



# Vanishing of Black Hole Love Numbers

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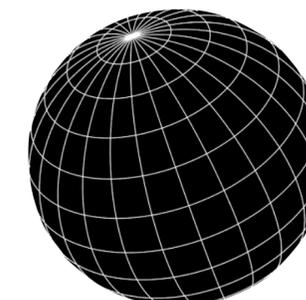
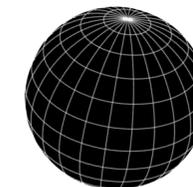


Oscar Combaluzier-Szteinsznaider  
Universite Paris Cite, APC

[Combaluzier-Szteinsznaider, Hui, Santoni, Solomon & **Wong**, 2410.10952]

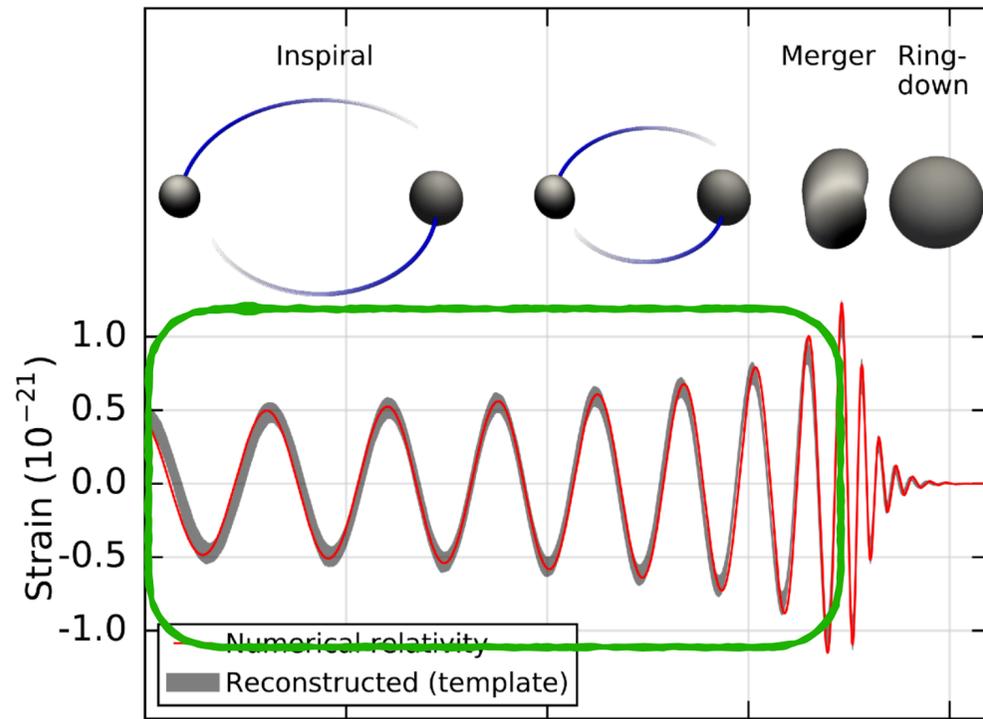
[De Luca, Khoury & **Wong**, 2305.14444 Phys.Rev.D]

