

Matrix Product States

Matrix Product States (MPS)

n -qubit state: $|\psi\rangle = \sum_{\vec{m}} a_{\vec{m}} |\vec{m}\rangle$ $|\vec{m}\rangle = |m_n \dots m_2 m_1\rangle$ $m_i = 0, 1$ $i = 1, \dots, n$

qubits “physical space”

$$a_{\vec{m}} = \langle L | A_n^{m_n} \dots A_2^{m_2} A_1^{m_1} | R \rangle$$

$A_i^{m_i}$ $\chi \times \chi$ matrices basis $|j\rangle$ $j = 0, 1, \dots, \chi - 1$

qudit “bond space”

χ “bond dimension”

“auxiliary”, “ancilla”, “memory”, “virtual”

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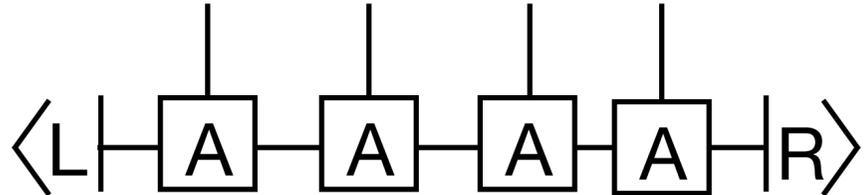
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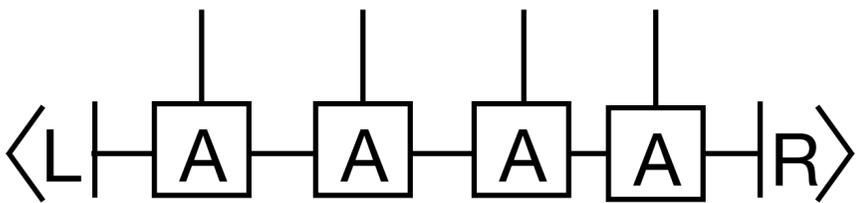
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Can represent any state $|\psi\rangle$ this way, for sufficiently large χ — can be useful if χ is not too large

Entanglement entropy $S \sim \log \chi$ Energy of first excited state - Energy of ground state > 0 for $n \rightarrow \infty$

For the ground state of a **gapped Hamiltonian**: $S \sim \text{area}$ (i.e., $S \sim n^{D-1}$, $D = \#$ space dimensions)

\Rightarrow In 1 space dimension, $S = \text{constant}$ (independent of n)

\Rightarrow $\chi = \text{constant}$ (independent of n)

$$|\psi\rangle = \sum_{\vec{m}} a_{\vec{m}} |\vec{m}\rangle \quad a_{\vec{m}} = \langle L | A_n^{m_n} \dots A_2^{m_2} A_1^{m_1} | R \rangle$$

For GHZ state:

$$|\text{GHZ}_n\rangle = \frac{1}{\sqrt{2}} (|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$$

$$A_i^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad A_i^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\langle L | = (1 \quad 1)$$

$$|R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = H|0\rangle$$

$$\chi = 2$$

translational invariant

Note

$$(A^0)^2 = A^0, \quad (A^1)^2 = A^1, \quad A^0 A^1 = 0 = A^1 A^0 \quad \Rightarrow \quad \text{m's are either all 0's or all 1's!}$$

$$\Rightarrow a_{\vec{m}} = \begin{cases} \frac{1}{\sqrt{2}}, & \text{if } \vec{m} = 0\dots 0 \quad \text{or} \quad \vec{m} = 1\dots 1 \\ 0, & \text{otherwise} \end{cases} \quad \checkmark$$

Also

$$A^m |j\rangle = \delta_{m,j} |j\rangle$$

Sequential state preparation

$$\sum_m A_i^{m\dagger} A_i^m = \mathbb{I} \quad (\text{left}) \text{ canonical}$$

\Rightarrow

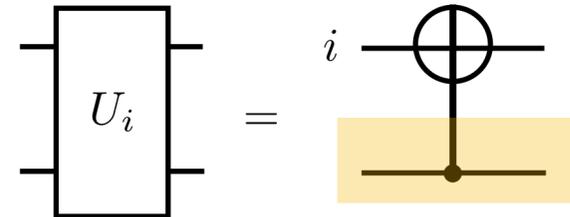
$$U_i |\underline{j}\rangle |0\rangle_i = \sum_m \underbrace{(A_i^m |\underline{j}\rangle)}_{\text{GHZ: } \delta_{m,j} |\underline{j}\rangle} |m\rangle_i = |\underline{j}\rangle |j\rangle_i$$

$$\begin{array}{c} j = 0, 1 \\ \hline j = 0, 1, \dots, \chi - 1 \end{array}$$

U_i unitary

\Rightarrow can use U_i to prepare the state sequentially:

$$\mathcal{U} = \prod_{i=1}^{\overbrace{n}} U_i$$



$$\mathcal{U} |R\rangle |0\rangle^{\otimes n} = \sum_{\vec{m}} A_n^{m_n} \dots A_1^{m_1} |R\rangle |\vec{m}\rangle$$

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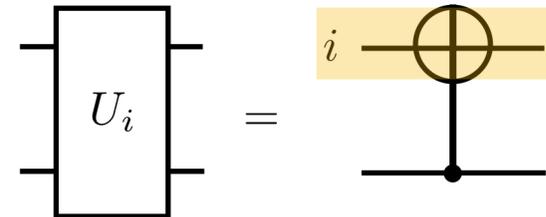
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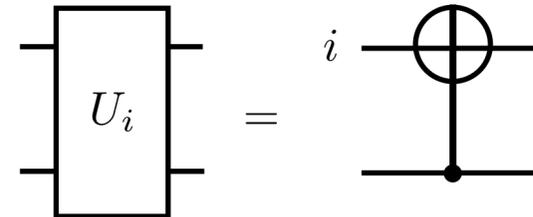
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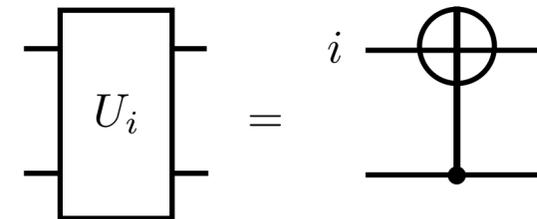
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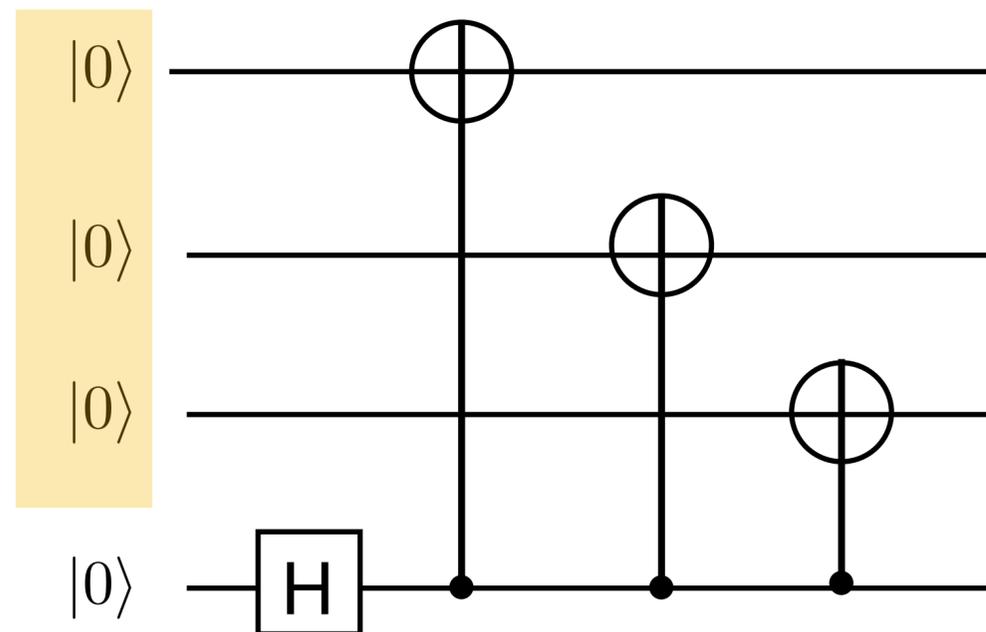
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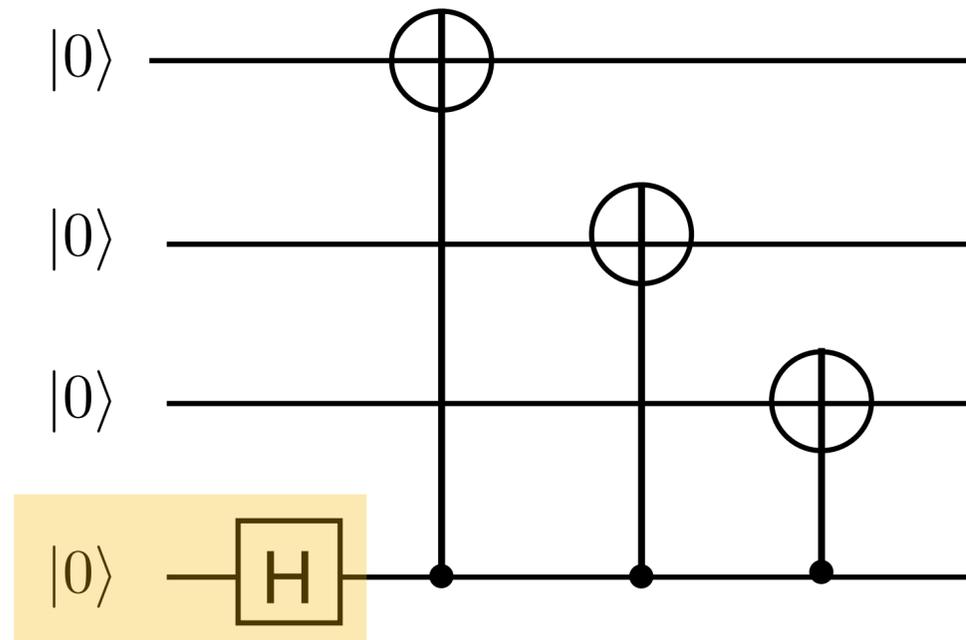
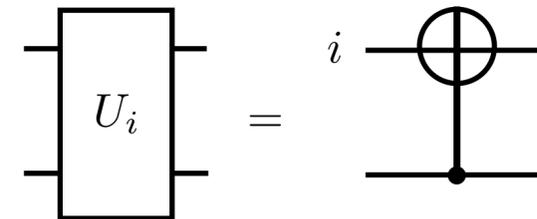
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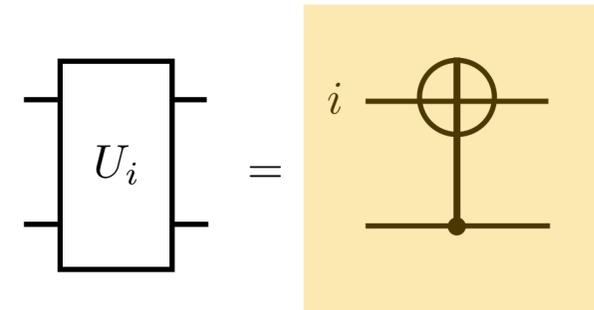
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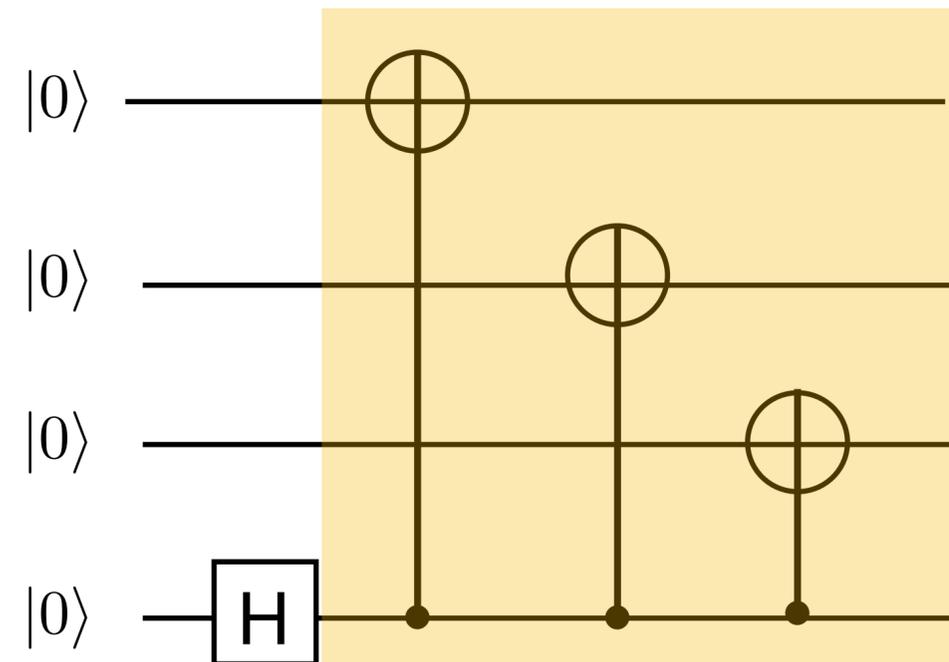
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GHZ: $\delta_{m,j} |\underline{j}\rangle$

