Phase diagram of QCD matter with magnetic field: domain-wall Skyrmion chain in chiral soliton lattice

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JHEP 12 (2023) 032

Outline

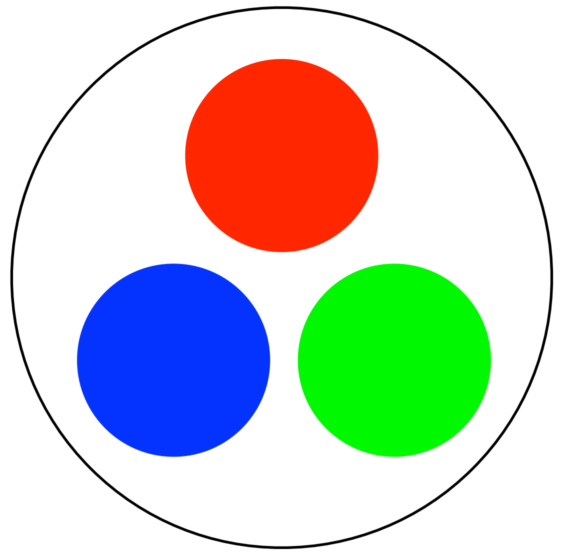
- Introduction

- Chiral soliton lattice Son and Stephanov (2008); Brauner and Yamamoto (2017)

- Domain wall Skyrmion Eto, KN and Nitta, JHEP 12 (2023) 032

Baryons and mesons

- **Baryon = Particle composed of quarks**



$$\simeq \quad \epsilon^{a_1 a_2 a_3} q^{f_1}_{a_1} q^{f_2}_{a_2} q^{f_3}_{a_3}$$

- **Pions = Nambu-Goldstone (NG) bosons :**

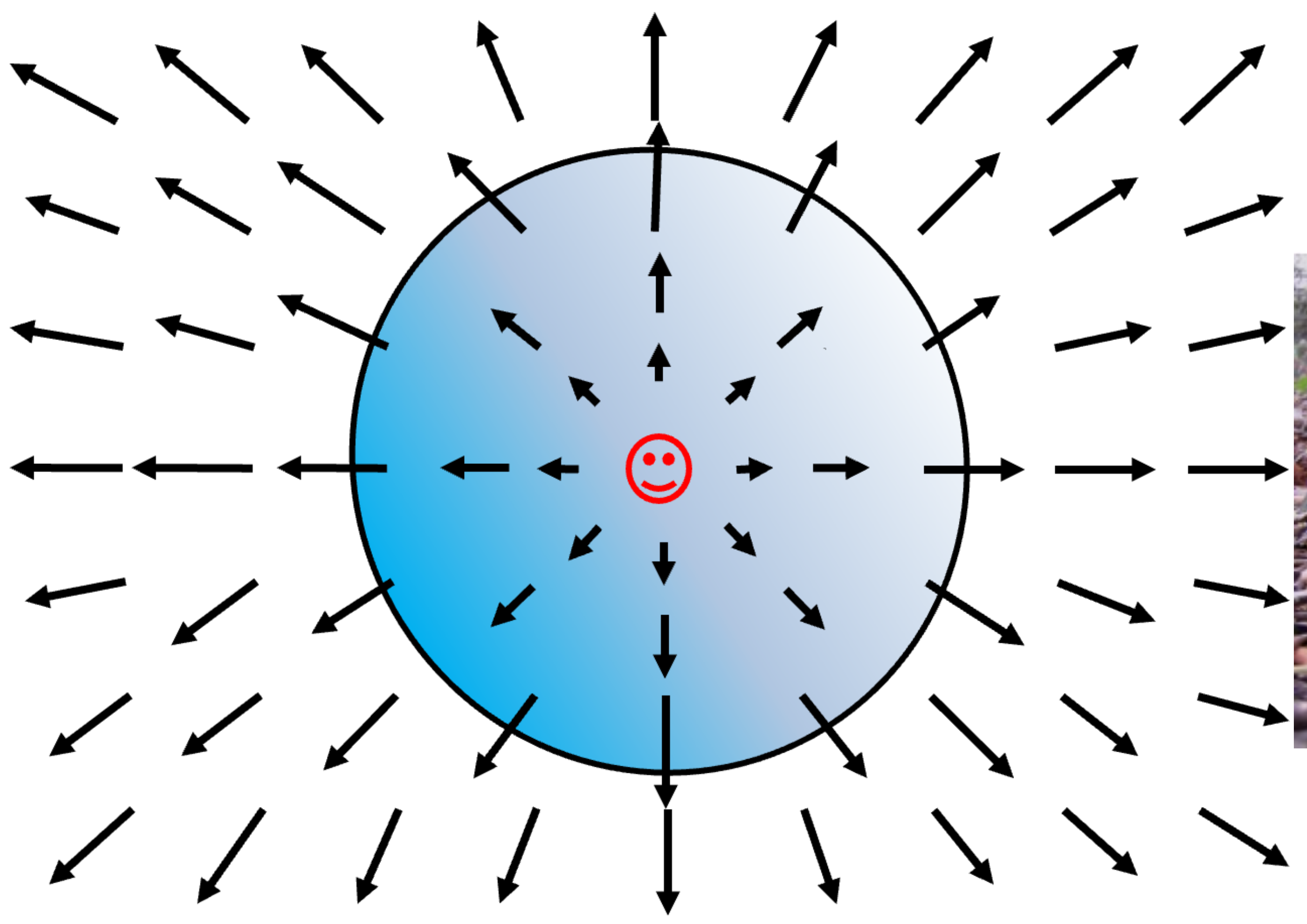
$$\Sigma(x) = \exp\left(\frac{\mathrm{i}\pi_a \tau_a}{f_\pi}\right)$$

Skyrmion

- Can the baryons be made by pions (rather than quarks)?

- Baryon as soliton = Skyrmion

Skyrme (1961)



Hedgehog

- Topological charge :