& work in progress

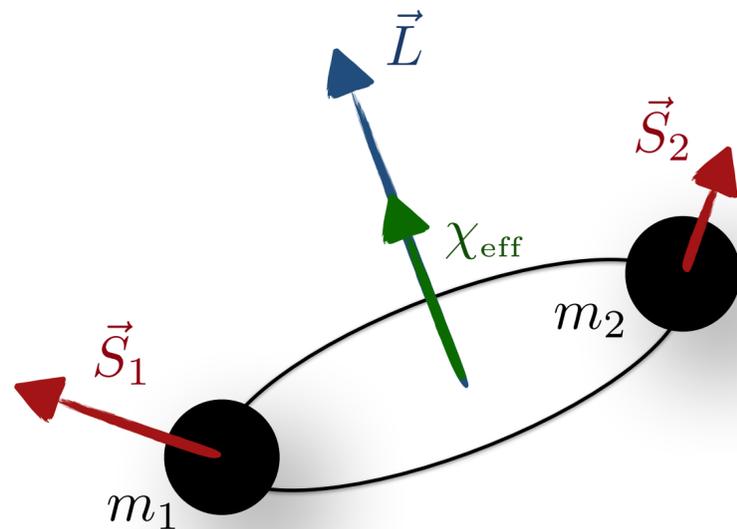


# Inspiral Phase of Mergers

- Observable: gravitational waveform from binary mergers



Credit: LVKC



Chirp mass: 
$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

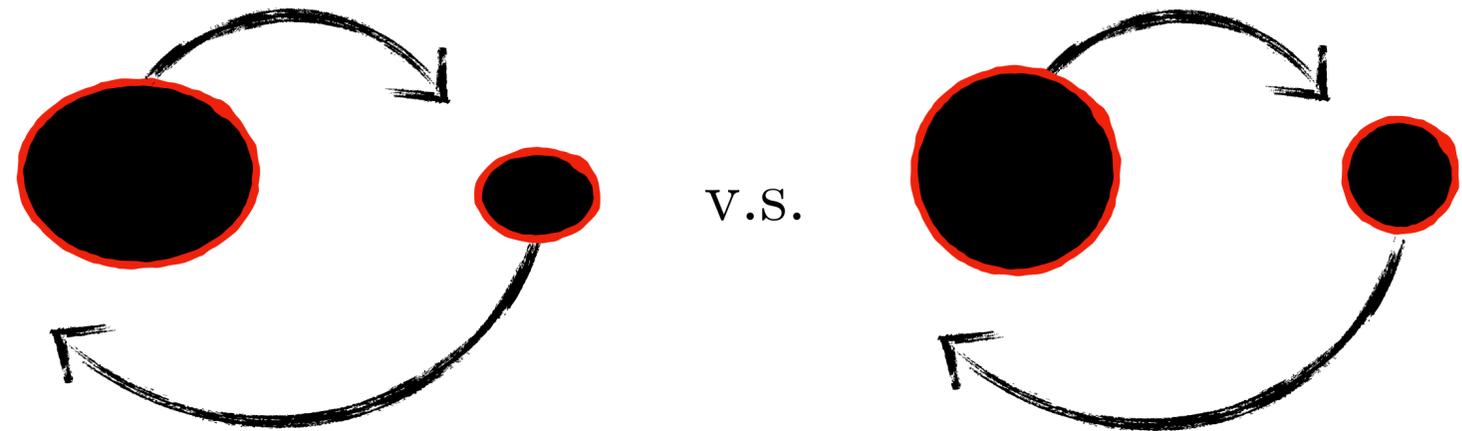
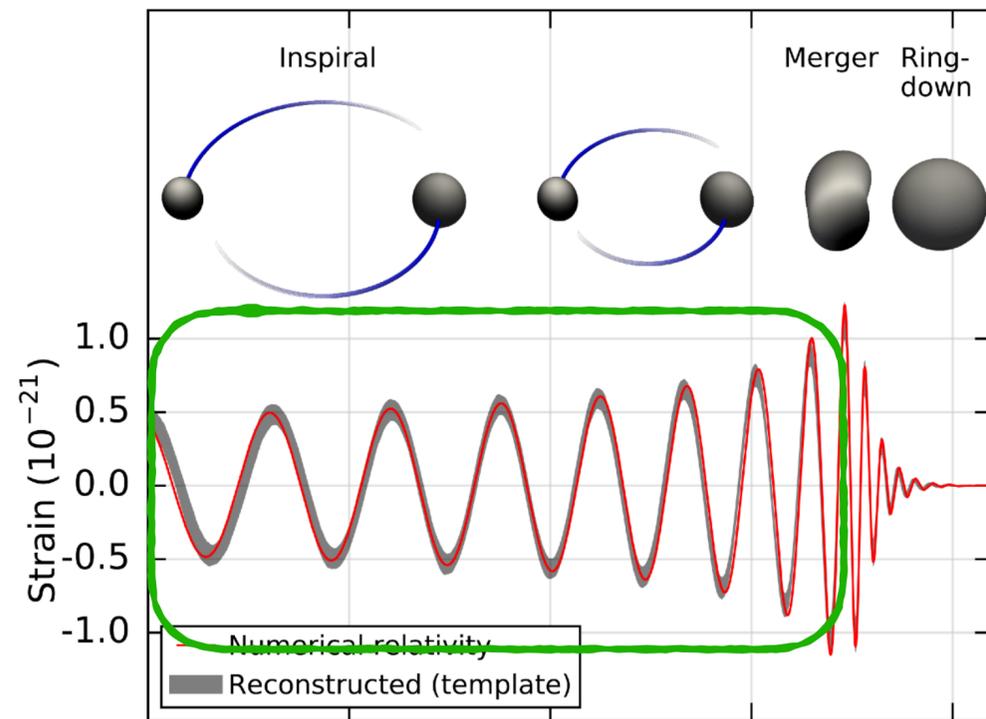
Mass ratio: 
$$q = m_2 / m_1$$

Effective spin: 
$$\chi_{\text{eff}} = \frac{\vec{S}_1 / m_1 + \vec{S}_2 / m_2}{m_1 + m_2} \cdot \hat{L}$$

Love number: 
$$\tilde{\Lambda} = g(m_1, m_2) k_2^{(1,2)}$$

# Inspiral Phase of Mergers

- Love number (14106?): tidal disruption



NS-NS mergers deform each other

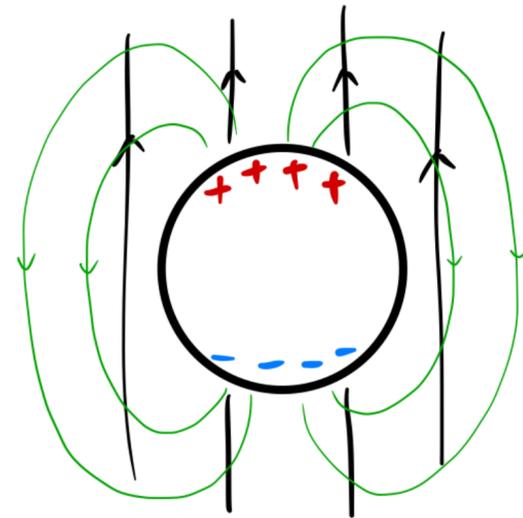
What about black holes?

