Probing Quadratic Gravity with Binary Inspirals

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The Einstein-Hilbert Action

The Einstein-Hilbert Action gives us the Einstein Field Equations

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa} + \mathcal{L}_M\right]$$

$$\text{ELE} \Rightarrow R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}$$

$$\text{Expanded} \Rightarrow \Box h_{\mu\nu} = 2\kappa\tau_{\mu\nu}$$

- $\kappa = 8\pi G = 8\pi/M_p^2$
- R is the only independent scalar which we can construct (up to second derivatives) of the metric
- The metric is split up as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ to find the wave equation

Why Modify General Relativity?

General Relativity is the simplest theory coupling spacetime curvature to matter

Good reason to look at modified theories

- Interaction with quantum matter, should be a limit from any quantum theory of gravity

We can consider modified theories by adding terms to the Hilbert action, as long as they

- Are diffeomorphism invariant, scalar, etc
- Limit correctly to GR, and Newtonian gravity

What effect do these modifications have?

- Must look at strong gravity
- $\Rightarrow$ Binary Systems are an ideal testing ground

The Post-Newtonian Formalism

The Post-Newtonian (PN) formalism is an iterative expansion scheme in v/c, for arbitrarily precise solutions to the Einstein field equations

- Requires slow moving, weakly stressed sources (valid for inspiralling binary black holes up to $v/c = 0.5$)
- Naturally includes non-linearity and higher multipole characteristics
- Convention is to just track $1/c^n$, and call those terms "$\frac{n}{2}$PN order"

0PN order is called the "Newtonian" order, and GR only affects dynamics at higher orders

$$h_{\mu\nu} = h^{(0)}_{\mu\nu} + \frac{1}{c}h^{(1)}_{\mu\nu} + \frac{1}{c^2}h^{(2)}_{\mu\nu} + \dots$$

Project Aim

To investigate gravitational waves from binary systems in the early inspiral phase, given by an effective field theory applicable only in the low energy/curvature regime

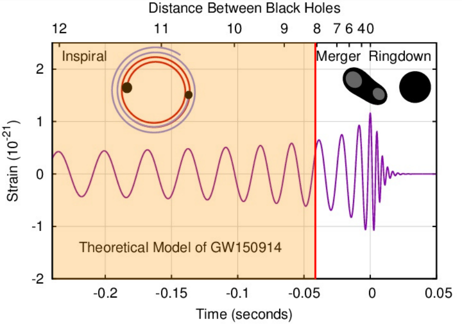

Figure 1: The orange section illustrates the early inspiral phase [1]

Modified Action

We can modify GR by adding in all independent terms up to 4th derivatives of the metric [2]

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa} + \beta R^2 + \gamma R^{\mu\nu}R_{\mu\nu}\right]$$

- These are unavoidable from one-loop renormalisation of matter with semi-classical gravity

We can interpret the extra degrees of freedom from the quadratic terms as a massive spin-0, and a massive spin-2 field

Modelling The Binary Inspiral

We model a binary system of two point particles with masses m_a, and 4-velocities v_a^μ

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{2\kappa} - \frac{1}{2}\left(\partial_\mu \pi^{\alpha\beta}\partial^\mu \pi_{\alpha\beta} + m_\pi^2 \pi^{\alpha\beta}\pi_{\alpha\beta}\right) - \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi + m_\phi^2 \phi^2\right)\right] + \sum_{a=1}^{2} m_a \int dt \sqrt{-\tilde{g}_{\mu\nu} v_a^\mu v_a^\nu}$$

where ϕ is a massive spin-0 field, $\pi_{\alpha\beta}$ is a massive spin-2 field, and the metric is conformally constructed as

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \sqrt{2\kappa}\eta_{\mu\nu}\phi + \sqrt{4\kappa}\pi_{\mu\nu}$$

Linearised Equations Of Motion

We can find the linearised field equations for ϕ, and $\pi_{\mu\nu}$, also cutting off the source terms at lowest PN order. This gives Yukawa-like solutions

$$\phi(x) = \sqrt{\frac{G}{4\pi}} \sum_{a=1}^{2} m_a \frac{e^{-m_\phi \left|\vec{x} - \vec{y}_a(t_r)\right|}}{\left|\vec{x} - \vec{y}_a(t_r)\right|}$$

$$\pi_{\mu\nu}(x) = -\sqrt{\frac{G}{2\pi}} \sum_{a=1}^{2} m_a \left(v_{\mu a} v_{\nu a} + \frac{1}{4} \eta_{\mu\nu} \right) \frac{e^{-m_\pi \left|\vec{x} - \vec{y}_a(t_r)\right|}}{\left|\vec{x} - \vec{y}_a(t_r)\right|}$$

