

Thermal regularization of t -channel singularities of scattering processes

Michał Iglicki

University of Warsaw



based on

B. Grzadkowski, M. Iglicki, S. Mrówczyński, [Nucl.Phys.B 984 \(2022\) 115967](#)

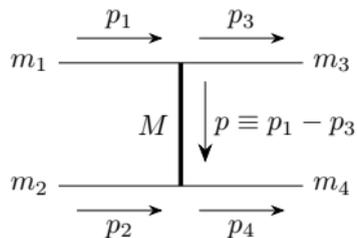
M. Iglicki, [JHEP 06 \(2023\) 006](#)

Theory and Experiment in High Energy Physics (workshop #1)

Bratislava, 27 July 2023

- The singularity
- Known approaches to the problem
- Our approach
- Result discussion
- Summary

t -channel singularity: definition



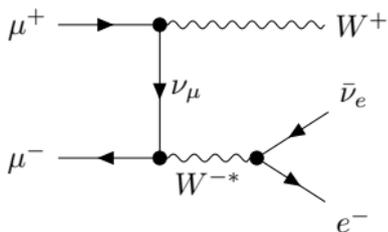
$$\mathcal{M} \sim \frac{1}{t - M^2}, \quad t \equiv p^2$$

$t = M^2 \Rightarrow$ singular matrix element
 \Rightarrow infinite cross section

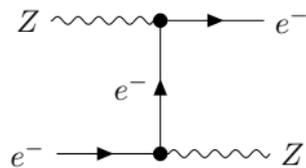
- IR regularization not applicable if $M > 0$
 - Dyson resummation not helpful if $\Gamma = 0$
- } \Rightarrow massive, stable mediator required

t -channel singularity: examples

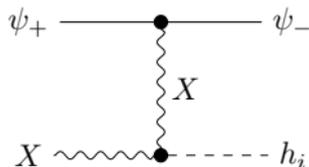
SM: muon colliders



SM: weak Compton scattering

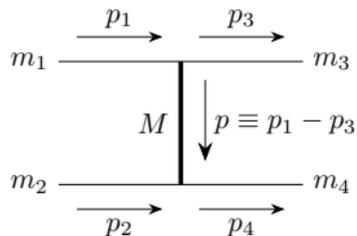


BSM: dark matter in the early Universe



(VFDM model: A. Ahmed et al., [Eur.Phys.J.C 78 \(2018\) 11, 905](#))

$2 \leftrightarrow 2$ process: when does the t -channel singularity occur?



$$\mathcal{M} \sim \frac{1}{t - M^2}$$

$$t \equiv p^2 = (p_1 - p_3)^2$$

$$s \equiv (p_1 + p_2)^2 = (p_3 + p_4)^2$$

- singularity: $t = M^2$ (massive, stable mediator required)
- cross section
- thermally averaged cross section

$$\sigma(s) \supset \int_{t_{\min}(s)}^{t_{\max}(s)} \frac{dt}{(t - M^2)^2}$$

$$\langle \sigma v \rangle(T) \supset \int \sigma(s) f(E_1, E_2, T) ds$$

- singularity condition

$$t_{\min}(s) < M^2 < t_{\max}(s)$$

$$t_{\min} = m_1^2 + m_3^2 - 2E_1 E_3 - 2|\mathbf{p}_1||\mathbf{p}_3|$$

$$t_{\max} = m_1^2 + m_3^2 - 2E_1 E_3 + 2|\mathbf{p}_1||\mathbf{p}_3|$$

$2 \leftrightarrow 2$ process: when does the t -channel singularity occur?

