}^\dagger)^2 \vert 0\rangle$
$N=3$	$\Sigma_u^{+\prime}$	$(a_{1+}^\dagger a_{2-}^\dagger + a_{1-}^\dagger a_{2+}^\dagger) \vert 0\rangle$	
	Σ_u^-	$(a_{1+}^\dagger a_{2-}^\dagger - a_{1-}^\dagger a_{2+}^\dagger) \vert 0\rangle$	
	Π_u'	$a_{3+}^\dagger \vert 0\rangle$	$a_{3-}^\dagger \vert 0\rangle$
	Π_u''	$(a_{1+}^\dagger)^2 a_{1-}^\dagger \vert 0\rangle$	$a_{1+}^\dagger (a_{1-}^\dagger)^2 \vert 0\rangle$
	Δ_u	$a_{1+}^\dagger a_{2+}^\dagger \vert 0\rangle$	$a_{1-}^\dagger a_{2-}^\dagger \vert 0\rangle$
	ϕ_u	$(a_{1+}^\dagger)^3 \vert 0\rangle$	$(a_{1-}^\dagger)^3 \vert 0\rangle$
$N=4$	$\Sigma_g^{+\prime\prime}$	$a_{2+}^\dagger a_{2-}^\dagger \vert 0\rangle$	
	$\Sigma_g^{+\prime\prime\prime}$	$(a_{1+}^\dagger)^2 (a_{1-}^\dagger)^2 \vert 0\rangle$	
	$\Sigma_g^{+(\mathrm{iv})}$	$(a_{1+}^\dagger a_{3-}^\dagger + a_{1-}^\dagger a_{3+}^\dagger) \vert 0\rangle$	
	Σ_g^-	$(a_{1+}^\dagger a_{3-}^\dagger - a_{1-}^\dagger a_{3+}^\dagger) \vert 0\rangle$	
	Π_g'	$a_{4+}^\dagger \vert 0\rangle$	$a_{4-}^\dagger \vert 0\rangle$
	Π_g''	$(a_{1+}^\dagger)^2 a_{2-}^\dagger \vert 0\rangle$	$(a_{1-}^\dagger)^2 a_{2+}^\dagger \vert 0\rangle$
	Π_g'''	$a_{1+}^\dagger a_{1-}^\dagger a_{2+}^\dagger \vert 0\rangle$	$a_{1+}^\dagger a_{1-}^\dagger a_{2-}^\dagger \vert 0\rangle$
	Δ_g'	$a_{1+}^\dagger a_{3+}^\dagger \vert 0\rangle$	$a_{1-}^\dagger a_{3-}^\dagger \vert 0\rangle$
