# Reassessing the flux tube formation via center-vortex ensembles in the lattice

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# 4d ensembles of percolating center vortices and chains

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  - In the lattice, center vortices attached to monopoles, forming chains, account for 97% of the cases

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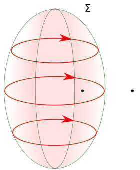


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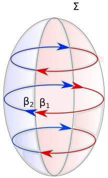

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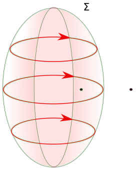


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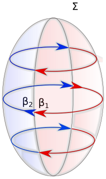


  - relevance of both percolating center vortices and chains to form a confining flux tube

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- Abelian profiles (LEO & Vercauteren, 2016) (LEO & Simões, 2019)

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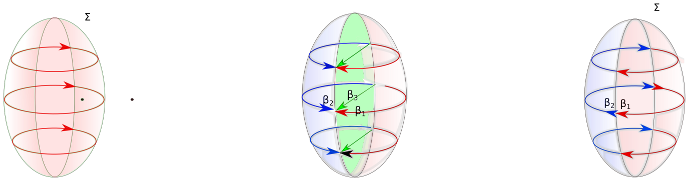

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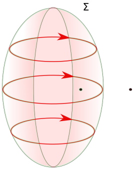
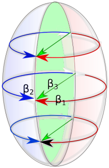
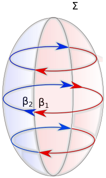
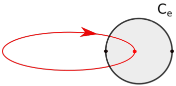
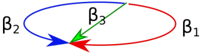


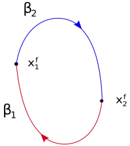




- *Abelian projected configurations in the wave functional formalism (Junior, Reinhardt & LEO, 2022)*



# 4d Mixed ensembles in the wave functional formalism

- elementary center-vortex loops carrying fundamental magnetic weights $\beta_1, \ldots, \beta_N$, with $N$-matching: $a = 2\pi\beta \cdot T\, \partial_i \chi + \ldots$

$$\Psi(A) = \sum_{\{\gamma\}} \psi_{\{\gamma\}}\, \delta(A - a(\{\gamma\})) \qquad , \qquad A_i(\mathbf{x}) \qquad , \qquad \mathbf{x} \in \mathbb{R}^3$$

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$$\tilde{\Psi}(E) = \int D\Phi\, e^{-S[\Phi,\Lambda]} \quad , \quad |D(\Lambda)\Phi|^2 + m^2 \mathrm{Tr}\, \Phi^\dagger \Phi + \mathrm{Tr}\, (\Phi^\dagger \Phi)^2 + \det \Phi + \mathrm{c.c.}$$

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- percolating phase: $\psi_\gamma$ with negative tension, positive stiffness and repulsive interactions $\to m^2 < 0$

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- In both cases: asymptotic Casimir law $\propto k(N-k)$

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- The Abelian projected case was only done at the level of the wave functional

# Matrix representation of surfaces

(Weingarten, 1980)

$$Z_0 = \sum_{\mathcal{S}} N^{\chi(\mathcal{S})} e^{-\sigma A(\mathcal{S})} \quad , \quad A(\mathcal{S}) = a^2 F \quad , \quad N \in \mathbb{N}$$

- $\mathcal{S}$ is formed by $F$ oriented plaquettes $p$ (faces)
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- The Weingarten representation

$$Z_0 = \int DV \, \exp\left(\gamma \sum_p \mathrm{Tr}\, V(p) - Q_0[V]\right) ,$$

$$Q_0[V] = \sum_{\{x,y\}} \mathcal{Q}_0(V(x,y)) \quad , \quad \mathcal{Q}_0(V) = \mathrm{Tr}\left(V^\dagger V\right)$$

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- this ensemble can also be thought of as colored surfaces $\mathcal{S}_\mathrm{c}$, with $N$ possible colors at each vertex

$$Z_0 = \sum_{\mathcal{S}_\mathrm{c}} e^{-\mu_0 A(\mathcal{S}_\mathrm{c})} = \sum_{\mathcal{S}} N^{V(\mathcal{S})} e^{-\mu_0 A(\mathcal{S})}$$

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$$Q[V] = \sum_{\{x,y\}} \mathcal{Q}(V(x,y)) \quad , \quad \mathcal{Q}(V) = \mathrm{Tr}\left(\eta V^{\dagger} V + \lambda (V^{\dagger} V)^2\right)$$

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$*$ Because of the Von Neumann trace inequality $|\mathrm{Tr}(AB)| \leq \sum_{i=1}^{N} \sigma_i(A)\sigma_i(B)$:

$$\mathrm{Tr}\, V(p) \leq \frac{1}{4} \mathrm{Tr}((A^\dagger A)^2 + (B^\dagger B)^2 + (C^\dagger C)^2 + (D^\dagger D)^2) \quad , \quad V(p) = ABCD$$