- singularity condition

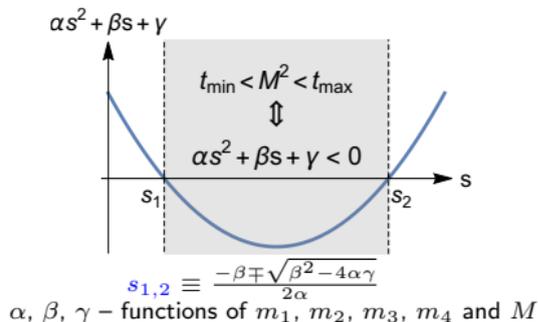
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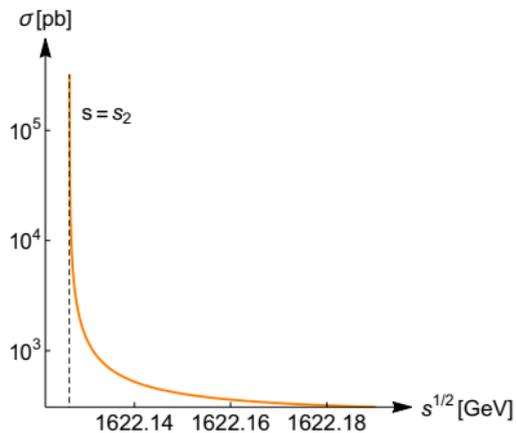
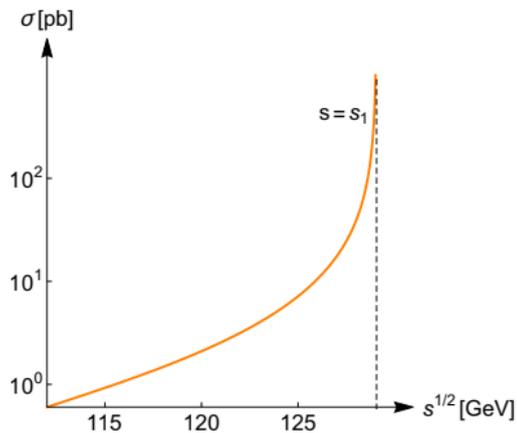
- in terms of the CM energy (\sqrt{s})

$$t_{\min}(s) < M^2 < t_{\max}(s)$$

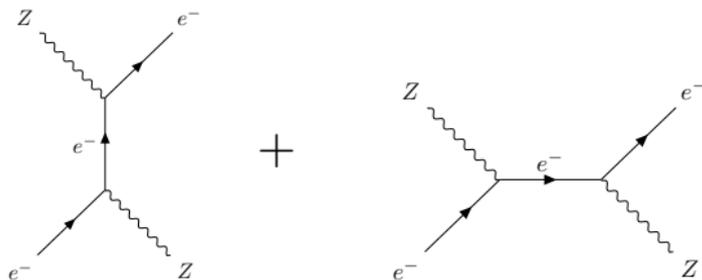
$$\Leftrightarrow s_1 < s < s_2$$



example: weak Compton scattering



$$Ze^- \rightarrow Ze^- =$$



$2 \leftrightarrow 2$ process: when does the t -channel singularity occur?

- singularity condition:

$$t_{\min}(s) < M^2 < t_{\max}(s)$$

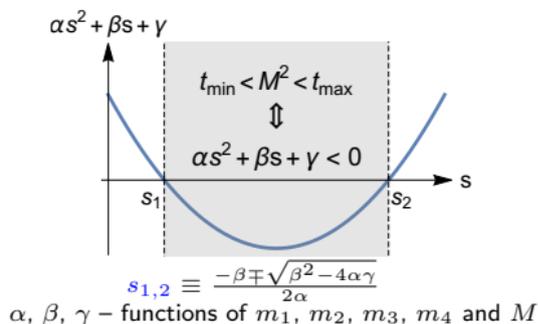
$$t_{\min} = m_1^2 + m_3^2 - 2E_1E_3 - 2|\mathbf{p}_1||\mathbf{p}_3|$$

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- in terms of the CM energy (\sqrt{s}):

$$t_{\min}(s) < M^2 < t_{\max}(s)$$

$$\Leftrightarrow s_1 < s < s_2$$



- thermally averaged cross section \leftarrow integration over $\sqrt{s} \in [\sqrt{s_{\min}}, \infty)$
(weighted by thermal distribution functions)
- conclusion for the cosmological case:
if $s_2 > s_{\min} \equiv \max\{(m_1 + m_2)^2, (m_3 + m_4)^2\}$,
singularity in the allowed range $\Rightarrow \langle \sigma v \rangle = \infty$

$$s_2 \equiv \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

α, β, γ – functions of m_1, m_2, m_3, m_4 and M

if $s_2 > s_{\min} \equiv \max\{(m_1 + m_2)^2, (m_3 + m_4)^2\}$, singularity in the allowed range