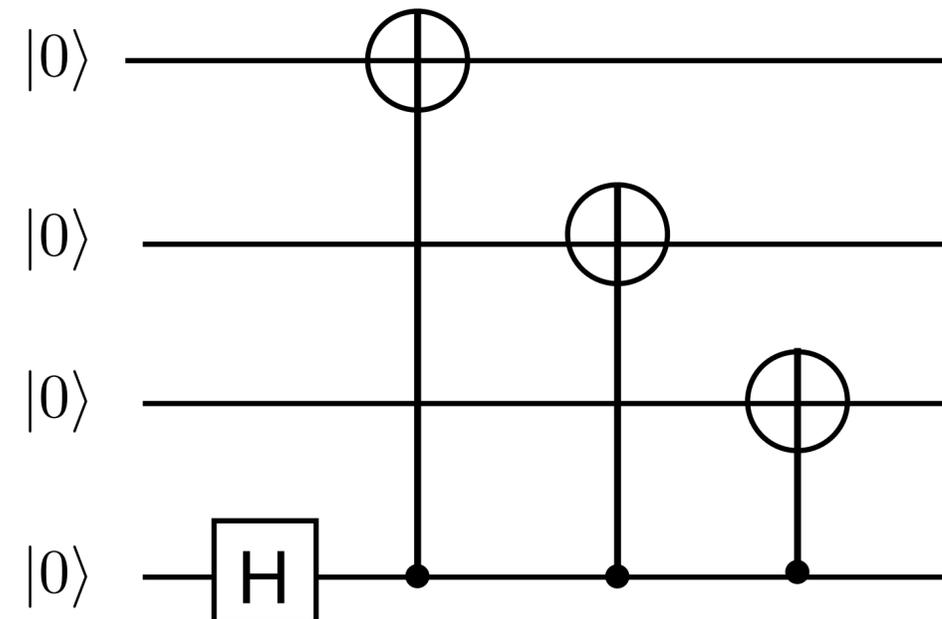
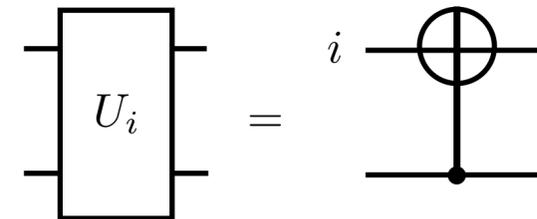
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For GHZ, the edge site may be used as ancilla (i.e., no need for independent ancilla qubit)

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Deriving the canonical MPS for a given state

Schollwöck 1008.3477

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$$|\psi\rangle = \sum_{\sigma_1, \dots, \sigma_L=0,1} c_{\sigma_1 \dots \sigma_L} |\sigma_1 \dots \sigma_L\rangle$$

L=4 GHZ

Schollwöck I008.3477

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$$Q^\dagger Q = \mathbb{I}$$

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Deriving the canonical MPS for a given state

$$|\psi\rangle = \sum_{\sigma_1, \dots, \sigma_L=0,1} c_{\sigma_1 \dots \sigma_L} |\sigma_1 \dots \sigma_L\rangle$$

- **reshape** into $2 \times 2^{L-1}$ matrix

$$c_{\sigma_1 \dots \sigma_L} = \Psi_{\sigma_1, (\sigma_2 \dots \sigma_L)}$$

- **QR decomposition (alternative: Singular Value Decomposition)**

$$\Psi_{\sigma_1, (\sigma_2 \dots \sigma_L)} = \sum_{a_1} Q_{\sigma_1, a_1} R_{a_1, (\sigma_2 \dots \sigma_L)}$$

- **reshape**

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L=4 GHZ

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$$|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle^{\otimes 4} + |1\rangle^{\otimes 4}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad 1 \times 16$$

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- **reshape**

$$= \sum_{a_2} A_{a_1, a_2}^{\sigma_2} \Psi_{(a_2, \sigma_3)}(\sigma_4 \dots \sigma_L)$$

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Deriving the canonical MPS for a given state

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- **reshape**

$$\text{etc.} = \sum_{a_2} A_{a_1, a_2}^{\sigma_2} \Psi_{(a_2, \sigma_3)(\sigma_4 \dots \sigma_L)}$$

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L=4 GHZ

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Deriving the canonical MPS for a given state

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No $\langle L|, |R\rangle$

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L=4 GHZ

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Deriving the canonical MPS for a given state

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etc.

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$$\sum_m A_i^{m\dagger} A_i^m = \mathbb{I}$$