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- The noninteracting Weingarten model is not related to a field theory
- However, consider the interacting Weingarten model when $\eta < 0$:

$$\gamma \sum_p \mathrm{Tr}\, V(p) - Q[V] = K[V] + U[V] \, ,$$

$$K[V] = 3\gamma \sum_{\{x,y\}} \mathrm{Tr}(V^\dagger(x,y)V(x,y))^2 - \gamma \sum_p \mathrm{Tr}\, V(p) \geq 0 \, ,$$

$$U[V] = \lambda' \sum_{\{x,y\}} \mathrm{Tr} \left( (V^\dagger(x,y)V(x,y) - \vartheta^2 I)^2 - \vartheta^4 I \right) \geq 0$$

- $\vartheta^2 = -\eta/(2\lambda')$

- $\eta < 0$ is realized for a tension $\mu$ below a critical value $\mu_c$

# The Goldstone modes for percolating surfaces

- when surfaces percolate and $\lambda' >> \gamma$, deviations away from the minima of the "potential" get suppressed because of a "mass" $\lambda' \vartheta^2$

$$V(x,y) \approx \vartheta\, U(x,y) \quad , \quad U(x,y) \in U(N)$$

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- the important role played by the excluded volume effects was clarified

# Abelian projection vs. Local magnetic colors

- at the level of the 4d partition function, the center-element average is

$$Z[B] \propto \int DV \, \exp\left(\gamma \sum_p \mathrm{Tr}\left(e^{iB(p)} V(p)\right) - Q[V]\right) \quad , \quad B(p) = 2\pi \beta_{\rm e} \cdot T \, s(p)$$

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- Abelian projection:
  - $N$ elementary center vortices carrying global defining weights $\beta_i \to$ $N$ complex variables $V_i(x,y)$ that generate each type ($Z[B] = \prod_i Z[b_i]$)

$$V = \begin{pmatrix} V_1 & 0 & \cdots & 0 \\ 0 & V_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V_N \end{pmatrix}$$

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- Local magnetic charge:
  - $N \times N$ complex matrix $V(x,y) \to N$ "magnetic colors" at each vertex

- The symmetry that is related with closed arrays:

$$V(x,y) \to U(x)V(x,y)U^{\dagger}(y)$$

$$without\ \ N-matching \Rightarrow \begin{cases} U(1)^N \\ U(N) \end{cases}$$

$$with\ \ N-matching \Rightarrow \begin{cases} U(1)^{N-1} \\ SU(N) \end{cases}$$

# Mixed ensemble of center vortices and chains

$$Z_{\rm mix}[B] \propto \int DVD\zeta\ \exp\left(-W_{\rm mix}[V,\zeta]\right)$$

$$W_{\rm mix}[V,\zeta] = W_{\rm c.v.}[V] + W_{\rm m}[\zeta, V]$$

$$W_{\rm m}[\zeta, V] = -\sum_{l}\langle\zeta^{\dagger} R\,\zeta\rangle + \sum_{x}\sum_{\alpha}\left(\tilde{\eta}\,|\zeta_{\alpha}|^{2} + \tilde{\lambda}\,|\zeta_{\alpha}|^{4}\right)$$

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- this generates "holonomies" $(L(\mathcal{C}) = na)$

$$e^{-\tilde{\mu}(L(\mathcal{C}_1)+L(\mathcal{C}_2)+\dots)}\, {\rm Tr}\, \Gamma(\mathcal{C}_1)\, {\rm Tr}\, \Gamma(\mathcal{C}_2) \dots$$

$$\text{monopole fields} = \begin{cases} \zeta_\alpha \to \phi_\alpha \in \mathbb{C} \quad , \quad \alpha = ij \\ \zeta_\alpha \text{ is complex adjoint} \end{cases}$$

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- $U(1)^{N-1}$ and $SU(N)$ as long as

$$\begin{cases} \phi_\alpha(x) \to e^{i\theta(x)\cdot\alpha_{ij}} \, \phi_\alpha(x) \quad , \quad \alpha_{ij} = \omega_i - \omega_j \\ \zeta_\alpha(x) \to U(x) \zeta_\alpha(x) U^{-1}(x) \end{cases}$$

# Percolating phase

i) center-vortex condensate

$$V(x,y) = \vartheta U(x,y) \quad , \quad U(x,y)U^{\dagger}(x,y) = I \quad , \quad \det U(x,y) = 1$$

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$$W_{\rm mix}[U,\zeta] \approx \gamma\vartheta^2 \sum_{p} \mathrm{Tr}\Big(I - e^{iB(p)}U(p)\Big)$$

$$+\vartheta^2 \sum_{x,\mu}\sum_{\alpha}(\Delta_\mu\zeta_\alpha)^{\dagger}\Delta_\mu\zeta_\alpha + \sum_{x}\sum_{\alpha}\left(a^2 m^2|\zeta_\alpha|^2 + \tilde{\lambda}\,|\zeta_\alpha|^4\right) + \dots \,,$$

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ii) softer transition where monopoles condense (for small enough $\tilde{\mu} \to m^2 < 0$)