\Leftrightarrow

$$m_1 > M + m_3 \text{ and } m_4 > M + m_2$$

or

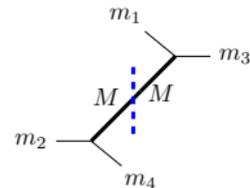
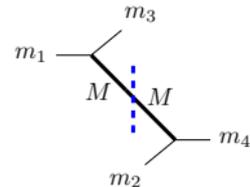
$$m_2 > M + m_4 \text{ and } m_3 > M + m_1$$

◇ Coleman-Norton theorem

S. Coleman & R. E. Norton, Nuovo Cim 38, 438–442 (1965)

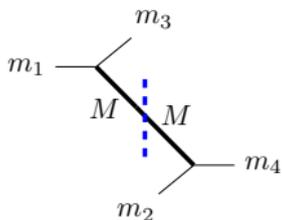
"It is shown that a Feynman amplitude has singularities on the physical boundary if and only if the relevant Feynman diagram can be interpreted as a picture of an energy- and momentum-conserving process occurring in space-time, with all internal particles real, on the mass shell, and moving forward in time"

note: one of the external states decays,
so it **cannot be an asymptotic state**



Known approaches to the problem

→ complex mass of unstable particles



idea: finite lifetime should affect the **wavefunction**

- at rest:
$$e^{im_1 t} \rightarrow e^{im_1 t} e^{-\Gamma_1 t}$$
$$= e^{i\tilde{m}_1 t}, \quad \tilde{m}_1 \equiv m_1 \left(1 + i \frac{\Gamma_1}{m_1} \right)$$
- after Lorentz boost: $p_1 \rightarrow \tilde{p}_1 \equiv p_1 \left(1 + i \frac{\Gamma_1}{m_1} \right)$

→ **problem**: $(\tilde{p}_1 - \tilde{p}_3)^2 \neq (\tilde{p}_4 - \tilde{p}_2)^2 \Rightarrow$ **lack of symmetry**
(momentum conservation...)

Known approaches to the problem

→ finite beam width

G. L. Kotkin et al., Yad. Fiz. 42 (1982) 692
G. L. Kotkin et al., Int. Journ. Mod. Phys. A 7 (1992) 4707
K. Melnikov & V. G. Serbo, Nucl.Phys. B483 (1997) 67
C. Dams & R. Kleiss, Eur.Phys.J.C29 (2003) 11
C. Dams & R. Kleiss, Eur.Phys.J. C36 (2004) 177

idea: at colliders, the beams have **finite size**

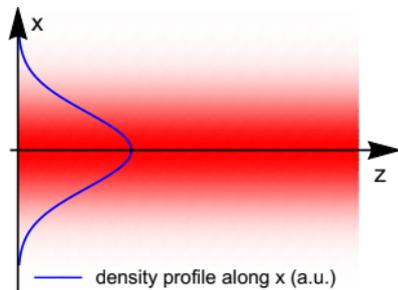


they **should not** be treated as plain waves

example:

Gaussian beam moving along z axis

$$n(x, y) \sim e^{-\frac{x^2+y^2}{2a^2}} \quad a - \text{beam width}$$



$$\int \frac{dt}{|t - M^2 + i\epsilon|^2} \rightarrow \int \frac{a^3 e^{-\frac{a^2 \kappa^2}{2}}}{(2\pi)^{3/2}} \frac{d^3 \kappa dt}{(t - M^2 + i\epsilon - \kappa \cdot \mathbf{q})(t - M^2 - i\epsilon + \kappa \cdot \mathbf{q})}$$
$$\sim \frac{\pi a}{|\mathbf{q}|}, \quad \mathbf{q} \equiv \left[\frac{E_3}{E_1} \mathbf{p}_1 - \mathbf{p}_3 \right]_{t=M^2}$$

