Unitarity in Multi-Higgs Production

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Talk based on J.Phys.G 49 (2022) 11, 115004, by Á. Curkó, and G. Cynolter.

Perturbative Standard Model

What is the scale of new physics, BSM?

- $M_H \simeq 125$ GeV, SM is weakly coupled up to M_{Pl}
- Asymptotic freedom,
  except $U_Y(1)$ and λ_H, Landau pole $\sim exp(1/\lambda)$ beyond M_{Pl}
- Metastability scale of EW vacuum 10^{10-12} GeV
- Problems with perturbative self-interacting scalars at high energies
  - Near kinematic threshold production of 'slow' multi scalars
  - Amplitude $h^* \to n \times h$ grows exponentially
  - $\to$ Unlimited $\sigma(s)$ violates the Froissart bound ($c \times \ln^2(s)$) and unitarity
- Classics 92-94, Brown, Son, Voloshin, Rubakov, Troitski
- Revival, V.V Khoze et al 2015 and 2017-19
- Recently Rattazzi et al.[2306.12489] Softening the UV/Hierarchy Problem with dynamics

SCALAR THEORY PART OF THE SM

Single, real scalar $\varphi(x)$, $\mathcal{L} = \frac{1}{2}\partial^\mu\varphi(x)\partial_\mu\varphi(x) - \frac{\lambda}{4}\left(\varphi(x)^2 - v^2\right)^2$. After SSB

$h(x) = \varphi(x) - v$, massive, prototype of the SM Higgs in unitary gauge

$$\mathcal{L} = \frac{1}{2}\partial^\mu h(x)\partial_\mu h(x) - \frac{m^2}{2}h^2(x) - \frac{\kappa}{3}h^3(x) - \frac{\lambda}{4}h^4(x), \qquad (1)$$

where $m = \sqrt{2\lambda}v$, $\kappa = 3\lambda v$ and $< h >= 0$ perturbative.

- Strange behaviour, decay large number(n) of scalars
- tree-level at kinematic threshold, factorial growth of graphs
- $\langle n|h(x)|0\rangle \sim n!\lambda^{n/2}$
- breakdown of pert. theory at $n \gtrsim 1/\sqrt{\lambda}$, at $E \gtrsim m/\sqrt{\lambda}$

TRANSITION RATE, $1^* \to n$

We are interested in the decay of excited, energetic $h^* \to nh$ process. Characterized by the Transition rate

$$R_n(p^2) = \int d\Pi_n |\mathcal{M}(1 \to n)|^2, \tag{2}$$

$\lambda n \gg 1$ limit, steepest descent method, semiclassical approach, pioneered by D.T.Son '96

$$R_n(p^2) \propto \exp\left[n \left(\log\left(\frac{\lambda n}{4}\right) - 1 + \frac{3}{2}\left(\log\left(\frac{\varepsilon}{3\pi}\right) + 1\right) - \frac{25}{12}\varepsilon + 0.85\sqrt{\lambda n} \right) \right], \tag{3}$$
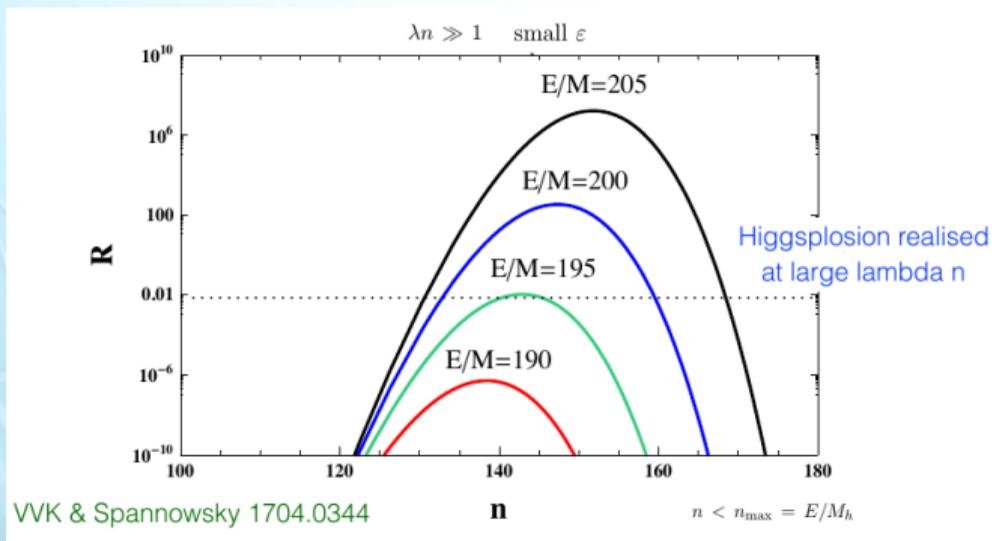
Terms are in order, tree-level, $\varepsilon = \frac{E - m \cdot n}{m \cdot n}$ $\varepsilon \neq 0$ tree correction, thin-wall (bubble) q-corr'n ε average kinetic energy of the final-state Higgses. Valid at

