

## Generalizations:

- higher spin                      RN, F Ravanini, D Raveh                      arXiv: 2402.03233
- higher rank                      RN, D Raveh                      arXiv: 2301.04989
- q-deformation                      D Raveh, RN                      arXiv: 2308.08392

**Bethe states**

# Bethe ansatz review





# Coordinate Bethe ansatz (1931)



Remarkable solution!

Reduces the problem to solving a system of polynomial equations “Bethe equations”

Only special models can be solved in this way “quantum integrable”

Many conserved commuting quantities

$$\left[ H, H^{(3)} \right] = 0$$

⋮

$$H^{(3)} = \sum_{n=1}^L \vec{\sigma}_n \cdot (\vec{\sigma}_{n+1} \times \vec{\sigma}_{n+2})$$

Eigenvectors are **multi-particle** (“magnon”) states

ground (0-particle) state:

$$|\psi_0\rangle = \left( \begin{array}{c} 1 \\ 0 \end{array} \right)^{\otimes L} = |\uparrow \cdots \uparrow\rangle$$

$$H|\psi_0\rangle = 0$$

1-particle state:

$$|\psi(k)\rangle = \sum_{x=1}^L e^{ikx} |\uparrow \cdots \underset{x}{\downarrow} \cdots \uparrow\rangle$$

$$H|\psi(k)\rangle = e(k)|\psi(k)\rangle$$

$$e(k) = 4 \sin^2\left(\frac{k}{2}\right)$$

1-particle energy

provided

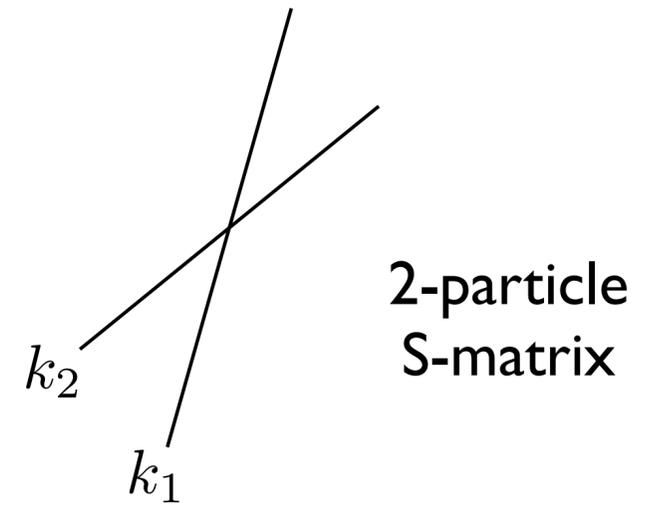
$$e^{ikL} = 1$$

2-particle state:

$$|\psi(k_1, k_2)\rangle = \sum_{1 \leq x_1 < x_2 \leq L} \left[ s(k_2, k_1) e^{i(k_1 x_1 + k_2 x_2)} - s(k_1, k_2) e^{i(k_2 x_1 + k_1 x_2)} \right] |\uparrow \cdots \downarrow_{x_1} \cdots \downarrow_{x_2} \cdots \uparrow\rangle$$

$$s(k, k') = 1 - 2e^{ik'} + e^{i(k+k')}$$

$$S(k_2, k_1) = -\frac{s(k_1, k_2)}{s(k_2, k_1)}$$



$$H|\psi(k_1, k_2)\rangle = E|\psi(k_1, k_2)\rangle \quad E = e(k_1) + e(k_2)$$

provided

$$\begin{cases} e^{ik_1 L} = S(k_1, k_2) \\ e^{ik_2 L} = S(k_2, k_1) \end{cases}$$

M-particle state:

$$|\psi(k_1, \dots, k_M)\rangle = \sum_{1 \leq x_1 < \dots < x_M \leq L} \sum_{\sigma \in \text{Perm}(1, \dots, M)} \varepsilon(\sigma) A(k_{\sigma(1)}, \dots, k_{\sigma(M)}) e^{i \sum_{j=1}^M k_{\sigma(j)} x_j} | \uparrow \dots \downarrow \dots \downarrow \dots \uparrow \rangle_{x_1 \quad x_M}$$