→ **problem**: inapplicable in **cosmological context**

Known approaches to the problem

→ Dyson resummation

$$-\frac{i\Delta}{p} = -\frac{i\Delta^{(0)}}{p} + \frac{i\Delta^{(0)}}{p} \textcircled{i\Pi} \frac{i\Delta^{(0)}}{p} + \frac{i\Delta^{(0)}}{p} \textcircled{i\Pi} \frac{i\Delta^{(0)}}{p} \textcircled{i\Pi} \frac{i\Delta^{(0)}}{p} + \dots$$

assumptions: $|\Pi|$ small, $p^2 \simeq M^2$

$$\text{scalar: } \frac{1}{p^2 - M^2} \rightarrow \frac{1}{p^2 - M^2 + \Pi}$$

$$\text{fermion: } \frac{\not{p} + M}{p^2 - M^2} \rightarrow \frac{\not{p} + M}{p^2 - M^2 + \text{Tr} \left[\frac{\not{p} + M}{2} \Pi \right]}$$

$$\text{vector: } \frac{-g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}}{p^2 - M^2} \rightarrow \frac{-g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}}{p^2 - M^2 + \frac{1}{3} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2} \right) \Pi_{\mu\nu}}$$

$$\Rightarrow \text{regulator: } \Sigma \equiv \begin{cases} \Im \Pi \\ \Im \left(\text{Tr} \left[\frac{\not{p} + M}{2} \Pi \right] \right) \\ \Im \left[\frac{1}{3} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2} \right) \Pi_{\mu\nu} \right] \end{cases}$$

→ **problem:** $\Sigma \xrightarrow[p^2 \rightarrow M^2]{\text{opt. th.}}$ decay width

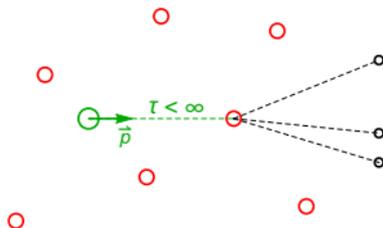
⇒ **no regularization for a stable mediator, but...**

- early Universe = hot gas
- every particle interacts with a thermal medium
- the mean life time cannot be infinite \Rightarrow effective width
- QFT in a thermal medium: Keldysh-Schwinger formalism

similar considerations (and results) by
H.A. Weldon, *Phys. Rev. D* 28 (1983) 2007



vacuum

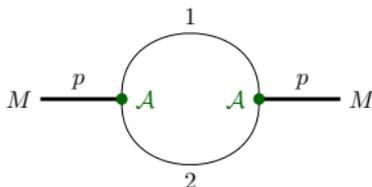


early Universe

One-loop self-energy

warning: hereafter, m_1 and m_2 are masses of the loop states

- one-loop contribution to mediator's self-energy



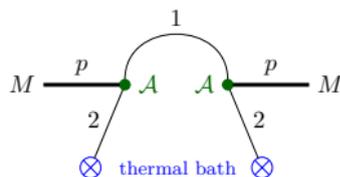
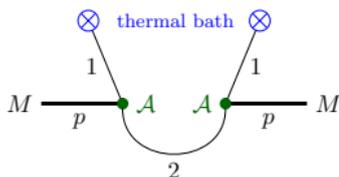
$$i\Pi(x, y) = i\Delta_1(x, y) \mathcal{A}(y) i\Delta_2(y, x) \mathcal{A}(x)$$

- non-zero imaginary part of the self-energy appears as a result of interactions with the thermal medium of particles

mediator + thermal bath
particle



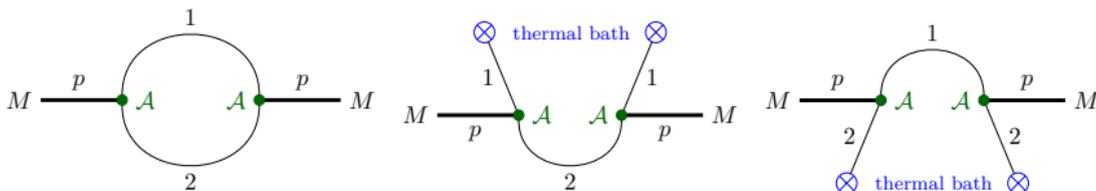
unstable intermediate state



Calculation of one-loop self-energy

- one-loop contribution to the self-energy

$$i\Pi(x, y) = i\Delta_1(x, y) \mathcal{A}(y) i\Delta_2(y, x) \mathcal{A}(x)$$



- non-zero imaginary part of the self-energy** appears as a result of **interactions with the thermal medium** of particles