	Δ_g''	$(a_{2+}^\dagger)^2 \vert 0\rangle$	$(a_{2-}^\dagger)^2 \vert 0\rangle$
	Δ_g'''	$(a_{1+}^\dagger)^3 a_{1-}^\dagger \vert 0\rangle$	$a_{1+}^\dagger (a_{1-}^\dagger)^3 \vert 0\rangle$
	Φ_g	$(a_{1+}^\dagger)^2 a_{2+}^\dagger \vert 0\rangle$	$(a_{1-}^\dagger)^2 a_{2-}^\dagger \vert 0\rangle$
	Γ_g	$(a_{1+}^\dagger)^4 \vert 0\rangle$	$(a_{1-}^\dagger)^4 \vert 0\rangle$



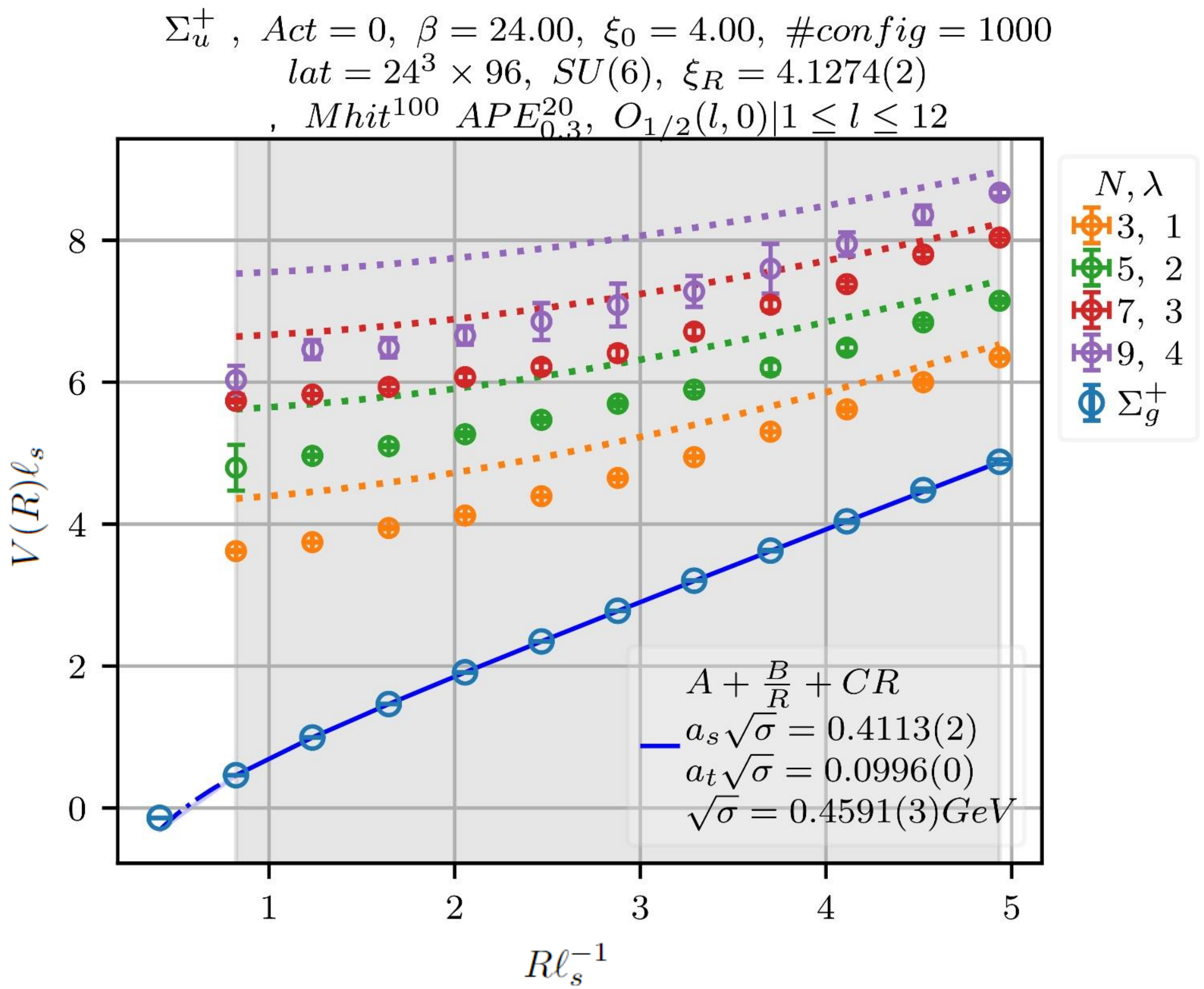
Open flux tube results: Σ_g^- representation in SU(6)

Excitation	Symmetry	State	
$N=0$	Σ_g^+	$\lvert 0\rangle$	
$N=1$	Π_u	$a_{1+}^\dagger \lvert 0\rangle$	$a_{1-}^\dagger \lvert 0\rangle$
$N=2$	$\Sigma_g^{+\prime}$	$a_{1+}^\dagger a_{1-}^\dagger \lvert 0\rangle$	
	Π_g	$a_{2+}^\dagger \lvert 0\rangle$	$a_{2-}^\dagger \lvert 0\rangle$
	Δ_g	$(a_{1+}^\dagger)^2 \lvert 0\rangle$	$(a_{1-}^\dagger)^2 \lvert 0\rangle$
$N=3$	$\Sigma_u^{+\prime}$	$(a_{1+}^\dagger a_{2-}^\dagger + a_{1-}^\dagger a_{2+}^\dagger)\lvert 0\rangle$	
	Σ_u^-	$(a_{1+}^\dagger a_{2-}^\dagger - a_{1-}^\dagger a_{2+}^\dagger)\lvert 0\rangle$	
	Π_u'	$a_{3+}^\dagger \lvert 0\rangle$	$a_{3-}^\dagger \lvert 0\rangle$
	Π_u''	$(a_{1+}^\dagger)^2 a_{1-}^\dagger \lvert 0\rangle$	$a_{1+}^\dagger (a_{1-}^\dagger)^2 \lvert 0\rangle$
	Δ_u	$a_{1+}^\dagger a_{2+}^\dagger \lvert 0\rangle$	$a_{1-}^\dagger a_{2-}^\dagger \lvert 0\rangle$
	ϕ_u	$(a_{1+}^\dagger)^3 \lvert 0\rangle$	$(a_{1-}^\dagger)^3 \lvert 0\rangle$