Bethe state

$$A(k_1, \dots, k_M) = \prod_{1 \leq j < l \leq M} s(k_l, k_j) \quad \varepsilon(\sigma) = \pm 1 \quad \text{signature of } \sigma$$

$$H|\psi(k_1, \dots, k_M)\rangle = E|\psi(k_1, \dots, k_M)\rangle \quad E = \sum_{j=1}^M e(k_j)$$

provided

$$e^{ik_j L} = \prod_{l=1; l \neq j}^M S(k_j, k_l), \quad j = 1, \dots, M$$

Bethe equations

$$\left( \frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} \right)^L = \prod_{\substack{l \neq j \\ l=1}}^M \frac{u_j - u_l + i}{u_j - u_l - i}$$

$$u_j = u(k_j)$$

$$u(k) = \frac{1}{2} \cot\left(\frac{k}{2}\right)$$

SU(2) symmetry  $\Rightarrow$  degeneracy  $L - 2M + 1$

$$M \leq \frac{L}{2}$$

Example:  $L = 4$

$$M = 0, 1, 2$$

$M$	$u_j$	$E$	degeneracy
0	-	0	5

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0	-	0	5
1	$\frac{1}{2}$	2	3
1	$-\frac{1}{2}$	2	3
1	0	4	3

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total:  $16 = 2^4 = 2^L$  “complete” ✓

# Bethe state preparation

## Closed periodic spin-1/2 XXZ chain

$$\mathcal{H} = -\frac{1}{2} \sum_{n=1}^L \left( \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \left( \sigma_n^z \sigma_{n+1}^z - \mathbb{I} \right) \right), \quad \vec{\sigma}_{L+1} = \vec{\sigma}_1$$

Assume Bethe roots  $\{k_1, \dots, k_M\}$  are known.

How to prepare corresponding Bethe state  $|B_M^L\rangle$  on quantum computer?

- |   |                         |   |
|---|-------------------------|---|
| • Van Dyke, Barron, Mayhall, Barnes, Economou 2021  | coordinate BA           | probabilistic; ancillas, real Bethe roots                 |
| • Sopena, Gordon, García-Martín, Sierra, López 2022 | algebraic BA            | deterministic; QR decompositions                          |
| • Ruiz, Sopena, Gordon, Sierra, López 2023          | algebraic/coordinate BA | analytical formulae for unitaries                         |
| • Ruiz, Sopena, López, Sierra, Pozsgay 2024         | algebraic BA            | F-basis   |
| • Raveh, RN 2024                                    | coordinate BA           | deterministic; no ancillas, <i>explicit</i>               |
| • Sahu, Vidal 2024                                  | coordinate BA           | deterministic; ancillas; log depth                        |
| • Yeo, Kim, Sohn, Jeong 2025                        | coordinate BA           | probabilistic; ancillas, real Bethe roots, constant depth |
| • Zi, Nie, Sun 2025                                 | [coordinate BA]         | deterministic; ancillas, constant depth                   |
| • Lutz, Piroli, Styliaris, Cirac 2025               | —                       | adiabatic; approximate                                    |

Basic idea:

set of all permutations of M 1's and L-M 0's

Bethe state

$$|B_M^L\rangle \propto \sum_{w \in P(L, M)} f(w) |w\rangle$$

Basic idea:

Bethe state

$$|B_M^L\rangle \propto \sum_{w \in P(L, M)} f(w) |w\rangle$$

$$f(w) = \sum_{\sigma \in \text{Perm}(1, \dots, M)}$$

$$\varepsilon(\sigma) A(k_{\sigma(1)}, \dots, k_{\sigma(M)}) e^{i \sum_{j=1}^M k_{\sigma(j)} x_j}$$

$$x_j \in \{1, \dots, L\}$$

positions of 1's in w

coordinate BA

Ex:

$$|B_2^4\rangle \propto f(0011) |0011\rangle + f(0101) |0101\rangle + f(1001) |1001\rangle \\ + f(0110) |0110\rangle + f(1010) |1010\rangle + f(1100) |1100\rangle$$

