Tensor Network Holography and Deep Learning

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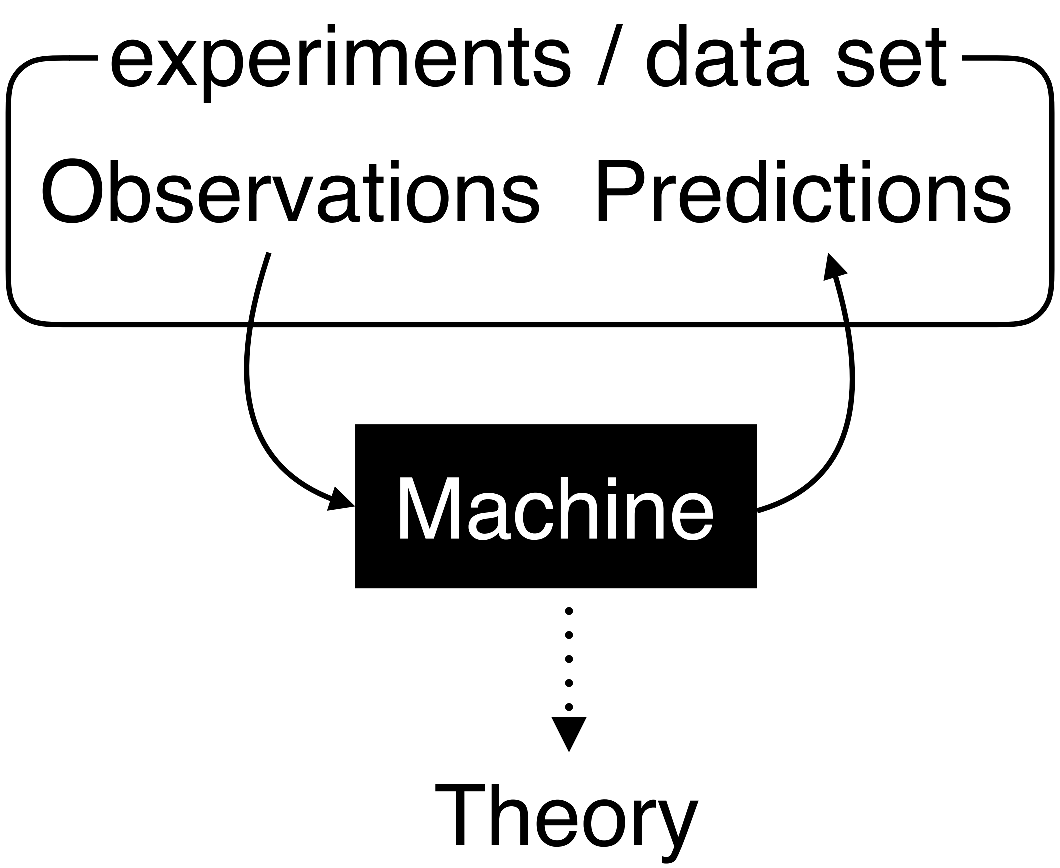
PRB **97**, 045153 (2018) arXiv:1709.01223

Workshop on Machine Learning in Geometry and Physics

TSIMF, June 2018

Machine Learning and Physics

- A general motivation: can artificial intelligence (AI) discover laws of physics?



- Interpretability of neural network is a hard problem in general. Let us start with simple cases and specific scenarios.

Machine Learning and Physics

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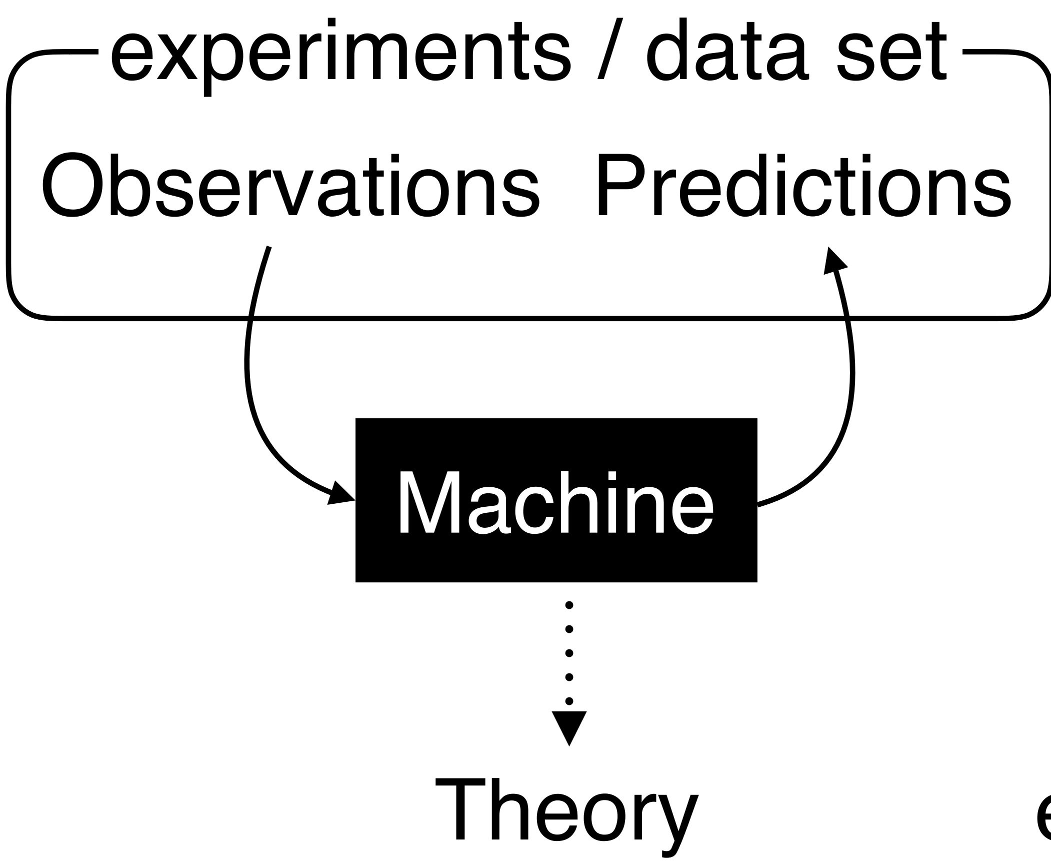
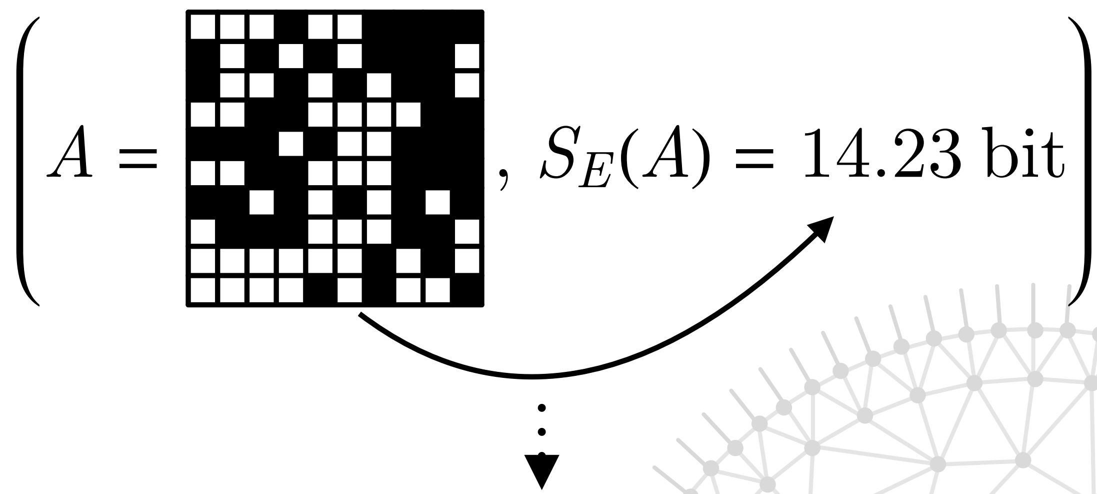


Entanglement Feature Learning

data about entanglement in quantum many-body state



emergent holographic geometry (as neural network connectivity)

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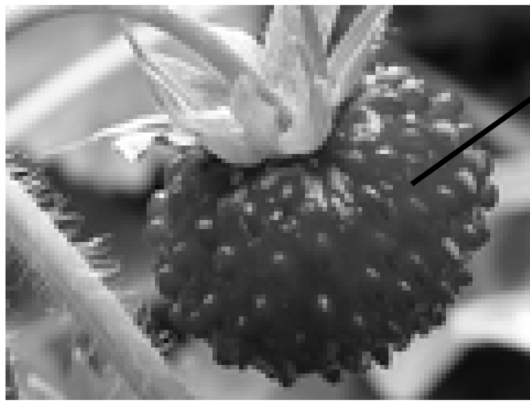
Quantum Field Theory as Image Dataset

- A field: a mapping from spacetime to some target manifold



Scalar fields

Vector fields

…

- A field theory: a model that assigns an action (= negative log likelihood) to every field configuration.

$P[$  $] \propto e^{-S[$  $;\, g\ldots]}$

action

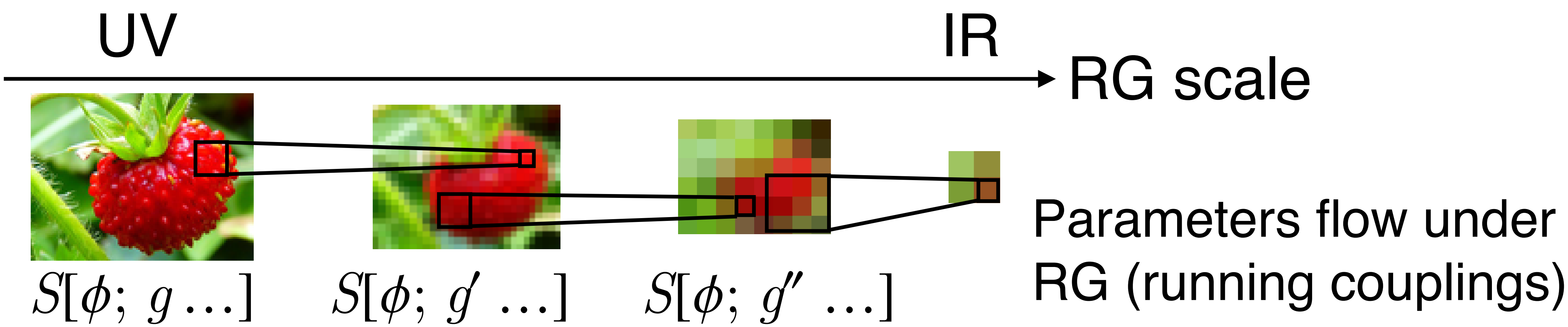
some model parameters (coupling constants)

- To evaluate any physical observables (typically correlation functions), we integrate over fields → path integral.