$$\Pi^+(p, T) = \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \left[\Delta_1^+(k+p) \mathcal{A} \Delta_2^{\text{sym}}(k, T) \mathcal{A} + \Delta_1^{\text{sym}}(k, T) \mathcal{A} \Delta_2^-(k-p) \mathcal{A} \right]$$

$$\Delta_i^{\text{sym}}(k, T) \equiv \frac{i\pi}{E_i} \left(\delta(E_i - k_0) + \delta(E_i + k_0) \right) \times [2\eta_i f(E_i, T) - 1] \times (\text{numerator})$$

$$\Delta_i^\pm(p) \equiv \frac{(\text{numerator})}{p^2 - m_i^2 \pm i \text{sgn}(p_0) \varepsilon}, \quad E_i \equiv \sqrt{\mathbf{k}^2 + m_i^2},$$

$$f(E_i, T) = (e^{E_i/T} + \eta_i)^{-1}, \quad \eta_i \equiv +1 \text{ for fermions, } -1 \text{ for bosons}$$

after tedious calculations... (assumption: $m_1 > m_2 + M$)

$$\begin{aligned}\Sigma(|\mathbf{p}|, T) &\equiv \Im \Pi^+(|\mathbf{p}|, T) \\ &= \frac{1}{16\pi} \frac{X_0}{\beta|\mathbf{p}|} \left[\ln \frac{e^{\beta(b+a)} + \eta_1}{e^{\beta(b-a)} + \eta_1} - \ln \frac{e^{\beta(b+a)} e^{-\beta E_p} + \eta_2}{e^{\beta(b-a)} e^{-\beta E_p} + \eta_2} \right] \\ &= \frac{1}{16\pi} \frac{X_0}{\beta|\mathbf{p}|} \ln \left[1 + \eta_2 \frac{e^{-\beta(b-a)} e^{\beta E_p} (1 - e^{-2\beta a}) (1 - \eta_1 \eta_2 e^{-\beta E_p})}{(1 + \eta_1 e^{-\beta(b-a)}) (1 + \eta_2 e^{-\beta(b+a)} e^{\beta E_p})} \right]\end{aligned}$$

$$a \equiv \frac{\lambda(m_1^2, m_2^2, M^2)^{1/2}}{2M^2} |\mathbf{p}|, \quad b \equiv \frac{m_1^2 - m_2^2 + M^2}{2M^2} E_p, \quad E_p \equiv \sqrt{\mathbf{p}^2 + M^2}$$

$$\lambda(m_1^2, m_2^2, M^2) \equiv [m_1^2 - (m_2 + M)^2] [m_1^2 - (m_2 - M)^2]$$

$$\eta_i \equiv +1 \text{ for fermions, } -1 \text{ for bosons,} \quad X_0 = \eta_2 |\mathcal{M}|_{\text{dec}}^2 \times \begin{cases} 1 & \text{scalar} \\ 1/2 & \text{fermion} \\ 1/3 & \text{vector} \end{cases}$$

effective width:

$$\Gamma_{\text{eff}}(|\mathbf{p}|, T) \equiv M^{-1} \Sigma(|\mathbf{p}|, T)$$

↓

Breit-Wigner propagator:

$$\frac{1}{(t - M^2)^2} \rightarrow \frac{1}{(t - M^2)^2 + M^2 \Gamma_{\text{eff}}(|\mathbf{p}|, T)^2}$$

Result discussion: general properties

$$\Sigma(|\mathbf{p}|, T) = \frac{1}{16\pi} \frac{X_0}{\beta|\mathbf{p}|} \ln \left[1 + \eta_2 \frac{e^{-\beta(b-a)} e^{\beta E_p} (1 - e^{-2\beta a}) (1 - \eta_1 \eta_2 e^{-\beta E_p})}{(1 + \eta_1 e^{-\beta(b-a)}) (1 + \eta_2 e^{-\beta(b+a)} e^{\beta E_p})} \right]$$

$$a \equiv \frac{\lambda(m_1^2, m_2^2, M^2)^{1/2}}{2M^2} |\mathbf{p}|, \quad b \equiv \frac{m_1^2 - m_2^2 + M^2}{2M^2} E_p, \quad E_p \equiv \sqrt{\mathbf{p}^2 + M^2}$$
$$\lambda(m_1^2, m_2^2, M^2) \equiv [m_1^2 - (m_2 + M)^2] [m_1^2 - (m_2 - M)^2]$$

$$\eta_i \equiv +1 \text{ for fermions, } -1 \text{ for bosons,} \quad X_0 = \eta_2 |\mathcal{M}|_{\text{dec}}^2 \times \begin{cases} 1 & \text{scalar} \\ 1/2 & \text{fermion} \\ 1/3 & \text{vector} \end{cases}$$

observations:

- $b > a + E_p$, since $E_p > |\mathbf{p}|$ and $m_1^2 - m_2^2 - M^2 > \lambda^{1/2}$
- $a > 0$ and $E_p > 0$
- a , b and E_p do not depend on T
- $\text{sgn}(\text{logarithmic part}) = \eta_2 = \text{sgn}(X_0) \Rightarrow \Sigma > 0$

Result discussion: limiting cases

$$\Sigma(|\mathbf{p}|, T) = \frac{1}{16\pi} \frac{X_0}{\beta|\mathbf{p}|} \ln \left[1 + \eta_2 \frac{e^{-\beta(b-a)} e^{\beta E_p} (1 - e^{-2\beta a}) (1 - \eta_1 \eta_2 e^{-\beta E_p})}{(1 + \eta_1 e^{-\beta(b-a)}) (1 + \eta_2 e^{-\beta(b+a)} e^{\beta E_p})} \right]$$

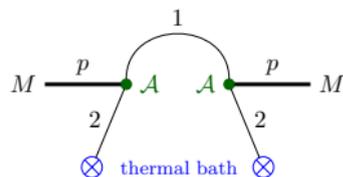
- $m_1 = m_2 + M$ (no decay)

$$a \equiv \frac{\lambda(m_1^2, m_2^2, M^2)^{1/2}}{2M^2} |\mathbf{p}| \rightarrow 0 \quad \Rightarrow \quad \Sigma \rightarrow 0$$

- $\beta \rightarrow \infty$ (zero temperature) or $\mathbf{p} \rightarrow \infty$

$$\beta a, \beta b, \beta E_p \rightarrow \infty \quad \Rightarrow \quad \ln[1 + \dots] \rightarrow 0 \quad \Rightarrow \quad \Sigma \rightarrow 0$$

- ★ minimal energy E_2 needed to produce particle 1 on-shell increases with $|\mathbf{p}|$
 \Rightarrow **statistical suppression**
- ★ zero temperature \leftrightarrow **no medium**: $f(E, T) \rightarrow 0$

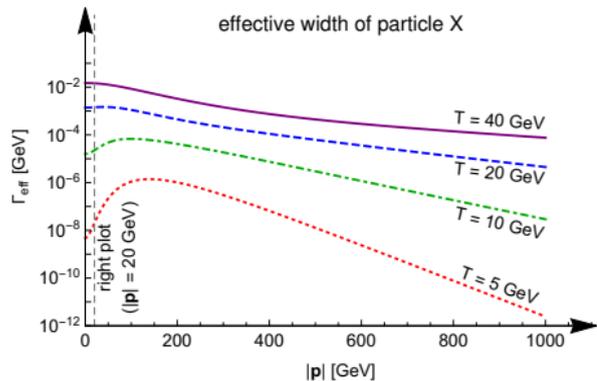


- $\mathbf{p} \rightarrow 0$ (mediator at rest)

$$\Sigma \rightarrow \frac{X_0}{16\pi M} \frac{\lambda(m_1^2, m_2^2, M^2)^{1/2}}{M^2} \frac{\eta_2 e^{-\beta(b_0-M)} (1 - \eta_1 \eta_2 e^{-\beta M})}{(1 + \eta_1 e^{-\beta b_0}) (1 + \eta_2 e^{-\beta(b_0-M)})} \quad \text{finite result}$$