**This talk: non-linear Love numbers**

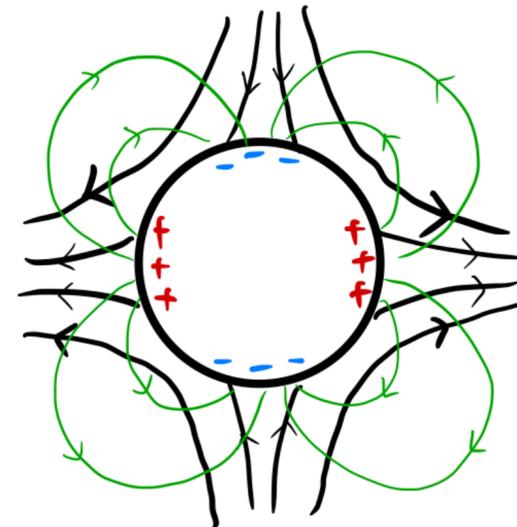
Different waveform from binary BHs

# Love Numbers

- E&M counterpart: Electric Polarizability & Magnetic Susceptibility



dipole  $\ell = 1$



quadrupole  $\ell = 2$

$c_\ell$ : electric polarizability of the object (electric Love number)

$$V(\vec{x}) = \sum_{\ell m} V_{\ell m} \left( r^\ell + \frac{c_\ell}{r^{\ell+1}} \right) Y_{\ell m}(\hat{\Omega})$$

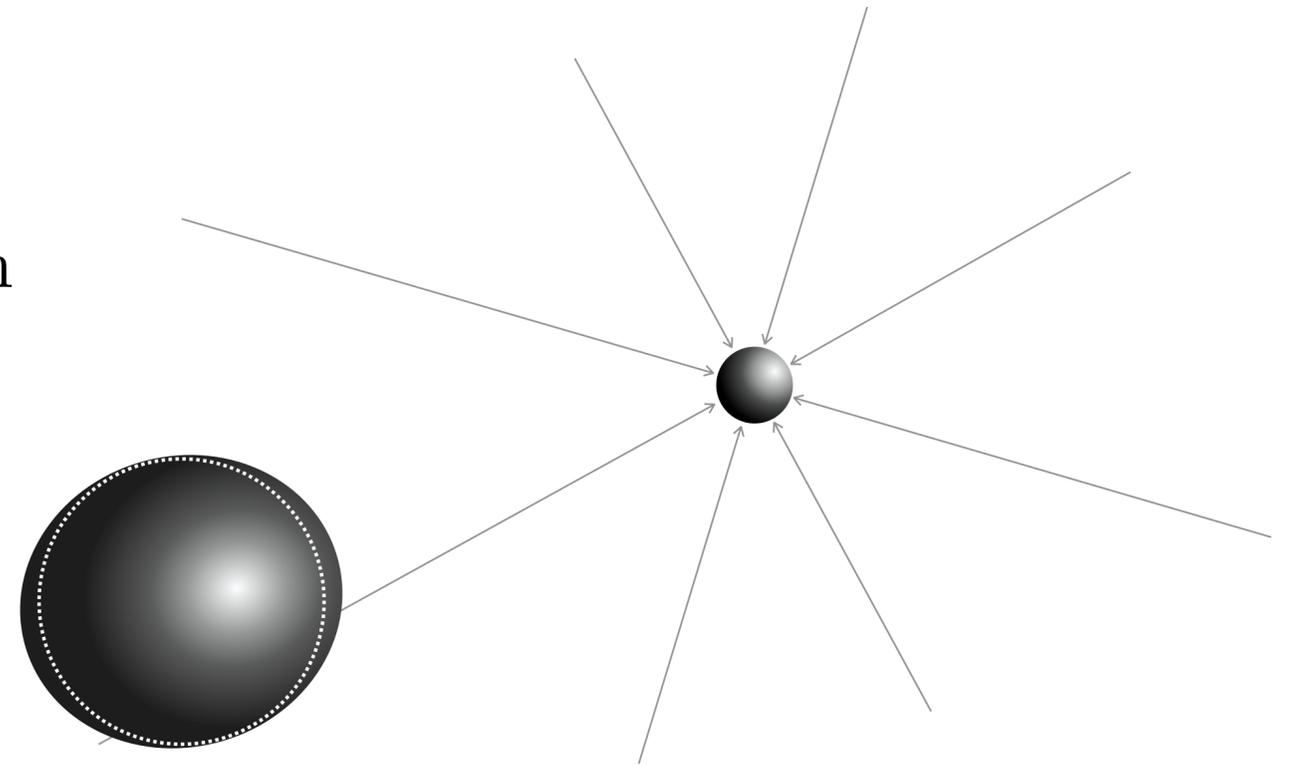
external source field

induced multipole

# Tidal Deformation (Tensor Love Number)

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- Gravitational deformation of compact objects
- Love numbers: a way to quantify tidal deformation