Basic idea:

coordinate BA

Bethe state

$$|B_M^L\rangle \propto \sum_{w \in P(L, M)} f(w) |w\rangle$$

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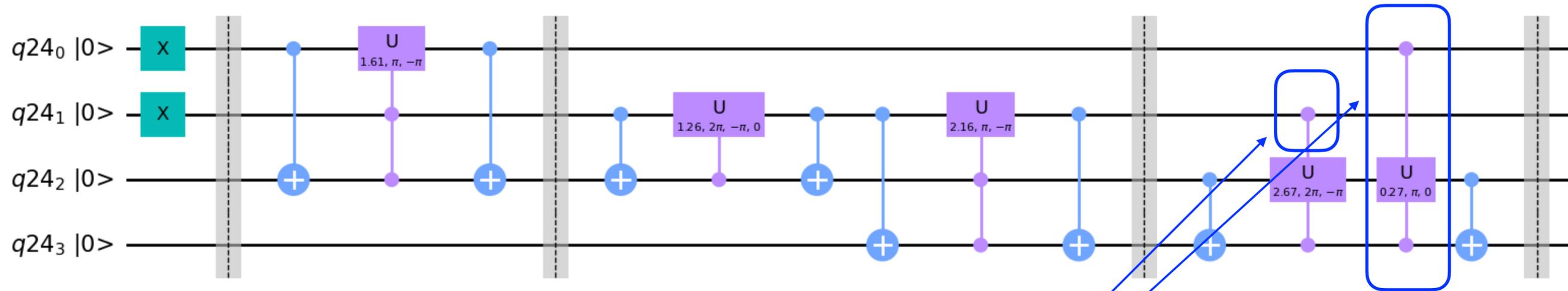
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$\binom{L}{M}$  coefficients — computed classically

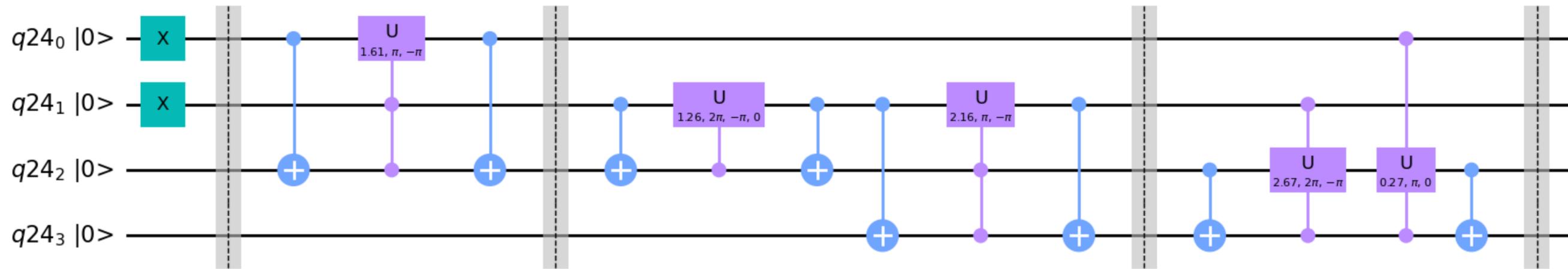
Recursion ( ~ Dicke states)

Ex: (L,M) = (4,2)



Same as circuit for Dicke state  $|D_2^4\rangle$  except for angles & these gates

Ex: (L,M) = (4,2)



circuit size =  $\mathcal{O}\left(\binom{L}{M}\right)$  = # coefficients  $\therefore$  expect  $\sim$  optimal

For  $M=L/2$   $\binom{L}{L/2} \sim \frac{2^L}{\sqrt{\pi L/2}}$  😞

Bethe roots from VQE

Bethe equations are generally hard to solve. Can quantum computers help?