- $\frac{1}{\lambda} \to \infty$, $n \to \infty$ $\lambda n =$ const.$\gg 1$
- $\varepsilon =$ const $\ll 1$

Still exponential $\leftrightarrow$ Unitarity ?

TRANSITION RATE, $1^* \rightarrow n$, exponential grow

$$R_n(p^2) \propto \exp\left[n\left(\log\left(\frac{\lambda n}{4}\right) - 1 + \frac{3}{2}\left(\log\left(\frac{\varepsilon}{3\pi}\right) + 1\right) - \frac{25}{12}\varepsilon + 0.85\sqrt{\lambda n}\right)\right],$$

VVK & Spannowsky 1704.0344

TRANSITION RATE, $1^* \to n$

$$R_n(p^2) \propto \exp\left[n\left(\log\left(\frac{\lambda n}{4}\right) - 1 + \frac{3}{2}\left(\log\left(\frac{\varepsilon}{3\pi}\right) + 1\right) - \frac{25}{12}\varepsilon + 0.85\sqrt{\lambda n}\right)\right],$$

Applied to SM $\lambda_{SM} \simeq 0.125$, $\sqrt{p^2} = (100 - 200) \cdot m_H$, changing $n \gg 1$

$\lambda n \gg 1$, ε small, but not fixed $\to R_n(p^2) \sim \exp(a\sqrt{p^2})$ with $a > 0$.

Origin of the problem

- limit in saddle point method invalid

- Missing terms   tree-level ε not small

                     quantum corrections for $\varepsilon \neq 0$

Use a not perturbative method

Summed up, self-consistent equation - expect unitarity

Calculate $R_n(p^2)$ from Schwinger-Dyson equations

SCHWINGER-DYSON EQUATIONS IN SSB ϕ^4 MODEL

SD system of coupled equations derived from the identity, ϕ^4 V.Sauli(2005)

$$\int \mathcal{D}\varphi \frac{\delta\theta[\varphi]}{\delta\varphi(x)} = 0, \text{ where } \theta[\varphi] = \exp\left\{i\int d^4x\left[\mathcal{L}(\varphi(x)) + J(x)\varphi(x)\right]\right\},$$

$$\Pi(p^2) = \kappa \int \frac{d^4k_1}{(2\pi)^4} G_2^c(-k_1)G_2^c(k_1+p)\Gamma_3(k_1,p)+$$

$$+ \lambda \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} G_2^c(k_1)G_2^c(-k_1-k_2)G_2^c(k_2+p)\Gamma_4(-k_1,k_1+k_2,p)+$$

$$+ 3\lambda \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} G_2^c(-k_1)G_2^c(-k_2)G_2^c(k_1+k_2+p)\Gamma_3(k_1+p,k_2)G_2^c(k_1+p)\Gamma_3(k_1,p),$$

