Fractional instantons, anomalies, and the gaugino condensate

Erich Poppitz, U. of Toronto

with Mohamed Anber, Durham U.

2210.13568, 2307.07495, and 240*.xxxxx

- **problems of confinement and dynamical mass generation are difficult, strong coupling (incl. recent) interest in various toroidal compactifications of 4d theories, which allow for** $\bf{calculus}$ be semiclassical studies of confinement on $\mathbb{R} \times \mathbb{T}^3,\, \mathbb{R}^2 \times \mathbb{T}^2,\, \mathbb{R}^3 \times \mathbb{S}^1 \dots$
	- García Perez, González-Arroyo…1990+; Ünsal…2007+; Tanizaki-Ünsal…2020+
		-

the big picture:

not the real world... but argue for /shown/ continuous connection to \mathbb{R}^4

-
-
- *raison d'être* **to speak here: many things I use are relevant for non-SUSY theories** ∃
- e.g. fractional instantons \simeq center vortices/monopole-instantons

… this talk:

particular focus on \mathbb{T}^4 , fractional instantons, and the gaugino condensate

I realize this is not a SUSY conference, so apologies!

missed in the 1980s

renewed interest in due to generalized anomalies 4

to see need spacetime with noncontractible 2-cycles

Gaiotto,Kapustin,Komargodski,Seiberg 2014-

MATHEMATICAL PHYSICS

A New Kind of Symmetry Shakes Up Physics

 \blacksquare 23 | |

So-called "higher symmetries" are illuminating everything from particle decays to the behavior of complex quantum systems.

The symmetries of 20th-century physics were built on points. Higher symmetries are based on one-dimensional lines.

Samuel Velasco/Quanta Magazine

no way to review generalized anomalies, or many details, will only give flavour disclaimer: so, get to the point: chiral U(1) broken to \mathbb{Z}_{2N} **by anomaly**

 \mathbb{Z}_{2N} spontaneously broken to \mathbb{Z}_2 by bilinear gaugino condensate ($\lambda^2(x) \equiv$ tr $\lambda^{\alpha}(x)\lambda_{\alpha}(x)$)

the "mother" of all exact results in SUSY

$$
\langle \lambda^2 \rangle = e^{i \frac{2\pi k}{N}} c \Lambda^3, k = 1, ..., N, c = 16\pi^2
$$

1983-1999: Novikov, Shifman, Vainshtein, Zakharov; Amati, Konishi, Rossi, Veneziano; Affleck, Dine, Seiberg; Cordes; Finnell, Pouliot (SQCD \rightarrow SYM on R^4); Davies, Hollowood, Khoze, Mattis;... 2014 Anber, Teeple, EP (SYM on $R^3 \times S^1 \rightarrow$ SYM on R^4)

semiclassical weakly-coupled instanton calculations + power of SUSY

recent independent large-N lattice determination! 2406.08955

SYM in 4d: SU(N) + 1 massless adjoint Weyl fermion λ^a_α (SUSY emergent when $m^a_\lambda=0$) $\frac{a}{\alpha}$ (SUSY emergent when $m_\lambda = 0$

> Bonnano, García Perez, González-Arroyo, Okawa et al

chiral U(1) broken to \mathbb{Z}_{2N} by anomaly **so, get to the point:**

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here, I will discuss the calculation of the condensate on \mathbb{T}^4

why, if all agree so well?

- **SYM in 4d: SU(N) + 1 massless adjoint Weyl fermion** *λ* **(SUSY emergent when m=0)** *a α*
- \mathbb{Z}_{2N} spontaneously broken to \mathbb{Z}_2 by bilinear gaugino condensate ($\lambda^2(x) \equiv$ tr $\lambda^\alpha(x) \lambda_\alpha(x)$)

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here, I will talk about the calculation of the condensate on \mathbb{T}^4 **why, if all agree so well?**

1. because we can: new developments allow us to do the \mathbb{T}^4 calculation - first attempt in 1984, Cohen and Gomez, *could not and did not* compute "c" 4

 $\mathbb{R} \times \mathbb{T}^3$, $\mathbb{R}^2 \times \mathbb{T}^2$, $\mathbb{R}^3 \times \mathbb{S}^1 \dots$: all argue for continuity to \mathbb{R}^4 *SYM the only theory where exact agreement should hold -and one should get that one case straight*