(left) canonical

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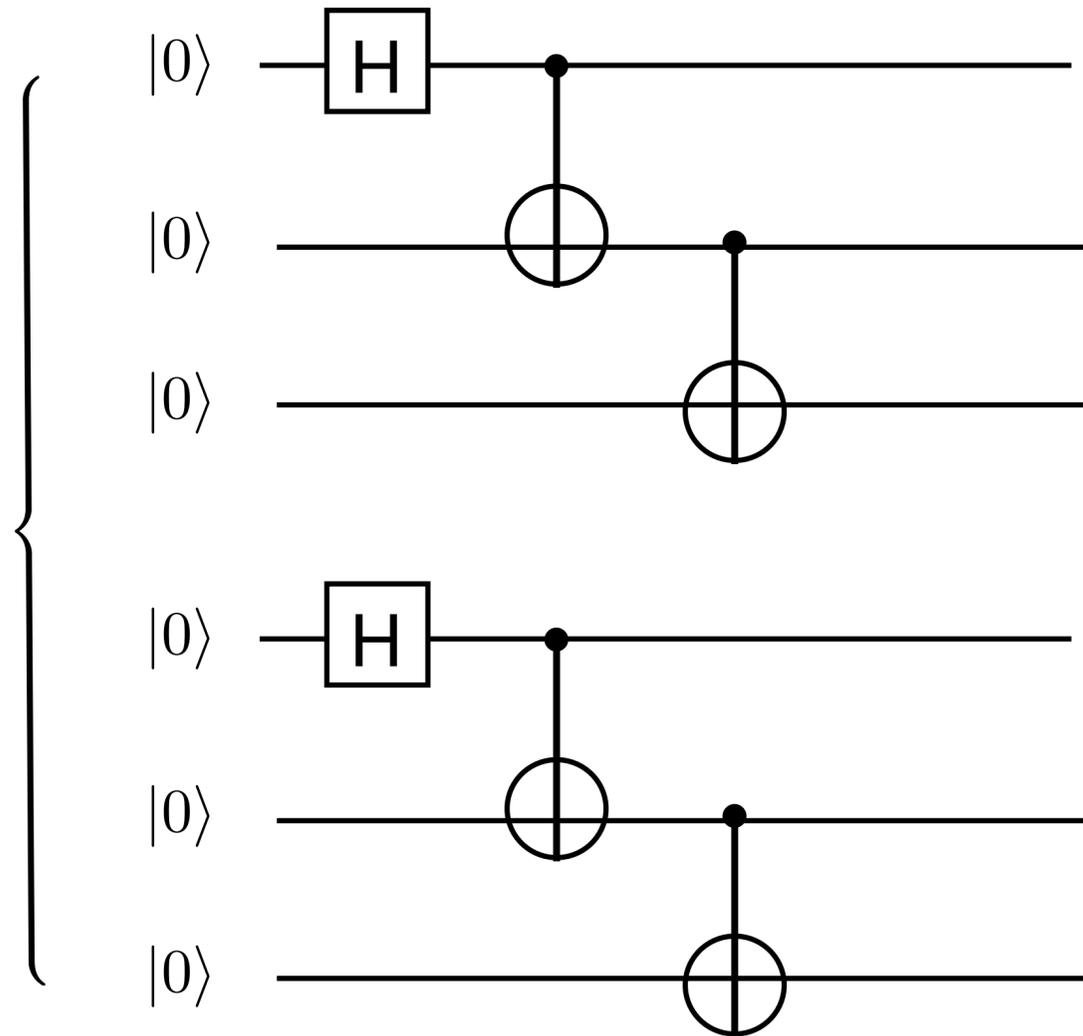
$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad 4 \times 2$$

Fusion measurements

GHZ states in constant depth via fusion measurements

Smith et al 2210.17548

Idea: “fuse” two short GHZ states into one long GHZ state using **Bell measurement** Probabilistic; correct as needed



$$|\psi_r\rangle = \sum_{\vec{m}} A^{m_n} \dots A^{m_1} |R\rangle |\vec{m}\rangle$$

$$|R\rangle = H|0\rangle$$

$$= \sum_{j, \vec{m}} |j\rangle \langle j| A^{m_n} \dots A^{m_1} |R\rangle |\vec{m}\rangle$$

$$= \sum_{j, \vec{m}} \langle j| A^{m_n} \dots A^{m_1} |R\rangle |j\rangle |\vec{m}\rangle$$

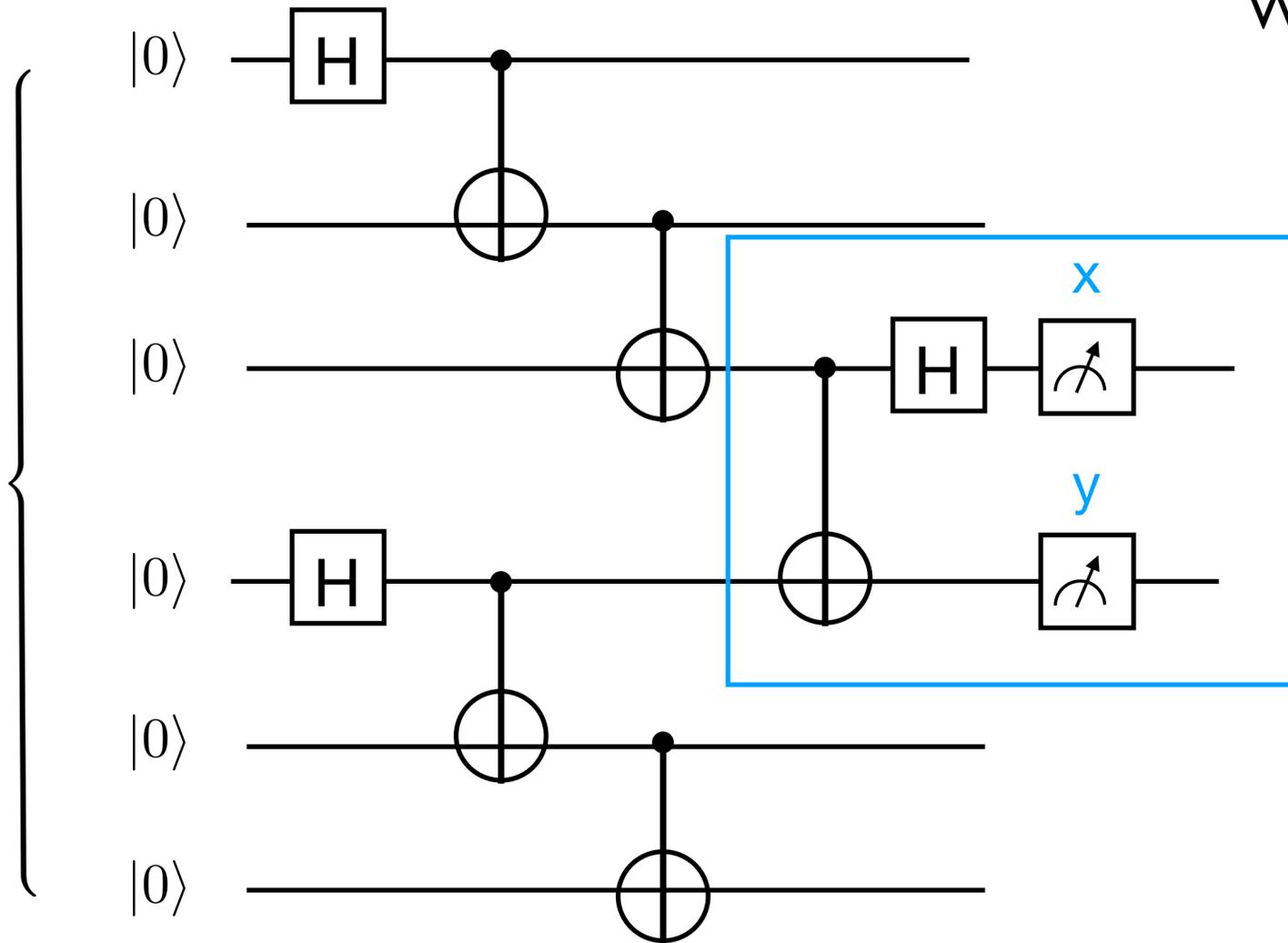
$$|\psi_l\rangle = \sum_{\vec{m}} A^{m_N} \dots A^{m_{n+1}} |R\rangle |\vec{m}\rangle$$

$$= \sum_{i, \vec{m}} \langle i| A^{m_N} \dots A^{m_{n+1}} |R\rangle |i\rangle |\vec{m}\rangle$$

$$|\Psi\rangle = |\psi_l\rangle |\psi_r\rangle = \sum_{i, j, \vec{m}} \langle i| A^{m_N} \dots A^{m_{n+1}} |R\rangle \langle j| A^{m_n} \dots A^{m_1} |R\rangle |ij\rangle |\vec{m}\rangle$$

$$|\Psi\rangle = \sum_{i,j,\vec{m}} \langle i|A^{m_N} \dots A^{m_{n+1}}|R\rangle \langle j|A^{m_n} \dots A^{m_1}|R\rangle |ij\rangle |\vec{m}\rangle$$

We want to “fuse” the two short GHZ states into one long GHZ state



Bell measurement

$$|\Psi\rangle \rightarrow |\Psi'\rangle = |\phi\rangle \langle \phi | \Psi\rangle$$

Bell state

$$|\phi\rangle = \sum_{i,j} \phi_{ij} |ij\rangle$$

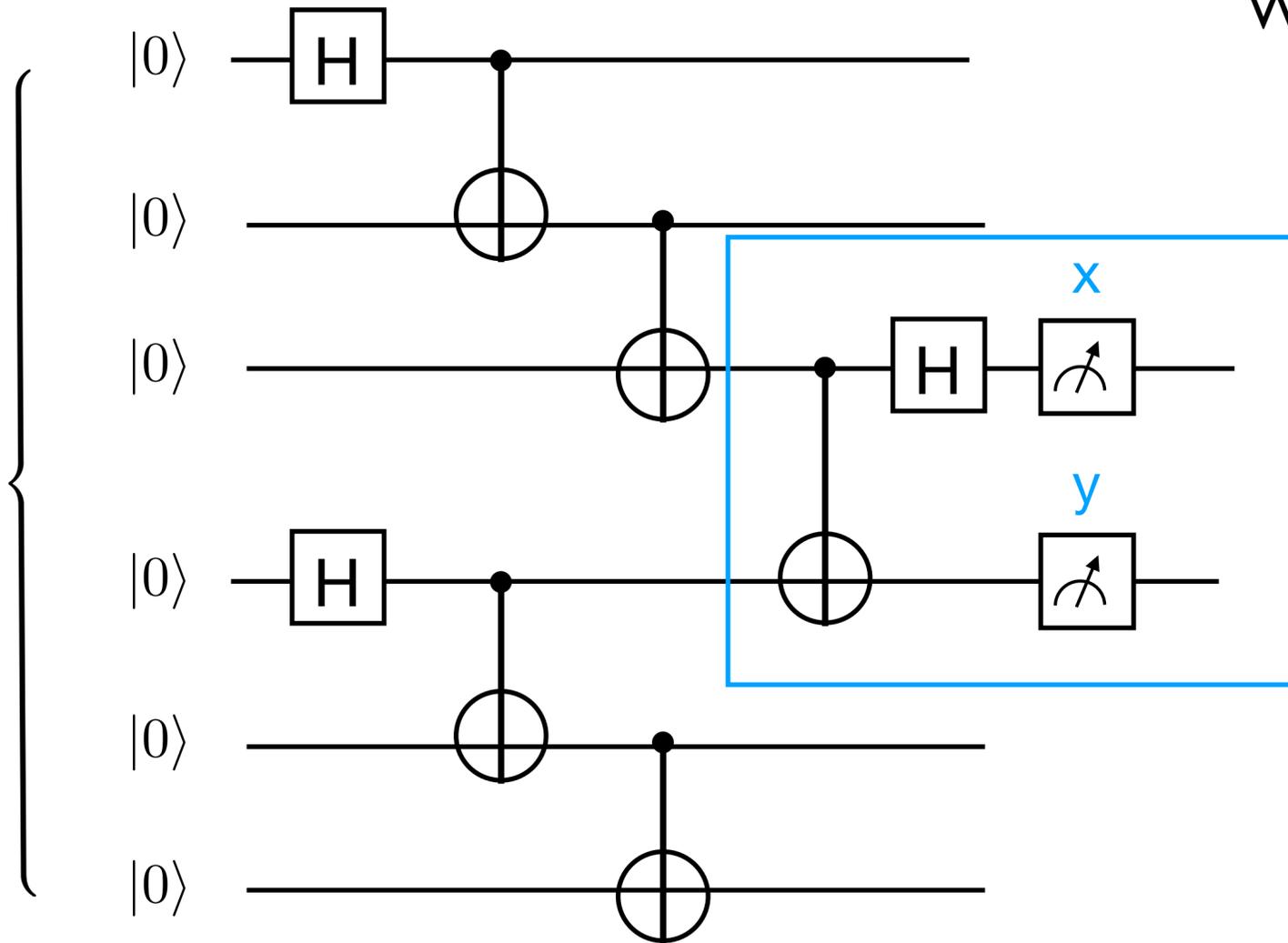
$$|\Psi'\rangle = \sum_{i,j,\vec{m}} \langle R|A^{m_N} \dots A^{m_{n+1}} |i\rangle \phi_{ij} \langle j|A^{m_n} \dots A^{m_1}|R\rangle |\vec{m}\rangle |\phi\rangle$$