$N=4$	$\Sigma_g^{+\prime\prime}$	$a_{2+}^\dagger a_{2-}^\dagger \lvert 0\rangle$	
	$\Sigma_g^{+\prime\prime\prime}$	$(a_{1+}^\dagger)^2 (a_{1-}^\dagger)^2 \lvert 0\rangle$	
	$\Sigma_g^{+(\mathrm{iv})}$	$(a_{1+}^\dagger a_{3-}^\dagger + a_{1-}^\dagger a_{3+}^\dagger)\lvert 0\rangle$	
	Σ_g^-	$(a_{1+}^\dagger a_{3-}^\dagger - a_{1-}^\dagger a_{3+}^\dagger)\lvert 0\rangle$	
	Π_g'	$a_{4+}^\dagger \lvert 0\rangle$	$a_{4-}^\dagger \lvert 0\rangle$
	Π_g''	$(a_{1+}^\dagger)^2 a_{2-}^\dagger \lvert 0\rangle$	$(a_{1-}^\dagger)^2 a_{2+}^\dagger \lvert 0\rangle$
	Π_g'''	$a_{1+}^\dagger a_{1-}^\dagger a_{2+}^\dagger \lvert 0\rangle$	$a_{1+}^\dagger a_{1-}^\dagger a_{2-}^\dagger \lvert 0\rangle$
	Δ_g'	$a_{1+}^\dagger a_{3+}^\dagger \lvert 0\rangle$	$a_{1-}^\dagger a_{3-}^\dagger \lvert 0\rangle$
	Δ_g''	$(a_{2+}^\dagger)^2 \lvert 0\rangle$	$(a_{2-}^\dagger)^2 \lvert 0\rangle$
	Δ_g'''	$(a_{1+}^\dagger)^3 a_{1-}^\dagger \lvert 0\rangle$	$a_{1+}^\dagger (a_{1-}^\dagger)^3 \lvert 0\rangle$
	Φ_g	$(a_{1+}^\dagger)^2 a_{2+}^\dagger \lvert 0\rangle$	$(a_{1-}^\dagger)^2 a_{2-}^\dagger \lvert 0\rangle$
	Γ_g	$(a_{1+}^\dagger)^4 \lvert 0\rangle$	$(a_{1-}^\dagger)^4 \lvert 0\rangle$



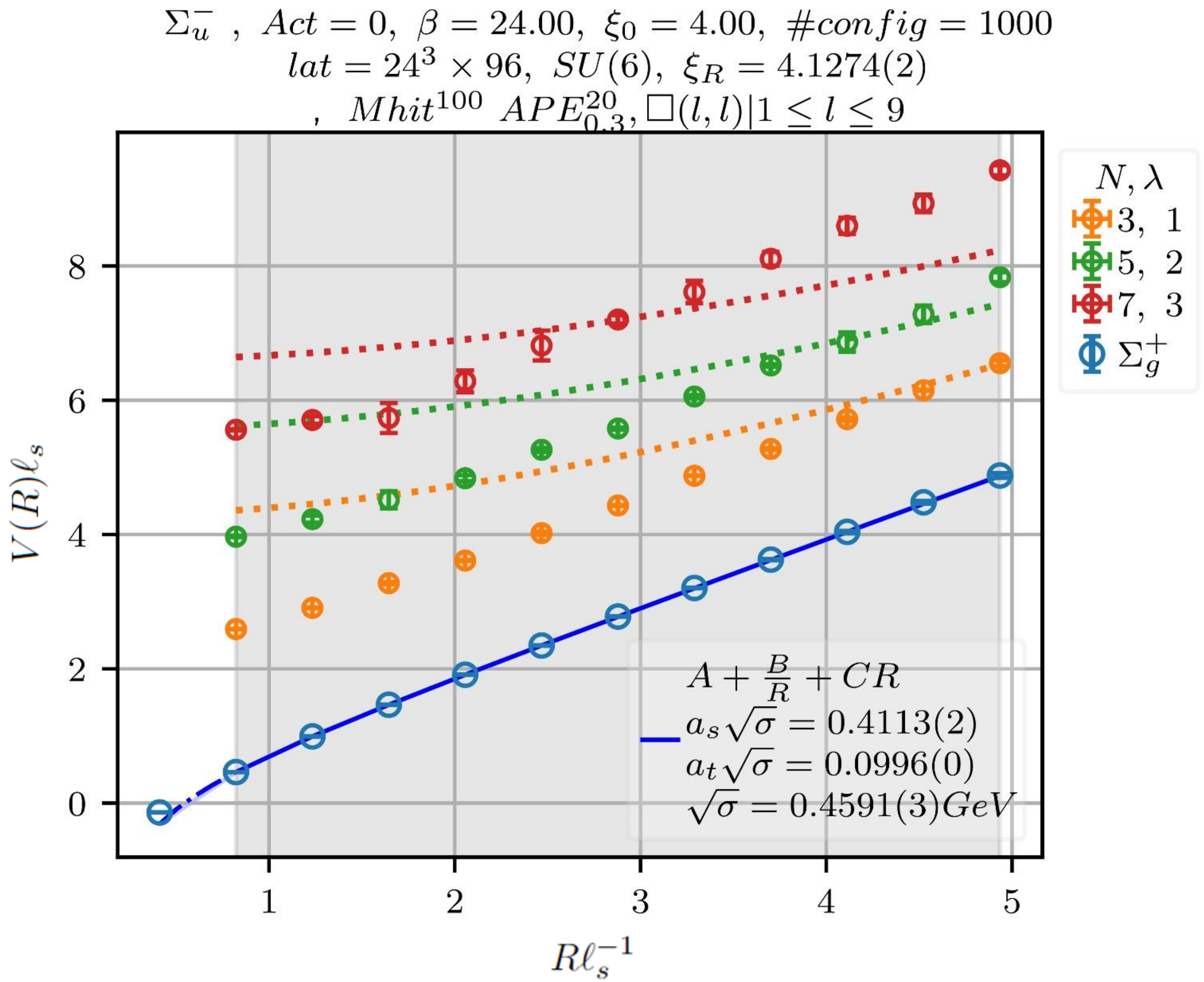
Open flux tube results: Σ_u^+ representation in SU(6)

Excitation	Symmetry	State	
$N=0$	Σ_g^+	$\vert 0\rangle$	
$N=1$	Π_u	$a_{1+}^\dagger\vert 0\rangle$	$a_{1-}^\dagger\vert 0\rangle$