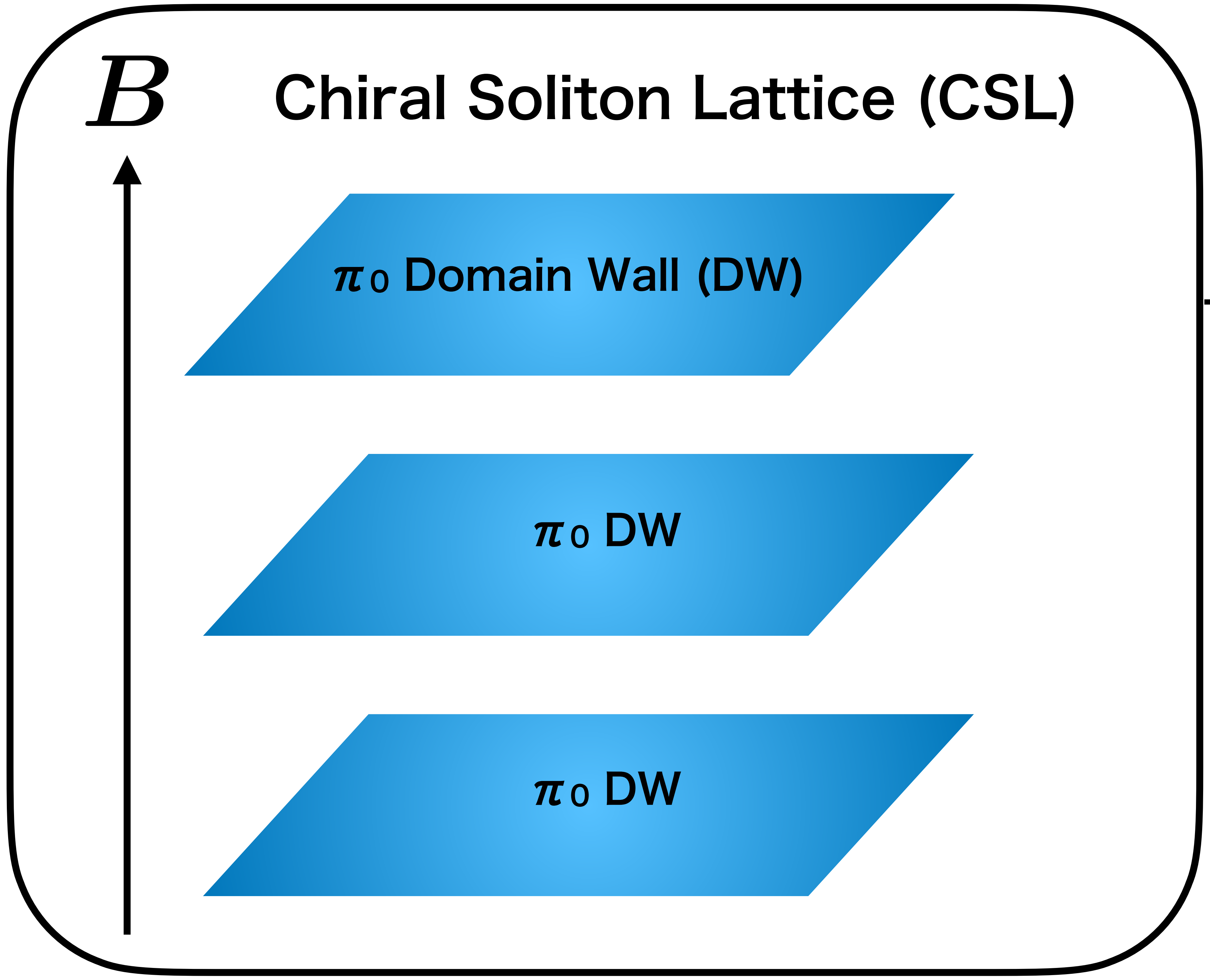
$$j^\mu_{\mathrm{B}} = \frac{\epsilon^{\mu\nu\alpha\beta}}{24\pi^2}\mathrm{tr}(\Sigma\partial_\nu\Sigma^\dagger\Sigma\partial_\alpha\Sigma^\dagger\Sigma\partial_\beta\Sigma^\dagger)$$

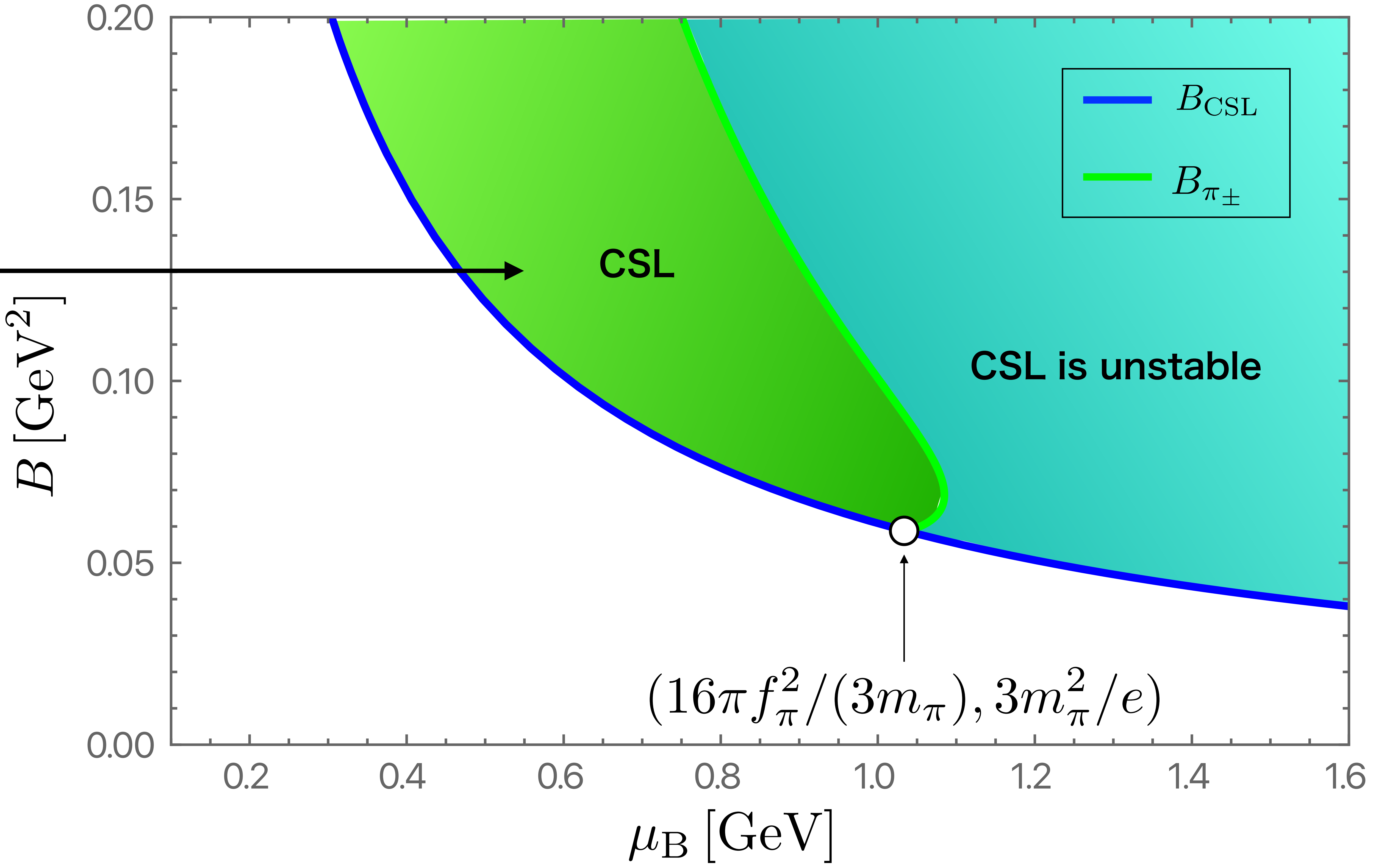
- R^3 surrounds the configuration space of the pions S^3 : $\pi_3(S^3)$

