On some problems of modifying gravity

Alexey Golovnev
Centre for Theoretical Physics
British University in Egypt

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Gravity is very successfully described by the General Relativity theory of Albert Einstein. It is one of the best and most beautiful theories we have. Still, we are stubbornly trying to modify it.

There are mysteries in cosmology. What are the Dark Sectors? Was there inflation, and if yes then how? And if the problems such as H_0 tension are real, what are we making out of that?

On top of that, there are singularities, inherent and unavoidable. They are mostly hidden whenever one can imagine. But don't we want to have a better understanding of what is going on?

And let alone the puzzle of quantum gravity, together with our pathological belief in the mathematically horrendous quantum field theory approach.

And the amazing news we get is that it is extremely difficult to meaningfully modify the theory of General Relativity.

Simple models such as $f(R)$ are almost nothing new, and can be reformulated as an extra universal force mediated by a scalar field on top of the usual gravity. Deeper attempts at modifying it require exquisite care to not encounter with ghosts, or other bad instabilities, or total lack of well-posedness, or no reasonable cosmology available, or.... you name it!

And having the miserable lack of an undoubtful success, it makes all the good sense to try whatever crazy modification or a new geometry one can think of. And let it lead us to a better understanding.

One possible idea is to modify the class of variations.
Recall unimodular gravity, for example.

Another option is mimetic gravity, or mimetic dark matter.
The idea is to represent the physical metric $g_{\mu\nu}$ in the Einstein-Hilbert variational principle as

$$g_{\mu\nu} = \tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi).$$

The conformal mode of $\tilde{g}$ is stripped off any physical significance, and its role is relegated to the scalar player. The latter has derivatives though, and therefore the equations of motion appear more general. Namely, an effective ideal pressureless fluid with purely potential flow is added.

One problem is that the caustic singularities become fundamental, not just mere artefacts of some approximation.

Many generalisations were tried, for example with higher derivatives of the scalar in the action.
Some singularity avoidance mechanism was constructed, though with a subtle use of changing branches of a multi-valued function.

Another option is a more general metric transformation:

$$g_{\mu\nu} = C(\phi, \tilde{X}) \cdot \tilde{g}_{\mu\nu} + D(\phi, \tilde{X}) \cdot (\partial_\mu \phi)(\partial_\nu \phi)$$

with $\tilde{X} \equiv \tilde{g}^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi)$.

Roughly speaking, the metric transformation is invertible as long as

$$\frac{\partial}{\partial \tilde{X}} \left(\frac{C}{\tilde{X}} + D \right) \neq 0.$$

Therefore, if

$$\frac{C}{\tilde{X}} + D = h(\phi),$$

then it is nowhere invertible, like in the classical mimetic gravity case, and reproduces precisely that.

If not to have it universal, the non-invertibility condition can be taken as a differential equation

$$C = \tilde{X}\frac{\partial}{\partial\tilde{X}}C + \tilde{X}^2\frac{\partial}{\partial\tilde{X}}D$$

for the scalar field.

Outside these loci, the model will be just the same as GR. With such scalar field configurations though, it will again offer more solutions, precisely of mimetic dark matter type.
It is nice, of course. But looks rather scary, in the sense of an ill-defined number of degrees of freedom.

One possible approach of a serious change in geometrical foundations is (modified) teleparallel gravity. The teleparallel framework works in terms of torsion instead of curvature.

In the tetrad-based description of gravity, one can naturally have torsionful connections without curvature or non-metricity by

$$\Gamma^{\alpha}_{\mu\nu} = e^{\alpha}_{A} \partial_{\mu} e^{A}_{\nu}.$$

At least locally, every connection of this sort can be written like this, for some particular tetrad.

If we go beyond TEGR, or just reproducing GR, this framework is about more than just a metric. In general, different tetrads for the same metric are physically different objects.

To fix the notations, recall that the quest for TEGR action can start from observing that a metric-compatible connection $\Gamma^{\alpha}_{\mu\nu}$ with torsion differs from the Levi-Civita one $\overset{(0)}{\Gamma}{}^{\alpha}_{\mu\nu}$ by a contortion tensor:

$$\Gamma^{\alpha}_{\mu\nu} = \overset{(0)}{\Gamma}{}^{\alpha}_{\mu\nu}(g) + K^{\alpha}{}_{\mu\nu}$$

which is defined in terms of the torsion tensor $T^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}$ as

$$K_{\alpha\mu\nu} = \frac{1}{2}\left(T_{\alpha\mu\nu} + T_{\nu\alpha\mu} + T_{\mu\alpha\nu}\right).$$

It is antisymmetric in the lateral indices because I ascribe the left lower index of a connection coefficient to the derivative, e.g. $\bigtriangledown_{\mu} T^{\nu} \equiv \partial_{\mu} T^{\nu} + \Gamma^{\nu}_{\mu\alpha} T^{\alpha}$.