$N=2$	$\Sigma_g^{+\prime}$	$a_{1+}^\dagger a_{1-}^\dagger\vert 0\rangle$	
	Π_g	$a_{2+}^\dagger\vert 0\rangle$	$a_{2-}^\dagger\vert 0\rangle$
	Δ_g	$(a_{1+}^\dagger)^2\vert 0\rangle$	$(a_{1-}^\dagger)^2\vert 0\rangle$
$N=3$	$\Sigma_u^{+\prime}$	$(a_{1+}^\dagger a_{2-}^\dagger + a_{1-}^\dagger a_{2+}^\dagger)\vert 0\rangle$	
	Σ_u^-	$(a_{1+}^\dagger a_{2-}^\dagger - a_{1-}^\dagger a_{2+}^\dagger)\vert 0\rangle$	
	Π_u'	$a_{3+}^\dagger\vert 0\rangle$	$a_{3-}^\dagger\vert 0\rangle$
	Π_u''	$(a_{1+}^\dagger)^2 a_{1-}^\dagger\vert 0\rangle$	$a_{1+}^\dagger(a_{1-}^\dagger)^2\vert 0\rangle$
	Δ_u	$a_{1+}^\dagger a_{2+}^\dagger\vert 0\rangle$	$a_{1-}^\dagger a_{2-}^\dagger\vert 0\rangle$
	ϕ_u	$(a_{1+}^\dagger)^3\vert 0\rangle$	$(a_{1-}^\dagger)^3\vert 0\rangle$
$N=4$	$\Sigma_g^{+\prime\prime}$	$a_{2+}^\dagger a_{2-}^\dagger\vert 0\rangle$	
	$\Sigma_g^{+\prime\prime\prime}$	$(a_{1+}^\dagger)^2(a_{1-}^\dagger)^2\vert 0\rangle$	
	$\Sigma_g^{+(\mathrm{iv})}$	$(a_{1+}^\dagger a_{3-}^\dagger + a_{1-}^\dagger a_{3+}^\dagger)\vert 0\rangle$	
	Σ_g^-	$(a_{1+}^\dagger a_{3-}^\dagger - a_{1-}^\dagger a_{3+}^\dagger)\vert 0\rangle$	
	Π_g'	$a_{4+}^\dagger\vert 0\rangle$	$a_{4-}^\dagger\vert 0\rangle$
	Π_g''	$(a_{1+}^\dagger)^2 a_{2-}^\dagger\vert 0\rangle$	$(a_{1-}^\dagger)^2 a_{2+}^\dagger\vert 0\rangle$
	Π_g'''	$a_{1+}^\dagger a_{1-}^\dagger a_{2+}^\dagger\vert 0\rangle$	$a_{1+}^\dagger a_{1-}^\dagger a_{2-}^\dagger\vert 0\rangle$