Solitonic phase in QCD

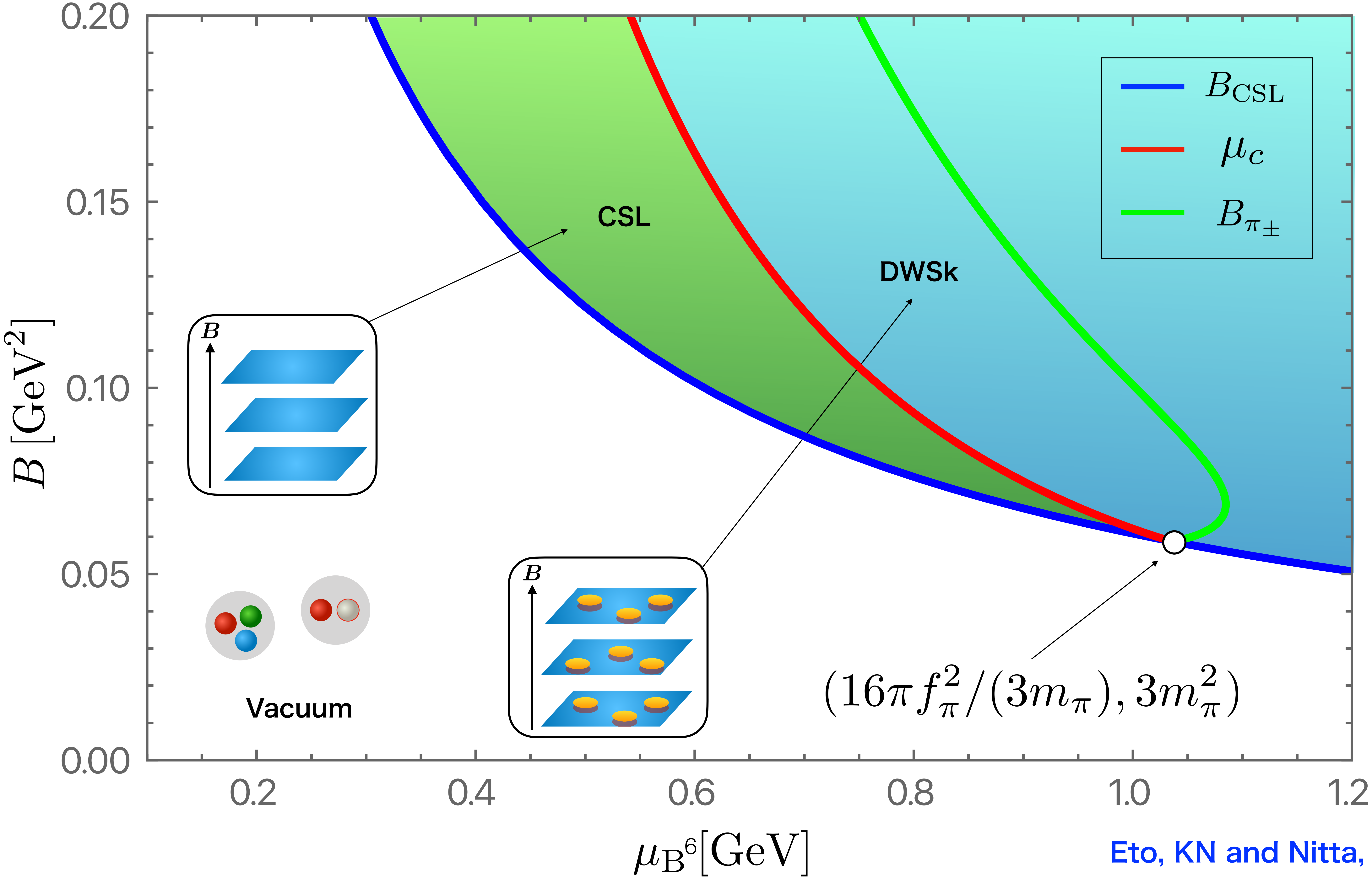
- **Energy of solitons is generally larger than that of uniform state since the solitons are spatially localized configurations.**

- **Solitons appear in the ground state of QCD at finite B and μ_B.** Brauner and Yamamoto (2017)





What I want to explain today



Eto, KN and Nitta, JHEP 12 (2023) 032

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ChPT w/ topological terms

- **Normal terms :** $\mathcal{L}_{\mathrm{ChPT}} = \frac{f_\pi^2}{4}\mathrm{tr}\left(D_\mu \Sigma D^\mu \Sigma^\dagger\right) - \frac{f_\pi^2 m_\pi^2}{4}(2 - \Sigma - \Sigma^\dagger)$

- **Baryon current in terms of mesons = $\pi_3(S^3)$ topological charge**

$$j_{\mathrm{B}}^\mu = \underbrace{-\frac{\epsilon^{\mu\nu\alpha\beta}}{24\pi^2}\mathrm{tr}\{L_\nu L_\alpha L_\beta}_{\text{Skyrimon charge}} - \underbrace{3ie\partial_\nu\left[A_\alpha Q(L_\beta + R_\beta)\right]}_{\mathrm{U(1)_{em}\ gauged\ part}}\}$$

$$L_\mu \equiv \Sigma\partial_\mu\Sigma^\dagger\,,\ R_\mu \equiv \partial_\mu\Sigma^\dagger\Sigma$$
$$Q = \mathrm{diag}(2/3, -1/3)$$

- “trial and error” $\mathrm{U(1)_{em}}$ gauging while preserving the baryon number conservation

Son and Stephanov (2008); Goldstone and Wilczek (1981)

- **Effective Lagrangian :** $\mathcal{L}_{\mathrm{B}} = -A_{\mathrm{B}}^\mu j_{\mathrm{B}\mu}\,, \quad A_{\mathrm{B}}^\mu = (\mu_{\mathrm{B}}, \mathbf{0})$

Son and Stephanov (2008)

sine-Gordon theory with the topological term

- I first ignore $\pi_\pm$: $\Sigma = \mathrm{e}^{\mathrm{i}\phi_3\tau_3}$
- Reduced Hamiltonian (B is oriented in z-direction) :

$$\mathcal{H} = \frac{f_\pi^2}{2}(\partial_z\phi_3)^2 + f_\pi^2 m_\pi^2(1-\cos\phi_3) - \frac{e\mu_\mathrm{B}}{4\pi^2}B\partial_z\phi_3$$

- The last term stems from the 2nd term of the skyrmion term.

$$\mathcal{L}_\mathrm{B} = -\mu_\mathrm{B}\frac{\epsilon^{0ijk}}{24\pi^2}\mathrm{tr}\{\cancel{L_i L_j L_k} - \underline{3\mathrm{i}e\partial_i\left[A_j Q(L_k+R_k)\right]}\}$$

- B≠0 → Finite 1st derivative term → Favor ϕ inhomogenity
- What is a ground state at finite B?

Brauner and Yamamoto (2017)

Chiral Soliton Lattice

- **EOM = Pendulum:** $\partial_z^2 \phi_3 = m_\pi^2 \sin\phi_3$

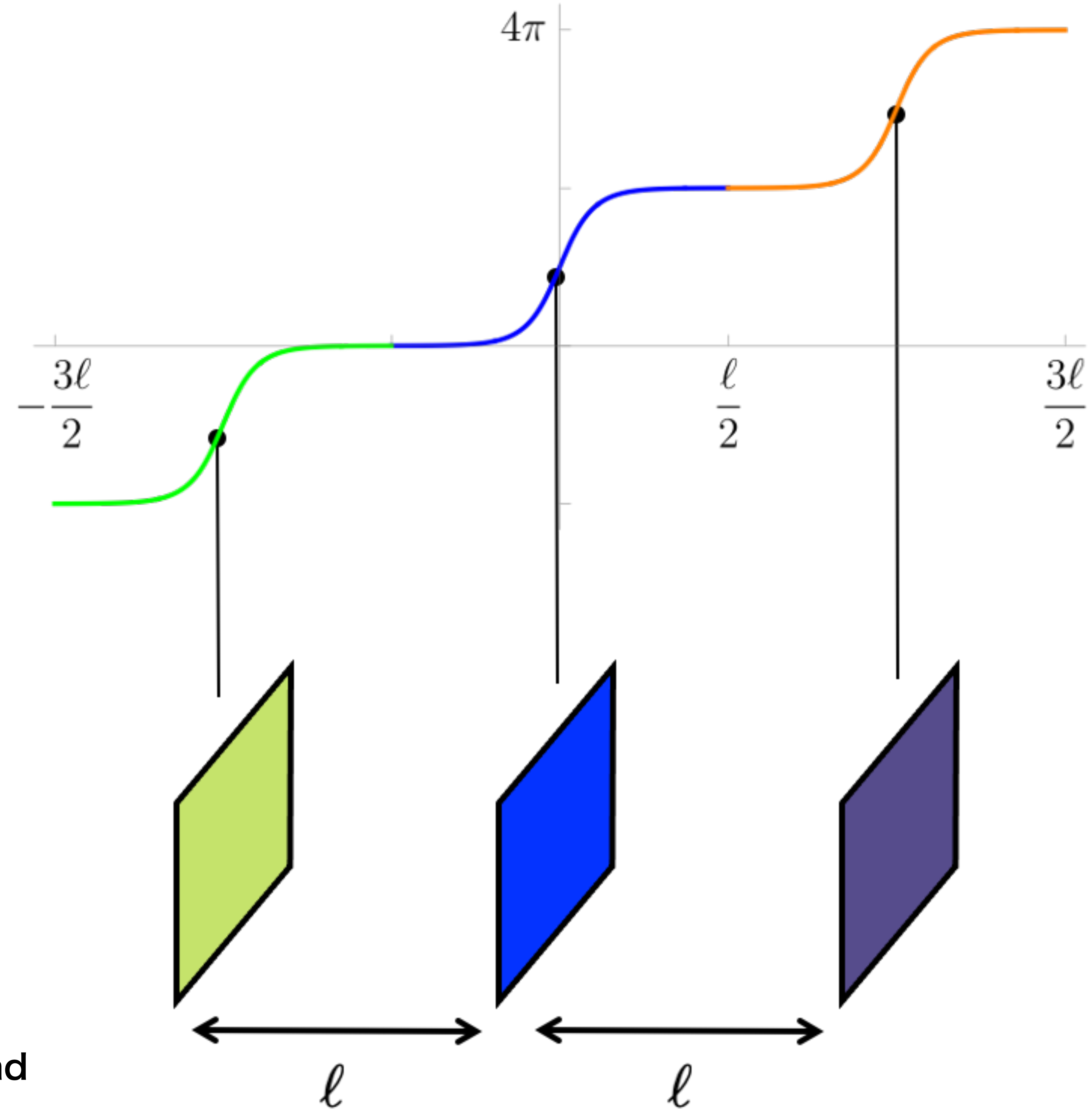
- **Analytic solution :** $\bar{\phi} = 2\mathrm{am}\,(z/\kappa, \kappa) + \pi$