The curvature tensor

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_\mu \Gamma^{\alpha}_{\nu\beta} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\rho}_{\nu\beta} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\rho}_{\mu\beta}$$

for the two different connections obviously has a quadratic in K expression in the difference. Then making necessary contractions, such as $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ and $R = g^{\mu\nu}R_{\mu\nu}$, we can come to

$$\overset{(0)}{R} + \mathbb{T} + 2 \overset{(0)}{\bigtriangledown}_\mu T^\mu = 0$$

since the connection Γ has zero curvature. Here $T_\mu \equiv T^{\alpha}{}_{\mu\alpha}$ is the torsion vector while the torsion scalar

$$\mathbb{T} \equiv \frac{1}{2} S_{\alpha\mu\nu} T^{\alpha\mu\nu}$$

is given in terms of the superpotential

$$S_{\alpha\mu\nu} \equiv K_{\mu\alpha\nu} + g_{\alpha\mu} T_\nu - g_{\alpha\nu} T_\mu$$

which is antisymmetric in the last two indices, the same as the torsion tensor.

Due to the basic relation above, the Einstein-Hilbert action $-\int d^4x\sqrt{-g}\overset{(0)}{R}$ is equivalent to the TEGR one, $\int d^4x\|e\|\mathbb{T}$. They are the same, up to the surface term $\mathbb{B} \equiv 2\overset{(0)}{\nabla}_\mu T^\mu$.

Of course, this equivalence disappears when we go to modified gravity, for example the $f(T)$ gravity:

$$S = \int f(\mathbb{T}) \cdot \|e\| d^4x.$$

Actually, the work of varying this action can be simplified a lot by using this observation.

But many problems await us!
The pesky strong coupling issues...

After some little exercise, the equation of motion can be written as

$$f_T \overset{(0)}{G}_{\mu\nu} + \frac{1}{2}\left(f - f_T \mathbb{T}\right) g_{\mu\nu} + f_{TT} S_{\mu\nu\alpha} \partial^\alpha \mathbb{T} = \kappa \mathcal{T}_{\mu\nu}$$

with $\mathcal{T}_{\mu\nu}$ being the energy-momentum tensor of the matter.
This is a very convenient form of equations!
Addition of a flat spin connection does not make any change to it.

If $f_{TT} \neq 0$, then the antisymmetric part of the equations takes the form of

$$(S_{\mu\nu\alpha} - S_{\nu\mu\alpha})\partial^\alpha \mathbb{T} = 0.$$

It can be thought of as related to Lorentzian degrees of freedom.

And we see that solutions with constant $\mathbb{T}$ are very special and do not go beyond the usual GR, unless we are to study perturbations around them.

The number of degrees of freedom is not very well known. And the main reason is a variable rank of the algebra of Poisson brackets of constraints.
But, what is for sure, is that there must be at least one extra mode.

Still, the trivial Minkowski $e^A_\mu = \delta^A_\mu$ is obviously in a strong coupling regime for it. Indeed, then $\mathbb{T} \propto (\partial\delta e)^2$, and for the quadratic action we just take $f(\mathbb{T}) = f_0 + f_1\mathbb{T} + \mathcal{O}(\mathbb{T}^2)$ which means accidental restoration of the full Lorentz symmetry, and linearised GR.
This no contradiction to experiments is highly problematic.

Moreover, the strong coupling issue is there also for the standard cosmology.

One can use the following ansatz for perturbations

$$
\begin{aligned}
e_0^{\emptyset} &= a(\tau)\cdot(1+\phi)\\
e_i^{\emptyset} &= a(\tau)\cdot(\partial_i\beta+u_i)\\
e_0^{a} &= a(\tau)\cdot(\partial_a\zeta+v_a)\\
e_j^{a} &= a(\tau)\cdot\left((1-\psi)\delta_j^a+\partial^2_{aj}\sigma+\epsilon_{ajk}\partial_k s+\partial_j c_a+\epsilon_{ajk}w_k+\frac{1}{2}h_{aj}\right).
\end{aligned}
$$

with the usual metric perturbations given by

$$
\begin{aligned}
g_{00} &= -a^2(\tau)\cdot(1+2\phi)\\
g_{0i} &= a^2(\tau)\cdot(\partial_i(\zeta-\beta)+v_i-u_i)\\
g_{ij} &= a^2(\tau)\cdot\left((1-2\psi)\delta_{ij}+2\partial^2_{ij}\sigma++\partial_i c_j+\partial_j c_i+h_{ij}\right).
\end{aligned}
$$

and the new (Lorentzian) components given are scalar $\beta+\zeta$, pseudoscalar s, vector u_i+v_i, and pseudovector w_j.