Variational Quantum Eigensolver (VQE): hybrid quantum/classical algorithm for estimating the ground-state energy  $E_0$  of a Hamiltonian  $\mathcal{H}$  using the variational theorem

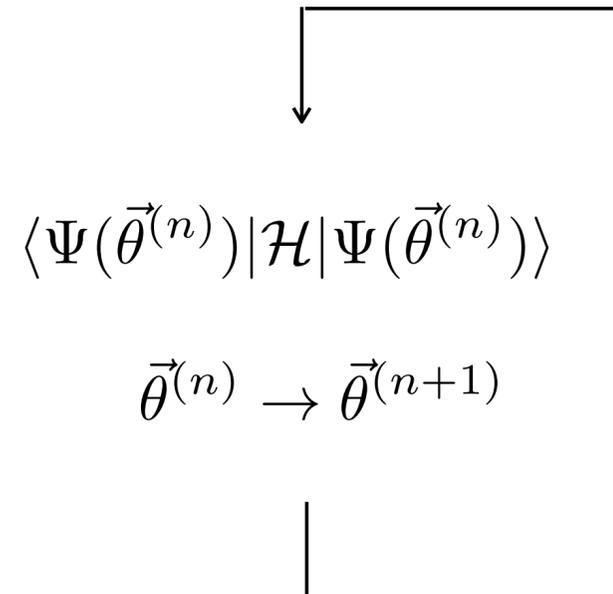
normalized trial state  $|\Psi(\vec{\theta})\rangle$        $\vec{\theta}$  parameters

iteration:  $\vec{\theta}^{(0)}$

classical simulators

~~quantum~~

classical



variational theorem  $\Rightarrow \langle \mathcal{H} \rangle \geq E_0$

Estimate Bethe roots using VQE, taking exact Bethe states as trial states, and treating Bethe roots  $\vec{k}$  as variational parameters

Raveh, RN 2404.18244

To test this idea, we instead used classical simulators.

## Classical simulators:

- **Qiskit Statevector simulator:** performs matrix arithmetic to compute *exact* expectation values
- **Qiskit Aer simulator:** noiseless simulation using 10,000 shots (trials)

## Ground-state Bethe roots:

- Closed chain  $\Delta = 2$  [Newton's method - Mathematica](#)

$L$	$M$	Energy	True roots
2	1	-2	3.14159
4	2	-2.73205	$\pm 1.94553$
6	3	-3.85577	$\pm 1.49862, 3.14159$

[Bethe roots real](#)

## Ground-state Bethe roots:

- Closed chain  $\Delta = 2$

$L$	$M$	Energy	True roots	Statevector roots	Aer roots
2	1	-2	3.14159	3.1415	3.1487
4	2	-2.73205	$\pm 1.94553$	1.9455, -1.9455	1.9623, -1.9503
6	3	-3.85577	$\pm 1.49862, 3.14159$	1.4986, -1.4986, 3.1416	1.5477, -1.4830, 3.1796

Bethe roots real

## Excited-state Bethe roots:

Instead of minimizing  $\langle \mathcal{H} \rangle$ , we now minimize the *variance*

[Zhang, Chen, Yuan, Yin (2020)]

$$\langle (\mathcal{H} - \langle \mathcal{H} \rangle)^2 \rangle = \langle \mathcal{H}^2 \rangle - \langle \mathcal{H} \rangle^2 \geq 0 \quad (= 0 \text{ for exact eigenstate})$$

- Closed chain  $\Delta = 2$

(selected)

$L$	$M$	Energy	True roots	Statevector roots	Aer roots
2	1	0	0	0	0.0019
3	1	-1	2.0944	2.0943	2.0938
4	2	0.732051	$\pm 0.831443i$	$\pm 0.8314i$	$\pm 0.8557i$
5	2	0.716341	$0.628319 \pm 0.835459i$	$0.6276 \pm 0.8349i$	$0.6263 \pm 0.8858i$
6	2	-1.75395	1.37766, 2.81114	1.3776, 2.8109	1.3998, 2.8293
6	3	1.18614	$0.244998 \pm 1.41247i,$ 1.6044	$0.2451 \pm 1.4120i,$ 1.6023	$0.2451 \pm 1.3341i,$ 1.3152

complex Bethe roots

# Conclusions

- The quantum state preparation problem is interesting and challenging
- MPS is useful for preparing quantum states with low bond dimension (entanglement)
- Can use intermediate measurements/feedforward operations to reduce circuit depth
- Much room for improvement of Bethe state preparation

Thank you for your attention!