$\phi =$  $: \ \mathbb{E}\, \phi(x_1)\, \phi(x_2) \propto \int \mathcal{D}[\phi]\, \phi(x_1)\, \phi(x_2)\, e^{-S[\phi; g\, \ldots]}$

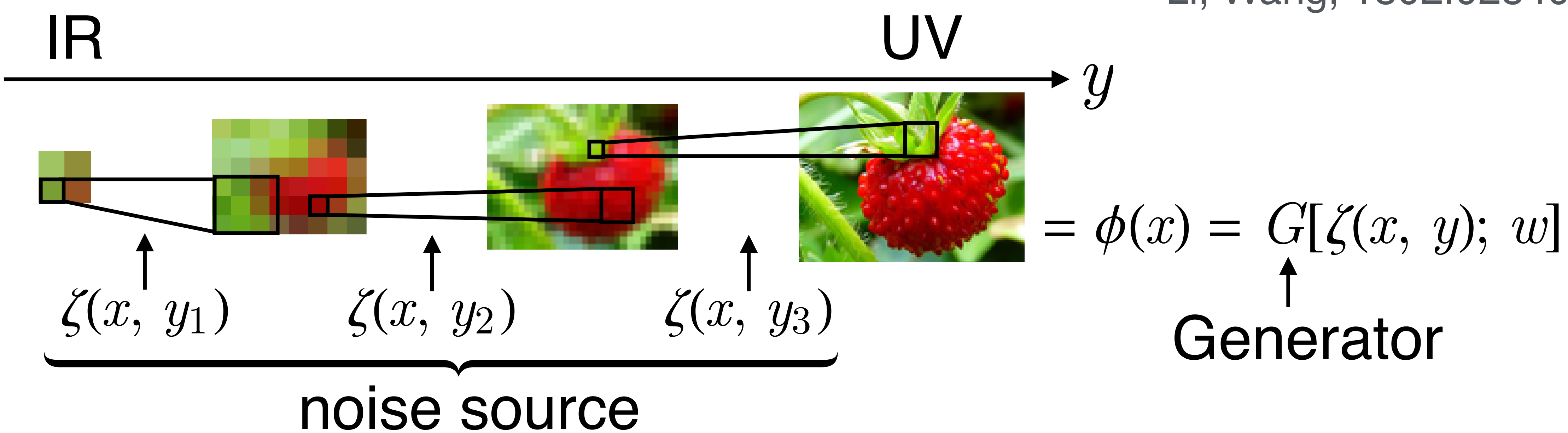
Renormalization Group as Generative Model

- Renormalization "group" (RG): progressively coarse-graining the field, similar in spirit to a convolutional neural network



- "Reversing" the RG: a generative model

Beny, 1301.3124
Li, Wang, 1802.02840

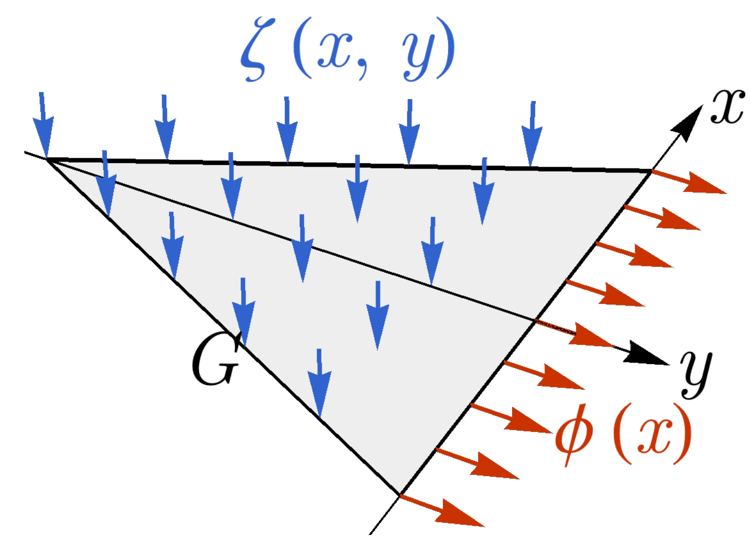


Machine Learning Holography

- Deforms noise to QFT

$$\zeta(x,\ y) \quad \longrightarrow \quad \phi(x) = G[\zeta(x,\ y);\ w]$$

$$P[\zeta] \propto e^{-\zeta^2} \longrightarrow P[\phi] = \det\left(\frac{\delta\ G}{\delta\ \zeta}\right)^{-1} P[\zeta] \propto e^{-S[\phi]}$$



- Machine Learning "Holography"
  - Given action $S[\phi]$, prepare training set $\phi \approx e^{-S[\phi]}$
  - Objective function: $\mathrm{KL}\left(e^{-S[\phi]} \,||\, \det(\delta_\zeta\ G)^{-1}\ P[\zeta]\right)$ (or GAN …)
  - Trains the generator (via unsupervised learning)
- A toy holographic duality (that captures some ideas)

$$Z = \mathrm{Tr}_{[\phi]}\ e^{-S[\phi]} \quad \longleftrightarrow \quad Z = \mathrm{Tr}_{[\zeta]} \det(\delta_\zeta\ G)^{-1}\ P[\zeta]$$

QFT of $\phi(x)$

(Classical) gravity + matter

- massive matter $\zeta(x,\ y)$
- couples to background $G(\cdot\,;\ w)$

Machine Learning Holography

- Holographic duality as a correspondence between

Features of dataset (in the visible layer)	$\longleftrightarrow$	Deep generative model (as the entire neural net)
Holographic boundary		**Holographic bulk**
d-dim spacetime $\phi(x)$		(d+1)-dim spacetime $\zeta(x,\ y)$

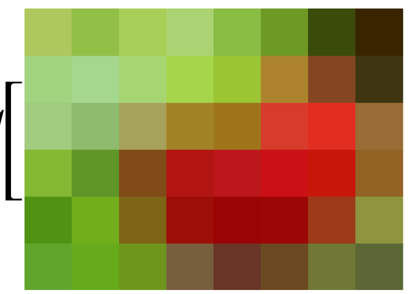
- The emergent holographic dimension = RG scale $\left(z = y^{-1}\right)$
  = The "deep" dimension of the neural network
- Geometry of the holographic bulk
  = neural network architecture (network connectivity)
- Distance determined by network weights w

Larger in weight = Shorter in distance

- Holography duality is most well studied as AdS/CFT.

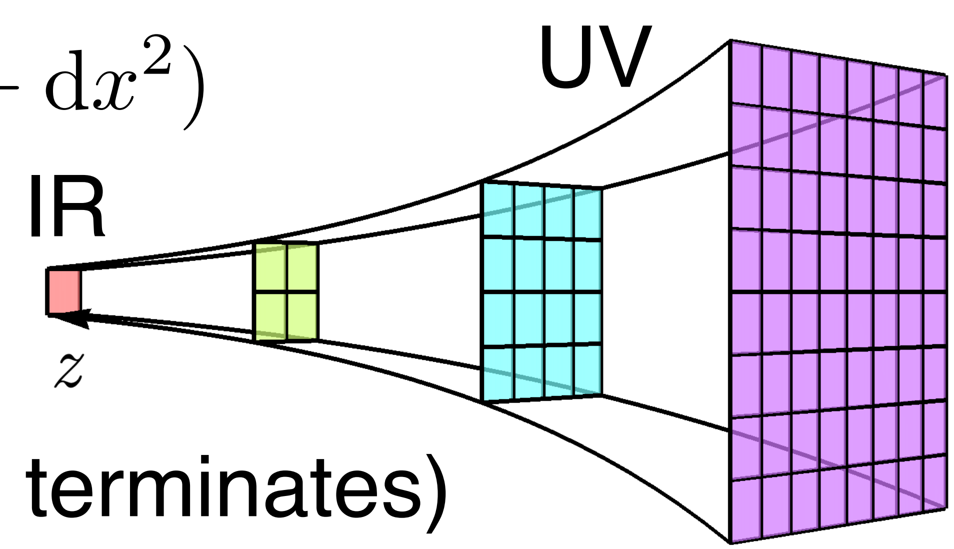
Machine Learning Holography

- What is special about conformal field theory (CFT)?
  - CFT is a QFT that respects conformal symmetry (invariant under scaling)

$S\Big[$  $, g\ldots\Big] = S\Big[$  $, g\ldots\Big] = S\Big[$  $, g\ldots\Big] = \ldots$

  - Conformal symmetry implies "translation" symmetry in the holographic bulk → all weights in the deep neural network are equal → holographic distance = network distance.