2. the semiclassical objects (instantons on twisted torus) are closely related to both center vortices and monopoles, argued to be responsible for confinement/mass gap/chiral symmetry breaking as opposed to BPST/ADHM instantons used in \mathbb{R}^4 calculation García Perez-González-Arroyo et al, more recent: Wandler-EP '22; Hayashi-Tanizaki; Güvendik-Schäfer-Ünsal; Wandler '24

 $\langle \lambda^2 \rangle = e^{i \frac{2\pi k}{N}} c \Lambda^3, k = 1,...,N, c = 16\pi^2$

here, I will talk about the calculation of the condensate on \mathbb{T}^4 **why, if all agree so well?**

3. we'll see that calculation raises interesting questions about semiclassics, boiling down to the basic definition of path integrals … not quite understood!

1. because we can: new developments allow us to do the \mathbb{T}^4 calculation - first attempt in 1984, Cohen and Gomez, *could not and did not* compute "c" 4

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(taking one particular vacuum)

$\langle \lambda^2 \rangle = c \Lambda^3$ in fact, on \mathbb{T}^4 we'll be able to do more than *h* In be able to do h *nore* than Here, (= *^L*4) is the extent of the Euclidean time direction, (1)*^F* is inserted to impose

SUSY Ward identities: $\langle \lambda^2(x_1) \lambda^2(x_2) \dots \lambda^2$ generator *T* ˆ

 \Rightarrow x-independence $/$ + clustering $/$ *x*⁴ twist of the boundary conditions by a center symmetry transformation (*n*³⁴ = 1). A

verified in weak-coupling calculation of $\langle \lambda^{2k} \rangle$ For *^O* ⁼ ^Q*^k* werified in weak-coupling calculation of $\langle \lambda^{2n} \rangle$ in SQCD on R

we will also calculate $\langle \lambda^{2k} \rangle$ on small \mathbb{T}^4 , gcd(N,k)=1; agrees with \mathbb{R}^4 classically in this paper. For brevity, in what follows we denote *^O* = (tr 2)*^k* and write (5.2) *C*(*x*1*, ..., xk*) = $\mathcal{L}^{\text{max}}_{\text{max}}$ I also calc ate $\langle \lambda^{2k} \rangle$ on small \mathbb{T}^4 .

SUSY Ward identities:
$$
\langle \lambda^2(x_1) \lambda^2(x_2) \dots \lambda^2(x_k) \rangle \equiv \langle \lambda^{2k} \rangle = (c \Lambda^3)^k
$$

verified in weak-coupling calculation of $\langle \lambda^{2K} \rangle$ in SQCD on \mathbb{R}^4 using ADHM **i**
1 transfer the path is precisely the path integral (1.1) computed semi-
1.1) computed semi-transfer the pat $\langle \lambda^{2k} \rangle$ in SQCD on \mathbb{R}^4

 $f(16\pi^2\Lambda^3)^k$ for $\mathcal{N}^{-1} = N$ \sim all \mathbb{T}^4 , \mathfrak{g} ⁸⇡² *Ng*2 ◆*^k* ^Z ^Y *C*0=1 $2e$ s with \mathbb{R}^4 $(16\pi^2\Lambda^3)^k$ for $\mathcal{N}^{-1} = N^2$ \leftarrow rational noint rationale… nagging point

$$
\langle (\text{tr }\lambda^2)^k \rangle \Big| = \mathcal{N}^{-1} \, N^2 \, (16\pi^2)
$$

1. semiclassical objects contributing to gaugino \sim condensate on the torus are related to center **vortices and monopoles, argued responsible A for chiral symmetry breaking and confinement condensate on the torus are related to center**

we will also calculate $\langle \lambda^{2k} \rangle$ on small \mathbb{T}^4 , gcd(N,k)=1; agrees with \mathbb{R}^4 classically in this paper. For brevity, in what follows we denote *^O* = (tr 2)*^k* and write (5.2) $\langle (\text{tr }\lambda^2)^k \rangle \Big| = \mathcal{N}^{-1} N^2$ $\sqrt{\frac{1}{2}}$ *Y tr()(*xi*) ate $\langle \lambda^{2k} \rangle$ on small \mathbb{T}^4 . ₂
2<u>∴16</u>
2∴16≌2∞16M3 <u>and</u> E <u>also \mathbb{R}^4 </u> **2. 3.**

where the energy state $\mathbf x$ is taken to be the inverse state. (just state… won't describe relation…other talks?)