$$\Psi_{\text{RW}} = c_1 r^{\ell+1} \left[ 1 + \dots + k_{\text{RW}}^{(\ell)} \left( \frac{r}{r_h} \right)^{-2\ell-1} + \dots \right]$$

external source field

induced tidal deformation

# Black Hole Perturbation Theory

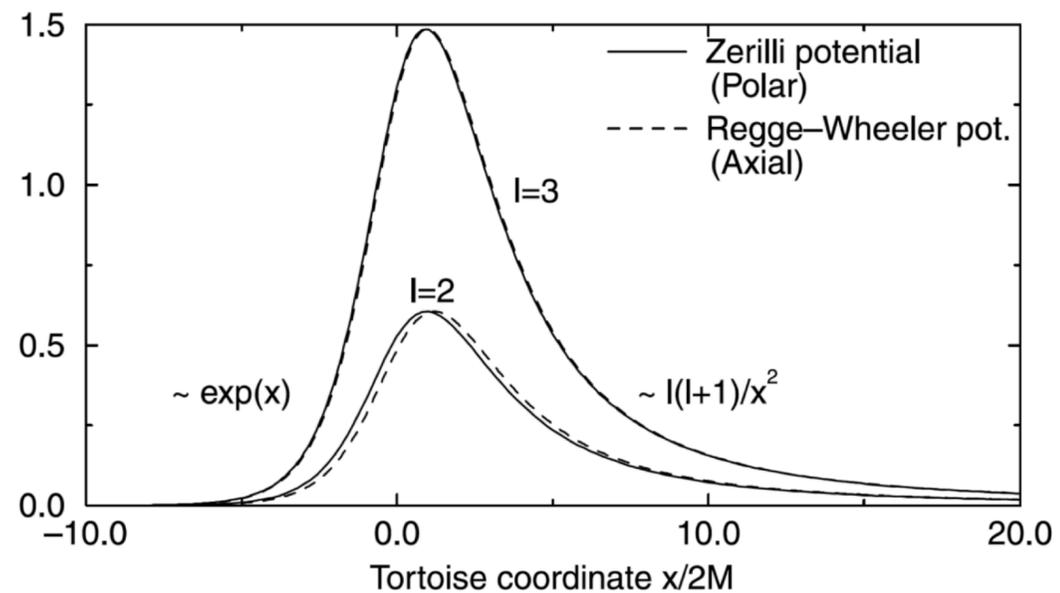
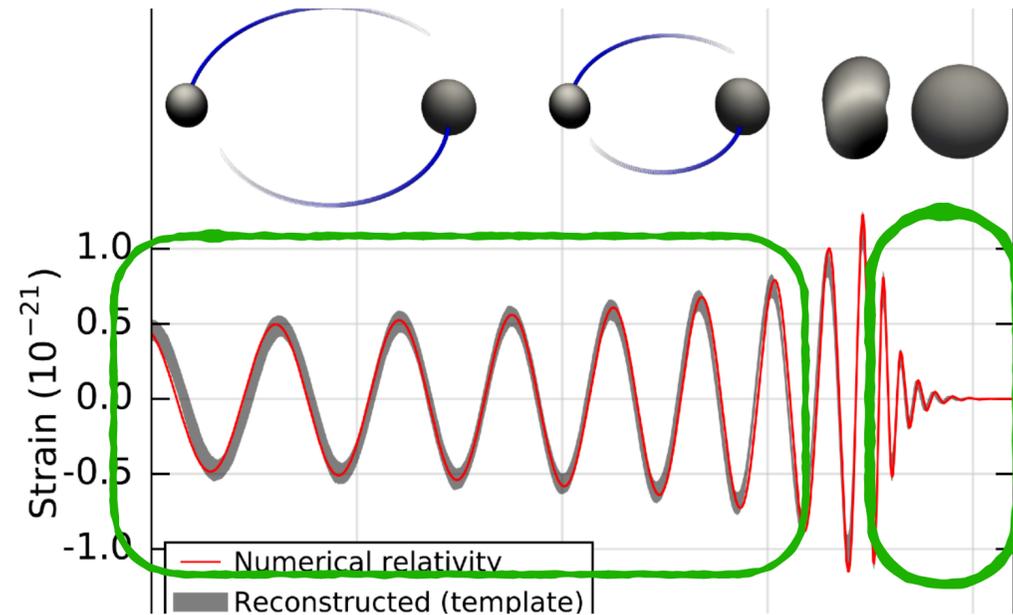
- Small perturbations around a non-rotating BH