$$M \propto \sum_{i,j} |i\rangle \phi_{ij} \langle j|$$

$$|\Psi'\rangle \propto \sum_{\vec{m}} \langle R|A^{m_N} \dots A^{m_{n+1}} M A^{m_n} \dots A^{m_1}|R\rangle |\vec{m}\rangle |\phi\rangle$$

$$|\Psi\rangle = \sum_{i,j,\vec{m}} \langle i|A^{m_N} \dots A^{m_{n+1}}|R\rangle \langle j|A^{m_n} \dots A^{m_1}|R\rangle |ij\rangle |\vec{m}\rangle$$

We want to “fuse” the two short GHZ states into one long GHZ state



Bell measurement

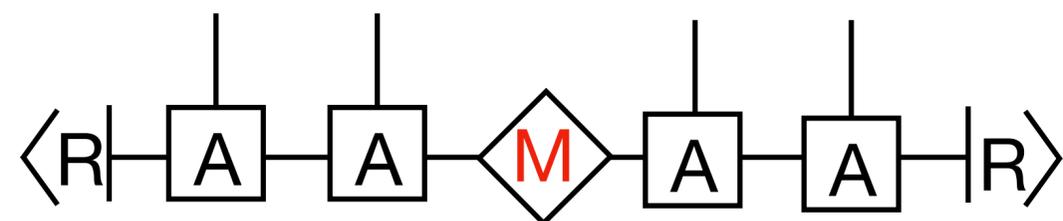
Bell state

$$|\Psi\rangle \rightarrow |\Psi'\rangle = |\phi\rangle \langle \phi | \Psi\rangle$$

$$|\phi\rangle = \sum_{i,j} \phi_{ij} |ij\rangle$$

$$|\Psi'\rangle = \sum_{i,j,\vec{m}} \langle R|A^{m_N} \dots A^{m_{n+1}}|i\rangle \phi_{ij} \langle j|A^{m_n} \dots A^{m_1}|R\rangle |\vec{m}\rangle |\phi\rangle$$

$$M \propto \sum_{i,j} |i\rangle \phi_{ij} \langle j|$$

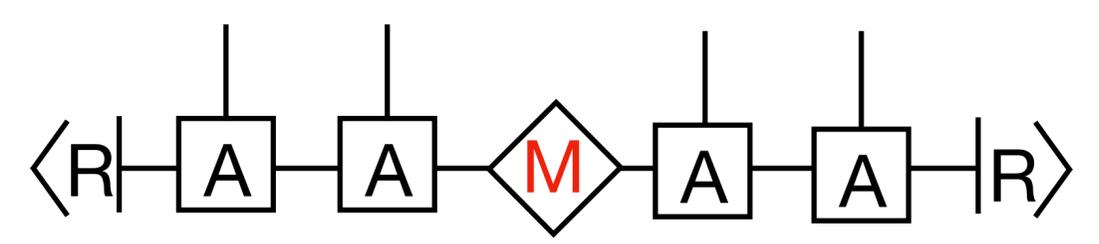


We want to remove this “defect”!

$$|\Psi'\rangle \propto \sum_{\vec{m}} \langle R|A^{m_N} \dots A^{m_{n+1}} M A^{m_n} \dots A^{m_1}|R\rangle |\vec{m}\rangle |\phi\rangle$$

$ \phi\rangle = \sum_{i,j} \phi_{ij} ij\rangle$	$M \propto \sum_{i,j} i\rangle \phi_{ij} \langle j $	Correction (of “physical” qubits)
$ B_{00}\rangle = \frac{1}{\sqrt{2}}(00\rangle + 11\rangle)$	$ 0\rangle\langle 0 + 1\rangle\langle 1 = \mathbb{I}$	\mathbb{I}
$ B_{10}\rangle = \frac{1}{\sqrt{2}}(00\rangle - 11\rangle)$	$ 0\rangle\langle 0 - 1\rangle\langle 1 = Z$	$?$
$ B_{01}\rangle = \frac{1}{\sqrt{2}}(01\rangle + 10\rangle)$	$ 0\rangle\langle 1 + 1\rangle\langle 0 = X$	$?$
$ B_{11}\rangle = \frac{1}{\sqrt{2}}(01\rangle - 10\rangle)$	$ 0\rangle\langle 1 - 1\rangle\langle 0 = ZX$	$?$

Exploit symmetries of MPS!



Symmetries of MPS:

$$Z \text{---} \boxed{A} \text{---} Z = \text{---} \boxed{A} \text{---}$$

$$Z \text{---} \boxed{A} \text{---} = \text{---} \boxed{A} \text{---} Z = \text{---} \boxed{A} \text{---} Z$$

$$m = 0, 1$$

$$Z A^m Z = A^m$$

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Z A^m = A^m Z = \sum_{m'} A^{m'} Z_{m'm}$$

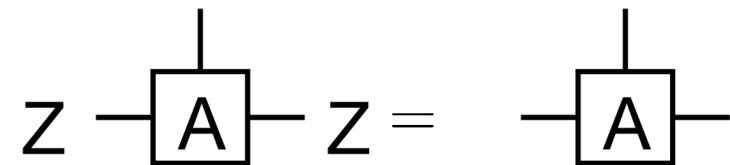
For $M = Z$:

$$\langle R | \boxed{A} \text{---} \boxed{A} \text{---} \boxed{Z} \text{---} \boxed{A} \text{---} \boxed{A} | R \rangle = \langle R | \boxed{A} \text{---} \boxed{A} \text{---} \boxed{A} \text{---} \boxed{A} | R \rangle$$

correction!

$ \phi\rangle = \sum_{i,j} \phi_{ij} ij\rangle$	$M \propto \sum_{i,j} i\rangle \phi_{ij} \langle j $	Correction (of “physical” qubits)
$ B_{00}\rangle = \frac{1}{\sqrt{2}}(00\rangle + 11\rangle)$	$ 0\rangle\langle 0 + 1\rangle\langle 1 = \mathbb{I}$	\mathbb{I}
$ B_{10}\rangle = \frac{1}{\sqrt{2}}(00\rangle - 11\rangle)$	$ 0\rangle\langle 0 - 1\rangle\langle 1 = Z$	$Z \quad (\text{one})$
$ B_{01}\rangle = \frac{1}{\sqrt{2}}(01\rangle + 10\rangle)$	$ 0\rangle\langle 1 + 1\rangle\langle 0 = X$	$?$
$ B_{11}\rangle = \frac{1}{\sqrt{2}}(01\rangle - 10\rangle)$	$ 0\rangle\langle 1 - 1\rangle\langle 0 = ZX$	$?$

Symmetries of MPS:

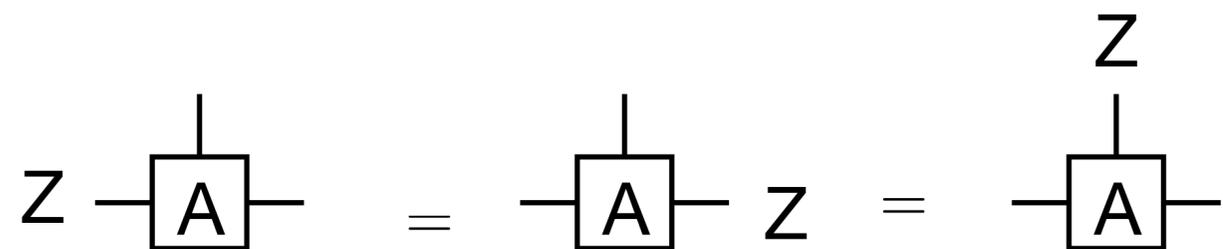


$$m = 0, 1$$

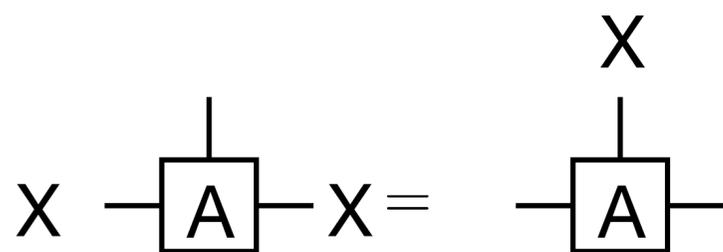
$$Z A^m Z = A^m$$

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$



$$Z A^m = A^m Z = \sum_{m'} A^{m'} Z_{m'm}$$

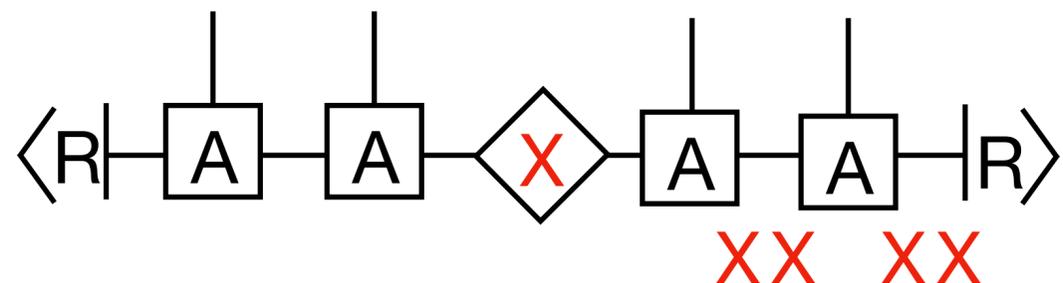


$$X A^m X = \sum_{m'} A^{m'} X_{m'm}$$

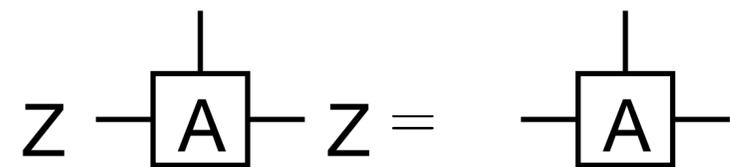
$$|R\rangle = H|0\rangle$$

$$X|R\rangle = |R\rangle$$

For $M = X$:



Symmetries of MPS:

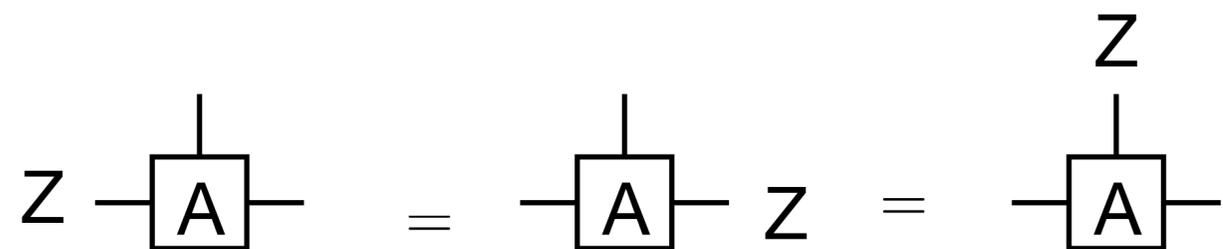


$$m = 0, 1$$

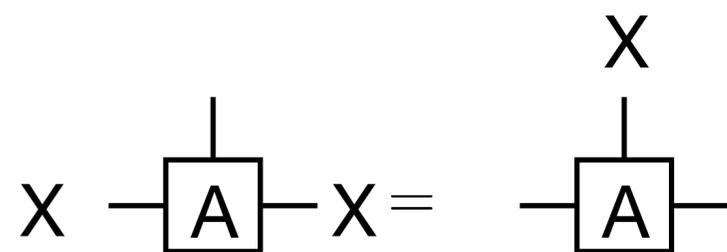
$$Z A^m Z = A^m$$

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$



$$Z A^m = A^m Z = \sum_{m'} A^{m'} Z_{m'm}$$

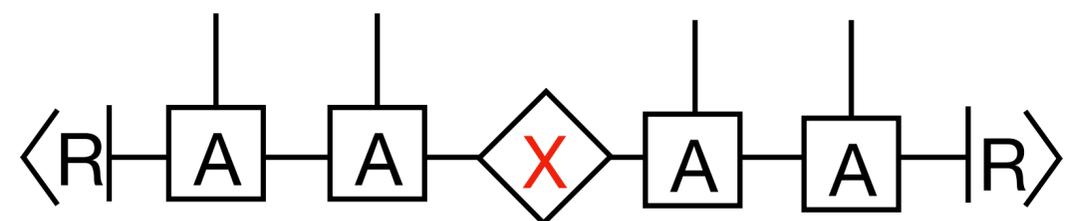


$$X A^m X = \sum_{m'} A^{m'} X_{m'm}$$

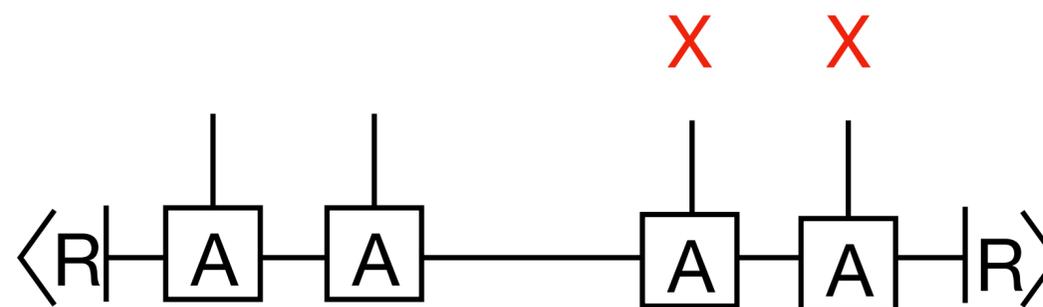
$$|R\rangle = H|0\rangle$$

$$X|R\rangle = |R\rangle$$

For $M = X$:



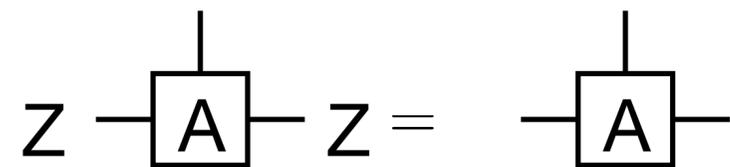
=



correction!

$ \phi\rangle = \sum_{i,j} \phi_{ij} ij\rangle$	$M \propto \sum_{i,j} i\rangle \phi_{ij} \langle j $	Correction (of “physical” qubits)
$ B_{00}\rangle = \frac{1}{\sqrt{2}}(00\rangle + 11\rangle)$	$ 0\rangle\langle 0 + 1\rangle\langle 1 = \mathbb{I}$	\mathbb{I}
$ B_{10}\rangle = \frac{1}{\sqrt{2}}(00\rangle - 11\rangle)$	$ 0\rangle\langle 0 - 1\rangle\langle 1 = Z$	Z (one)
$ B_{01}\rangle = \frac{1}{\sqrt{2}}(01\rangle + 10\rangle)$	$ 0\rangle\langle 1 + 1\rangle\langle 0 = X$	X (multiple)
$ B_{11}\rangle = \frac{1}{\sqrt{2}}(01\rangle - 10\rangle)$	$ 0\rangle\langle 1 - 1\rangle\langle 0 = ZX$	$?$

Symmetries of MPS:

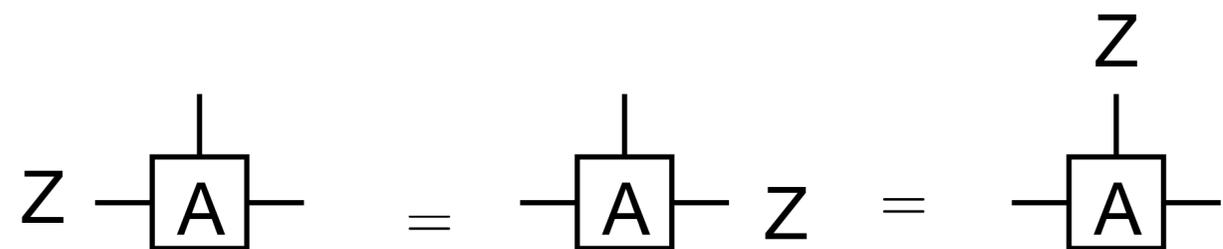


$$m = 0, 1$$

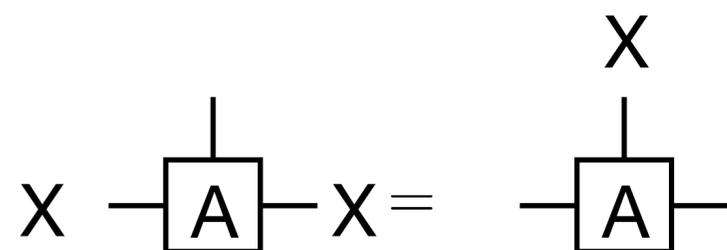
$$Z A^m Z = A^m$$

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$



$$Z A^m = A^m Z = \sum_{m'} A^{m'} Z_{m'm}$$

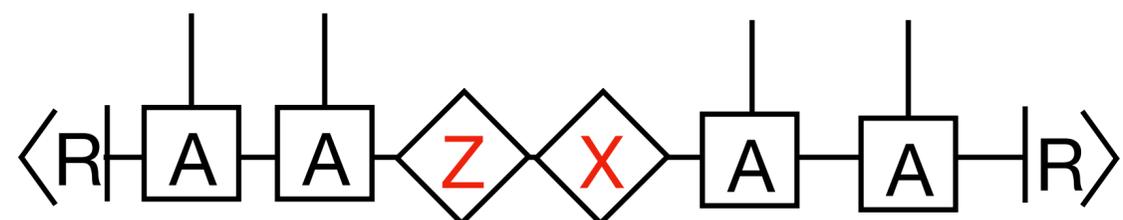


$$X A^m X = \sum_{m'} A^{m'} X_{m'm}$$

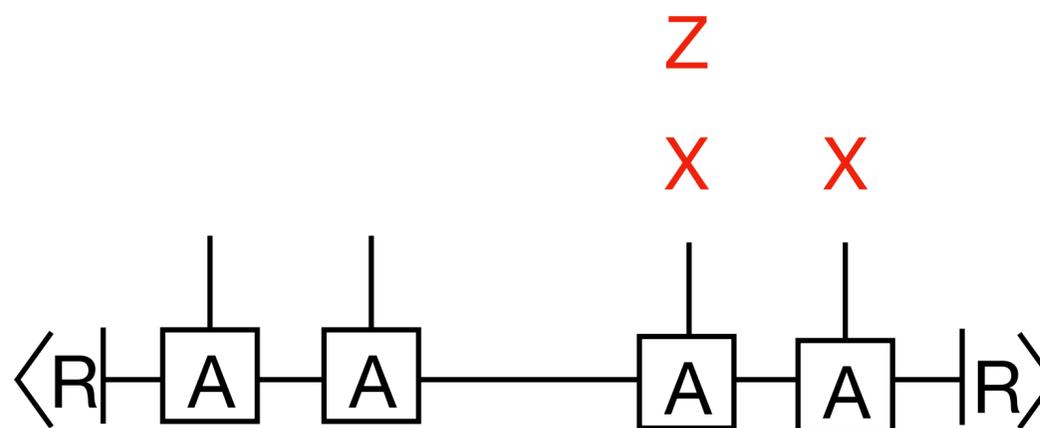
$$|R\rangle = H|0\rangle$$

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For $M = Z X$:

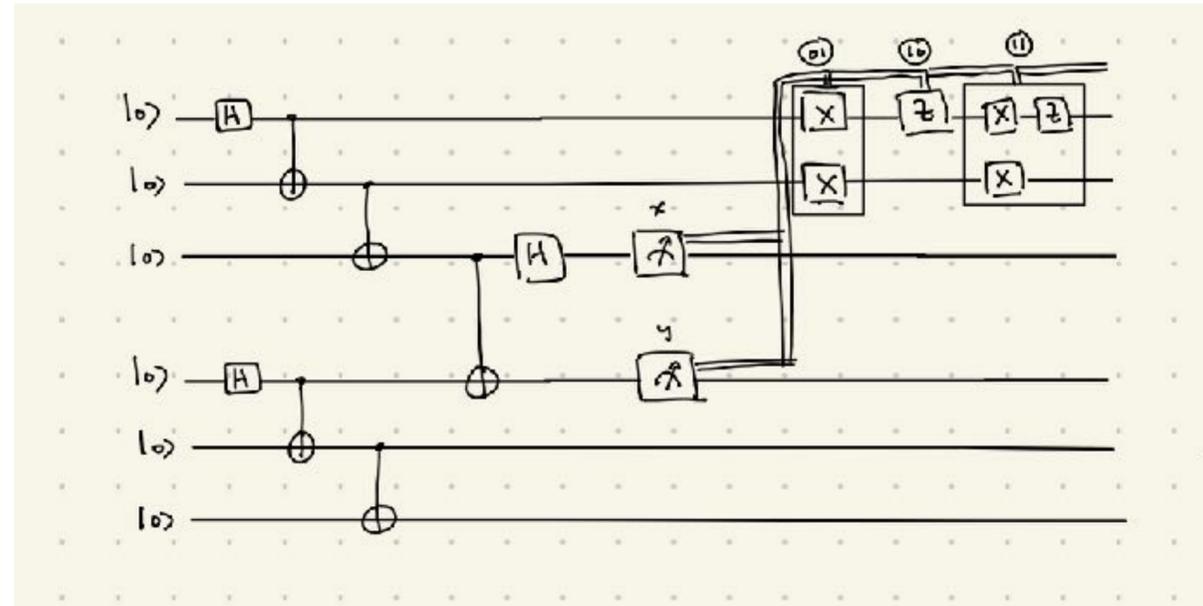


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correction!

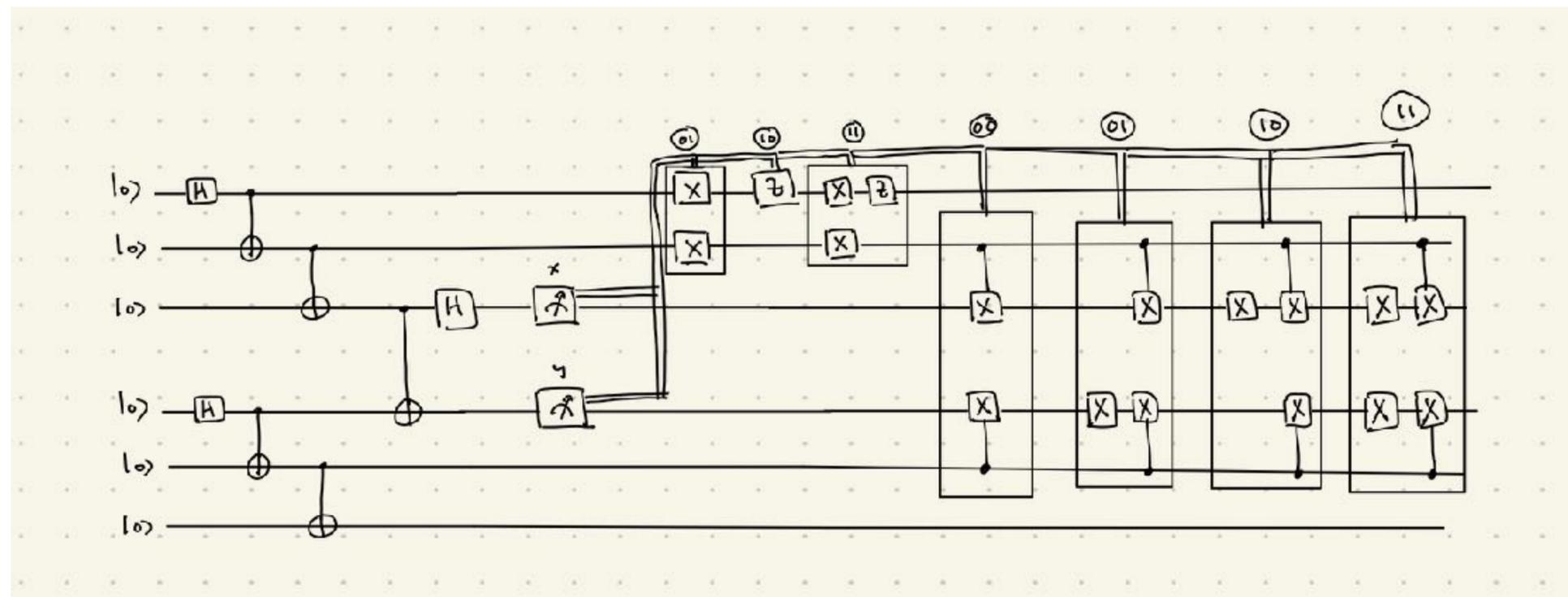
$ \phi\rangle = \sum_{i,j} \phi_{ij} ij\rangle$	$M \propto \sum_{i,j} i\rangle \phi_{ij} \langle j $	Correction (of “physical” qubits)
$ B_{00}\rangle = \frac{1}{\sqrt{2}}(00\rangle + 11\rangle)$	$ 0\rangle\langle 0 + 1\rangle\langle 1 = \mathbb{I}$	\mathbb{I}
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$ B_{01}\rangle = \frac{1}{\sqrt{2}}(01\rangle + 10\rangle)$	$ 0\rangle\langle 1 + 1\rangle\langle 0 = X$	$X \quad (\text{multiple})$
$ B_{11}\rangle = \frac{1}{\sqrt{2}}(01\rangle - 10\rangle)$	$ 0\rangle\langle 1 - 1\rangle\langle 0 = ZX$	<div style="border: 2px solid red; border-radius: 15px; padding: 10px; display: inline-block;"> $\begin{cases} X & (\text{multiple}) \\ Z & (\text{one}) \end{cases}$ </div>



The “physical” qubits are now corrected; but the measured qubits are not yet in the desired GHZ state.

To correct the measured qubits: Re-initialize the measured qubits; then apply CNOTS

Complete circuit:



Can be simplified; similar to Bäumer et al

Fusion measurement has been used recently to prepare *generic* states in constant depth!

- H Buhrman, et al 2307.14840
- Yeo et al 2501.02929
- Zi et al 2503.16208

AKLT states

AKLT (Affleck-Kennedy-Lieb-Tasaki) model

$$H = \sum_{i=1}^{n-1} \left[\underbrace{\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3} (\vec{S}_i \cdot \vec{S}_{i+1})^2}_{h_{i,i+1}} \right] \quad \vec{S}_i \quad \text{spin-1} \quad \text{open BC}$$

Not integrable; but the ground state is known exactly!