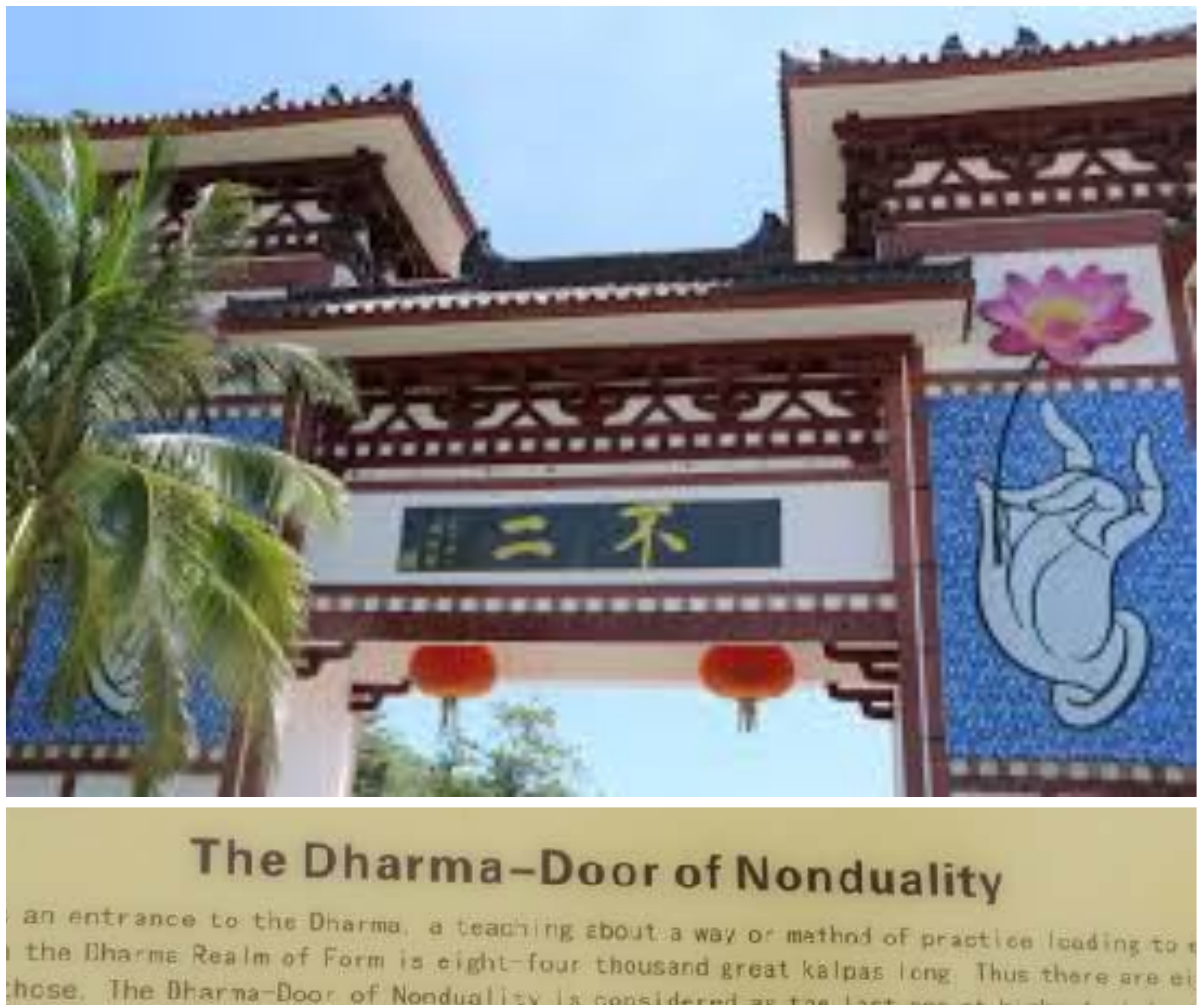
AdS geometry: $\mathrm{d}s^2 = \frac{1}{z^2}(\mathrm{d}z^2 + \mathrm{d}x^2)$



- CFT = a deep holographic bulk
- Mass deformation from a CFT introduces $z_{\text{cut}} \sim -\ln m$ (where RG terminates)

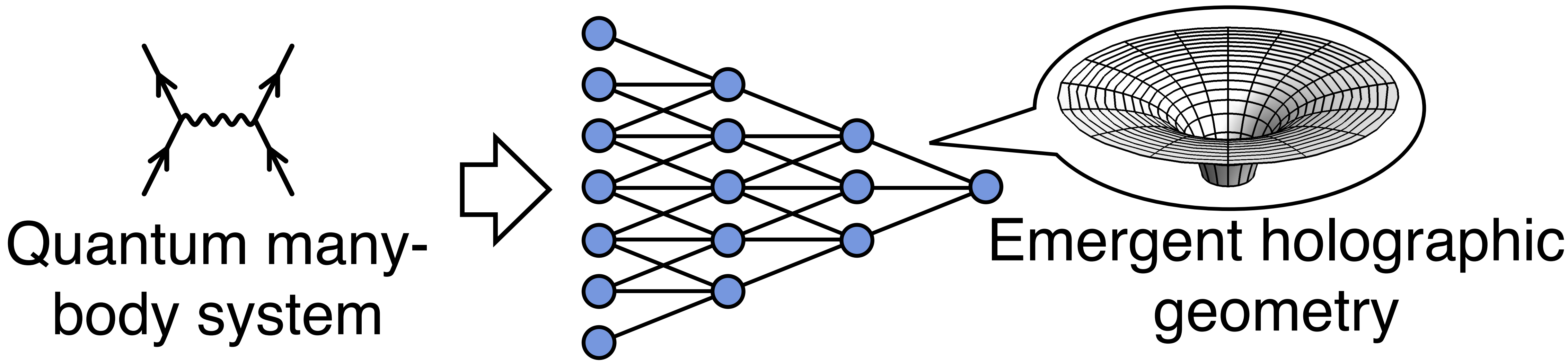
A Nonduality or the Only Approach

- Machine learning holography: not a duality in the strict sense, but may provide a unified approach to understand renormalization, holography and deep learning.



Geometry Emerges from Learning

- The general theme: geometry emerges from learning.



- To be more concrete, let us focus on a simpler problem:

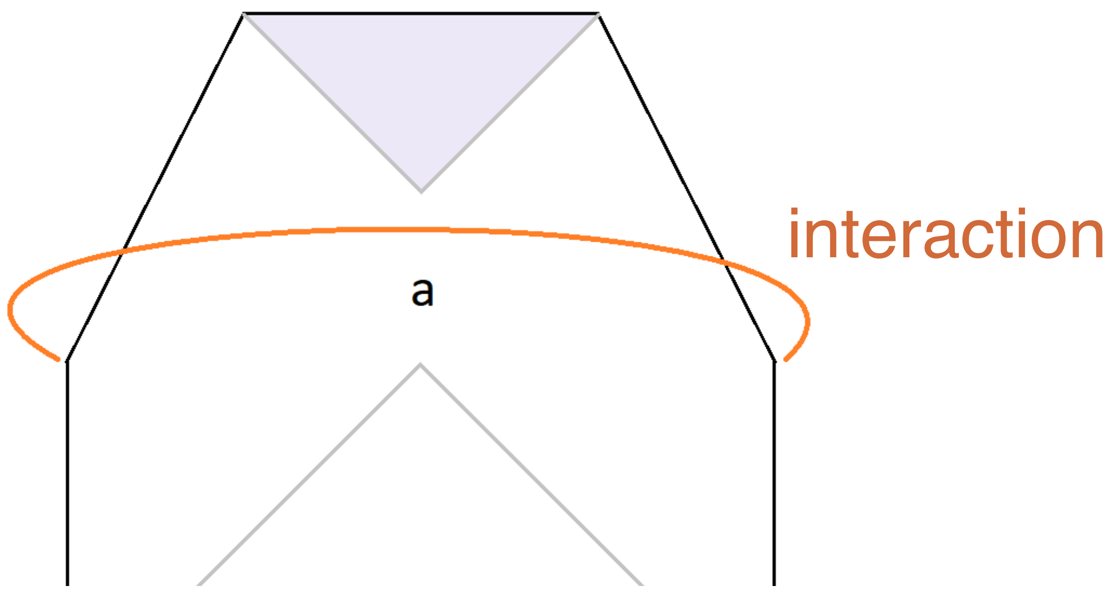
  Holography on a equal-time slice, instead of the spacetime

  (Try to determine geometry, instead of the full gravity theory)

  - Only need to look at data about quantum entanglement
  - Intuition: (1) both entanglement and geometry are universal (operator independent) (2) ER = EPR

Worm Hole and Quantum Entanglement

- ER = EPR Maldacena, Susskind 1306.0533, Susskind 1604.02589
  - ER: Einstein-Rosen bridge (worm hole)
  - EPR: Einstein-Podolsky-Rosen pair (entanglement)
- Quantum Teleportation
  - Traversable worm hole (in AdS)



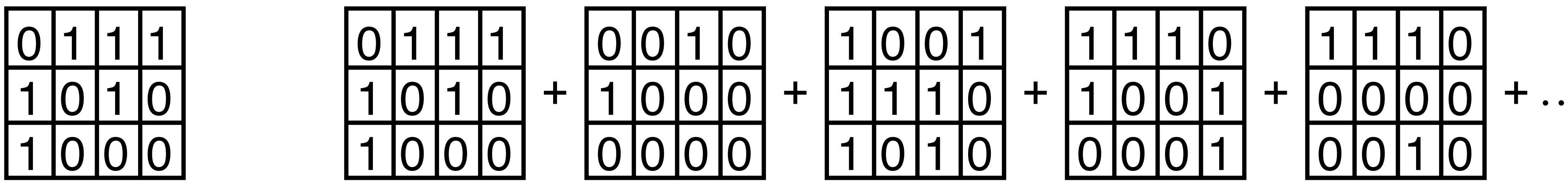
Gao, Jafferis, Wall 1608.05687;
Maldacena, Stanford, Yang, 1704.05333

from Quantum Magazine

  - Quantum entanglement + classical communication

Quantum Entanglement

- Quantum entanglement: a non-local quantum information sharing scheme



Classical

Quantum

  - Even we know everything about the quantum system, we know nothing about the state of each quantum bit (qubit).
  - Quantum information distributed among qubits → unable to separate qubits apart
- Any attempt to single out a subset of qubits from the entangled quantum system → lost of quantum information → production of entanglement entropy.