$$ds^2 = -f_t(r)dt^2 + \frac{1}{f_r(r)}dr^2 + r^2d\Omega^2$$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}^{\text{even}} + h_{\mu\nu}^{\text{odd}}$$

$$h_{\mu\nu}^{\text{odd}} = \sum_{\ell, m} \begin{pmatrix} 0 & 0 & h_0^{\ell m} \epsilon_A^B D_B \\ 0 & 0 & h_1^{\ell m} \epsilon_A^B D_B \\ h_0^{\ell m} \epsilon_A^B D_B & h_1^{\ell m} \epsilon_A^B D_B & h_2^{\ell m} \epsilon_{(A}^C D_B) D_C \end{pmatrix} Y_\ell^m(\theta, \varphi)$$

$$h_{\mu\nu}^{\text{even}} = \sum_{\ell, m} \begin{pmatrix} f_t H_0^{\ell m} & H_1^{\ell m} & \alpha^{\ell m} D_A \\ H_1^{\ell m} & \frac{1}{f_r} H_2^{\ell m} & \beta^{\ell m} D_A \\ \alpha^{\ell m} D_A & \beta^{\ell m} D_A & r^2 K^{\ell m} \gamma_{AB} + Q^{\ell m} D_A D_B \end{pmatrix} Y_\ell^m(\theta, \varphi)$$



Regge Wheeler/Zerrili equations:

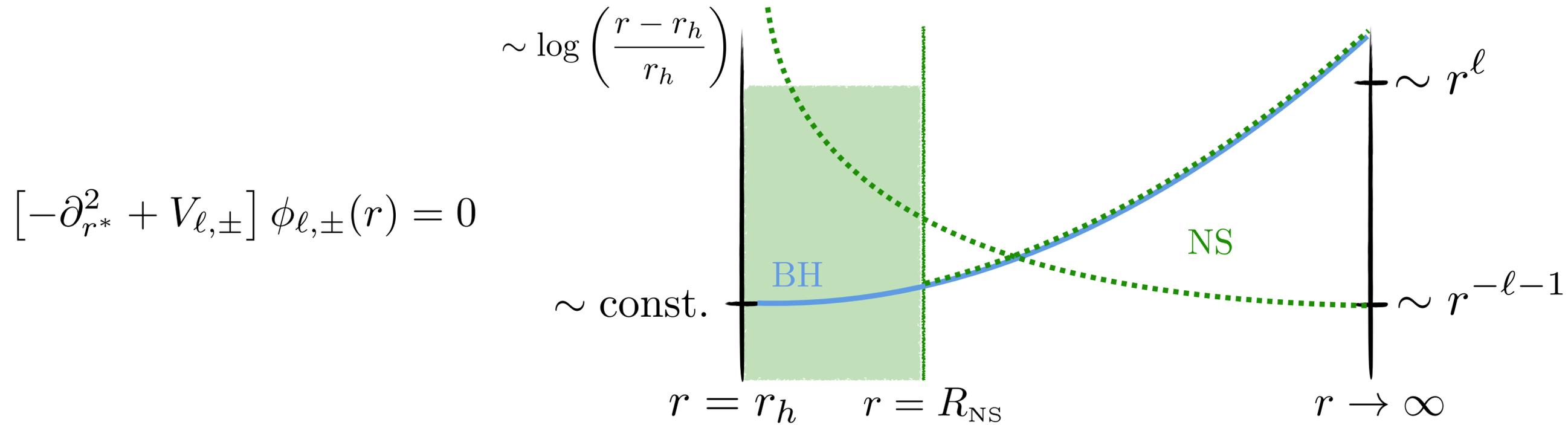
$$[\partial_t^2 - \partial_{r^*}^2 + V_\pm(r)] \phi_\pm(t, r^*) = 0$$

For static Love number problems

Kerr: Teukolsky equation

# TLN of Black Holes and Stars

*Surprising fact of GR: BHs have no Love (linear)*



- Black holes: regularity at the horizon kills the tail

accidental symmetry: [\[Hui, Joyce, Penco, Santoni & Solomon '21\]](#) [\[Achour, Livine, Mukohyama & Uzan '22\]](#)  
[\[Charalambous, Dubovsky & Ivanov '22\]](#) and many more

- Stars: match solutions of perturbation's EOM in the interior and exterior of the star, regularity at the center is required

Non-linearities in tidal deformation

# Non-linear Response

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We have been discussing linear response theory. However,

*GR is non-linear*

$$\left( \frac{\partial^2}{\partial r_{\star}^2} - V_{\text{RW}}(r) \right) \Psi_{\text{RW}}^{(2)} = S(\Psi_{\text{RW}}^{(1)2})$$



- How to define non-linear Love number?
- How does nonlinearity impact tidal deformation of BHs?