$\Pi(p^2)$ is the self-energy
$G_2^c(k)$ is the connected two-point function
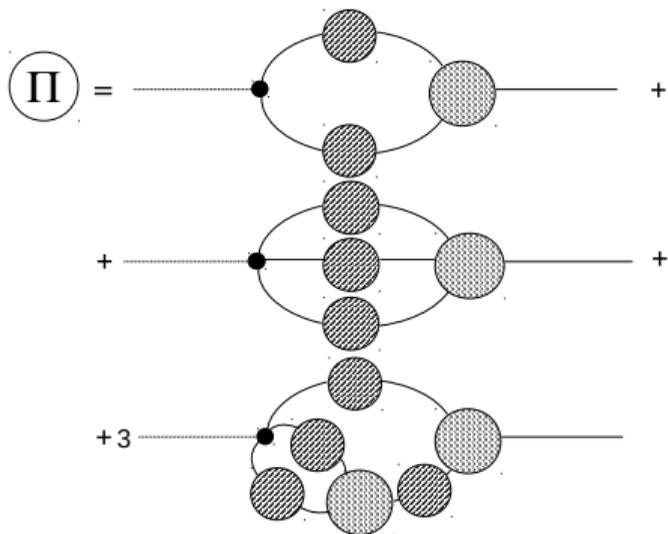$\Gamma_n(k_1, k_2, \ldots k_{n-1})$ is the n-point vertex function
$m = \sqrt{2\lambda}v, \; \kappa = 3\lambda v$

SCHWINGER-DYSON EQUATIONS GRAPHICALY

$\Pi(p^2)$ self-energy

$G_2^c(k)$ connected two-point function

$\Gamma_n(k_1, k_2, \ldots k_{n-1})$ n-point vertex fn

Complete SD system of eq'n, extra eq'ns for vertex function, small λ, κ tree level can be used

SCHWINGER-DYSON EQUATIONS SIMPLIFIED

Small λ, κ vertex functions $\begin{aligned}\Gamma_3(k_1, k_2) &= -i2\kappa + \dots, \\ \Gamma_4(k_1, k_2, k_3) &= -i6\lambda + \dots\end{aligned}$ Arrive at

$$\Pi(p^2) = -i2\kappa^2 \int \frac{d^4 k_1}{(2\pi)^4} G_2^c(-k_1) G_2^c(k_1 + p) -$$

$$-i6\lambda^2 \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} G_2^c(k_1) G_2^c(-k_1 - k_2) G_2^c(k_2 + p) -$$

$$-12\lambda\kappa^2 \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} G_2^c(-k_1) G_2^c(-k_2) G_2^c(k_1 + k_2 + p) G_2^c(k_1 + p),$$

Bubble+setting sun +... Solve with spectral decomposition method
$\kappa = 3\lambda v$ has mass dimension
Third diagram(line) negligible, the prefactor $\frac{12\lambda\kappa^2}{(4\pi)^4} \ll \frac{2\kappa^2}{(4\pi)^2}|_{\text{bubble}}$ by 3 order.

SPECTRAL DECOMPOSITION

Generic decomposition of a scalar two-point function

$$G(p^2) = \underbrace{\frac{iZ}{p^2 - m^2 + i\epsilon}}_{\text{pole}} + \underbrace{\int_{\omega_{th}}^{\infty} d\omega \frac{i\sigma(\omega)}{p^2 - \omega + i\epsilon}}_{\text{continuum}} \qquad \omega_{th} = 4m^2, \quad \omega \sim s$$

Z is the wave-function renormalization, $\sigma(\omega)$ spectral function for the propagator

Divergence structure

$$\text{bubble} \sim \log\Lambda \text{ logarithmic 1 subtraction}$$

$$\text{setting sun} \sim \Lambda^2 \text{ quadratic need 2 subtractions}$$

$$\Pi_R(p^2) = \Pi(p^2) - \Pi(m^2) - \frac{d\Pi(p^2)}{dp^2}\Big|_{p^2=m^2}(p^2 - m^2) \text{ Renormalized self energy}$$

$$\Pi_R(p^2 = m^2) = 0, \quad \Pi_R'(p^2 = m^2) = Z - 1 \text{ Renormalization cond., (Nonmin mom. subtr.)}$$