  κ : Elliptic modulus

- **Period :** $\phi(z+\ell) = \phi(z) + 2\pi$

  $\ell = 2\kappa K(\kappa)$

  $K(\kappa)$: The complete elliptic integral of the first kind



Brauner and Yamamoto (2017)

Minimization of the total energy

- **Minimizing the total energy gives us the optimal κ.**

$$\mathcal{E}_{\rm tot} = \int_0^\ell \mathrm{d}z \left[\underbrace{\frac{f_\pi^2}{2}(\partial_z\phi)^2 + f_\pi^2 m_\pi^2 (1-\cos\phi)}_{\text{positive}} - \underbrace{\frac{\mu_{\rm B}}{4\pi^2} B\partial_z\phi}_{\text{negative!}} \right]$$

$\phi(\ell) - \phi(0) = 2\pi$

- **Energy minimization condition :**

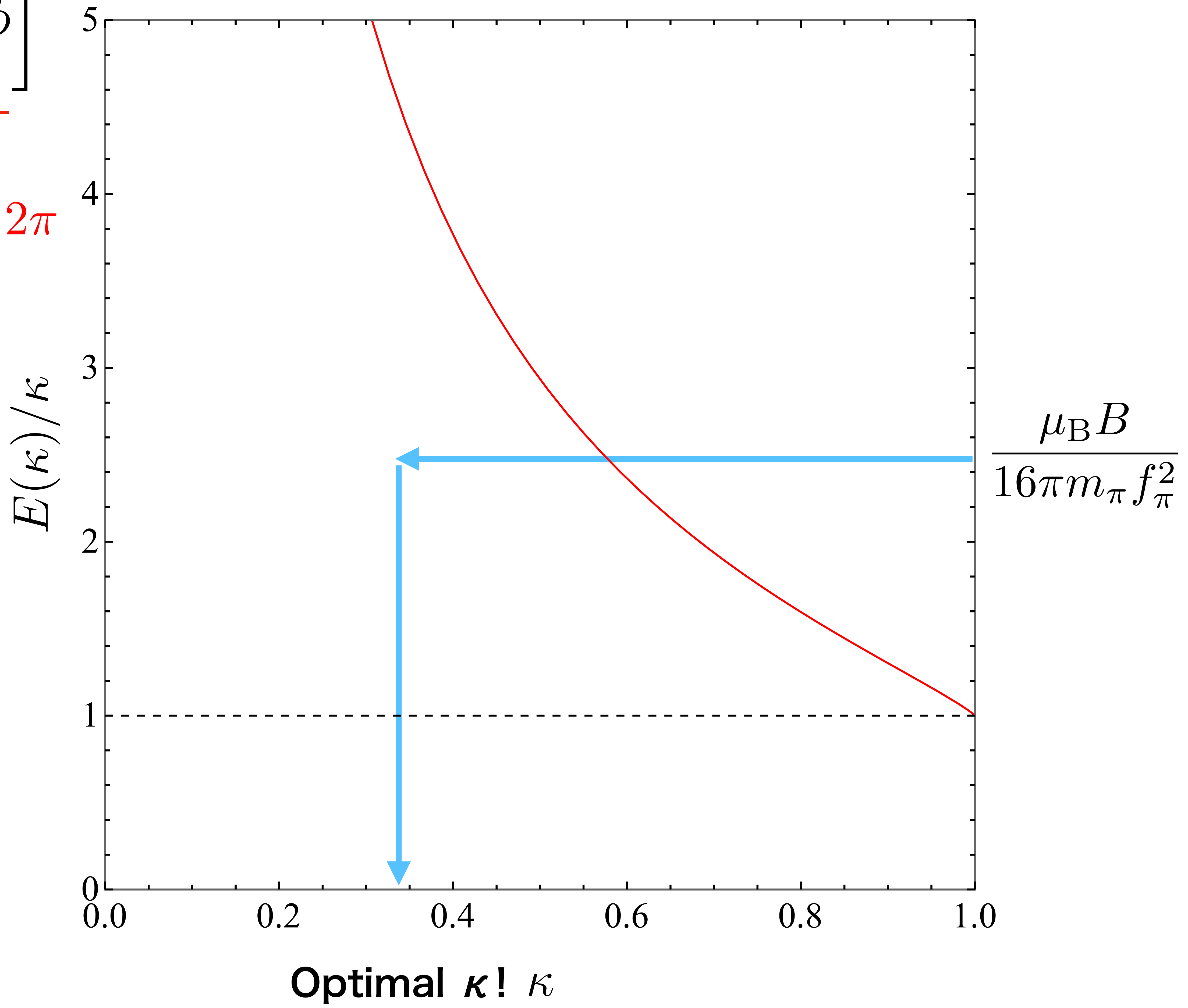
$$\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{\mathcal{E}_{\rm tot}}{\ell}\right) \to \frac{E(\kappa)}{\kappa} = \frac{\mu_{\rm B} B}{16\pi m_\pi f_\pi^2}$$