	Δ_g'	$a_{1+}^\dagger a_{3+}^\dagger\vert 0\rangle$	$a_{1-}^\dagger a_{3-}^\dagger\vert 0\rangle$
	Δ_g''	$(a_{2+}^\dagger)^2\vert 0\rangle$	$(a_{2-}^\dagger)^2\vert 0\rangle$
	Δ_g'''	$(a_{1+}^\dagger)^3 a_{1-}^\dagger\vert 0\rangle$	$a_{1+}^\dagger(a_{1-}^\dagger)^3\vert 0\rangle$
	Φ_g	$(a_{1+}^\dagger)^2 a_{2+}^\dagger\vert 0\rangle$	$(a_{1-}^\dagger)^2 a_{2-}^\dagger\vert 0\rangle$
	Γ_g	$(a_{1+}^\dagger)^4\vert 0\rangle$	$(a_{1-}^\dagger)^4\vert 0\rangle$



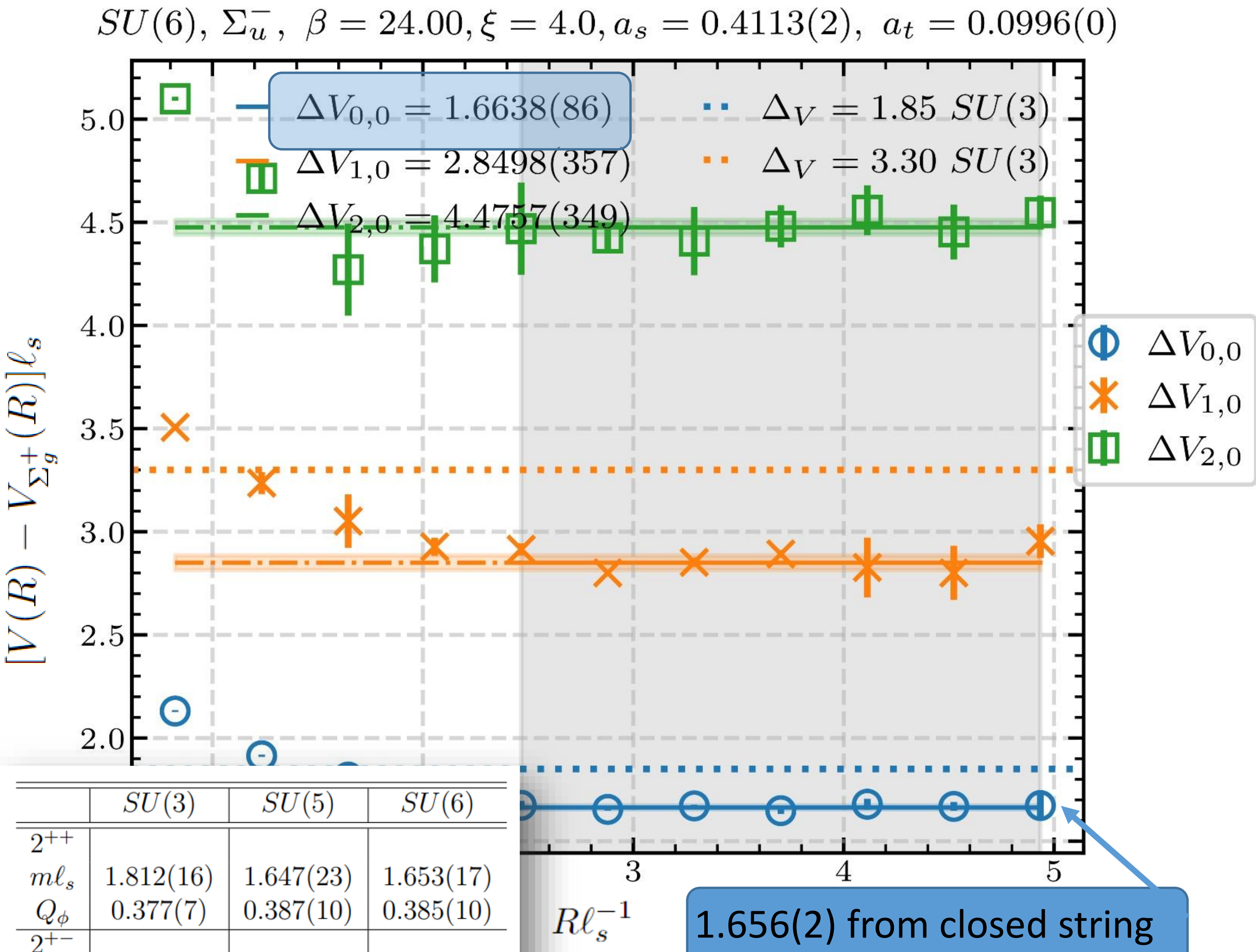
Open flux tube results: Σ_u^- representation in SU(6)

Excitation	Symmetry	State	
$N=0$	Σ_g^+	$\lvert 0\rangle$	
$N=1$	Π_u	$a_{1+}^\dagger \lvert 0\rangle$	$a_{1-}^\dagger \lvert 0\rangle$
$N=2$	$\Sigma_g^{+\prime}$	$a_{1+}^\dagger a_{1-}^\dagger \lvert 0\rangle$	
	Π_g	$a_{2+}^\dagger \lvert 0\rangle$	$a_{2-}^\dagger \lvert 0\rangle$
	Δ_g	$(a_{1+}^\dagger)^2 \lvert 0\rangle$	$(a_{1-}^\dagger)^2 \lvert 0\rangle$
$N=3$	$\Sigma_u^{+\prime}$	$(a_{1+}^\dagger a_{2-}^\dagger + a_{1-}^\dagger a_{2+}^\dagger)\lvert 0\rangle$	
	Σ_u^-	$(a_{1+}^\dagger a_{2-}^\dagger - a_{1-}^\dagger a_{2+}^\dagger)\lvert 0\rangle$	
	Π_u'	$a_{3+}^\dagger \lvert 0\rangle$	$a_{3-}^\dagger \lvert 0\rangle$
	Π_u''	$(a_{1+}^\dagger)^2 a_{1-}^\dagger \lvert 0\rangle$	$a_{1+}^\dagger (a_{1-}^\dagger)^2 \lvert 0\rangle$