# Non-linear Response

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We have been discussing linear response theory. However,

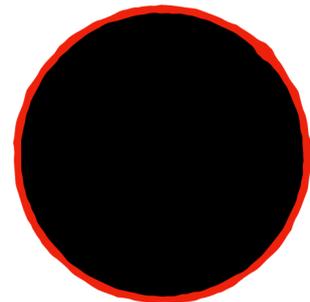
*GR is non-linear*

$$\left( \frac{\partial^2}{\partial r_{\star}^2} - V_{\text{RW}}(r) \right) \Psi_{\text{RW}}^{(2)} = S(\Psi_{\text{RW}}^{(1)2})$$



- How does nonlinearity impact tidal deformation of BHs?

However, we all know that non-linear GR is hard. As a prototype, we study scalar tidal deformation



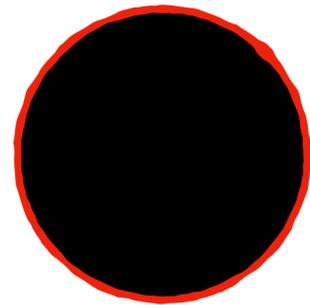
$$\square \phi = \alpha \mathcal{S}(\phi)$$

# Non-linear Response

Non-linearities of materials

$$P^i = \chi^{(1) i}_j E^j + \chi^{(2) i}_{jk} E^j E^k + \chi^{(3) i}_{jkl} E^j E^k E^l + \dots$$

And non-linearities in the bulk



$$\square\phi = \alpha\mathcal{S}(\phi)$$

Worldline of the compact object (BH)

All these terms describe the properties of the compact object, such as mass and Love Numbers

Both are captured by the point-particle EFT:

$$S = -\frac{1}{2} \int d^4x \sqrt{-g} [-(\partial\phi)^2 + \alpha\mathcal{O}(\phi)] + \int d\tau e \left[ \frac{1}{2} e^{-2} \dot{x}^\mu \dot{x}_\mu - \frac{m^2}{2} + g\phi + \sum_{\ell=1}^{\infty} \frac{\lambda_\ell}{2\ell!} (\partial_{(a_1} \cdots \partial_{a_\ell)_T} \phi)^2 + \dots \right]$$

# Non-linear Scalar Response

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In general:

Power-law interactions:

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} (\partial\phi)^2 + \frac{\alpha}{n} \phi^n \right]$$

Higher derivative terms:

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} (\partial\phi)^2 + \alpha (\partial\phi)^4 \right]$$

**A tail is generated at spatial infinity!**

$\mathcal{O}\left(\frac{1}{r^m}\right)$  tail due to non-linear response

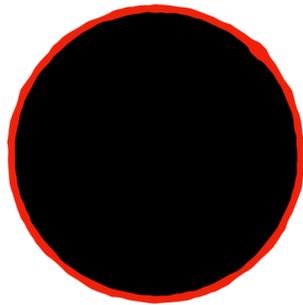
(Tidal response appear in the form of power law tail)

# Non-linear Scalar Response

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- Non-linear sigma model

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} G_{IJ}(\phi) \partial_\mu \phi^I \partial^\mu \phi^J \right]$$



$$\square \phi^I + \Gamma_{KL}^I \partial_\mu \phi^K \partial^\mu \phi^L = 0$$

$$\mathcal{O} \left( \frac{1}{r^m} \right)$$

*No falloff tails at spatial infinity in all order in perturbation theory*

# Non-linear Scalar Response

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- Why is this interesting?

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} G_{IJ}(\phi) \partial_\mu \phi^I \partial^\mu \phi^J \right]$$

- Is there a symmetry reason?
- Two derivatives per term, interacting, diff in  $\phi$  resembles diff in GR
- GR in 4D with one Killing vector reduces to a  $O(2,1)$  non-linear sigma model with a 3D metric [\[Sanchez 1981\]](#)

$$\nabla^2 \phi^I + \Gamma^I_{JK} \nabla_i \phi^J \nabla^i \phi^K = 0$$

$${}^{(3)}R_{ij} = T_{ij}(\phi)$$

- Next problem: GR (or Yang-Mills)? The pp-EFT point of view?