SD EQUATION WITH SPECTRAL DECOMPOSITION $\sigma(\omega)$, $\rho(\omega)$

Double subtracted Renormalized self energy

$$\Pi_R(p^2) = \int_{\omega_{th}}^{\infty} d\omega \frac{\rho(\omega)}{p^2 - \omega + i\epsilon} \left(\frac{p^2 - m^2}{\omega - m^2}\right)^2 = \int_{\omega_{th}}^{\infty} d\omega \frac{\tilde{\rho}(\omega, p^2; m^2)}{p^2 - \omega + i\epsilon}$$

Defines $\rho(\omega) = -\text{Im}\Pi_R(p^2)/\pi$ spectral func. of the self energy and double subtracted $\tilde{\rho}(\omega, p^2; m^2)$.

$$\rho(\omega) = \frac{2\kappa^2}{(4\pi)^2} \left[Z^2 X(m^2, m^2, \omega) + 2Z \int_{\omega_{th}}^{\infty} d\alpha_1 X(\alpha_1, m^2, \omega)\sigma(\alpha_1) + \right.$$

$$+ \int_{\omega_{th}}^{\infty} d\alpha_1 \int_{\omega_{th}}^{\infty} d\alpha_2 X(\alpha_1, \alpha_2, \omega)\sigma(\alpha_1)\sigma(\alpha_2) \left. \right] +$$

$$+ \frac{6\lambda^2}{(4\pi)^4} \left[Z^3 Y(m^2, m^2, m^2, \omega) + 3Z^2 \int_{\omega_{th}}^{\infty} d\alpha_1 Y(\alpha_1, m^2, m^2, \omega)\sigma(\alpha_1) + ... \right]$$

BUBBLE+ SETTING SUN SDE WITH SPECTRAL DECOMP'N*

$$X(\alpha_1, \alpha_2, \omega) = \frac{\sqrt{\lambda(\alpha_1, \alpha_2, \omega)}}{\omega} \Theta(\omega - (\sqrt{\alpha_1} + \sqrt{\alpha_2})^2), \quad \lambda^{Kallen}(x, y, z) = x^2 - 2xy + y^2 - ...$$

$$\rho(\omega) = \frac{2\kappa^2}{(4\pi)^2} \left[Z^2 X(m^2, m^2, \omega) + 2Z \int_{\omega_{th}}^{\infty} d\alpha_1 X(\alpha_1, m^2, \omega) \sigma(\alpha_1) + \right.$$

$$+ \int_{\omega_{th}}^{\infty} d\alpha_1 \int_{\omega_{th}}^{\infty} d\alpha_2 X(\alpha_1, \alpha_2, \omega) \sigma(\alpha_1) \sigma(\alpha_2) \Bigg] +$$

$$+ \frac{6\lambda^2}{(4\pi)^4} \left[Z^3 Y(m^2, m^2, m^2, \omega) + 3Z^2 \int_{\omega_{th}}^{\infty} d\alpha_1 Y(\alpha_1, m^2, m^2, \omega) \sigma(\alpha_1) + \right.$$

$$+ 3Z \int_{\omega_{th}}^{\infty} d\alpha_1 \int_{\omega_{th}}^{\infty} d\alpha_2 Y(\alpha_1, \alpha_2, m^2, \omega) \sigma(\alpha_1) \sigma(\alpha_2) +$$

$$+ \int_{\omega_{th}}^{\infty} d\alpha_1 \int_{\omega_{th}}^{\infty} d\alpha_2 \int_{\omega_{th}}^{\infty} d\alpha_3 Y(\alpha_1, \alpha_2, \alpha_3, \omega) \sigma(\alpha_1) \sigma(\alpha_2) \sigma(\alpha_3) \Bigg],$$