SU(2) invariant $[H, \vec{S}] = 0$ $\vec{S} = \sum_{i=1}^n \vec{S}_i$ total spin

For 2 sites: Clebsch-Gordan $\Rightarrow 1 \otimes 1 = 0 \oplus 1 \oplus 2$

$$P^{(2)} \equiv \frac{1}{2} h_{12} + \frac{1}{3} \mathbb{I} \quad \left(P^{(2)} \right)^2 = P^{(2)} \quad \text{projector onto spin-2}$$

The 2-site Hamiltonian is essentially $P^{(2)}$

Ground state

Regard each spin-1 as a pair of spin-1/2

$$|s\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

singlet state (spin-0)

$$\Pi \equiv |0\rangle\langle\uparrow\uparrow| + |1\rangle\frac{1}{\sqrt{2}}(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|) + |2\rangle\langle\downarrow\downarrow|$$

triplet states

projector to spin-1

Ground state

Regard each spin-1 as a pair of spin-1/2

$$|s\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

singlet state (spin-0)

$$\Pi \equiv |0\rangle\langle\uparrow\uparrow| + |1\rangle\frac{1}{\sqrt{2}}(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|) + |2\rangle\langle\downarrow\downarrow|$$

\mathbb{C}^3 basis states

projector to spin-1

Ground state

Regard each spin-1 as a pair of spin-1/2

$$|s\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{singlet state (spin-0)}$$

$$\Pi \equiv |0\rangle\langle\uparrow\uparrow| + |1\rangle\frac{1}{\sqrt{2}}(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|) + |2\rangle\langle\downarrow\downarrow| \quad \text{projector to spin-1}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1 \ 0 \ 0 \ 0) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} (0 \ 1 \ 1 \ 0) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (0 \ 0 \ 0 \ 1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Check: $\Pi|s\rangle = 0 \quad \checkmark$

diagrammatic
representation

Ground state

- • Regard each spin-1 as a pair of spin-1/2

• — • $|s\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$ singlet state (spin-0)

○ $\Pi \equiv |0\rangle\langle\uparrow\uparrow| + |1\rangle\frac{1}{\sqrt{2}}(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|) + |2\rangle\langle\downarrow\downarrow|$ projector to spin-1

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1 \ 0 \ 0 \ 0) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} (0 \ 1 \ 1 \ 0) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (0 \ 0 \ 0 \ 1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Check: $\Pi|s\rangle = 0 \quad \checkmark$



$$|\psi\rangle = (\Pi \otimes \Pi) \left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \otimes |s\rangle \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \right)$$

diagrammatic representation

Ground state

- • Regard each spin-1 as a pair of spin-1/2

•—• $|s\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$ singlet state (spin-0)

○ $\Pi \equiv |0\rangle\langle\uparrow\uparrow| + |1\rangle\frac{1}{\sqrt{2}}(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|) + |2\rangle\langle\downarrow\downarrow|$ projector to spin-1

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1 \ 0 \ 0 \ 0) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} (0 \ 1 \ 1 \ 0) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (0 \ 0 \ 0 \ 1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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spin-1/2 "edge states" !

diagrammatic
representation

Ground state

- • Regard each spin-1 as a pair of spin-1/2

• — • $|s\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$ singlet state (spin-0)

○ $\Pi \equiv |0\rangle\langle\uparrow\uparrow| + |1\rangle\frac{1}{\sqrt{2}}(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|) + |2\rangle\langle\downarrow\downarrow|$ projector to spin-1

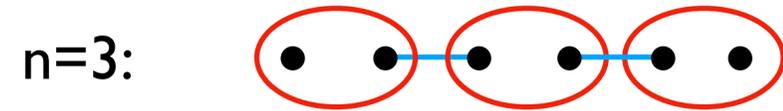
$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1 \ 0 \ 0 \ 0) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} (0 \ 1 \ 1 \ 0) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (0 \ 0 \ 0 \ 1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Check: $\Pi|s\rangle = 0 \quad \checkmark$

n=2: ○ ● — ● ○ “valence bond” connects neighboring sites

$$|\psi\rangle = (\Pi \otimes \Pi) \left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \otimes |s\rangle \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \right) \quad \alpha_1, \alpha_2, \beta_1, \beta_2 \text{ arbitrary!} \quad \text{spin-1/2 “edge states” !}$$

$P^{(2)}|\psi\rangle = 0 \Rightarrow$ ground state (Only spin-2 states have $P^{(2)} \neq 0$: $(\Pi \otimes \Pi)(\text{triplet} \otimes \text{triplet})$)



$$|\psi\rangle = (\Pi \otimes \Pi \otimes \Pi) \left(\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \otimes |s\rangle \otimes |s\rangle \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \right) \right)$$

general n :

$$|\psi\rangle = \underbrace{(\Pi \otimes \Pi \otimes \dots \otimes \Pi)}_n \left(\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \otimes \underbrace{|s\rangle \otimes |s\rangle \otimes \dots \otimes |s\rangle}_{n-2} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \right) \right)$$

$$E_0 = -\frac{2}{3}(n-1)$$

4-fold degenerate

How to prepare?

MPS

$$|\psi\rangle = \sum_{\vec{m}} a_{\vec{m}} |\vec{m}\rangle \quad |\vec{m}\rangle = |m_n \dots m_2 m_1\rangle \quad m_i = 0, 1, 2 \quad \text{qutrits} \quad \text{“physical space”}$$

$i = 1, \dots, n$

$$a_{\vec{m}} = \langle L | A_n^{m_n} \dots A_2^{m_2} A_1^{m_1} | R \rangle$$

$$\langle L | = (\beta_1 \quad \beta_2) \quad |R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$A_i^0 = \sqrt{\frac{2}{3}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad A_i^1 = -\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad A_i^2 = -\sqrt{\frac{2}{3}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\chi = 2$$

translational invariant

$$\sum_m A_i^{m\dagger} A_i^m = \mathbb{I} \quad \text{(left) canonical}$$

sequential preparation: depth $\sim n$

constant depth via fusion measurements

Smith et al 2210.17548

use symmetries of MPS to remove defects

Dicke states

Dicke states

$|D_k^n\rangle$: completely symmetric state of $|1\rangle$'s and $|0\rangle$'s

k

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Dicke states

$|D_k^n\rangle$: completely symmetric state of $|1\rangle$'s and $|0\rangle$'s
 \uparrow
 k

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

total # qubits = n

Dicke states

$|D_k^n\rangle$: completely symmetric state of $|1\rangle$'s and $|0\rangle$'s

\uparrow \uparrow
 k $n - k$

total # qubits = n

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Ex:

$$|D_2^4\rangle = \frac{1}{\sqrt{6}} (|1100\rangle + |1010\rangle + |0110\rangle + |1001\rangle + |0101\rangle + |0011\rangle)$$

$$|0011\rangle = |0\rangle \otimes |0\rangle \otimes |1\rangle \otimes |1\rangle$$

Dicke states

$$|D_k^n\rangle : \text{completely symmetric state of } |1\rangle\text{'s and } |0\rangle\text{'s} \quad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \\ k & n-k & \end{array} \quad \text{total \# qubits} = n$$

Ex:

$$|D_2^4\rangle = \frac{1}{\sqrt{6}} (|1100\rangle + |1010\rangle + |0110\rangle + |1001\rangle + |0101\rangle + |0011\rangle)$$

$$|0011\rangle = |0\rangle \otimes |0\rangle \otimes |1\rangle \otimes |1\rangle$$

$$|D_k^n\rangle \propto (S^-)^k |0\rangle^{\otimes n} \quad S^- = S^x - iS^y \quad \vec{S} = \sum_{i=1}^n \vec{S}_i \quad \vec{S}_i = \frac{1}{2}\vec{\sigma}_i$$

Exact ground states of ferromagnetic Heisenberg & Lipkin-Meshkov-Glick Hamiltonians

$$-\sum_i \vec{S}_i \cdot \vec{S}_{i+1} \quad -\vec{S}^2 = -\sum_{i,j} \vec{S}_i \cdot \vec{S}_j$$

How to prepare on quantum computer?

Cannot implement $|D_k^n\rangle \propto (\mathbb{S}^-)^k |0\rangle^{\otimes n}$ \mathbb{S}^- is not unitary

Instead:

$$|e_k^n\rangle = |0\rangle^{\otimes(n-k)} |1\rangle^{\otimes k} \quad \text{“reference” state (product)}$$

We seek:

$$U_n |e_k^n\rangle = |D_k^n\rangle$$

“Dicke operator”

- unitary
- independent of k

key idea: recursion!

[Bärtschi, Eidenbenz 2019]

$$\begin{aligned} \text{Ex: } |D_2^4\rangle &= \frac{1}{\sqrt{6}} (|110\underline{0}\rangle + |101\underline{0}\rangle + |011\underline{0}\rangle + |100\underline{1}\rangle + |010\underline{1}\rangle + |001\underline{1}\rangle) \\ &\quad \underbrace{(|110\rangle + |101\rangle + |011\rangle)}_{(|110\rangle + |101\rangle + |011\rangle) \otimes |\underline{0}\rangle} \quad \underbrace{(|100\rangle + |010\rangle + |001\rangle)}_{(|100\rangle + |010\rangle + |001\rangle) \otimes |\underline{1}\rangle} \end{aligned}$$

key idea: recursion!

[Bärtschi, Eidenbenz 2019]

$$\text{Ex: } |D_2^4\rangle = \frac{1}{\sqrt{6}} (|1100\rangle + |1010\rangle + |0110\rangle + |1001\rangle + |0101\rangle + |0011\rangle)$$

$$\underbrace{(|110\rangle + |101\rangle + |011\rangle)}_{(|110\rangle + |101\rangle + |011\rangle) \otimes |0\rangle} \quad \underbrace{(|100\rangle + |010\rangle + |001\rangle)}_{(|100\rangle + |010\rangle + |001\rangle) \otimes |1\rangle}$$

$$|D_2^4\rangle = \sqrt{\frac{1}{2}} |D_2^3\rangle \otimes |0\rangle + \sqrt{\frac{1}{2}} |D_1^3\rangle \otimes |1\rangle$$

$$|D_k^n\rangle = \sqrt{\frac{n-k}{n}} |D_k^{n-1}\rangle \otimes |0\rangle + \sqrt{\frac{k}{n}} |D_{k-1}^{n-1}\rangle \otimes |1\rangle$$

Use $|D_k^n\rangle = U_n |e_k\rangle$ on both sides:

$$U_n |e_k^n\rangle = (U_{n-1} \otimes \mathbb{I}) \underbrace{\left(\sqrt{\frac{n-k}{n}} |e_k^{n-1}\rangle \otimes |0\rangle + \sqrt{\frac{k}{n}} |e_{k-1}^{n-1}\rangle \otimes |1\rangle \right)}$$

$$\equiv W_n |e_k^n\rangle$$

- unitary
- independent of k

$$W_n |e_k^n\rangle = \sqrt{\frac{n-k}{n}} |e_k^{n-1}\rangle \otimes |0\rangle + \sqrt{\frac{k}{n}} |e_{k-1}^{n-1}\rangle \otimes |1\rangle$$

$$U_n = (U_{n-1} \otimes \mathbb{I}) W_n$$

\Rightarrow

$$U_n = \prod_{m=2}^{\overset{\curvearrowright}{n}} \left(W_m \otimes \mathbb{I}^{\otimes(n-m)} \right)$$

Suffices to construct W_m 's !