	Δ_u	$a_{1+}^\dagger a_{2+}^\dagger \lvert 0\rangle$	$a_{1-}^\dagger a_{2-}^\dagger \lvert 0\rangle$
	ϕ_u	$(a_{1+}^\dagger)^3 \lvert 0\rangle$	$(a_{1-}^\dagger)^3 \lvert 0\rangle$
$N=4$	$\Sigma_g^{+\prime\prime}$	$a_{2+}^\dagger a_{2-}^\dagger \lvert 0\rangle$	
	$\Sigma_g^{+\prime\prime\prime}$	$(a_{1+}^\dagger)^2 (a_{1-}^\dagger)^2 \lvert 0\rangle$	
	$\Sigma_g^{+(\mathrm{iv})}$	$(a_{1+}^\dagger a_{3-}^\dagger + a_{1-}^\dagger a_{3+}^\dagger)\lvert 0\rangle$	
	Σ_g^-	$(a_{1+}^\dagger a_{3-}^\dagger - a_{1-}^\dagger a_{3+}^\dagger)\lvert 0\rangle$	
	Π_g'	$a_{4+}^\dagger \lvert 0\rangle$	$a_{4-}^\dagger \lvert 0\rangle$
	Π_g''	$(a_{1+}^\dagger)^2 a_{2-}^\dagger \lvert 0\rangle$	$(a_{1-}^\dagger)^2 a_{2+}^\dagger \lvert 0\rangle$
	Π_g'''	$a_{1+}^\dagger a_{1-}^\dagger a_{2+}^\dagger \lvert 0\rangle$	$a_{1+}^\dagger a_{1-}^\dagger a_{2-}^\dagger \lvert 0\rangle$
	Δ_g'	$a_{1+}^\dagger a_{3+}^\dagger \lvert 0\rangle$	$a_{1-}^\dagger a_{3-}^\dagger \lvert 0\rangle$
	Δ_g''	$(a_{2+}^\dagger)^2 \lvert 0\rangle$	$(a_{2-}^\dagger)^2 \lvert 0\rangle$
	Δ_g'''	$(a_{1+}^\dagger)^3 a_{1-}^\dagger \lvert 0\rangle$	$a_{1+}^\dagger (a_{1-}^\dagger)^3 \lvert 0\rangle$
	Φ_g	$(a_{1+}^\dagger)^2 a_{2+}^\dagger \lvert 0\rangle$	$(a_{1-}^\dagger)^2 a_{2-}^\dagger \lvert 0\rangle$
	Γ_g	$(a_{1+}^\dagger)^4 \lvert 0\rangle$	$(a_{1-}^\dagger)^4 \lvert 0\rangle$



Open flux tube results: Σ_u^- representation in SU(6)