$$b_0 \equiv (m_1^2 - m_2^2 + M^2)/(2M) > M$$

Numerical example: effective width

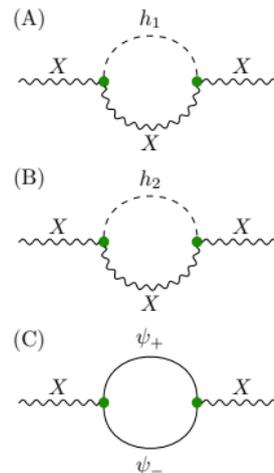
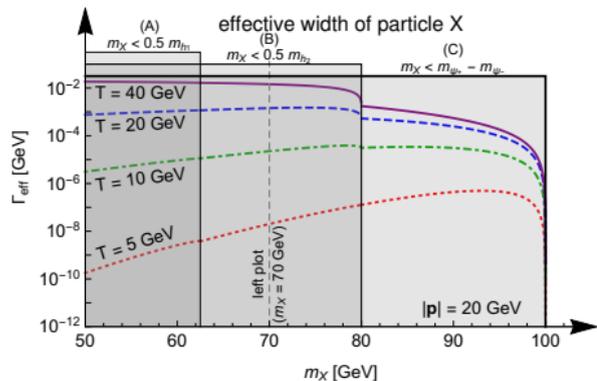


$$m_X = 70 \text{ GeV} \quad (\text{upper plot})$$

$$m_{\psi_+} = 130 \text{ GeV} \quad m_{\psi_-} = 30 \text{ GeV}$$

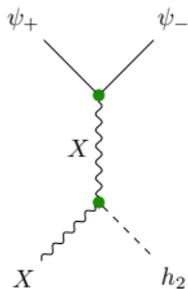
$$m_{h_1} = 125 \text{ GeV} \quad m_{h_2} = 160 \text{ GeV}$$

$$g_x = 0.1 \quad \sin \alpha = 0.1$$



VFDM model: A. Ahmed et al., [Eur.Phys.J.C 78 \(2018\) 11, 905](#)

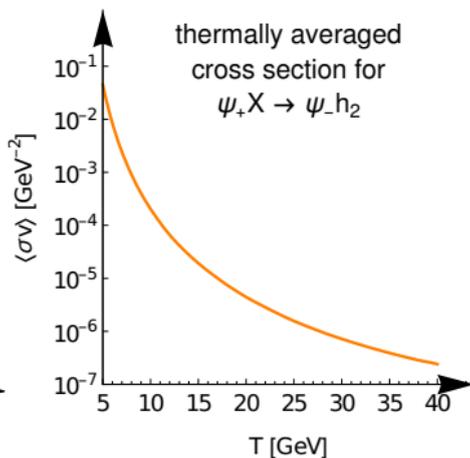
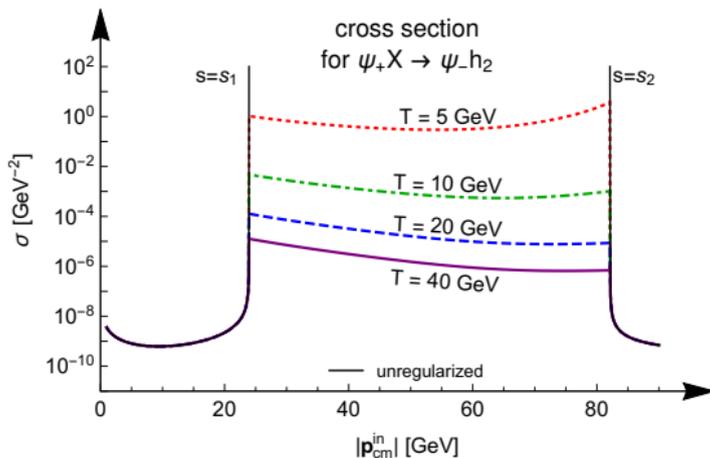
Numerical example: thermally averaged cross section



$$\langle \sigma v \rangle_{12 \rightarrow 34}(T) = \int d\Phi_1 d\Phi_2 f(E_1, E_2, T)$$

$$\times \int d\Phi_3 d\Phi_4 |\mathcal{M}|_{\text{dec}}^2 \frac{(2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4)}{(t - M^2)^2 + M^2 \Gamma_{\text{eff}}(|\mathbf{p}|, T)^2}$$

$$d\Phi_i \equiv \frac{d^3 p_i}{(2\pi)^3 2E_i} \quad \text{-- phase-space element}$$