# Vanishing Non-linear Tensor Love Number in GR

- Using the point particle EFT, the non-linear Love numbers are defined as the coefficients of the higher order operators composed by

$$E_{\mu\nu} = C_{\mu\rho\nu\sigma} u^\rho u^\sigma \quad B_{\mu\nu} = \tilde{C}_{\mu\rho\nu\sigma} u^\rho u^\sigma$$

$$S_{\text{int}} = \sum_{l=2}^{\infty} \sum_{n=1}^{\infty} \int d\tau \sum_{\substack{l_1 \dots l_n \\ l=l_1 \otimes \dots \otimes l_n}} \lambda_{l_1 \dots l_n}^{(n)} F(E_{i_L} E_{i_{L_1}} \dots E_{i_{L_n}}) = \sum_{l=2}^{\infty} \int d\tau \left[ \lambda_l^{(1)} E_{i_L} E^{i_L} + \sum_{\substack{l_1, l_2 \\ |\ell_2 - \ell_1| \leq \ell \leq \ell_1 + \ell_2}} \lambda_{l_1 l_2}^{(2)} F(E_{i_L} E_{i_{L_1}} E_{i_{L_2}}) + \dots \right],$$

+ terms that contains the parity odd part

After matching with the calculation in GR, it turns out that, for all the higher order operators composed purely of  $E$ , at each order in field and each combination of  $(\ell_1, \ell_2, \dots, \ell_n)$ , there is *at least one* vanishing coefficient.

Worldline of the compact object (BH)

# Vanishing Non-linear Tensor Love Number in GR

Consider the most general static, axisymmetric spacetime

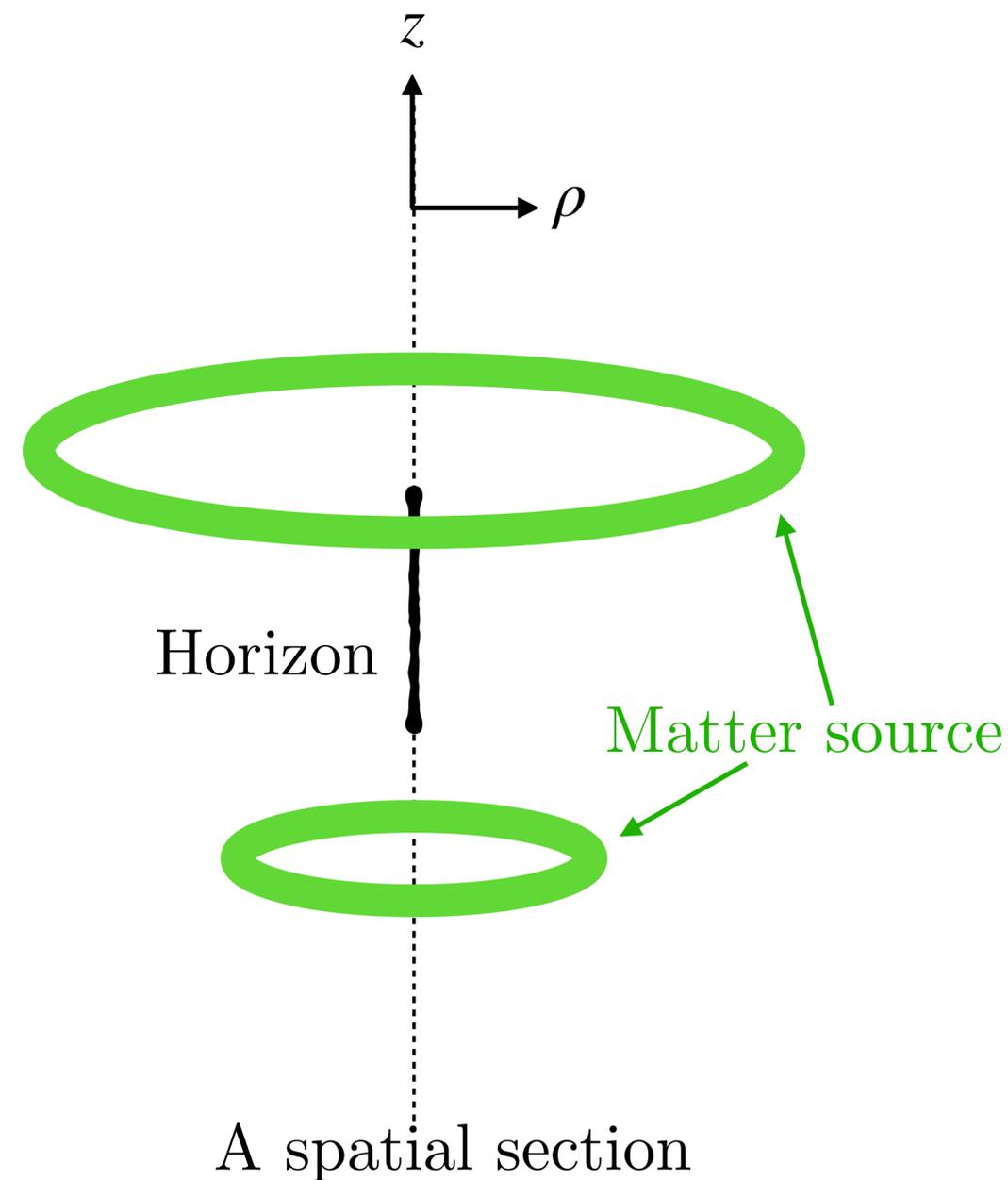
$$ds^2 = -e^{-\psi(z,\rho)} dt^2 + e^{\psi(z,\rho)} \left[ e^{2\gamma(z,\rho)} (d\rho^2 + dz^2) + \rho^2 d\phi^2 \right]$$