$E(\kappa)$: **The complete elliptic integral of the 2nd kind**

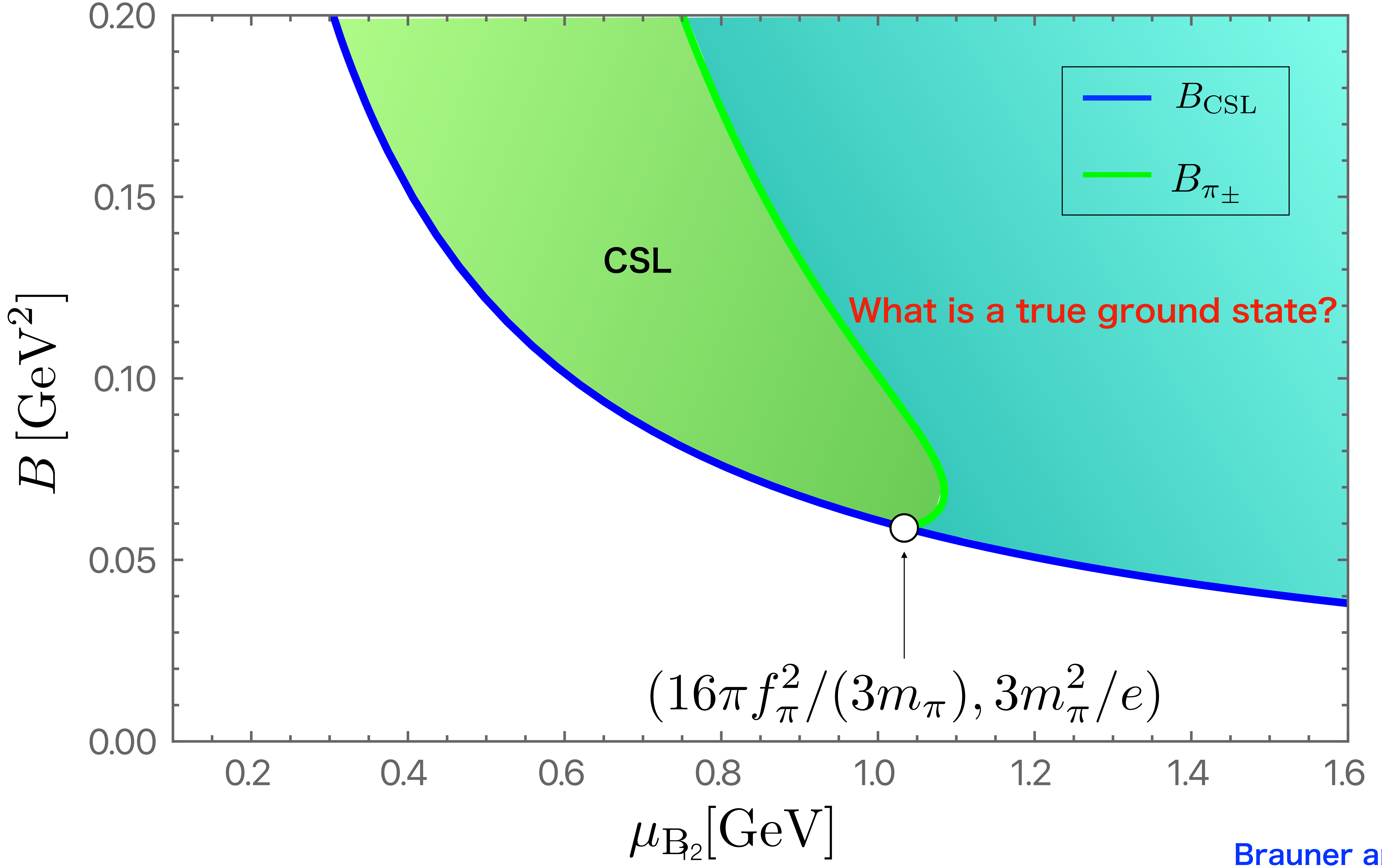
- **Critical magnetic field :** $B_{\rm CSL} = 16\pi f_\pi^2 m_\pi / \mu_{\rm B}$

  Brauner and Yamamoto (2017)

- **The energy density with the minimization condition is smaller than that of ϕ_3=0.**



μ_B-B phase diagram



Brauner and Yamamoto (2017)

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- Domain wall Skyrmion   Eto, KN and Nitta, JHEP 12 (2023) 032

What we overlooked

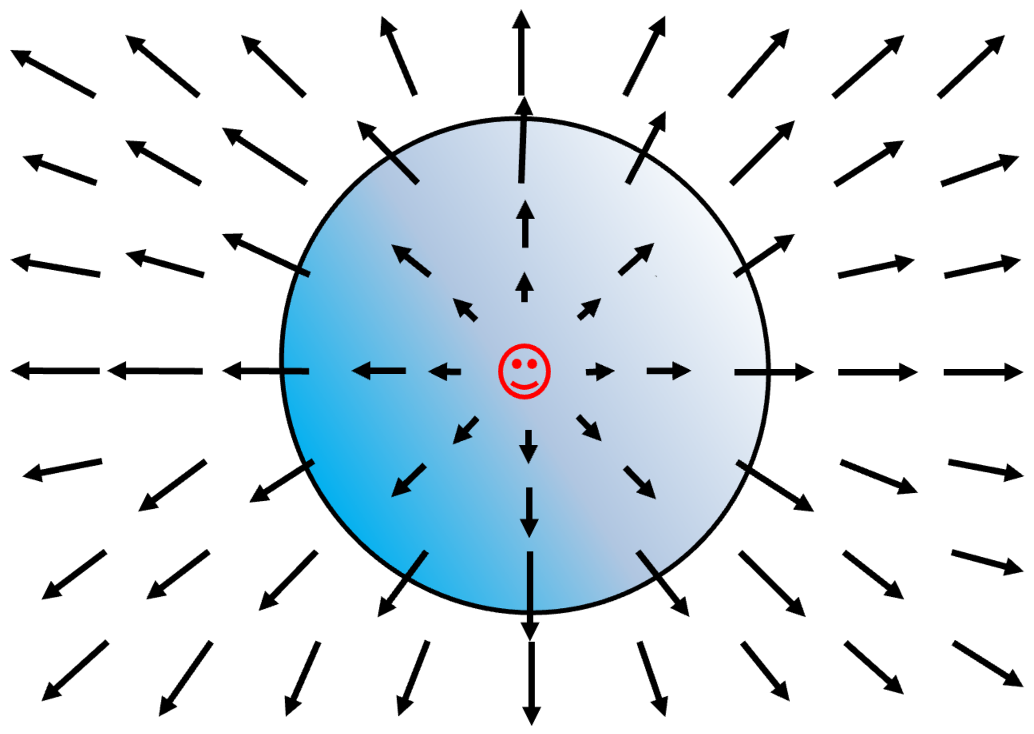
- The baryon current contains the Skyrmion charge, which is $O(p^2)$.

$$\mathcal{L}_\mathrm{B} = -\mu_\mathrm{B}\frac{\epsilon^{0ijk}}{24\pi^2}\mathrm{tr}\{\underline{L_iL_jL_k} - \underbrace{3ie\partial_i\left[A_jQ(L_k+R_k)\right]}_{\mu_\mathrm{B}B\partial_z\phi_3 \subset \mathcal{L}_\mathrm{B}}\}$$

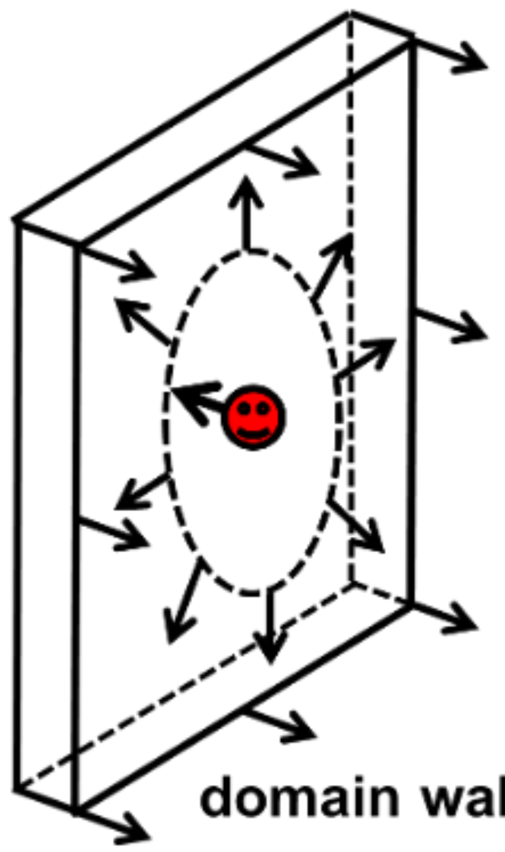
- In the unstable region, $\pi_\pm$ is important element.
- The Skyrmion charge term becomes finite only when $\pi_\pm$ is considered.

- When Σ has Skymion number, 1st term decreases the energy density!

Skyrmion : $\pi_3(S^3)$



Baby Skyrmion : $\pi_2(S^2)$



become stable at finite μ_B!