Quantum Entanglement

- Quantify entanglement by entanglement entropy
  - Given a quantum many-body state $|\Psi\rangle$
  - Or equivalently a pure state density matrix $\rho = |\Psi\rangle\langle\Psi|$
  - For every bipartition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\overline{A}}$ define reduced density matrix

$$\rho_A = \mathrm{Tr}_{\overline{A}}\,\rho = \mathrm{Tr}_{\overline{A}}\,|\Psi\rangle\langle\Psi|$$

  - (Renyi) Entanglement Entropy

$$S_E^{(n)}(A) = \frac{1}{1-n}\ln \mathrm{Tr}_A\,\rho_A^n$$

    - $n \to 1$ limit: von Neumann Entropy $S_E^{(1)}(A) = -\mathrm{Tr}_A\,\rho_A \ln\,\rho_A$
  - Entanglement mutual information

$$I^{(n)}(A,\,B) = S_E^{(n)}(A) + S_E^{(n)}(B) - S_E^{(n)}(A \cup B)$$

Entanglement Features

- More specifically, consider a many-"qudit" system

$$|\Psi\rangle = \sum_{[s_i]} \Psi(s_1, s_2, \ldots) \, |s_1 \; s_2 \ldots\rangle \qquad s_i = 1, \ldots, d$$

- Entanglement features: entanglement entropies over all possible regions A, to all orders of Renyi index n

You, Gu
1803.10425

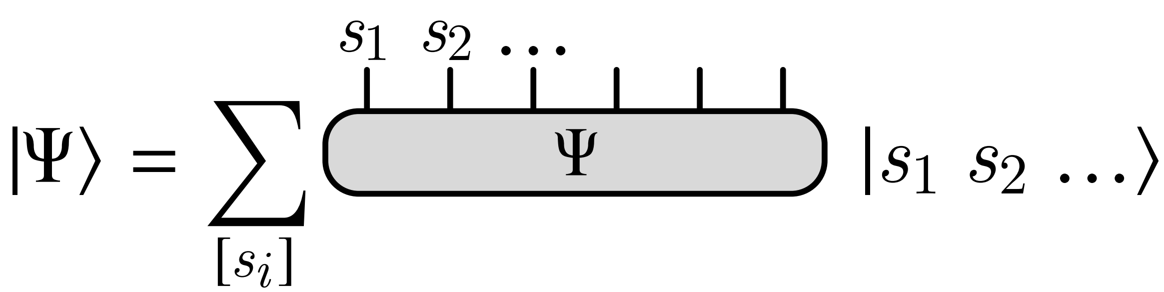
  - A full characterization of entanglement properties
  - For N qudits, there will be 2^N different choices of regions
  - Each choice of region A specifies by an Ising configuration

$A =$  $\longleftrightarrow \; [\tau]: \;\; \tau_v = \begin{cases} +1 & v \in A \\ -1 & v \in \bar{A} \end{cases}$

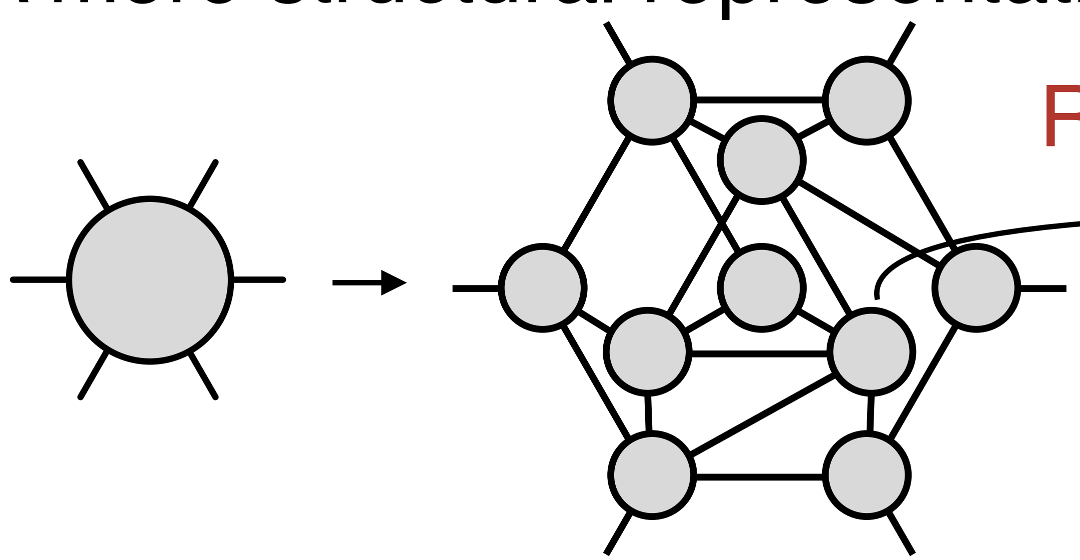
  - Associated entanglement entropy $S_E^{(n)}(A) \equiv S_E^{(n)}[\tau]$
  - To make life easier, we will focus on $S_E^{(2)}(A) \equiv S_E^{(2)}[\tau]$

Random Tensor Network

- These 2^N entanglement entropies are correlated, what is the underlying relation among $S_E^{(2)}[\tau]$? (What is the structure of many-body entanglement?)
- Our toolbox: random tensor network
  - Many-body wave function as a big tensor in general



  - A more structural representation: tensor network



Random tensor network (RTN)

every tensor drawn randomly

- circle - tensor
- link - index contraction

Random Tensor Network

- Why random tensors?
  Intuition from holography: entanglement (as a local unitary invariant property) mainly has to do with the structure of the tensor network not the tensor content.
- Entanglement entropy of RTN state

Hayden, Nezami, Qi, Thomas, Walter, Yang (2016)

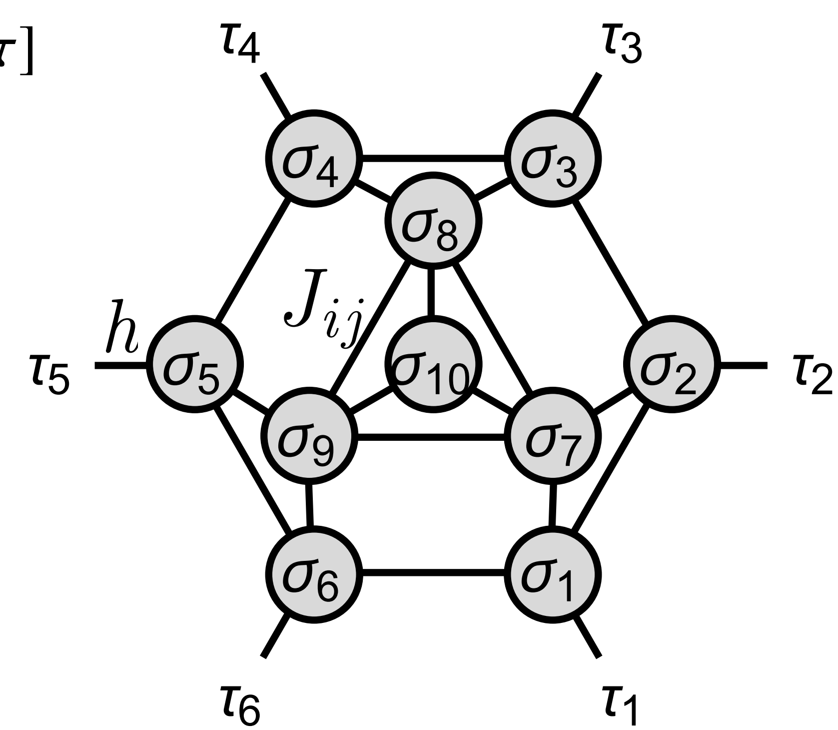
  - Given region A, $\rho_A = \mathrm{Tr}_{\overline{A}} \left|\Psi_{\mathrm{RTN}}\right\rangle \left\langle\Psi_{\mathrm{RTN}}\right|$
  - 2nd Renyi entropy ~ free energy of an Ising model

$$\mathbb{E}\, e^{-S_E^{(2)}(A)} = \mathbb{E}\,\mathrm{Tr}\,\rho_A^2 \propto \sum_{[\sigma]} e^{-E[\tau,\sigma]} = e^{-F[\tau]}$$

  - Energy function

$$E[\tau,\sigma] = -\sum_{\langle ij\rangle} J_{ij}\,\sigma_i\,\sigma_j - h\sum_{i\in\partial}\tau_i\,\sigma_i$$

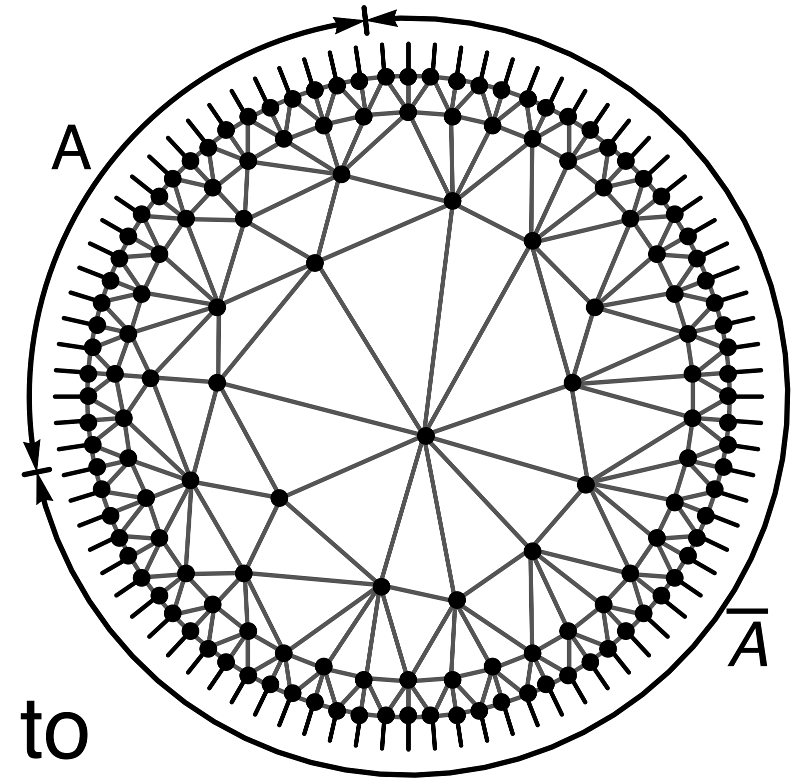
  - Coupling $J_{ij} = \frac{1}{2}\ln d_{ij}\,, h = \frac{1}{2}\ln d$



Ising Model and Minimal Cut

- To make sense of entanglement - Ising model mapping
  - Entanglement region → boundary pinning fields
  - Ferromagnetic Ising model

$$E[\tau, \sigma] = -\sum_{\langle ij \rangle} J_{ij}\,\sigma_i\,\sigma_j - h \sum_{i \in \partial} \tau_i\,\sigma_i$$



  - In the large *J* limit (large bond dim), Ising domain wall through the bulk automatically finds the minimal cut.
  - Length of minimal cut $|\gamma_A|$ proportional to domain wall energy $E_A - E_0 = 2\,J\,|\gamma_A|$ (assuming uniform *J*)
- Entanglement = Area Ryu, Takayanagi (2006)

$$S_E(A) = \frac{1}{4\,G_N}\,|\gamma_A| \longrightarrow S_E^{(2)}(A) = F[\tau] - F[\tau = +1]$$