Excitation	Symmetry	State	
$N=0$	Σ_g^+	$\|0\rangle$	
$N=1$	Π_u	$a^\dagger_{1+}\|0\rangle$	$a^\dagger_{1-}\|0\rangle$
$N=2$	$\Sigma_g^{+\prime}$	$a^\dagger_{1+}a^\dagger_{1-}\|0\rangle$	
	Π_g	$a^\dagger_{2+}\|0\rangle$	$a^\dagger_{2-}\|0\rangle$
	Δ_g	$(a^\dagger_{1+})^2\|0\rangle$	$(a^\dagger_{1-})^2\|0\rangle$
$N=3$	$\Sigma_u^{+\prime}$	$(a^\dagger_{1+}a^\dagger_{2-}+a^\dagger_{1-}a^\dagger_{2+})\|0\rangle$	
	Σ_u^-	$(a^\dagger_{1+}a^\dagger_{2-}-a^\dagger_{1-}a^\dagger_{2+})\|0\rangle$	
	Π_u'	$a^\dagger_{3+}\|0\rangle$	$a^\dagger_{3-}\|0\rangle$
	Π_u''	$(a^\dagger_{1+})^2a^\dagger_{1-}\|0\rangle$	$a^\dagger_{1+}(a^\dagger_{1-})^2\|0\rangle$
	Δ_u	$a^\dagger_{1+}a^\dagger_{2+}\|0\rangle$	$a^\dagger_{1-}a^\dagger_{2-}\|0\rangle$
	ϕ_u	$(a^\dagger_{1+})^3\|0\rangle$	$(a^\dagger_{1-})^3\|0\rangle$
$N=4$	$\Sigma_g^{+\prime\prime}$	$a^\dagger_{2+}a^\dagger_{2-}\|0\rangle$	
	$\Sigma_g^{+\prime\prime\prime}$	$(a^\dagger_{1+})^2(a^\dagger_{1-})^2\|0\rangle$	
	$\Sigma_g^{+(\mathrm{iv})}$	$(a^\dagger_{1+}a^\dagger_{3-}+a^\dagger_{1-}a^\dagger_{3+})\|0\rangle$	
	Σ_g^-	$(a^\dagger_{1+}a^\dagger_{3-}-a^\dagger_{1-}a^\dagger_{3+})\|0\rangle$	
	Π_g'	$a^\dagger_{4+}\|0\rangle$	$a^\dagger_{4-}\|0\rangle$
	Π_g''	$(a^\dagger_{1+})^2a^\dagger_{2-}\|0\rangle$	$(a^\dagger_{1-})^2a^\dagger_{2+}\|0\rangle$
	Π_g'''	$a^\dagger_{1+}a^\dagger_{1-}a^\dagger_{2+}\|0\rangle$	$a^\dagger_{1+}a^\dagger_{1-}a^\dagger_{2-}\|0\rangle$