Constructing W_m 's

Strategy: look for operators $I_{m,l}$ that perform $W_m |e_l^m\rangle$ for fixed l

$$|e_l^m\rangle = |0\rangle^{\otimes(m-l)} |1\rangle^{\otimes l}$$

$$I_{m,l'} |e_l^m\rangle = \begin{cases} |e_l^m\rangle & l' < l \\ W_m |e_l^m\rangle & l' = l \end{cases}$$

and

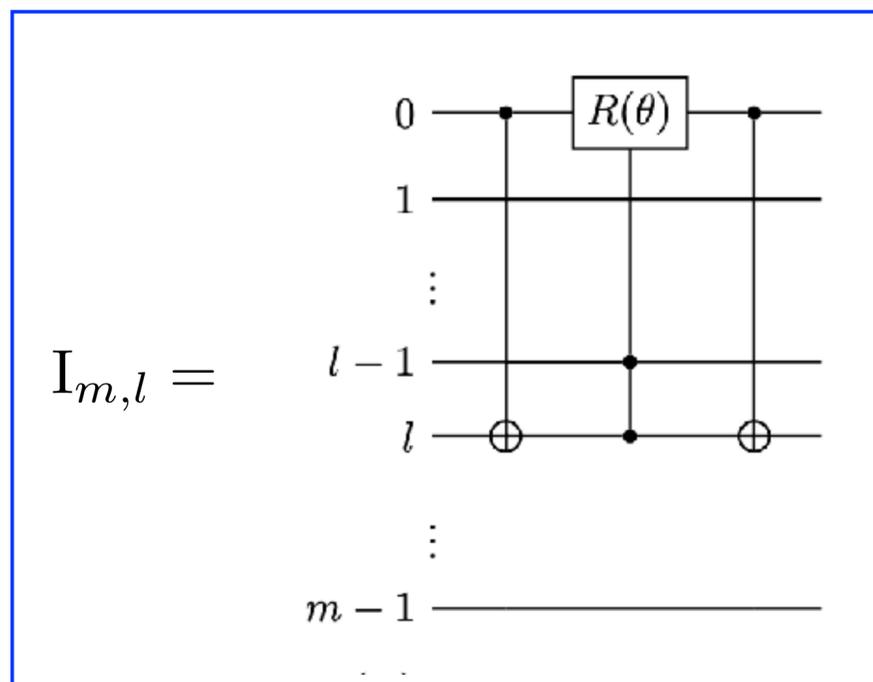
$$I_{m,l'} (I_{m,l} |e_l^m\rangle) = (I_{m,l} |e_l^m\rangle) \quad \text{for } l' > l$$

ordered so as to not interfere with each other

Then

$$W_m = \prod_{l=1}^{\overbrace{m-1}} I_{m,l}$$

Suffices to construct $I_{m,l}$'s !

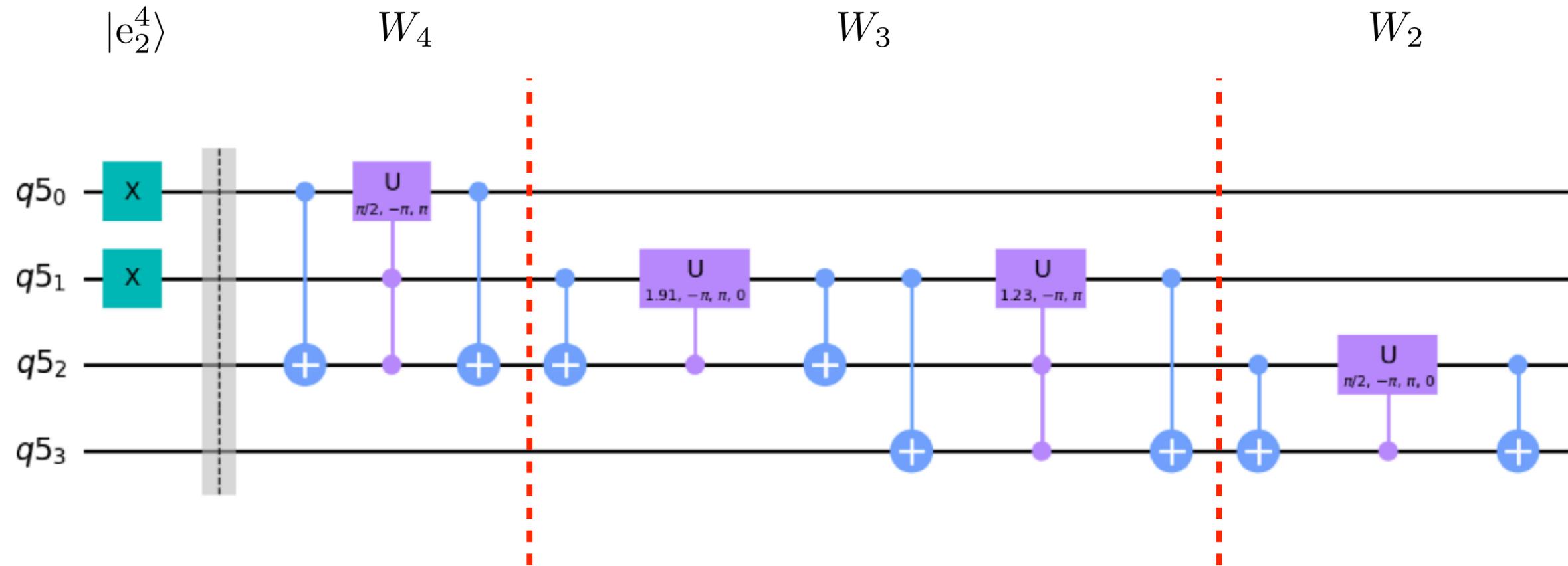


$$R(\theta) = \begin{pmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}$$

$$\cos(\frac{\theta}{2}) = \sqrt{\frac{l}{m}}$$

circuit size & depth = $\mathcal{O}(k(n-k))$

Ex: $|D_2^4\rangle$



Related construction: sequential preparation based on matrix product state (MPS) representation

Raveh, RN 2408.04729

$$|D_k^n\rangle = \sum_{\vec{m}} a_{\vec{m}} |\vec{m}\rangle$$

$$|\vec{m}\rangle = |m_n \dots m_2 m_1\rangle \quad m_i = 0, 1 \quad \text{qubits} \quad \text{“physical space”}$$
$$i = 1, \dots, n$$

$$= \sum_{\vec{m}} \langle \underline{k} | A_n^{m_n} \dots A_2^{m_2} A_1^{m_1} | \underline{0} \rangle |\vec{m}\rangle$$

$$A_i^{m_i} \quad (k+1) \times (k+1) \quad \text{matrices}$$

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$$|j\rangle \quad j = 0, 1, \dots, k$$

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$$A_i^{m_i} \quad (k+1) \times (k+1) \quad \text{matrices} \quad |\underline{j}\rangle \quad j = 0, 1, \dots, k \quad \text{qudit} \quad \text{“bond space”}$$

$$\langle \underline{j}' | A_i^m | \underline{j} \rangle = \gamma_{j,m}^{(i)} \delta_{j',j+m}, \quad \gamma_{j,m}^{(i)} = \sqrt{\frac{\binom{n-i}{k-j-m}}{\binom{n-i+1}{k-j}}}$$

- quasi-diagonal

Related construction: sequential preparation based on **matrix product state (MPS) representation**

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- quasi-diagonal
- minimal bond dimension
- exact

$$\sum_m A_i^{m\dagger} A_i^m = \mathbb{I} \quad \text{canonical} \quad \Rightarrow \quad U_i |j\rangle |0\rangle_i = \sum_m (A_i^m |j\rangle) |m\rangle_i \quad U_i \text{ unitary}$$

Schön, et al 2005

\Rightarrow Can use U_i to prepare the state sequentially:

$$\mathcal{U} = \prod_{i=1}^{\overleftarrow{n}} U_i \quad \mathcal{U} |0\rangle |0\rangle^{\otimes n} = |\underline{k}\rangle |D_k^n\rangle$$

$\uparrow \quad \uparrow$
 decoupled!

deterministic!

$$\sum_m A_i^{m\dagger} A_i^m = \mathbb{I} \quad \text{canonical} \quad \Rightarrow \quad U_i |\underline{j}\rangle |0\rangle_i = \sum_m (A_i^m |\underline{j}\rangle) |m\rangle_i \quad U_i \text{ unitary} \quad \text{Schön, et al 2005}$$

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Quantum circuit is similar to Bäertschi-Eidenbenz

Other approaches:

- Y Wang, B Terhal 2104.14310
- H Buhrman, M Folkertsma, Loff, N Neumann 2307.14840
- L Bond, M Davis, J Minar, R Gerritsma, G Brennen, A Safavi-Naini 2312.06060
- L Piroli, G Styliaris, J Cirac 2403.07604
- J. Yu, S. Muleady, Y-X Wang, N. Schine, A. Gorshkov, A. Childs 2411.03428
- Z. Liu, A. Childs, D. Gottesman 2411.04019