Non-abelian soliton

- **The single soliton:** $\Sigma_0 = \mathrm{e}^{\mathrm{i}\sigma_3\theta}\,,\ \ \theta = 4\tan^{-1}\mathrm{e}^{m_\pi z}$

- **More general solution :** $\Sigma = g\Sigma_0 g^\dagger = \exp(\mathrm{i}\theta g\tau_3 g^\dagger)\,,\quad g \in \mathrm{SU}(2)_\mathrm{V}$

- **Σ_0 is invariant under U(1) transformation :** $g = \mathrm{e}^{\mathrm{i}\tau_3\theta}$

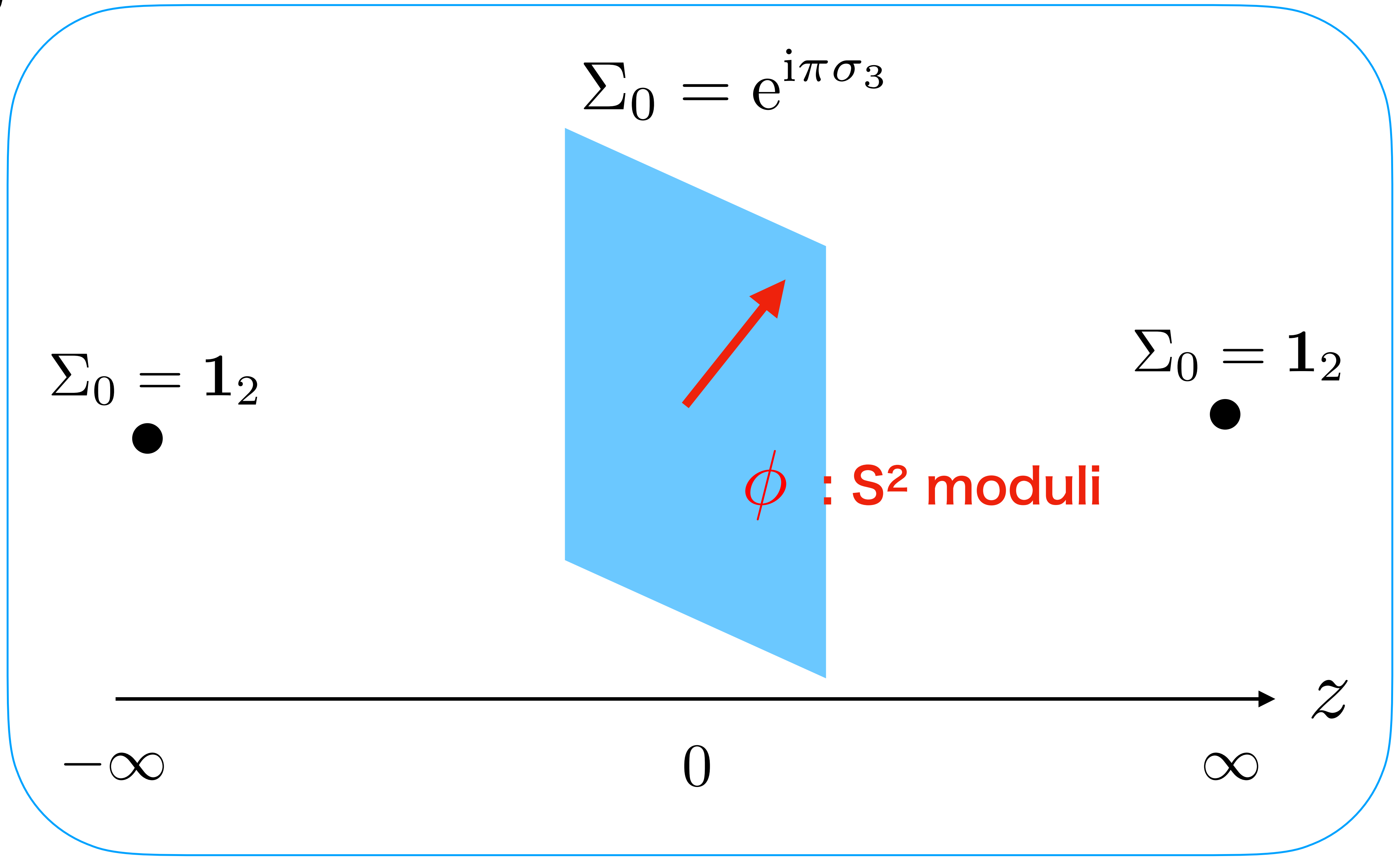
- **SSB of SU(2)$_\mathrm{V}$ → U(1)** ⟶ $\mathrm{SU}(2)/\mathrm{U}(1) \cong S^2$

  Moduli (NG mode)

- **The collective coordinate :** $\phi \in \mathbb{C}^2\,,\qquad \phi^\dagger\phi = 1$

  Nitta (2015); Eto and Nitta (2015)

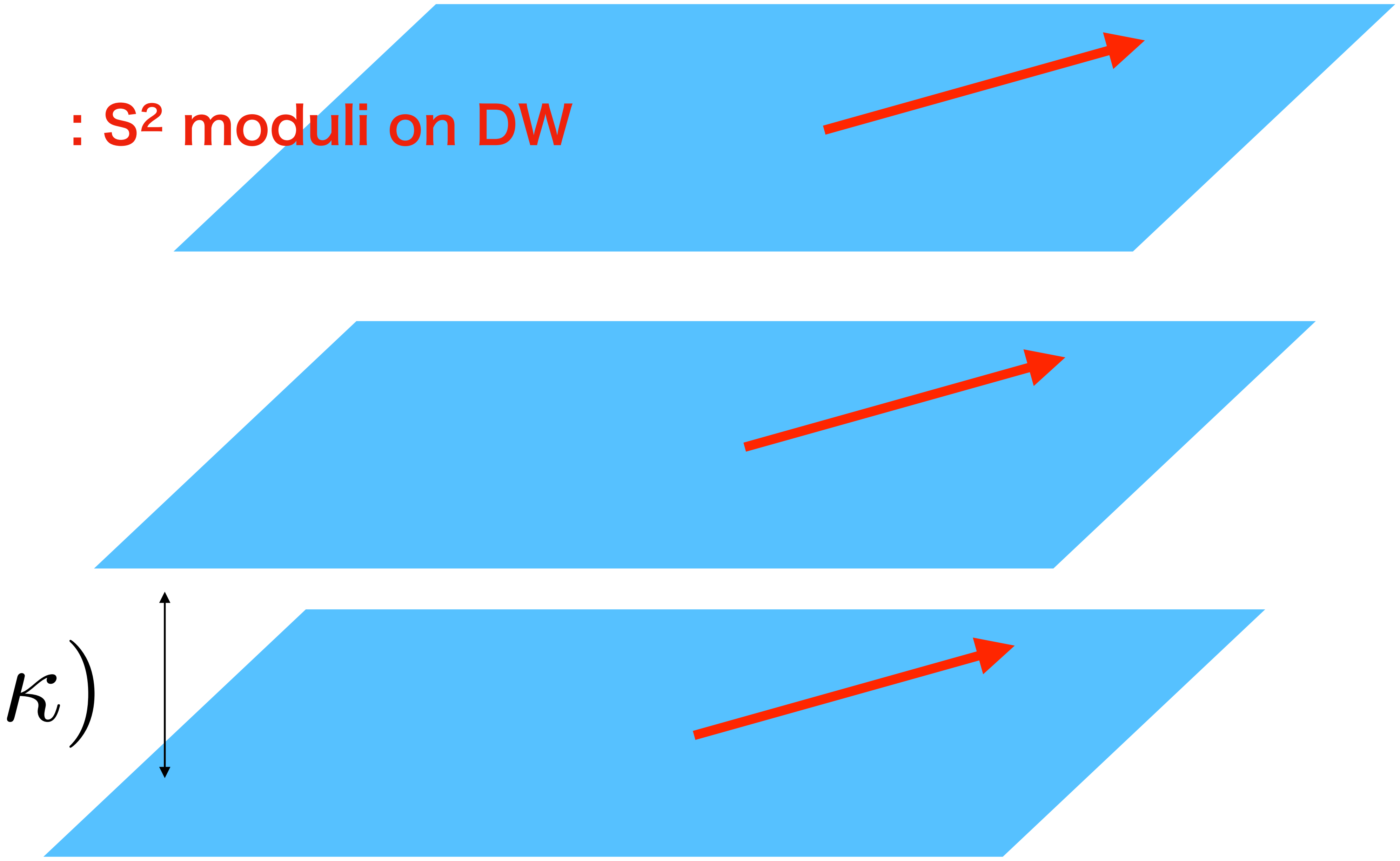
$$g\sigma_3 g^\dagger = 2\phi\phi^\dagger - 1$$