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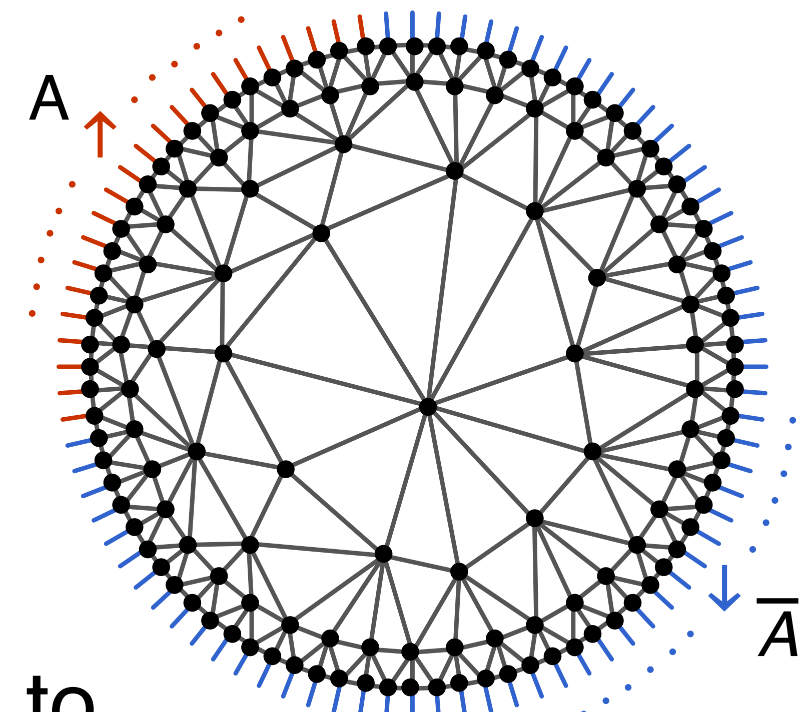

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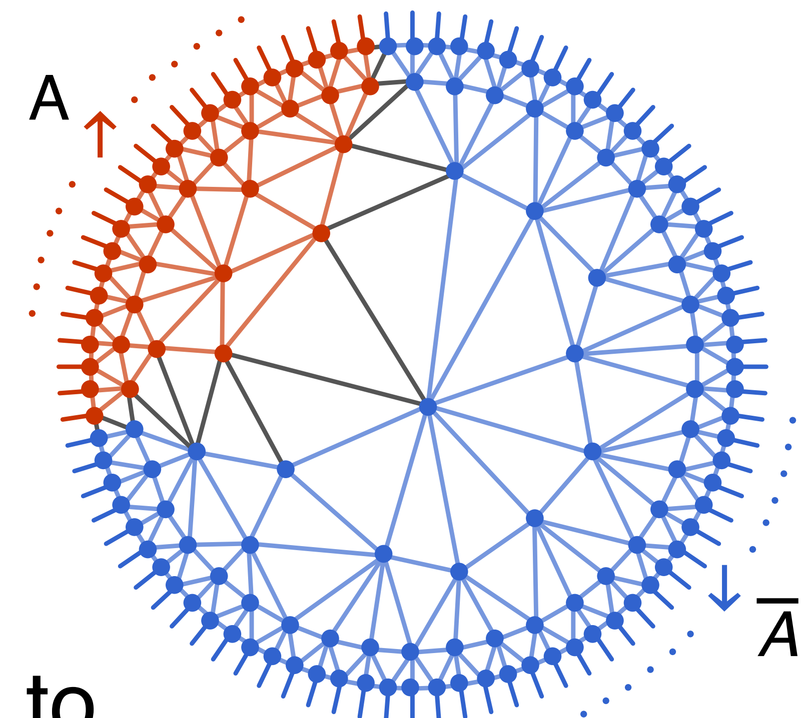

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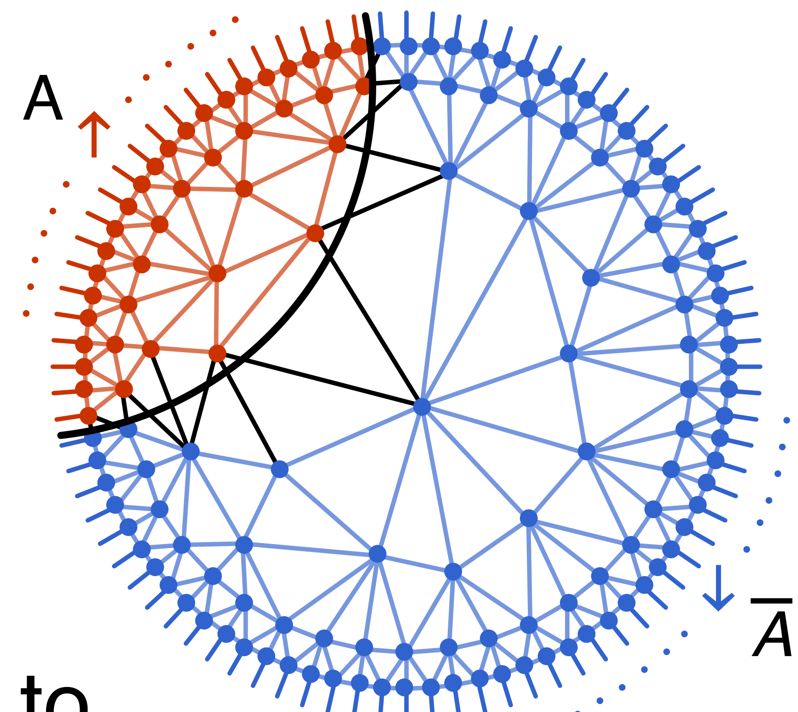

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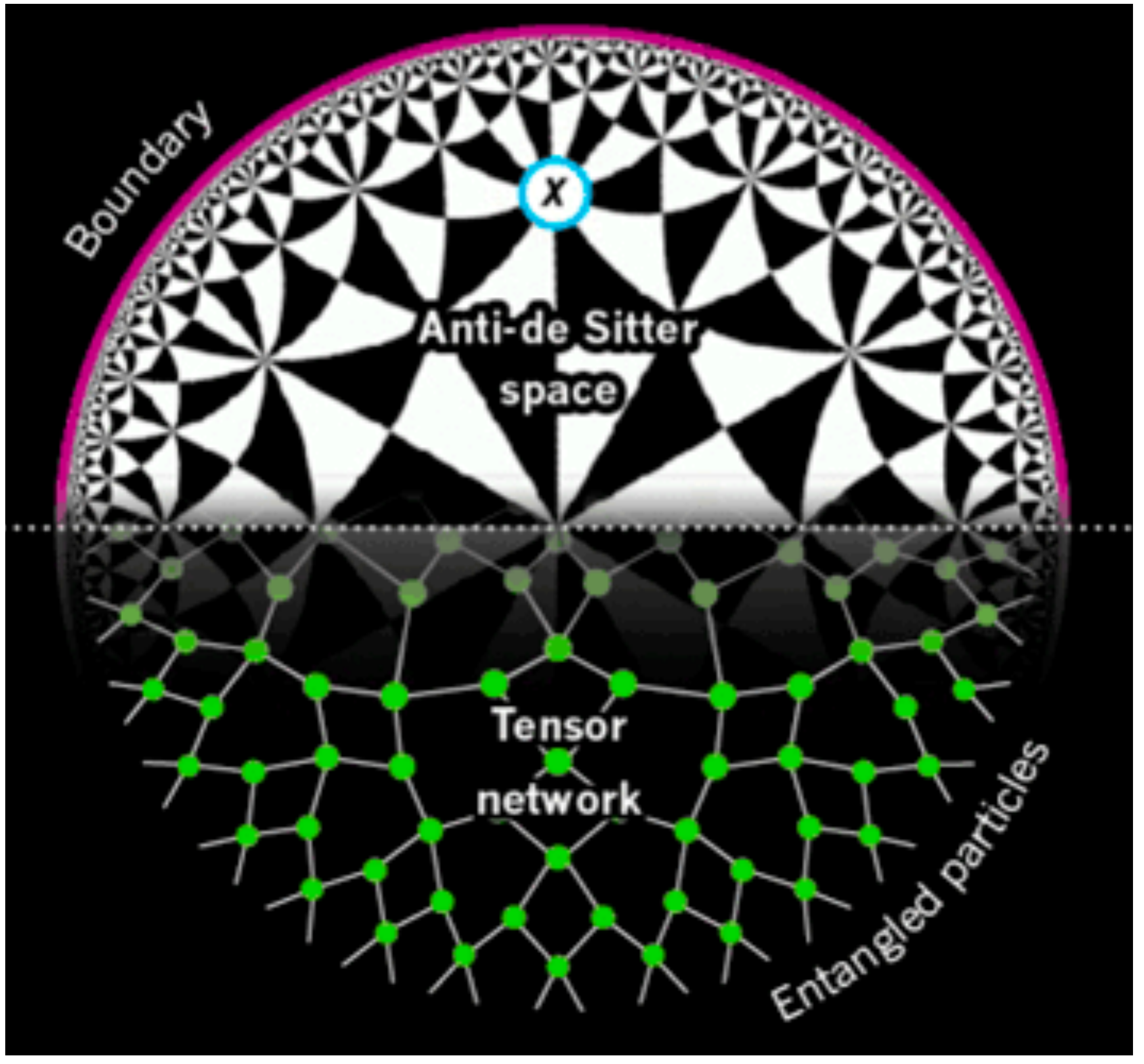
Tensor Network Holography