using the twisted partition function, a trace over the Hilbert space *Hm*³ : **thus, the main points of my talk:**

the main points relevant for the \mathbb{T}^4 calculation of $\langle \lambda^{2k} \rangle$

1. Hamiltonian: \mathbb{T}^3 **with 't Hooft twist** $m_3 = n_{12} = -k$, **1. Hamiltonian:** \mathbb{T}^3 with 't Hooft twist $m_3 = n_{12} = -k$, $gcd(N, k) = 1$

> \mathbb{Z}_{2N} chiral- \mathbb{Z}_N center anomaly: exact degeneracies on \mathbb{T}^3 ! 3

$$
\hat{T}_3 \; \hat{X}_{2N} \; \hat{T}_3^{-1} = e^{i\frac{2\pi}{N}k} \hat{X}_{2N}
$$

Cox, Wandler, EP 2021 Jx-states N-fold degenerate

$$
\hat{T}_3 | E, e_3, \dots \rangle = | E, e_3, \dots \rangle e^{i \frac{2\pi}{N} e_3} \qquad e_3 \text{ flu}
$$

 c_3 nox states in fore degenerate \sim 00x, wander, Error c_3 and c_4 implement of \mathbb{R}^4 limited of \mathbb{R}^4 limited and \mathbb{R}^4 $\overline{}$ ˆ ²*^N |E,* ~ *e*i has the same energy as *|E,* ~ *e*i. Since gcd(*N, k*) = 1, we conclude lowest E (=0, SUSY) degenerate flux states \Longrightarrow N clustering vacua in \mathbb{R}^4 limit

e e e a eigenstate COA, *vertuer*, *LP*
 e ˆ Cox, Wandler, EP 2021 ³ with eigenvalue *e*³ + *k*. But since

 $=$ λ^{2k}

periodic bidico.

and the value in the symmetry of the center symmetry of the center system of the center system N times value in one of the e_3 flux-states) k | $E = 0, e_3 = 0$

$$
\hat{T}_3 \; \hat{X}_{2N} \; \hat{T}_3^{-1} = e^{i\frac{2\pi}{N}k} \hat{X}_{2N}
$$

of **examines in a exact degenerate states:**
 of the degenerate states:
 of the degenerate states: anomaly => torus trace above adds absolute values $O = \lambda^{2\kappa}$ in
degenerate states: anomaly => torus trace above adds absolute values $\mathscr{O} = \lambda^{2k}$ in λ *degenerate states:*

$$
\langle \mathcal{O} \rangle \equiv \mathcal{N}^{-1} \operatorname{tr}_{\mathcal{H}_{m_3}} \left[\mathcal{O}e^{-\beta H} \hat{T}_3(-1)^F \right] \qquad \hat{T}_3 \text{ inserts } n_{34} = 1, \ \mathcal{N}\text{- convenience}
$$
\n
$$
\text{anomaly} \Rightarrow \text{torus trace above adds absolute values } \mathcal{O} = \lambda^{2k} \text{ in}
$$

$$
\hat{T}_3 | E, e_3, \dots \rangle = |E, e_3, \dots \rangle e^{i \frac{2\pi}{N} e_3}
$$
 e_3 flux-states N-fold degenerate

$$
\langle (\text{tr}\lambda^2)^k \rangle \Big|_{\beta \to \infty} = \mathcal{N}^{-1} N \langle E = 0, e_3 = 0 | (\text{tr}\lambda^2)^k | E = 0, e_3 = 0 \rangle
$$

³ with eigenvalue *e*³ + *k*. But since

1. Hamiltonian: \mathbb{T}^3 **with 't Hooft twist** $m_3 = n_{12} = -k$, **1. Hamiltonian:** \mathbb{T}^3 with 't Hooft twist $m_3 = n_{12} = -k$, $gcd(N, k) = 1$ the main points relevant for the \mathbb{T}^4 calculation of $\langle \lambda^{2k} \rangle$ the main points relevant for the \mathbb{F} calculation of $\langle \lambda^{2\kappa} \rangle$ the main points relevant for the \mathbb{T}^4 calculation of $\langle \lambda^{2k} \rangle$