The solutions look just like the regular GR potential, expect they have a exponential mass suppression. Also, note the signs, where the spin-0 field is attractive, while the spin-2 field is repulsive

The Conservation Equation

In order to calculate the waveform we can invoke the conservation equation

$$\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = 0$$

This provides us with the modified geodesic equation, this modification is due to the extra forces from the massive spin-0 and spin-2 fields. Then the relative acceleration to Newtonian order is

$$\vec{a} = -\frac{GM}{r^2}\hat{n}\left(1 + 2e^{-m_\phi r}(m_\phi r + 1) - 3e^{-m_\pi r}(m_\pi r + 1)\right)$$

where $r = |\vec{y}_1 - \vec{y}_2|$, and $\hat{n} = (\vec{y}_1 - \vec{y}_2)/r$

Energy Balance Equation

Using the acceleration, we can find an effective Lagrangian for the binary, and therefore the energy

$$E = -\frac{Gm_1 m_2}{r}\left(\frac{1}{2} + 2e^{-m_\phi r} - 3e^{-m_\pi r}\right)$$

We assume the balance equation that takes into account the energy loss due to the waves with the flux seen by an observer far away

$$\frac{dE}{dt} = -\mathcal{F}$$

Note that the far-field flux will be highly suppressed for the massive fields, and so we can use the usual GR flux

The Inspiral Waveform

Using the energy balance equation the leading corrections to the GR waveform phase of the binary inspiral is given as

$$\varphi = -\frac{x^{-5/2}}{32\nu}\left[1 + e^{-m_\phi \frac{GM}{x}}\left(\frac{5}{2} - \frac{5}{3}m_\phi\frac{GM}{x}\right) - e^{-m_\pi \frac{GM}{x}}\left(\frac{15}{4} - \frac{5}{2}m_\pi\frac{GM}{x}\right)\right]$$

where $x \equiv (GM\Omega)^{\frac{2}{3}}$ is a frequency-related parameter, and $\nu = \frac{m_1 m_2}{M^2}$ is the symmetric mass ratio

Inside the bracket, the terms with x^0 are associated with the Newtonian (quadrupole) order, and the x^{-1} are associated with the dipole order

The Stationary Phase Approximation

We can assume a GW signal with amplitude $A(t)$, and phase $\Phi(t)$ takes the form [5]

$$h(t) = 2A(t)\cos\Phi(t)$$

In the Stationary Phase Approximation (SPA), the frequency domain signal is given by the following

$$\tilde{h}(f) = \frac{\sqrt{2\pi}A(t_f)}{\sqrt{\ddot{\Phi}(t_f)}}e^{i\psi(f)}$$

$$\psi(f) = 2\pi f t_f - \pi/4 - \Phi(t_f)$$

where the parameter t_f is given by the time when $d\Phi(t)/dt = 2\pi f$. Generically $2\varphi = \Phi$

The Fourier Coefficients

Following the SPA prescription we obtain the relation for $\psi(f)$

$$\psi(f) = 2\pi f t_c - \phi_c - \pi/4 + \frac{3}{128\nu}\sum_{i=-4}^{0}\varphi_i (GM\pi f)^{(i-5)/3}$$

where t_c, and ϕ_c are the time, and phase at coalescence respectively. The relevant non zero coefficients φ_i for our analysis are given as

$$-1\text{PN} \quad \varphi_{-2} = \frac{451928}{27} m_\phi GM e^{-m_\phi \delta} - \frac{225964}{9} m_\pi GM e^{-m_\pi \delta}$$

where $\delta = 4GM\left(8\pi GMf\right)^{-2/3}$

Parameters of Quadratic Gravity

The absolute deviation of the -1PN phase has been constrained from gravitational wave data to be $|\delta\varphi_{-1PN}| < 10^{-2}$

$$\left|\frac{451928}{27} m_\phi G M e^{-m_\phi \delta} - \frac{225964}{9} m_\pi G M e^{-m_\pi \delta}\right| \lesssim 10^{-2}$$

Taking the typical values $f = 75\text{Hz}$, and $M = 30 M_\odot$, the masses should satisfy the inequalities separately

$$m_{\phi,\pi} \gtrsim 7.1 \cdot 10^{-12} \text{eV}$$

Rewritten as limits on the original dimensionless parameters of quadratic gravity

$$\begin{aligned} 0 \leq \quad & \gamma & \lesssim 1.5 \cdot 10^{78} \\ -\frac{\gamma}{4} \leq \quad & \beta & \lesssim -1.1 \cdot 10^{77} \end{aligned}$$