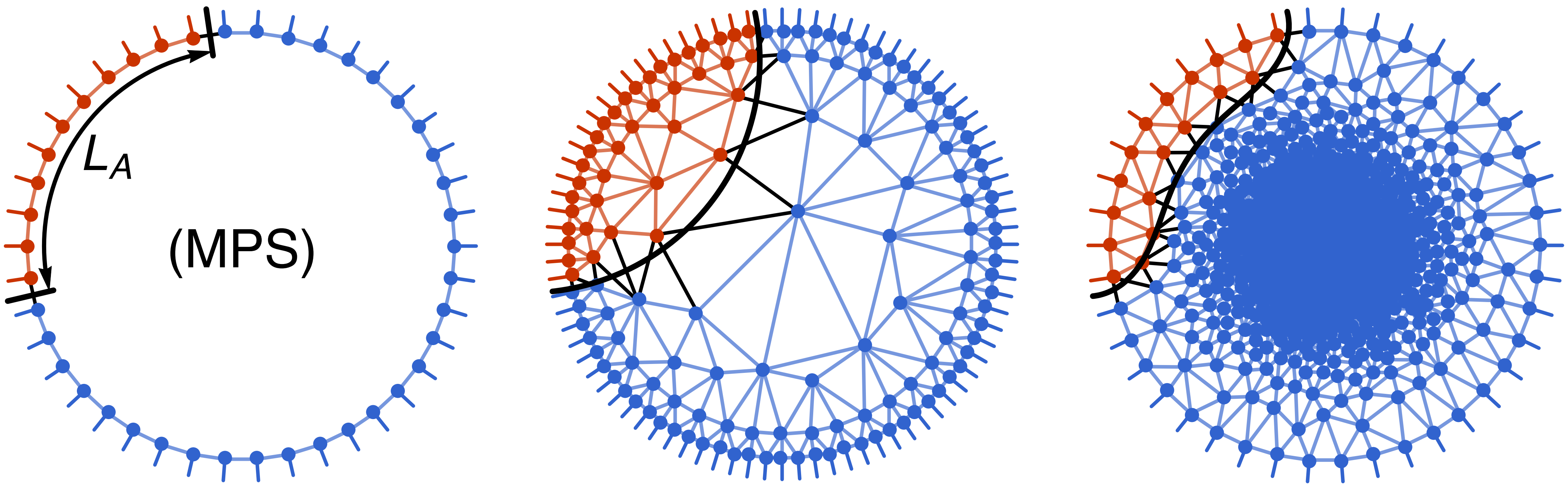
- Tensor network geometry = Holographic bulk geometry



Tensor Network Holography

- Tensor network geometry = Holographic bulk geometry
- Different geometries of tensor network leads to different entanglement features of quantum many-body state

Area-law	Logarithmic	Volume-law
$S_E(A) \sim \text{cosnt.}$	$S_E(A) \sim \ln L_A$	$S_E(A) \sim L_A$



- Given the entanglement feature, what is the optimal tensor network connection that best produce the same features?

Entanglement Feature Learning

- Given a many-body state $|\Psi\rangle$
- Draw the training sample $[\tau]$ from the following distribution

$$P_{\text{dat}}[\tau] \propto e^{-S^{(2)}_{E,\Psi}[\tau]}$$

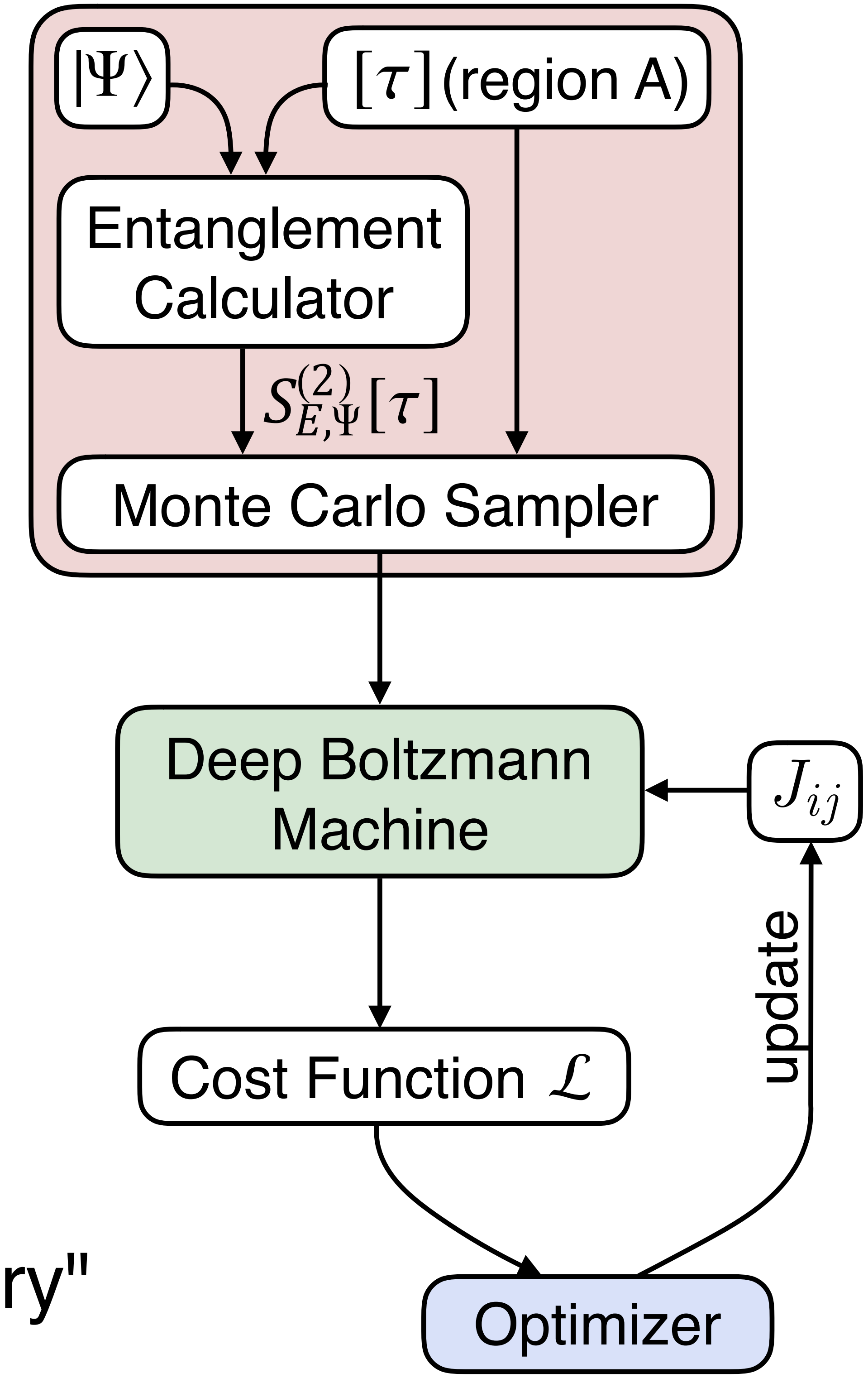
- Model the distribution by a deep Boltzmann machine

$$P_{\text{mdl}}[\tau;\, J] \propto \sum_{[\sigma]} e^{-E[\tau,\sigma;J]}$$

- Compute the free energy release

$$\mathcal{L}[J] = \mathrm{KL}(P_{\text{dat}} \,||\, P_{\text{mdl}}) + \text{reg.}$$

- Minimize $\mathcal{L}$ by updating J_{ij}
→ look for optimal network "geometry"



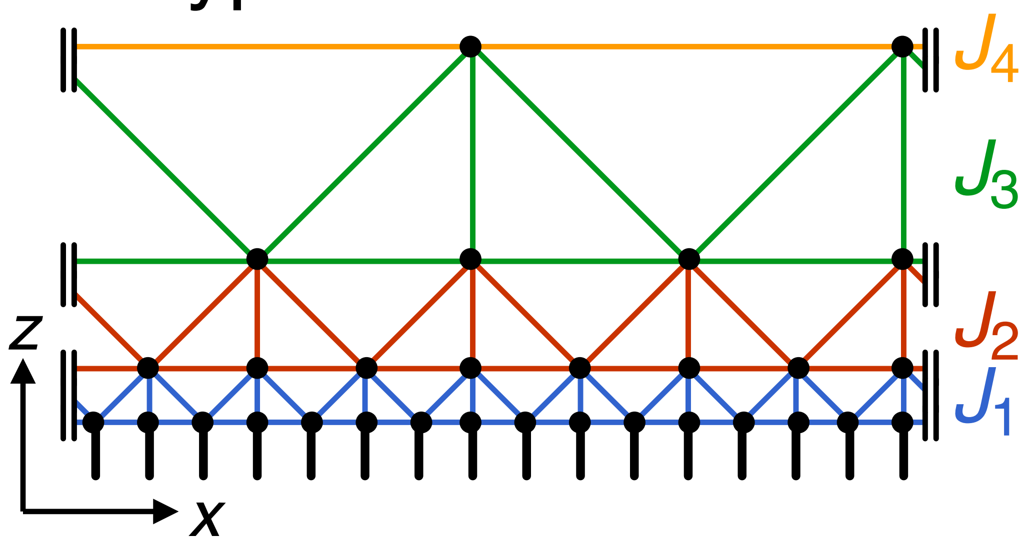
A Few Technical Details

- Generic deep Boltzmann machine → hard to train
- For learning 2-dim holographic bulk geometry of 1-dim quantum many-body system, natural to assume planar graph

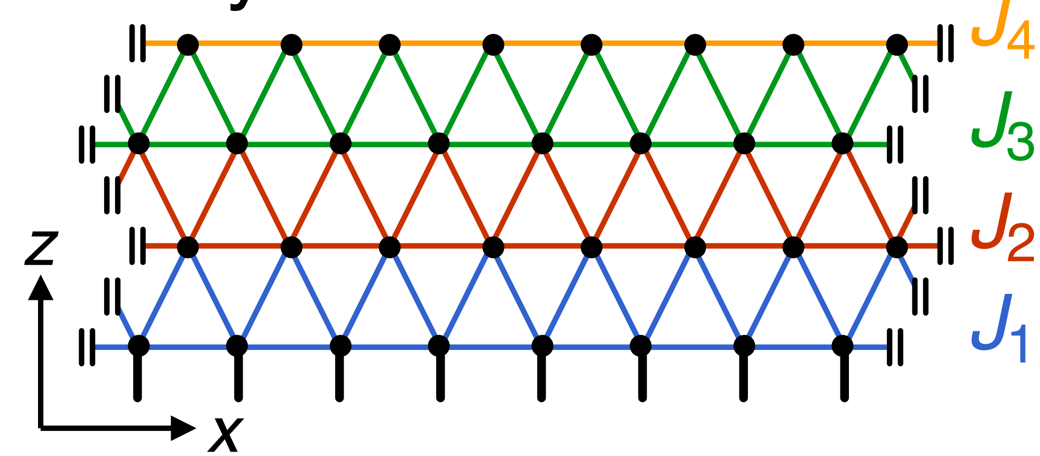
Globerson, Jaakkola 07, Schraudolph, Kamenetsky 08