of *e*3. This is an exact degeneracy (in addition to the degeneracy due to supersymmetry) of $\frac{d\mathbb{Z}_{2N}}{dx}$ chiral- $\frac{d\mathbb{Z}_{N}}{dx}$ center anomaly: \hat{T}_{2} , \hat{X}_{2N} , $\hat{T}_{2}^{-1}=\rho^{i\frac{2\pi}{N}k}$ **Exact degeneracies on If the Hamiltonian formalism, we consider the constant of operators** \mathbb{R}^n **.** \mathbb{Z}_{2N} chiral- \mathbb{Z}_N center anomaly: exact degeneracies on \mathbb{T}^3 ! 3

$$
\langle (\text{tr}\lambda^2)^k \rangle \Big|_{\beta,\text{L}\to\infty}^{\text{Hamiltonian}}
$$
 = \mathcal{N}^{-1} N $\langle (\text{tr}\lambda^2)^k \rangle$

³ with 't Hooft twist $m_3 = n_{12} = -k$, $gcd(N, k) = 1$

$=$ \mathcal{N}^{-1} N \langle E = 0,e₃ = 0|(tr λ^2) $k | E = 0, e_3 = 0 \rangle$

 SUSY Ward identity implies β, *L - independence*

) k ⟩ 1 vacuum on R⁴ - convenience, to cancel N to get single-vacuum value

 $-$ next: calculate semiclassically at small \mathbb{T}^4

1. Hamiltonian: \mathbb{T}^3 **with 't Hooft twist** $m_3 = n_{12} = -k$, **upshot:** the main points relevant for the \mathbb{T}^4 calculation of $\langle \lambda^{2k} \rangle$

$$
\langle (\text{tr} \lambda^2)^k \rangle \Big|_{\beta \to \infty} = \mathcal{N}^{-1} N \langle E :
$$

2. Hilbert space -> path integral **Filiperi space -> path integral** Here, (= *^L*4) is the extent of the Euclidean time direction, (1)*^F* is inserted to impose 2. Hilbert space -> path integral

$$
\langle \mathcal{O} \rangle \equiv \mathcal{N}^{-1} \operatorname{tr}_{\mathcal{H}_{m_3}} \left[\mathcal{O}e^{-\beta H} \hat{T}_3(-1)^F \right] \quad \hat{T}_3 \text{ inserts } n_{34} = 1
$$

with
$$
\mathcal{O} = \prod_{i=1}^k tr(\lambda \lambda)(x_i)
$$
:

$$
\langle (\operatorname{tr} \lambda^2)^k \rangle = \mathcal{N}^{-1} \sum_{\nu \in \mathbb{Z}} \int [DA_{\mu}][D\lambda][D\bar{\lambda}] \left[\prod_{i=1}^k \operatorname{tr}(\lambda \lambda)(x_i) \right] e^{-S_{S^2}}
$$

$Oe^{-\beta H} \hat{T}_3(-1)^F$ $\overline{1}$ $\langle {\cal O} \rangle \equiv {\cal N}^{-1} \; {\rm tr} \, {\cal H}_{m_3} \; \left| {\cal O}e^{-\beta H} \hat{T}_3 (-1)^F \right| \quad T_3$ inserts $n_{34} = 1$ ̂ *ⁱ*=1 tr()(*xi*)ⁱ in *SU*(*N*) super Yang-Mills theory on a small deformed ^T4. $(1 + r_{\alpha}) \quad [\rho_{\alpha} - \beta H \hat{T}_{\alpha}(-1)F] \quad \hat{T}_{\alpha}$ inserts $n_{\alpha} = 1$ λ ⁴ λ ⁴ λ ⁴ λ ⁴ \hat{H} \hat{H} \hat{H} . 1

3. Multi-fractional instantons on the twisted T^4 *^a^µ* ! *^a^µ* ⁺ *^wa, z^µ* ! *^z^µ ^C^a*

Anber, EP 2307.07495, 240*.xxxxx based on and extending 't Hooft 1982 García Perez, González-Arroyo, Pena 2000 The nonneggative integer *C^a* exists because of the gcd(*N, k*) = 1 condition. These González-Arroyo 2018 where *NC^a* = *a* (mod *k*)*, C^a* 2 Z+*.* (3.9)