EFT of the DW

- **Construct DW world effective theory via the moduli approximation.**

- **This EFT identifies S² moduli $\boldsymbol{\phi}$ as dof.**



- **Promote the moduli to a field on 2+1 dim world volume**

$$\phi \to \phi(x^\alpha)\,, \quad (\alpha = 0, 1, 2)$$

$$\ell = 2\kappa K(\kappa)$$

- **Integrating over the codimension z :** $$\mathcal{L}_{\rm EFT} = \int_{-\infty}^{\infty} \mathrm{d}z\, \underline{\mathcal{L}_{\rm ChPT} + \mathcal{L}_{\rm B}}$$

Substitution

$$\Sigma = \exp(2\mathrm{i}\theta\phi\phi^\dagger)u^{-\mathrm{i}\chi_3^{\rm CSL}} \qquad \chi_3^{\rm CSL} = 2\,\mathrm{am}\left(\frac{m_\pi z}{\kappa}, \kappa\right) + \pi$$

- **Effective Lagrangian :** $\mathcal{L}_{\mathrm{EFT}} = \mathcal{L}_{\mathrm{const}} + \mathcal{L}_{\mathrm{kin}} + \mathcal{L}_{\mathrm{topo}}$ Eto, KN and Nitta, JHEP 12 (2023) 032

- **DW tension :** $\mathcal{L}_{\mathrm{const}} = -\mathcal{E} + \dfrac{e\mu_{\mathrm{B}} B}{2\pi}$

- The condition that the DW tension is negative gives B_{CSL}.

- **Kinetic term :**

$$\mathcal{L}_{\mathrm{kin}} = \mathcal{C}(\kappa)[(\phi^\dagger D_\alpha \phi)^2 + D^\alpha \phi^\dagger D_\alpha \phi]$$

$$\mathcal{C}(\kappa) \equiv \frac{16 f_\pi^2}{3 m_\pi} \frac{(2-\kappa^2)E(\kappa) - 2(1-\kappa^2)K(\kappa)}{\kappa^3}$$

- **Topological terms :**

$$-\mu_{\mathrm{B}} \frac{\epsilon^{0ijk}}{24\pi^2} \int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{tr}\{L_i L_j L_k - 3\mathrm{i}e\partial_i [A_j Q(L_k + R_k)]\}$$

$$\mathcal{L}_{\mathrm{topo}} = -2\mu_{\mathrm{B}} q + \frac{e\mu_{\mathrm{B}}}{2\pi} \epsilon^{03jk} \partial_j [A_k (1 - n_3)]$$

Stabilizing the baby Skyrmion at finite μ_{B}

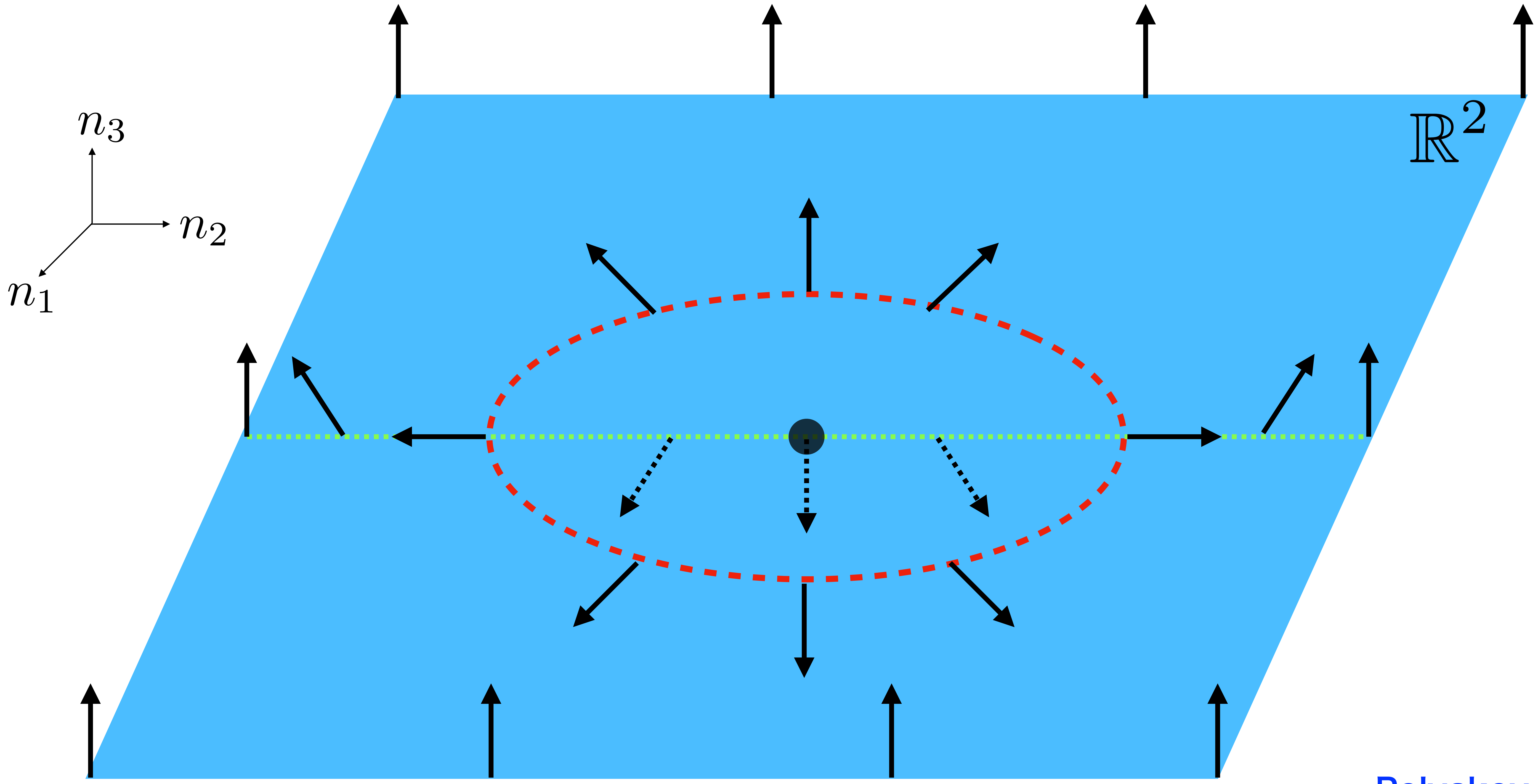
O(3) nonlinear sigma model

$n_a \equiv \phi^\dagger \sigma_a \phi \quad |\boldsymbol{n}| = 1$

$\pi_2(S^2)$ topological charge (density) : $$q \equiv -\frac{\mathrm{i}}{2\pi} \epsilon^{ij} \partial_i \phi^\dagger \partial_j \phi\,, \quad k = \int \mathrm{d}^2 x\, q \in \mathbb{Z}$$

Baby Skyrmion

- **Configuration on DW surrounding S²:** $\uparrow$ = n_a $\quad \boldsymbol{n}^2 = 1$



Polyakov and Belavin (1975)

Bogomol'nyi bound

- Baby Skyrmion naturally appears when minimizing the Hamiltonian.

$$\mathcal{H}_{\rm DW} = \frac{\mathcal{C}(\kappa)}{4}\partial_i \boldsymbol{n}\cdot\partial_i \boldsymbol{n} + 2\mu_{\rm B} q - \frac{e\mu_{\rm B}}{2\pi}\epsilon^{03jk}\partial_j[A_k(1-n_3)]$$

- Completing the square of the kinetic term is useful!

$$(\partial_i \boldsymbol{n})^2 = \frac{1}{2}\underbrace{(\partial_i \boldsymbol{n} \pm \epsilon_{ij}\boldsymbol{n}\times\partial_j \boldsymbol{n})^2}_{=0} \pm 8\pi q \geq \pm 8\pi q$$

$\rightarrow$ BPS equation $\rightarrow$ Baby Skyrmion!