Under infinitesimal diffeomorphisms $x^\mu \to x^\mu + \xi^\mu(x)$ with ξ^0 and $\xi^i \equiv \partial_i \xi + \tilde{\xi}_i$, one can simply derive the following transformation laws:

$$
\begin{aligned}
\phi &\longrightarrow \phi - \xi^{0\prime} - H\xi^0 \\
\psi &\longrightarrow \psi + H\xi^0 \\
\sigma &\longrightarrow \sigma - \xi \\
\beta &\longrightarrow \beta - \xi^0 \\
\zeta &\longrightarrow \zeta - \xi' \\
c_i &\longrightarrow c_i - \tilde{\xi}_i \\
v_i &\longrightarrow v_i - \tilde{\xi}_i'.
\end{aligned}
$$

Guage invariant combinations are obvious.

One natural choice: $\sigma = 0$ and $\beta = \zeta$ (conformal Newtonian gauge), and $c_i = 0$.

The pseudoscalar mode is still not present in the linear equations.

A very interesting (but unreliable) prediction is non-zero gravitational slip:

$$\phi - \psi = -\frac{12 f_{TT} H (H' - H^2)}{f_T} \zeta$$

where

$$\triangle \zeta = -3 \left(\psi' + H\phi - \frac{H' - H^2}{H} \psi \right).$$

But still, no new dynamical modes in linear perturbations! According to a recent work, the same is true even for spatially curved cosmologies.

This is a very amazing persistence of strong coupling.
In my opinion, the big trouble is that we have broken the local Lorentz invariance not strongly enough. Some random and strange net of "remnant symmetries" is still there.

For example, we can take another solution for Minkowski metric

$$e^a_\mu = \left(\begin{array}{cccc} \cosh(\lambda) & \sinh(\lambda) & 0 & 0 \\ \sinh(\lambda) & \cosh(\lambda) & 0 & 0 \\ 0 & 0 & \cos(\psi) & -\sin(\psi) \\ 0 & 0 & \sin(\psi) & \cos(\psi) \end{array} \right)$$

with arbitrary fixed functions $\lambda(t, x, y, z)$ and $\psi(t, x, y, z)$.
It has $\mathbb{T} = 0$, and therefore is a solution iff $f(0) = 0$.

It does not even form a group!

For linear Lorentzian perturbations one gets equations

$$\begin{aligned}
-\psi_z \mathbb{T}_t - \lambda_y \mathbb{T}_x + \lambda_x \mathbb{T}_y + \psi_t \mathbb{T}_z &= 0,\\
\psi_y \mathbb{T}_t - \lambda_z \mathbb{T}_x - \psi_t \mathbb{T}_y + \lambda_x \mathbb{T}_z &= 0,\\
-\lambda_y \mathbb{T}_t - \psi_z \mathbb{T}_x + \lambda_t \mathbb{T}_y + \psi_x \mathbb{T}_z &= 0,\\
-\lambda_z \mathbb{T}_t + \psi_y \mathbb{T}_x - \psi_x \mathbb{T}_y + \lambda_t \mathbb{T}_z &= 0
\end{aligned}$$

for linear variations of $\mathbb{T}$ around the zero value.

In generic enough a situation we get $\mathbb{T} =$const. However, in case of only a boost or only a rotation, perturbations of non-constant $\mathbb{T}$ are possible.

In particular, for $\lambda(z)$ and no rotation, we get a new mode with strange Cauchy data of $C_1(y, z)$ and $C_2(x, y, z)$.

One can look for generalisations. For example, a model of $f(\mathbb{T}, \mathbb{B})$ type. Those go beyond one of the main initial motivations for $f(\mathbb{T})$ gravity, for they produce 4-th order equations of motion.

It is unclear whether they can avoid the Ostrogradski-type ghosts, unless in the case of $f(\overset{(0)}{R})$. However, what is clear is that they inherit all the troubles of $f(\mathbb{T})$ gravity. Indeed, they obviously can be rewritten as $f(\mathbb{T}, \overset{(0)}{R})$, with all the issues of rather chaotic remnant symmetries.

There are also symmetric teleparallel options, in terms of non-metricity. They seem to have similar troubles, with diffeomorphism invariance violation being not strong enough, like the Lorentz one in metric teleparallel.

Conclusions

There are many reasons to try modifying gravity,
even in most crazy ways ever.

It brings a lot of pleasure,
and no less serious troubles...

Thank you!