# Abelian projection vs. Local magnetic colors

- *The Abelian projected model is embedded in the non-Abelian one:*

$$U = \begin{pmatrix} U_1 & 0 & \cdots & 0 \\ 0 & U_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U_N \end{pmatrix} \quad , \quad \prod_i U_i = 1 \quad , \quad \zeta_\alpha = \phi_\alpha E_\alpha$$

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- In both cases
  - Wilson loop average at asymptotic distances was modeled in the lattice:

$$\langle \mathcal{W}_{\rm D}(\mathcal{C}_e) \rangle = \mathcal{N} \sum_{\omega} e^{-S(\omega)} \frac{1}{\mathfrak{D}} \, \mathrm{Tr} \left[ \mathrm{D} \left( e^{i \frac{2\pi}{N}} I \right) \right]^{\mathrm{L}(\omega, \mathcal{C}_{\rm e})}$$

  - percolating center vortices $\rightarrow$ gauge fields
  - chains $\rightarrow$ include scalar monopole fields

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  - percolating center vortices $\to$ gauge fields
  - chains $\to$ include scalar monopole fields

- $N$-ality: the frustration is blind to the specific $\beta_{\rm e}$, it only depends on $k$.
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- As the continuum is approached:
  - the lowest lattice action must cancel the frustration
  - in the whole lattice, there are different possibilities, which are expected to depend on specific weights with $N$-ality $k$

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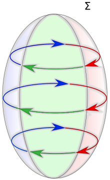


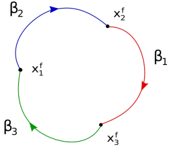

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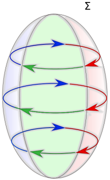
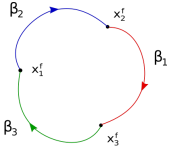



  - a monopole condensate can be formed such that the saddle-point is (Junior, LEO & Simões, 2023)

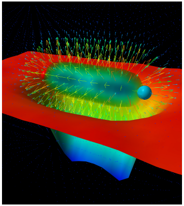
$$\zeta_\alpha(x) = \phi_\alpha(x) S(x) E_\alpha S^\dagger(x) \qquad , \qquad S \in SU(N)$$

# Abelian projection vs. Local magnetic colors

- for any $D(\cdot)$ with $N$-ality $k$, the lowest lattice action is expected to be governed by $\beta_{\rm e} = \beta_{k-{\rm A}}$ (Junior, LEO & Simões, 2020)

  - $\beta_{k-{\rm A}}$ rotates $k(N-k)$ monopole fields $\phi_\alpha \to$ Casimir law (among the possibilities: Lucini, Teper & Wenger, 2004)

  - $k$-independent widths ($k \neq 0$) (Lucini & Teper, 2001)



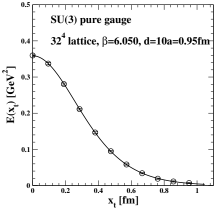
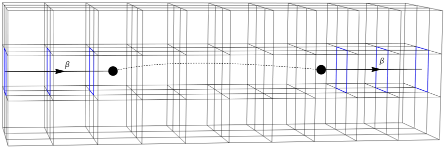
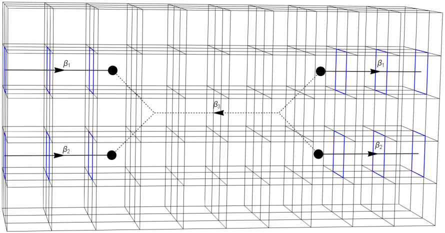



Figure: D. Leinweber, Visualizations of Quantum Chromodynamics, University of Adelaide © 2003, 2004 (left) - Interpolation of Abelian-like flux vs. $SU(3)$ lattice simulation, Cea, Cosmai, Cuteria & Papa (2017) (right) - see also Yanagihara, Iritani, Kitazawa, Asakawa, Hatsuda (2019), Yanagihara, Kitazawa (2019).

# Conclusions



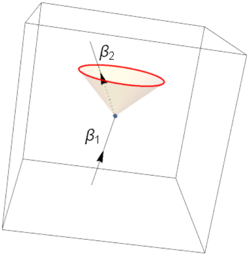

$SU(3)$

# Conclusions

- Relevant role layed by monopoles:

  *the probability to link depends on solid angles and the effect only depends on the N-ality of $D(\cdot)$*

  - Double Wilson-loops/tetraquarks are expected to be correctly described



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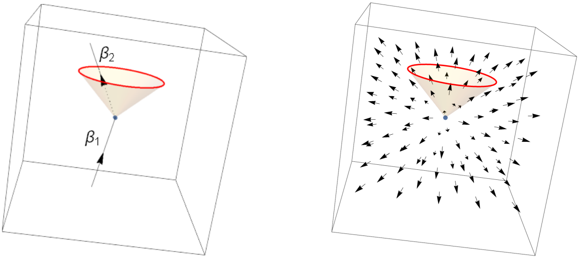


- This is in contrast with monopole ensembles, where:

  *the flux depends on solid angles but the effect depends on specific weights*

  - It is forbidden to change the weights

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- possibility to include the effect of thickness and understand the transition from the asymptotic to intermediate confining regions