- Bogomol'nyi bound : $E_{\rm DW} \geq \underline{2\pi\mathcal{C}(\kappa)|k| + 2\mu_{\rm B}k} - \underline{\frac{e\mu_{\rm B}}{2\pi}\int d^2x\,\epsilon^{03jk}\partial_j[A_k(1-n_3)]}$

Can it be negative here? Let us consider anti-lump!

Some constraints on the lump

Eto, KN and Nitta, JHEP 12 (2023) 032

Constraint on baby Skyrmion

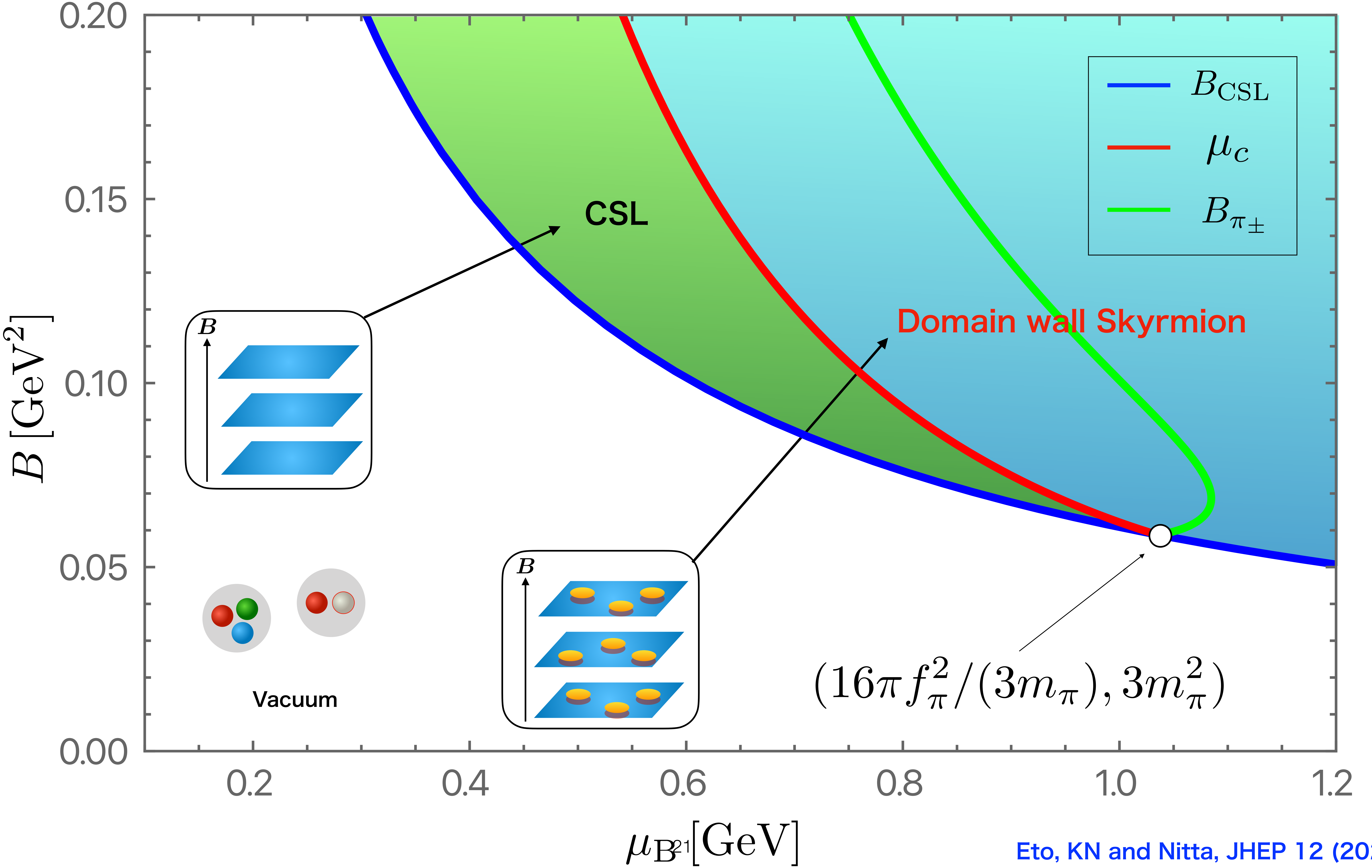
- k anti-Baby Skyrmion solution: $n_3 = \frac{1-|f|^2}{1+|f|^2}\,,\ f = \frac{b_{k-1}\bar{w}^{k-1} + \cdots + b_0}{\bar{w}^k + a_{k-1}\bar{w}^{k-1} + \cdots + a_0}$

- E_{DWSk}, for the k anti-baby Skyrmion: $E_{\mathrm{DWSk}} = \underline{2\pi C(\kappa)|k| - 2\mu_{\mathrm{B}}|k|} + e\mu_{\mathrm{B}} B |b_{k-1}|^2$

  Can it be negative here?

- In order to minimize E_{DWSk}, $b_{k-1}=0$.

- Critical baryon chemical potential: $\mu_{\mathrm{B}} \geq \mu_{\mathrm{c}} = \pi C(\kappa)$ Eto, KN and Nitta, JHEP 12 (2023) 032



Eto, KN and Nitta, JHEP 12 (2023) 032

Thank you for your attention!

Back up

Elliptic integrals and functions

- **The elliptic integral of the first kind :** $k' = \sqrt{1-k^2}$

$$K(k) = \int_0^{\pi/2} \mathrm{d}\theta \, \frac{1}{\sqrt{1-k^2\sin^2\theta}} \simeq \ln\frac{4}{k'^2} + \frac{k'^2}{4}\left(\ln\frac{4}{k'^2} - 1\right)$$

- **The elliptic integral of the second kind :**

$$E(k) = \int_0^{\pi/2} \mathrm{d}\theta \, \sqrt{1-k^2\sin^2\theta} \simeq 1 + \frac{k'^2}{2}\left(\ln\frac{4}{k'^2} - \frac{1}{2}\right)$$

Fluctuations of $\pi_\pm$

- Fluctuation around the CSL background :

- CSL is unstable against fluctuations of $\pi_\pm$ above $B^{\pi\pm}{}_{\mathrm{BEC}}$

$$\boxed{\frac{E(k)}{k} = \frac{\mu_{\mathrm{B}} B^{\pi\pm}_{\mathrm{BEC}}}{16\pi m_\pi f_\pi^2}} \qquad \boxed{B^{\pi\pm}_{\mathrm{BEC}} = \frac{m_\pi^2}{k^2}\left(2 - k^2 + 2\sqrt{1 - k^2 + k^4}\right)}$$

$$k = k(B^{\pi\pm}_{\mathrm{BEC}})$$

- Derive the effective action up to the 2nd of the fluctuations from the CSL
- Calculate the dispersion relation ω
- When $\omega^2<0$, the fluctuation is tachyonic and CSL becomes unstable.

Brauner and Yamamoto (2017

EOM of the fluctuations

- **Fluctuation around the CSL background :**

$$\omega^2\pi_+ = \underbrace{\left[-\partial_x^2 + B^2\left(x - \frac{p_y}{B}\right)^2\right]}_{\text{Giving the Landau quantization}}\pi_+ + (\partial_z^2 + 2\mathrm{i}\partial_z + m_\pi^2 \mathrm{e}^{\mathrm{i}\phi_3})\pi_+$$

- **Chiral limit :** $\omega^2 = p_z^2 - \underbrace{\frac{\mu_B B p_z}{2\pi^2 f_\pi^2}}_{\text{Deducing the energy!}} + (2n+1)B$

- **ω^2<0 :** $B_{\pi_\pm} = \frac{16\pi^4 f_\pi^2}{\mu_B^2}$

Brauner and Yamamoto (2017)