SCHWINGER-DYSON+ EXTRA RELATION

Relation from $1 = G(p^2)G^{-1}(p^2) = \underbrace{G(p^2)}_{\sigma(\omega)}(p^2 - m^2 - \underbrace{\Pi_R(p^2)}_{\rho(\omega)})$

$$\sigma(p^2) = \frac{Z\rho(p^2)}{(p^2 - m^2)^2} + \frac{1}{p^2 - m^2}\mathrm{P}\cdot\int_{\omega_{th}}^{\infty}d\omega\,\frac{\sigma(p^2)\tilde{\rho}(\omega, p^2; m^2) + \rho(p^2)\sigma(\omega)}{p^2 - \omega}$$

Sum rule can be derived, "particle conservation" pole+ cont.

$$Z + \int_{\omega_{th}}^{\infty}d\omega\,\sigma(\omega) = 1,\text{ actually }\sigma(\omega)\text{ small}\to Z\simeq 1$$

TRANSITION RATE, $\sum_n R_n(p^2) = ?$

Use LSZ reduction formula for 2-point functions

$$-Z\Pi(p^2) = \mathcal{M}(p \to p)$$

optical theorem

$$- \mathrm{Im}\left({}^{p^2}\!\!-\!\!\blacksquare\!\!-^{p^2} \right)$$

$$2\mathrm{Im}\mathcal{M}(p \to p) = \sum_n R_n(p^2) \text{ sum over all inner states through a cut}$$

From the renormalized self-energy get the transition rate ($\rho(\omega) = -\mathrm{Im}\Pi_R(p^2)/\pi$)

$$\sum_n R_n(p^2) = 2\pi Z \rho(p^2)$$

ASYMPTOTICS, DOMINANT SETTING SUN, CUTOFF

SD eq'ns determine the asymptotic behaviour of the renormalized self energy

$$\rho(\omega) = m^2 \lambda f_{\text{bubble}}(\omega, m^2) + \lambda^2 f_{\text{setting sun}}(\omega, m^2) + m^2 \lambda^2 f_3(\omega, m^2),$$
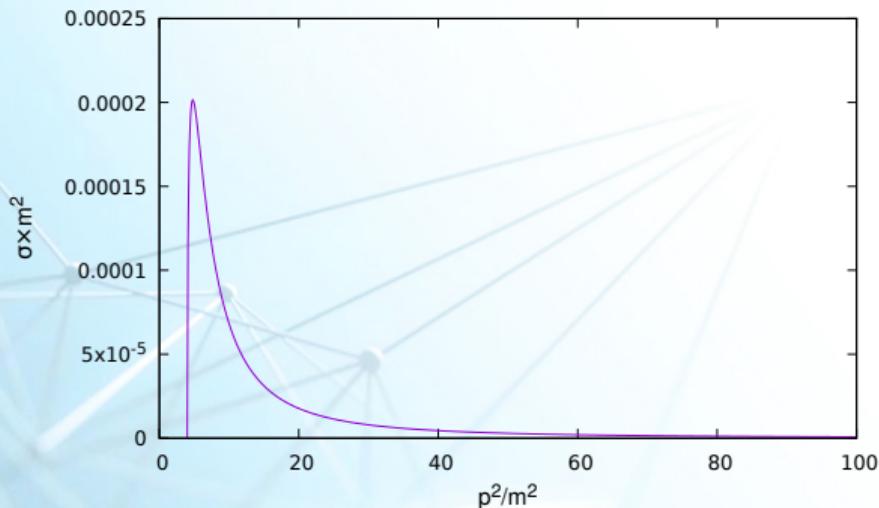
Dimensional analysis , $[\rho] = M^2$, only $f_{settingsun}(\omega, m^2)$ has mass dimension, for $\omega \gg m^2$
$f_{\text{setting sun}}(\omega, m^2) \to \omega$ setting sun is the dominant at high energy.