- Put together translation symmetry, for now we restrict to two types of architectures

Hyperbolic network



Cylindrical network



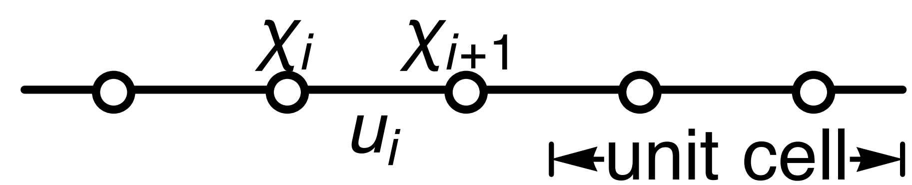
- Only a few parameters: J_z (z = layer depth)

Training Set: 1D Free Fermion Chain

- Computing entanglement entropies for generic many-body system is hard.
- To test the idea of EFL, we start with the simplest free fermion chain

Majorana fermions

$$H = -\sum_{a=1}^{N}\sum_{i=1}^{L} i\, u_i\, \chi_{a,i}\, \chi_{a,i+1} \qquad u_i = 1 + m\,(-1)^i$$

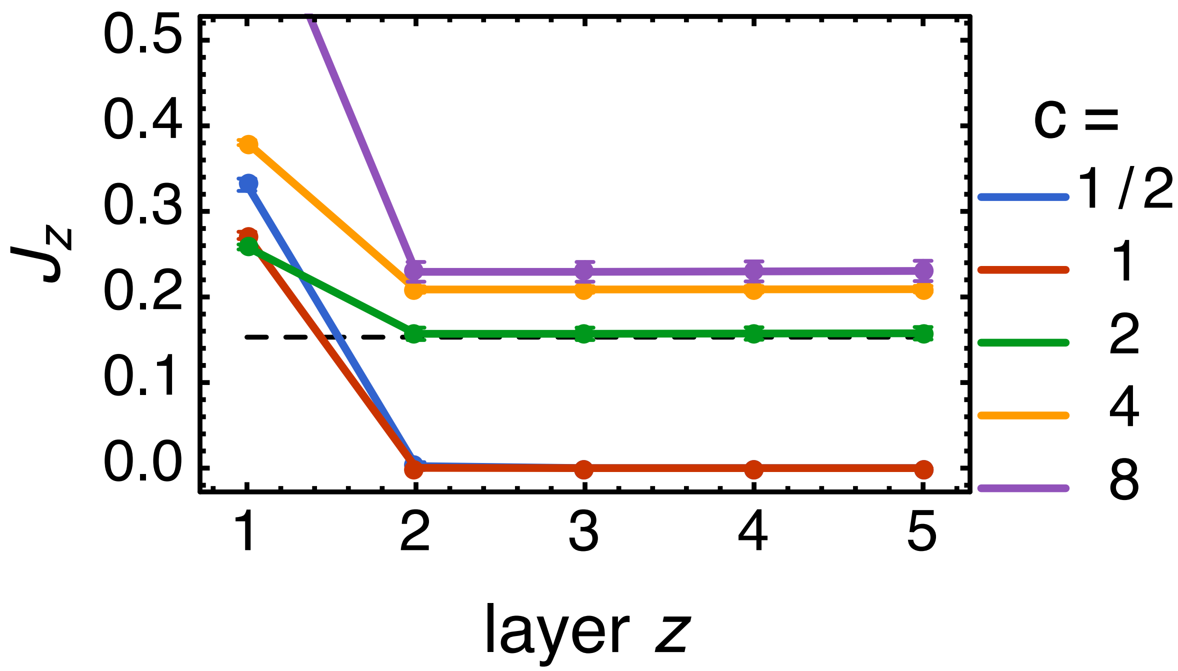


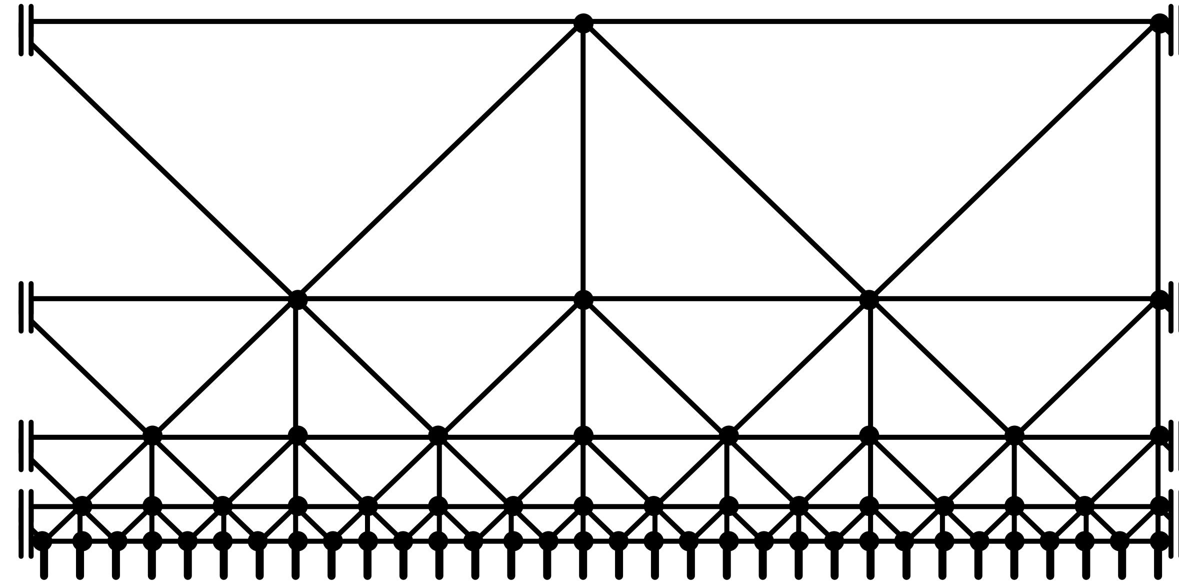
- Two key parameters:
  - Central charge $c = N/2$, counts fermion flavor number
  - Mass m: controls correlation length $\xi = 1/m$
- Entanglement entropy over any (multiple) regions can be calculated efficiently from the fermion Green's function

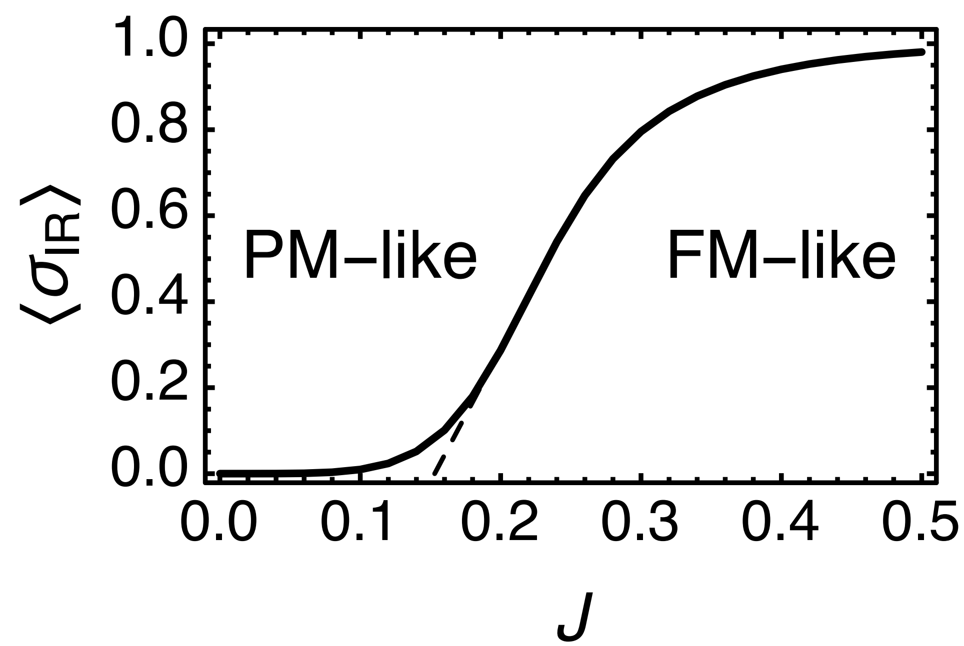
$$G_A = \frac{1}{2}\langle P_A\, \chi\, \chi^{\mathrm{T}}\, P_A\rangle \qquad S_E^{(2)}(A) = -\mathrm{Tr}\,\ln\left(G_A^2 + (1 - G_A)^2\right)$$

Choosing the Central Charge

- Fix mass $m = 0$ (critical fermion chain), hyperbolic network







- Ising coupling J_z develops layer by layer during training
- J_z varies with central charge → a FM-PM crossover in the bulk
- AdS/CFT: $c = 3\,\ell\,/\,(2\,G_N)$
  small c = strong-coupling (quantum) gravity/geometry

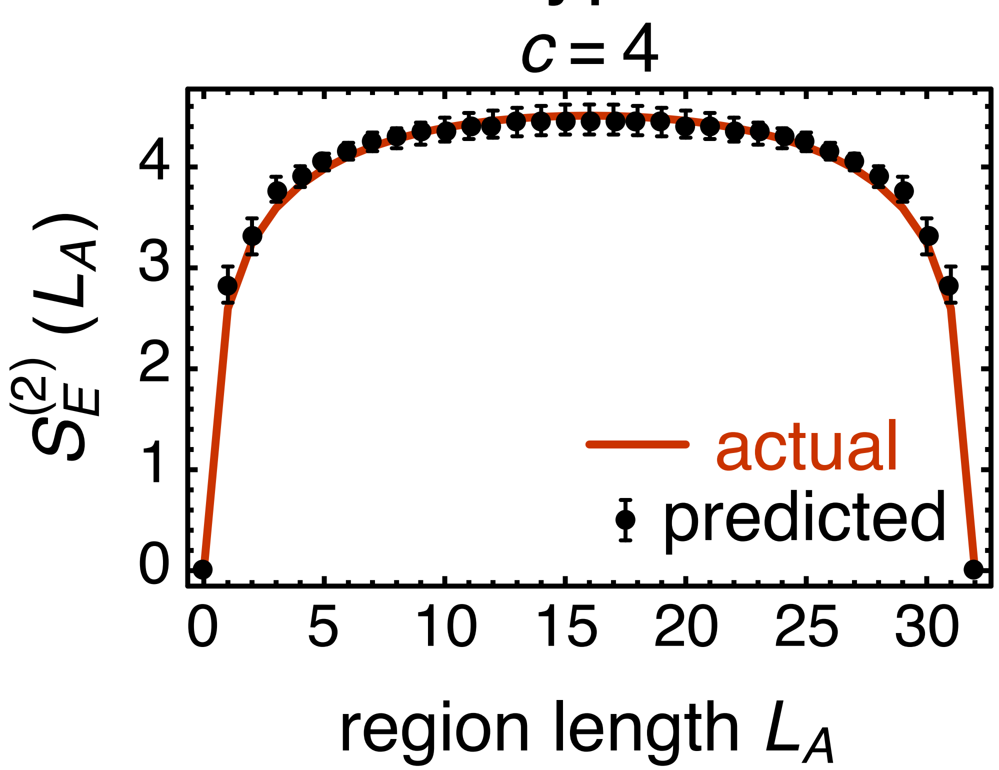
Single-Region Entanglement Entropy

- For critical fermion chain ($m = 0$), the single-region entanglement entropy follows the logarithmic scaling