	Δ_g'	$a^\dagger_{1+}a^\dagger_{3+}\|0\rangle$	$a^\dagger_{1-}a^\dagger_{3-}\|0\rangle$
	Δ_g''	$(a^\dagger_{2+})^2\|0\rangle$	$(a^\dagger_{2-})^2\|0\rangle$
	Δ_g'''	$(a^\dagger_{1+})^3a^\dagger_{1-}\|0\rangle$	$a^\dagger_{1+}(a^\dagger_{1-})^3\|0\rangle$
	Φ_g	$(a^\dagger_{1+})^2a^\dagger_{2+}\|0\rangle$	$(a^\dagger_{1-})^2a^\dagger_{2-}\|0\rangle$
	Γ_g	$(a^\dagger_{1+})^4\|0\rangle$	$(a^\dagger_{1-})^4\|0\rangle$



	$SU(3)$	$SU(5)$	$SU(6)$
2^{++}			
$m\ell_s$	1.812(16)	1.647(23)	1.653(17)
Q_ϕ	0.377(7)	0.387(10)	0.385(10)
2^{+-}			
$m\ell_s$	1.811(16)	1.648(23)	1.656(17)
Q_ϕ	0.354(6)	0.346(7)	0.337(10)

1.656(2) from closed string

Outlook

- We have obtained the spectrum of closed flux-tubes in 4D SU(N) Gauge Theories
  - The interaction involving axions can be approximated by $T\bar{T}$ deformation
  - We observe two towers of states – (phonon) + (axions + phonon) states
- We have obtained the spectrum of open flux-tubes in 4D $SU(N)$ Gauge Theories
  - The spectrum involves an axion with the same mass as that for the closed flux-tube
  - There is need for an Effective String Theory for Open Flux Tubes
  - TBA analysis for the Open Flux tube is needed
- Spectrum of closed flux tube supports that it can be described by worldsheet fields only

Thanks for your attention!!!

Thanks Derek Leinweber for the amazing visualizations