Summary

We were able to recast the quadratic gravity degrees of freedom as a massive spin-0 and spin-2 field alongside the usual massless spin-2 graviton, and derived linear, lowest order field equations

To Newtonian order, they respectively act as attractive, and repulsive Yukawa potentials modifying gravity

Found the -1PN correction to the GW phase of an inspiralling binary system in quadratic gravity

Placed constraints on quadratic gravity from real GW observations from the LIGO, and Virgo Collaborations

References

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[2] K. S. Stelle, Phys. Rev. D **16**, 953 (1977). doi:10.1103/PhysRevD.16.953

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Definition Of Terms

The mass terms are

$$m_\phi^2 = \frac{1}{3\kappa(4\beta + \gamma)}$$

$$m_\pi^2 = \frac{1}{2\kappa\gamma}$$

The Θ term is defined as

$$\Theta \equiv \frac{\nu}{5GM}(t_c - t)$$

The symmetric mass ratio terms are

$$M = m_1 + m_2$$

$$\mu = \frac{m_1 m_2}{M}$$

$$\nu = \frac{\mu}{M}$$

Quadratic Gravity As An Effective Field Theory

We cutoff our Lagrangian at quadratic order to avoid non renormalisability at the 2-loop level

Stelle [2] noted the negative norm states of the massive spin-2 field

- We must interpret this as an effective field theory

Quick calculation to show realm of validity

$$
\begin{aligned}
M_p^2 R > \alpha R^{\text{quad}} &\Rightarrow M_p^2 p^2 > \alpha p^4 (\text{In momentum space}) \\
&\Rightarrow M_p^2/r^2 > \alpha/r^4 \\
m_{\phi,\pi} \approx M_p^2/\alpha &\Rightarrow m_{\phi,\pi} r > 1
\end{aligned}
$$

We can then see that far-field plane waves $e^{-i(\omega t - \vec{k}\vec{x})}$ are suppressed

$$
\begin{aligned}
v^2 \approx GM/r < 1 < m_{\phi,\pi} r &\Rightarrow m_{\phi,\pi} > \Omega^2 \approx \omega^2 \\
&\Rightarrow k^2 = \omega^2 - m_{\phi,\pi}^2 < 0
\end{aligned}
$$

Recasting The Lagrangian

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa} + \beta R^2 + \gamma R^{\mu\nu}R_{\mu\nu}\right]$$

Setting $S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R$, and $\alpha = \beta + \frac{\gamma}{4}$

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa} + \alpha R^2 + \gamma S^{\mu\nu}S_{\mu\nu}\right]$$

Using Lagrange multipliers, and the following conformal transformation

$$\widetilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \qquad \Omega^2 = (1 + \sqrt{2\kappa}\phi)$$

$$S = \int d^4x\sqrt{-g}\left[\frac{\widetilde{R}}{2\kappa} + \pi^{\mu\nu}\widetilde{S}_{\mu\nu} - \frac{1}{4\gamma}\pi^{\mu\nu}\pi_{\mu\nu} - \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi + m_\phi^2\phi^2\right)\right]$$

Separating $\pi^{\mu\nu}$ from $\widetilde{h}^{\mu\nu}$ we obtain the final result

Details On The Waveform Calculations

We start from the balance equation

$$\frac{dE}{dt} = -\mathcal{F}$$

Using the chain rule our energy balance equation becomes

$$\frac{dE}{dx}\frac{dx}{d\varphi}\frac{d\varphi}{d\Theta}\frac{d\Theta}{dt} = -\mathcal{F}$$

We can calculate the third term by

$$\frac{d\varphi}{dt} = \Omega \quad \Rightarrow \quad \frac{d\varphi}{d\Theta} = -\frac{5}{\nu}x^{3/2}$$

The usual GR flux in terms of x

$$\mathcal{F} = \frac{32}{5G}\nu^2 x^5$$

Details On The Fourier Calculations

To get φ as a function of time t from x, we rewrite x as a function of time

$$\begin{aligned} x(t) &= f(\Theta(t)) \\ \Rightarrow \varphi(t) &= g(\Theta(t)) \end{aligned}$$

To find t_f we take derivatives

$$\begin{aligned} \frac{d\Phi(t)}{dt} &= 2\pi f \\ \Rightarrow \frac{d\varphi(t)}{dt} &= \pi f \\ \Rightarrow t_f &= \dots \end{aligned}$$

Insert everything back into $\psi(f)$ to obtain the Fourier coefficients