- ϕ^4 theory is trivial and to satisfy the sum rule $Z + \underbrace{\int d\omega \sigma(\omega)}_{\sim log(\omega) \text{ div.}} = 1$

- Introduce cutoff $\Lambda \gg E_{\text{Higgspersion}} \sim 100 m_H$. (weird, rapid Higgs decay starts here)
  Choose $\Lambda = (800m)^2$, corresponding to $E_{max} = 100$ TeV.
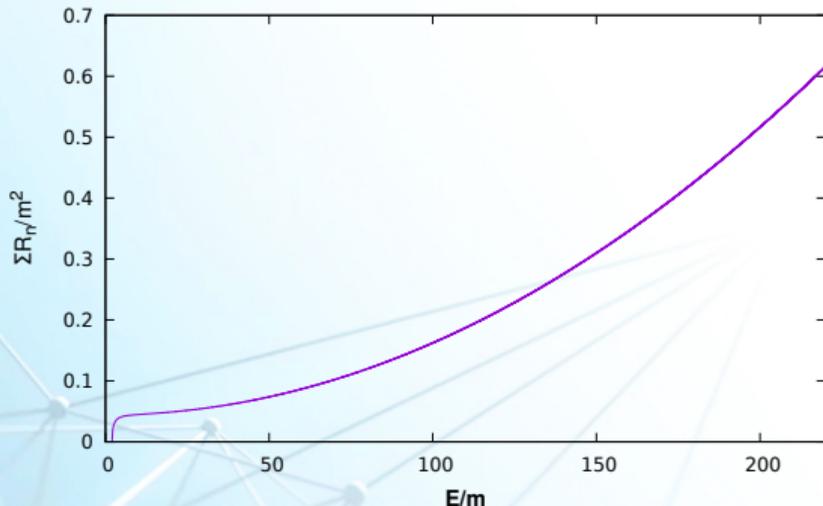
NUMERICAL ITERATIVE SOLUTION, SDE$+GG^{-1}=I$, $\sigma(p^2)$

$$Z_0 = 1(\sigma(\omega)=0) \to \rho_0 \to \sigma_1(\omega) \propto \rho_0 \xrightarrow{sumrule} Z_1 = 1 - \int_{th}^{\Lambda} \sigma_1(\omega)d\omega \xrightarrow[back]{go}$$

Convergence tested by the sum rule (with Z) or $\int_{th}^{\Lambda} d\omega\sigma(\omega) \sim O(10^{-5})$ for $\lambda = 0.12$, also $\lambda = 10$

NUMERICAL SOLUTION, TRANSITION RATE $\sim E^2$

Numerics show $\rho(p^2) \sim p^2$ at $E \gg m$ and dimensional analysis was fine
$\sum_n R_n(p^2) = 2\pi Z \rho(p^2) \sim E^2$ for $E \gg m$, grows with E^2 up to Λ

Exponentially growing $R_n(p^2)$ violates unitarity, here $R_n(p^2) \sim p^2$. Is it safe?

UNITARITY, THE HIGGS PROPAGATOR

The propagator is the geometric series of the 1PI self energy $\Pi(p^2)$

$$G(p^2) = \frac{i}{p^2 - m^2} \sum_{n=0}^{\infty} \left(-i\Pi(p^2) \frac{i}{p^2 - m^2} \right)^n = \frac{i}{p^2 - m^2 - \Pi(p^2)}.$$

- Can only summed up if the series converges
- Solution of Khoze et al. $\Pi \sim \exp(\sqrt{p^2})$ diverges, it cannot be resummed. [Belyaev]
- SDE solution $\Pi \sim \rho(p^2) \sim p^2$ and G decays with p^2
  Also showed by $\sigma(\omega)$ asymptotically decaying see fig. 1. (p 17.)
- SDE provides a well behaving spectral density
- Multi-Higgs production, growth of the σ is cured via 'Higgspersion' mechanism

THE HIGGSPERSION, HIGGS PRODUCTION

Via the optical theorem $\text{Im}\Pi = \Gamma(s) = \sum_n^\infty \Gamma_n(s) \propto \sum_n R_n(s)$ tames the G propagator, upper bound Example h^* production and decay in gg fusion, summed up! self energy

$$\sigma_{gg \to n \times h} \propto |\mathcal{M}_{gg \to h^*}|^2 \times \frac{1}{(p^2 - m^2 - \text{Re}\Pi(p^2))^2 + (\text{Im}\Pi(p^2))^2} \times R_n(p^2)$$

$$\simeq \log^4\left(\frac{m_t}{\sqrt{p^2}}\right) 4Z^2 \frac{R_n(p^2)}{(p^2 + \sum_n R_n(p^2))^2} < \log^4\left(\frac{m_t}{\sqrt{p^2}}\right) 4Z^2 \frac{1}{\sum_n R_n(p^2)} \xrightarrow[p^2 \to \infty]{} 0$$