*x*3=*x*4=0 strongly overlapping, liquid-like

To enhance visualization, the plot extends to double the periods in *x*¹ and *x*2. The graph reveals three lumps, each one described by the function *F* of (2.22) (itself defined in (B.6)) but with a di↵erent (Γ includes images of instanton under global center symmetry + modding by gauge equivalences)

$$
k \text{ turns on } k \text{ l of l
$$

each lump carries 2 gaugino zero modes $\sqrt{\sqrt{2}}$ (see 2307.07495)

center of mass motion + relative motion ($\Gamma_r^{SU(k)} = SU(k)$ root lattice)

 $\Gamma =$ $\sqrt{2}$ \prod 4 $\mu = 1$ $\mathbb{S}^1_\mu \times \Gamma^{SU(k)}_r$ *r* \mathbb{Z}_k \setminus $\int S_k$ (in) $\overline{\text{S}}$ Z*k* $\overline{1}$ Γ (in SUSY: | only!!!)

combined weight-lattice/c.m. shifts permutation of lumps = $SU(k)$ Weyl

<u>+ SUSY ward identities</u> *N* , and the gaugino condensate + SUSY ward identities

$$
\langle (\operatorname{tr} \lambda^2)^k \rangle = \mathcal{N}^{-1} \sum_{\nu \in \mathbb{Z}} \int [DA_{\mu}][DA][D\bar{\lambda}] \left[\prod_{i=1}^k \operatorname{tr}(\lambda \lambda)(x_i) \right] e^{-S_{SYM} - i\theta(\nu + \frac{k}{N})} \Big|_{n_{12} = -k, n_{34} = 1}
$$

$$
= \mathcal{N}^{-1} N^2 \left(\frac{16\pi^2 M_{\text{PV}}^3}{g^2} e^{-\frac{8\pi^2}{Ng^2}} \right)^k \int \prod_{C'=1}^k d\zeta_1^{C'} d\zeta_2^{C'} \zeta_2^{C'} \zeta_1^{C'}
$$

integral over bosonic and fermionic moduli; $M_{PV}^{n_B - \frac{1}{2}n_F} = M_{PV}^{4k-k} = M_{PV}^{3k}$

path instanton calculations on twisted \mathbb{T}^4 we need the subtleties in the subtleties in the subtleties in the subset of the tr()(*xi*) $\ddot{}$

$$
\frac{1 \text{ SUSYward identities}}{\text{also } R^4} \left\langle (\text{tr }\lambda^2)^k \right\rangle = \mathcal{N}^{-1} \ N^2 \left(16\pi^2 \Lambda^3 \right)^k
$$

home \mathbb{R}^4 instanton calculations on R4. We next turn to a discussion of the subtleties involved. \vert d also \mathbb{R}^4

 \mathbf{v} *i* \mathbf{v} is only the subserved $\mathbf{v} = \mathbf{v} \mathbf{v}$ if $\mathbf{v} = \mathbf{v} \mathbf{v}$ *^C*(*x*1*, .., xk*) on a small ^T⁴ in the semi-classical regime. The pre-coecient *^N* ¹ is a normal- $\frac{1}{2}$ PV $\frac{P}{2}$ **e** integral over bosonic and fermionic moduli; $M_{PV}^{n_B-\frac{1}{2}n_F}$ *PV* $=M_{PV}^{4k-k}=M_{PV}^{3k}$ where the energy scale \mathbf{r} is taken to be the inverse size of \mathbf{r} is taken to be the inverse size of \mathbf{r}

path integral on twisted
$$
\mathbb{T}^4
$$
 w/gcd(N,k)=1: $\langle (\text{tr }\lambda^2)^k \rangle = \mathcal{N}^{-1} N^2 (16\pi^2 \Lambda^3)^k$

 δ and fermion δ 2 $(16\pi^{2}\Lambda^{3})^{k}$ $\left| k \right\rangle \Big| = \; \; \; {\cal N}^{-1} \; N^2 \left(16 \pi^2 \Lambda^3 \right)^k$

 ${}_{PV}^{4k-k} = M_{PV}^{3k}$

compared to the k>1 ADHM
 Example to the k>1 ADHM
 Example to the following calculation on \mathbb{R}^4 this is (to calculation on \mathbb{R}^4 , this is (to us) infinitely simpler