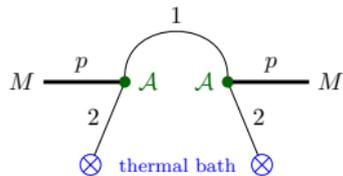
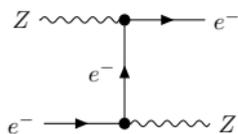


Summary

- t -channel singularity of $\langle\sigma v\rangle$ occurs if
 - the process can be seen as a sequence of decay and fusion processes



- the mediator is massive and stable
- the singularity is present both in SM and BSM physics
- known approaches are either unsatisfactory or inapplicable
- interactions with the medium result in a non-zero effective width that regularizes the singularity



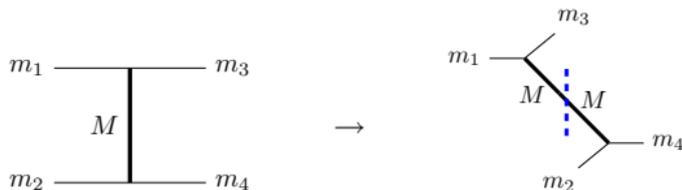
- the effective width depends on temperature and mediator's momentum (momentum transfer) and behaves in an expected, natural way

$$\Gamma_{\text{eff}} = \Gamma_{\text{eff}}(T, |\mathbf{p}|)$$

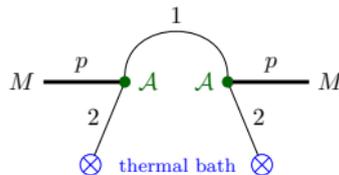
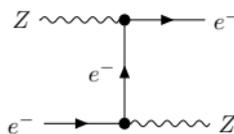
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thank you!



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- known approaches are either unsatisfactory or inapplicable
- interactions with the medium result in a non-zero effective width that regularizes the singularity



- the effective width depends on temperature and mediator's momentum (momentum transfer) and behaves in an expected, natural way

$$\Gamma_{\text{eff}} = \Gamma_{\text{eff}}(T, |\mathbf{p}|)$$

BACKUP SLIDES

Values of s_1, s_2 in terms of masses

- in terms of the CM energy (\sqrt{s}):

$$t_{\min}(s) < M^2 < t_{\max}(s)$$

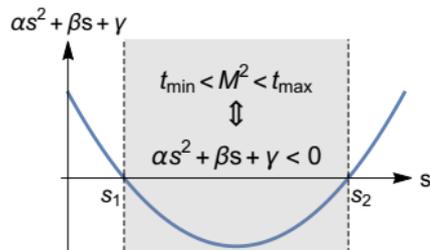
$$\Leftrightarrow s_1 < s < s_2$$

$$s_{1,2} \equiv \frac{-\beta \mp \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

$$\alpha \equiv M^2$$

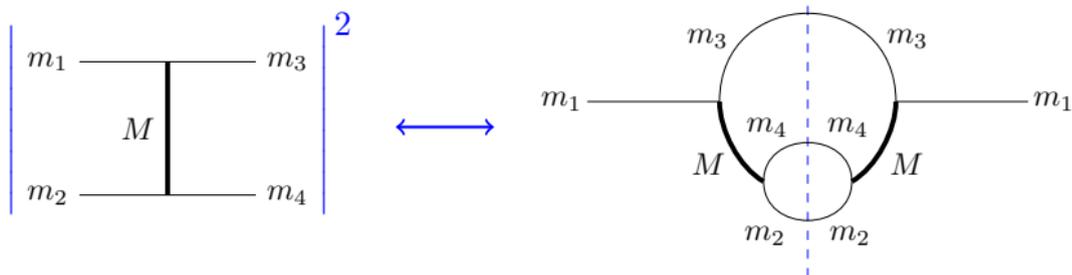
$$\beta \equiv M^4 - M^2(m_1^2 + m_2^2 + m_3^2 + m_4^2) + (m_1^2 - m_3^2)(m_2^2 - m_4^2)$$

$$\gamma \equiv M^2(m_1^2 - m_2^2)(m_3^2 - m_4^2) + (m_1^2 m_4^2 - m_2^2 m_3^2)(m_1^2 - m_2^2 - m_3^2 + m_4^2)$$



Process of interest in relation to other diagrams

- self-energy cut



- part of a larger diagram