As long as $\sum_n R_n(s) \propto s$ grows faster than $\log^4\left(\frac{m_t}{s}\right)$

- No violation of unitarity
- Multi-Higgs production, growth of the decay rate is cured via 'Higgspersion' mechanism, even faster growing scalar decay width.
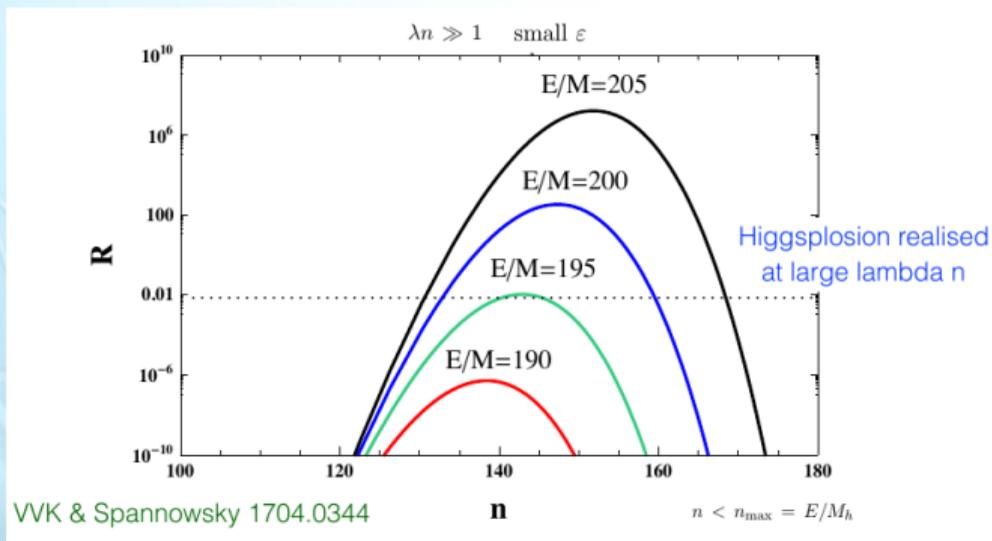
CONCLUSIONS

- We have investigated the debated transition rate $h^* \to nh$ in the ϕ^4 theory, SSB 3+1

- Successfully solved by iteration the Schwinger-Dyson equations, calculated the contribution of the dominant setting sun diagram (for $\lambda = 0.125, 10$)

- The self energy of the SDE solution can be bubble-resummed, the propagator also well behaved

- No unitarity violation within this approach.

- The transition rate growing with p^2 tamed by the resummed propagator $\sim p^{-2}$ in real physical processes. See no new scale in the SM.

BACKUP

TRANSITION RATE, $1^* \to n$, exponential grow

$$R_n(p^2) \propto \exp\left[n\left(\log\left(\frac{\lambda n}{4}\right) - 1 + \frac{3}{2}\left(\log\left(\frac{\varepsilon}{3\pi}\right) + 1\right) - \frac{25}{12}\varepsilon + 0.85\sqrt{\lambda n}\right)\right],$$

VVK & Spannowsky 1704.0344

Perturbative limit, simplified equation

Keeping only the leading terms in SD and constraint -Pertubative solution, dotted line

$$\rho'(\omega) = \frac{9\lambda m^2}{16\pi^2} Z^2 X(m^2, m^2, \omega) \quad \sigma'(\omega) = \frac{Z\rho'(\omega)}{(\omega - m^2)^2}$$

(a) $\lambda = 0.125$ **(b)** $\lambda = 10$