*^g*² *^e*

represents the path integral of the path integral (α is precisely the path integral (4.1) α for generalizes our zz ro calculation for
N=2, k=1 to all N,k with gcd(N,k)=1 *d*⇣*C*⁰ ¹ *d*⇣*C*⁰ ² ⇣*C*⁰ ² ⇣*C*⁰ generalizes our 2210 calculation for $N=2$, k=1 to all N,k with gcd $(N, k)=1$

 $SU(2)$ -> SU(N) same N^2 for all k!

$$
\langle (\text{tr }\lambda^2)^k \rangle \Big| = \mathcal{N}^{-1} \ N^2 \left(16\pi^2 \Lambda^3 \right)^k
$$
 also \mathbb{R}^4

 $\overline{}$

 $\overline{}$

C(*x*1*, ..., xk*) = amhi ning all... $USY:$ **combining all… SUSY:**

E,~ *e* \mathbb{R}^4 weak-coupling instanton/lattice (exact): $\qquad \qquad$ $\langle \text{tr} \lambda^{2\mathrm{k}} \rangle$

clearly, there is a consistent choice for all our N,k: $\mathcal{N}^{-1} = N^2$ clearly, there is a consistent choice for all our N,k: $\mathscr{N}^{-1}=N^2$

- **n** this is sa **• this is satisfying - and we** (will give rationale)
- and by the way, one hagging point remains that would be mice to understand... ● along the way, one nagging point remains that would be nice to understand...

the way, one nagging point remains that would be nice to understand... So far in this paper, we performed a computation of the Euclidean path integral (4.1) with 't

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time to pay the piper: $\frac{1}{2}$ **time to pay the piper:**

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 $SU(2)$ -> SU(N) same N^2 for all k!

atisfying - and we will end accepting it, as it works for all our N,k (tr 2)*^k* in degenerate flux states di↵ering by *k* units of *e*³ flux di↵er by a Z*^N* phase: 5 The Hamiltonian on T³ with a twist, the path integral, the normalization this is satisfying - and we will end accepting it, as it works for all our N,k

$$
\langle (\text{tr }\lambda^2)^k \rangle \Big| = \mathcal{N}^{-1} \, N^2 \, (16\pi^2 \Lambda^3)^k
$$

C(*x*1*, ..., xk*) = amhi ning all... $USY:$ **combining all… SUSY:**

$$
\mathbb{R}^4 \text{ weak-coupling instanton/lattice (exact):} \qquad \langle \text{tr} \lambda^{2k} \rangle_{\text{exact}} = (16\pi^2 \Lambda^3)^k
$$

time to pay the piper: time to pay the piper: $\frac{1}{2}$

$$
\mathbb{R}^4 \text{ weak-coupling instanton/lattice (exact):} \qquad \langle \text{tr} \lambda^{2k} \rangle_{\text{exact}} = (16\pi^2 \Lambda^3)^k
$$

$$
\langle \text{tr} \lambda^{2k} \rangle \Big|_{\beta, L \to \infty}^{\text{Hamilt.}} = \mathcal{N}^{-1} \text{ tr}_{m_3} \left(e^{-\beta H} \hat{T}_3(-1)^F \lambda^{2k} \right) = \mathcal{N}^{-1} \mathbf{N} \langle \lambda^{2k} \rangle \Big|_{1 \text{ vacuum on } \mathbb{R}^4}
$$

 $=$ tr_{m₃} thus, taking $\mathscr N$ to be the Witten index, $\;\mathscr N = \operatorname{tr}_{\mathbf m_3}\!\mathrm e^{-\beta\mathrm H}(-1)^{\mathrm F} = \mathbf N\;$ we cancel overall N. $\rho, E \times \infty$
 $\frac{1}{1}$ vacaam on as thus, taking $\mathscr N$ to be the Witten index, $\;\mathscr N= \text{tr}_{\text{m}_3}\text{e}^{-\beta\text{H}} (-1)^{\text{F}}=\text{N}\;$ we cancel overall $\textsf N$ obtained when comparing the result for ^h(tr)*k*ⁱ of (4.9) to the ^R⁴ result, here we reinterpret