The vacuum Einstein's equations are very simple

$$\nabla^2 \psi = \left( \partial_\rho^2 + \frac{1}{\rho} \partial_\rho + \partial_z^2 \right) \psi = 0 \quad \partial_\rho \gamma = \frac{\rho}{4} \left( (\partial_\rho \psi)^2 - (\partial_z \psi)^2 \right) \quad \partial_z \gamma = \frac{\rho}{2} \partial_\rho \psi \partial_z \psi$$

Together with regularity conditions for invariant quantities at the horizon, such as  $\nabla_\mu \xi^\nu \nabla^\mu \xi_\nu$ , where  $\xi$  is the time like killing vector, one finds that *there is no induced power law tails in the general solution!*

Through matching with the EFT, some of the  $\lambda^{(n)}$  are zero (for the parity even sector). The EFT is beyond axisymmetry, which means that they should vanish for more general spacetime.

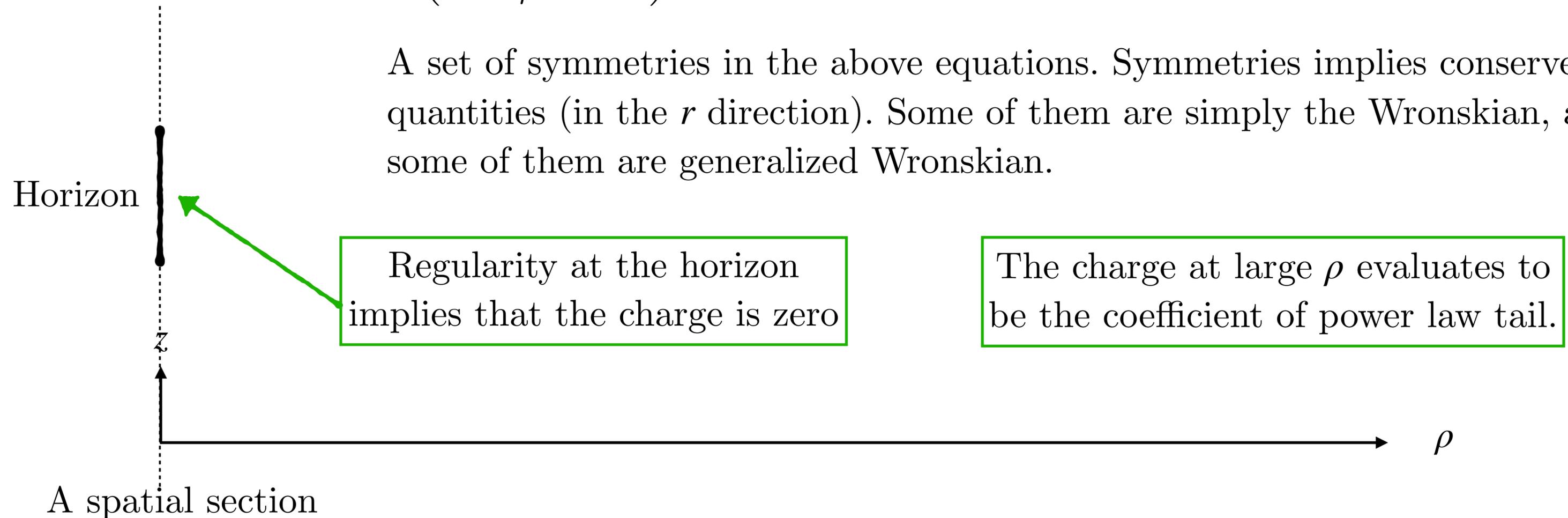


# Vanishing Non-linear Tensor Love Number in GR

A story of symmetry behind this

$$\nabla^2 \psi = \left( \partial_\rho^2 + \frac{1}{\rho} \partial_\rho + \partial_z^2 \right) \psi = 0 \quad \partial_\rho \gamma = \frac{\rho}{4} \left( (\partial_\rho \psi)^2 - (\partial_z \psi)^2 \right) \quad \partial_z \gamma = \frac{\rho}{2} \partial_\rho \psi \partial_z \psi$$

A set of symmetries in the above equations. Symmetries implies conserved quantities (in the  $r$  direction). Some of them are simply the Wronskian, and some of them are generalized Wronskian.



Symmetries:  $sl(2, \mathbb{R}) + \text{others}$

# Love Number in GR: Summary & Outlook

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- ◉ It seems that there is evidence that *all* non-linear Love numbers should be vanishing for non-rotating BHs
  - we found that in the purely even sector,
  - a direct computation shows that at the quadratic order, the parity odd Love numbers are also vanishing
- ◉ Natural questions to ask: rotating BHs, parity odd+even sector
- ◉ Can the non-linear sigma model or the use of symmetry lead us to somewhere?

Questions are welcome