E,~ *e*

clearly, there is a consistent choice for all our N,k: $\mathcal{N}^{-1} = N^2$ *m*₃ f_{or b} $\frac{1}{2}$ for brevity¹ in the set of the clearly, there is a consistent choice for all our N,k: $\mathscr{N}^{-1}=N^2$

from the Hamiltonian perspective, $\mathscr{N}^{-1} = N$ is the natural value to take; recall due to anomaly, we had Next, who is the *X*² tonian perspective, $\mathcal{N}^{-1} = N$ is the natural value to take; recall due to anomaly, we had
 from the Hamiltonian perspective, $\mathcal{N}^{-1} = N$ is the natural value to take; recall due to anomaly, we had

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combining all… SUSY: *C*(*x*1*, ..., xk*) = amhi ning all... $USY:$

$$
\langle (\text{tr}\,\lambda^2)^k \rangle \Big| = \mathcal{N}^{-1} \, N^2 \, (16\pi^2 \Lambda^3)^k
$$

$$
\mathbf{ex}, \quad \mathcal{N} = \text{tr}_{m_3=n_{12}} e^{-\beta H} (-1)^F = N
$$

Now, my final story: let $\mathcal N$ be the Witten index,

independent on *β*, *L* (E=0 only contribute,E>0 cancel due to SUSY), so calculate semiclassically at small L Witten 1982, ²⁰⁰⁰

calculated in Hamiltonian on \mathbb{T}^3 with twist

-
-

- twist removes zero modes of all fields, gap, only discrete set of E=0 states

- exactly N zero energy classical states: A=0 and x_3 center-symmetry transforms thereof center in x_1, x_2 acts trivially

$$
\text{by final story: let } \mathcal{N} \text{ be the Witten index}, \quad \mathcal{N} = \text{tr}_{m_3 = n_{12}} e^{-\beta H} (-1)^F = N
$$

Now, my final story: let N be the Witten index, $\mathcal{N} = \text{tr}_{m,-n} e^{-\beta H} (-1)^{F} = N$

- \sim the same discrepancy. We take this to imply that the discrepancy has a common origin, the discrepancy has a common origin, \sim $\overline{\mathsf{u}}$
- calculated in Hamiltonian on \mathbb{T}^3 with twist
	- twist removes zero modes of all fields gan only disc
		-
- $\begin{array}{c} \hbox{for all our conditions} \end{array}$ path integral expression for Witten index is

- exactly N zero energy classical states: A=0 and x_3 center-symmetry transforms thereof the path integral formally center in x_1, x_2 acts trivially - twist removes zero modes of all fields, gap, only discrete set of E=0 states

$$
\mathcal{N} = \sum_{\nu \in \mathbb{Z}} \int [DA_{\mu}][D\lambda][D\bar{\lambda}]
$$

- at small \mathbb{T}^4 , weak coupling: sum over zero action saddle points (each contributes 1 due to susy)
	- \rightarrow more formal argument using SUSY localization…worthy of pursuit… !
- there are exactly N^2 gauge nonequivalent zero action configurations González-Arroyo 1998
	- can be described abstractly, but also explicitly as $A=0$ and x_3, x_4 center-symmetry transforms thereof can be described abstractly, but also explicitly as A=0 and x_3, x_4 center-symmetry transforms thereof
		-
- thus, path integral has us take $\;\;\mathscr{N}^{-1} = N^2\;\;$, restoring all agreement with SUSY and \mathbb{R}^4 .

[*DD*] denotes aux. field… localization?

 $[DA_\mu][D\lambda][D\overline{\lambda}][DD]$ $e^{-S_{SYM}-i\theta\nu}$ $\begin{array}{c} \hline \end{array}$ $\begin{array}{c} \hline \end{array}$ n_{12} = $-k, n_{34}$ =0 (sum over integer topological charges)

We and the sum here, as only the sum to the sum of the sum here, is one to sum to the sum of the sum

 $\mathsf{exactly}\ N^2$ gauge nonequivalent zero action configurations González-Arrovo 1998 and badden <u>It gaago honoquratom zoro adtion cormguratione</u> conzule and o too

(essentially because Euclidean path integral can be taken to have either x_3, x_4 as time)

independent on *β*, *L* (E=0 only contribute,E>0 cancel due to SUSY), so calculate semiclassically at small L Witten 1982, ²⁰⁰⁰

center symmetry images of A=0 saddle?

likewise, N vs N^2 in gaugino condensate:

- due to center-symmetry images of Q=k/N saddles in x_3 vs in both x_3 and x_4 in path integral
	-
- path integral calculation of Witten index with twist was never done!
	- deep stuff? localization, complexification, middle dimensional cycles,… ???
		-

perhaps easier to answer for Witten index!

as we saw, understanding this has implications for semiclassics on twisted \mathbb{T}^4

The nagging point is: based on our understanding of Hamiltonian <—> path integral

N vs N^2 in "Witten index":

there should be agreement between the two

- due to center-symmetry transforms on A=0 saddles in x_3 vs in both x_3 and x_4 in path integral
	- what principle in Euclidean path integral says one should omit one of x_4/x_3

1. semiclassical objects contributing to gaugino \sim condensate on the torus are related to center **vortices and monopoles, argued responsible A for chiral symmetry breaking and confinement condensate on the torus are related to center**

where the energy scale in the energy scale of the inverse size (just stated; didn't describe any of this…other talks?)

using the twisted partition function, a trace over the Hilbert space *Hm*³ : **summary of the main points of my talk:**

two backups

wawaav uu vaav wuunuwaaswasu vaaw $Pf(\hat{m})$ and not on μ . It would be interesting to try to verify this by direct

¹³The idea of the proof is to first reduce to $\mu = 1$ by replacing \hat{m} by \hat{m}/μ and s by s/ μ . Then one shows that one can by an $SL(4, \mathbb{Z})$ transformation set $\hat{m}_{12} = 1$, after which by an $SL(4, \mathbb{Z})$ transformation one can set $\hat{m}_{ij} = 0$ for $i = 1, 2$ and $j = 3, 4$ and (therefore) $\widehat{m}_{34} = \mathrm{Pf}(\widehat{m}).$

2000 "Supersymmetric index in 4d gauge theories" 865

Edward Witten

study of path integrals, but we will not do that in the present paper. Section 4 of the paper is devoted to microscopic calculations verifying the predictions that we have presented up to this point, but these calculations will be done from a Hamiltonian point of view.

S to an integral over moduli; here: a discrete set of points... she (usually, localization leads to an integral over moduli; here: a discrete set of points… should be simpler?)

(formal) loc \blacksquare $\frac{1}{2}$ yields the same discrepancy. We take this to imply that the discrepancy has a common origin, (formal) localization argument:

$$
g^2 S_{SYM} = \delta^{\alpha} \mathcal{O}_{\alpha} + \delta_{\dot{\alpha}} \mathcal{O}^{\dot{\alpha}}, \text{ where } \mathcal{O}_{\alpha} = \frac{1}{8} \int d^4 x \, \Delta_{\alpha}, \ \mathcal{O}^{\dot{\alpha}} = \frac{1}{8} \int d^4 x \, \Delta^{\dot{\alpha}}
$$

$$
\Delta_\alpha \equiv \sigma_{\mu\nu\;\alpha}{}^\beta \;\lambda_\beta^a \;F_{\mu\nu}^a + \lambda_\alpha^a D^a, \, \Delta^{\dot\alpha} \equiv \bar\sigma_{\mu\nu}^{\quad \dot\alpha} \; \bar\lambda^{\dot\beta\; a} \; F_{\mu\nu}^a + \bar\lambda^{\dot\alpha\; a} D^a
$$

$$
\mathcal{N} = \sum_{\nu \in \mathbb{Z}} \int [DA_{\mu}][D\lambda][D\bar{\lambda}][DD] e^{-S_{SYM} - i\theta \nu}
$$

hence, calculate at $g^2\to 0$: integral localizes to sum over zero action saddles + 1-loop

$$
\frac{d\mathcal{N}}{dg^{-2}} = - \int_{\mathbb{T}^4 \text{ with } n_{12} \neq 0 \pmod{N}} DA D\lambda D\bar{\lambda} DD \left[\delta^{\alpha} (\mathcal{O}_{\alpha} e^{-S_{SYM}}) + \delta_{\dot{\alpha}} (\mathcal{O}^{\dot{\alpha}} e^{-S_{SYM}}) \right] = 0.
$$

 \overline{u} li: persymmetry variation integral (α is the path integral (5) is coupled and α integral (α) is complexed (α) in α 3181 MeInkowski space version 38 and 38 and 38 and 38 hence, calculate at $g^2 \rightarrow 0$: integral localizes to sum over zero action saddles + 1-loop