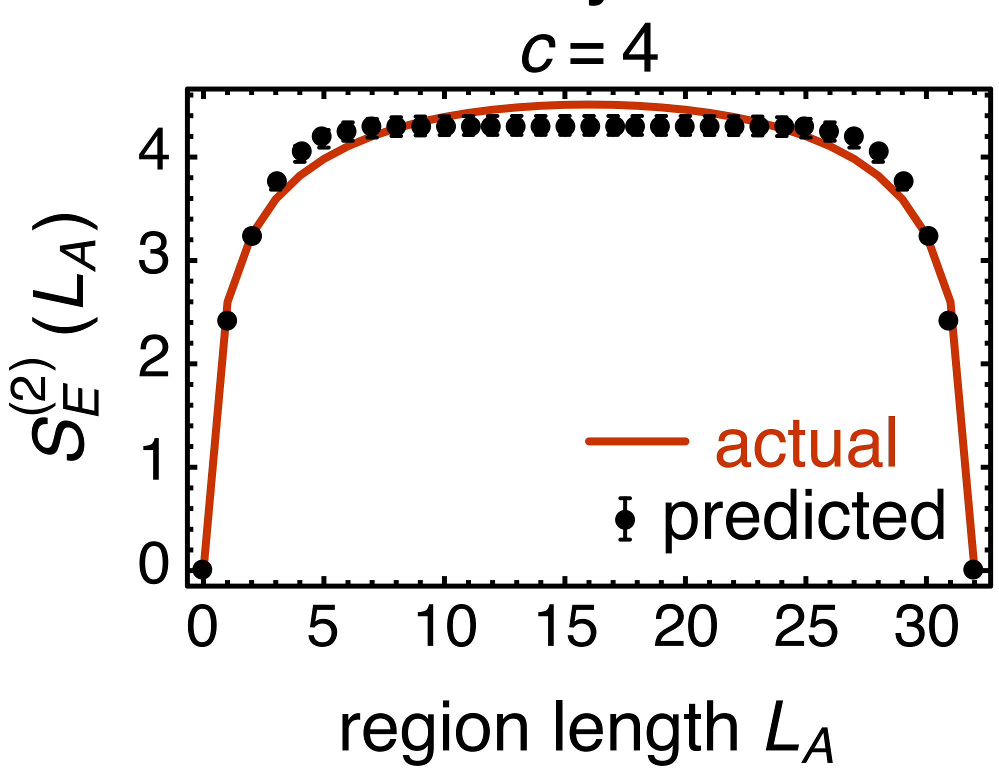
$$S_E^{(n)}(A) = \frac{c}{6}\left(1 + \frac{1}{n}\right) \ln L_A$$

Calabrese, Cardy (2004)

Trained on hyperbolic net



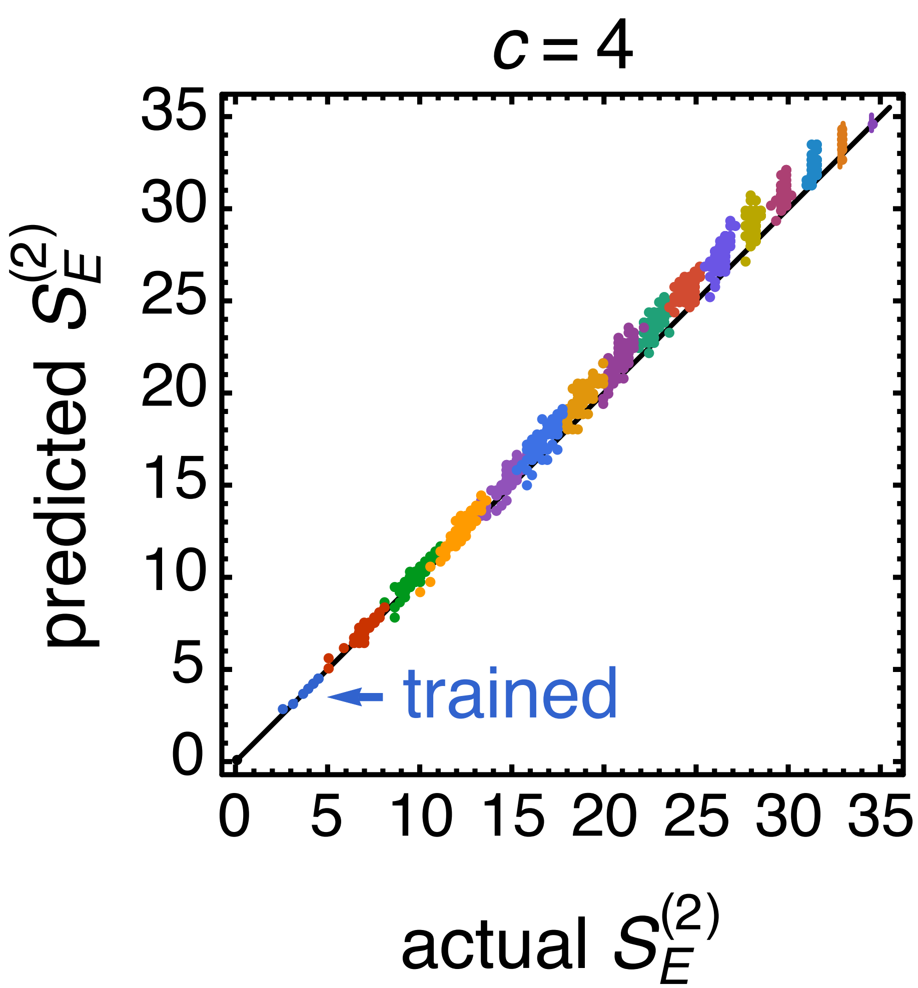
Trained on cylindrical net



- For CFT states, hyperbolic network provides a better fit of the logarithmic scaling of the entanglement entropy.

Multi-Region Entanglement Entropy

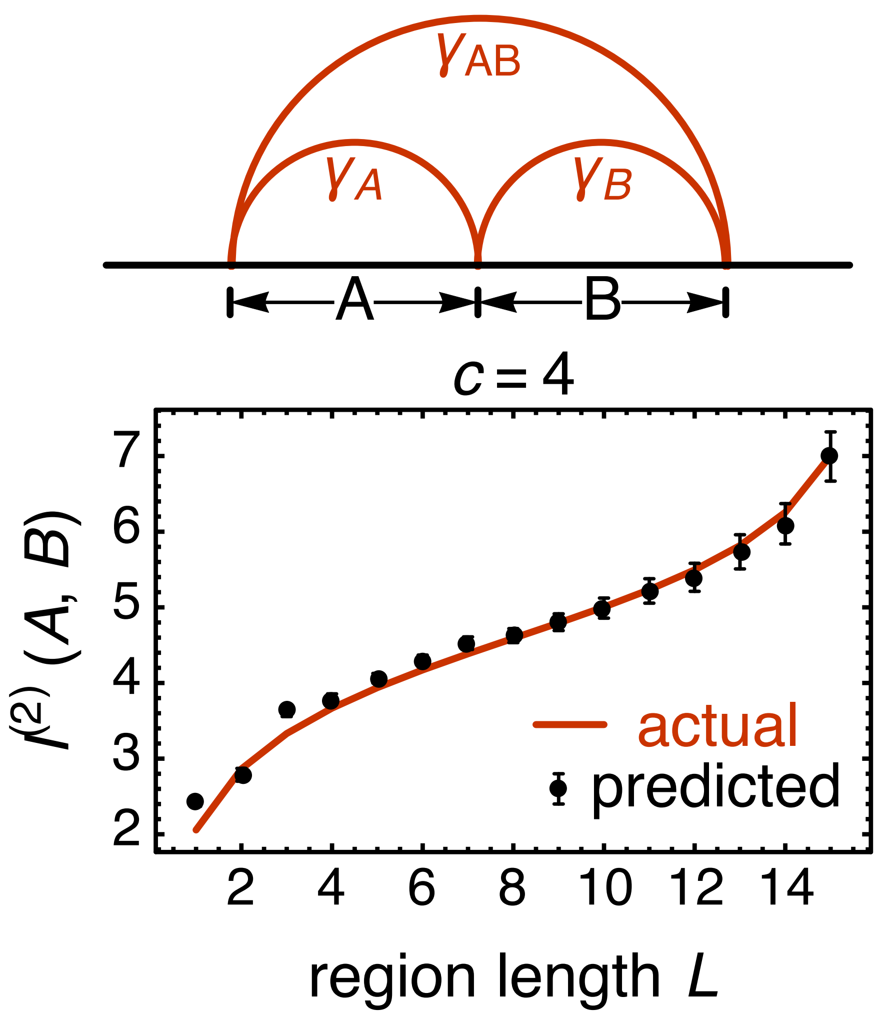
- Train a hyperbolic network using single-region entanglement entropies → Can the network predict multi-region entanglement entropies?
  - Different color - different number of entanglement regions
  - In the training phase, only single-region data is served
  - The trained machine was able to predict multi-region entanglement entropies (which was not in the training set) with accuracy ~ 95%



  - Not too surprising, as $S_E(A \cup B) = S_E(A) + S_E(B) - I(A, B)$
    The additive part is relatively easy to capture.
    What about the sub-additive (mutual information) part?

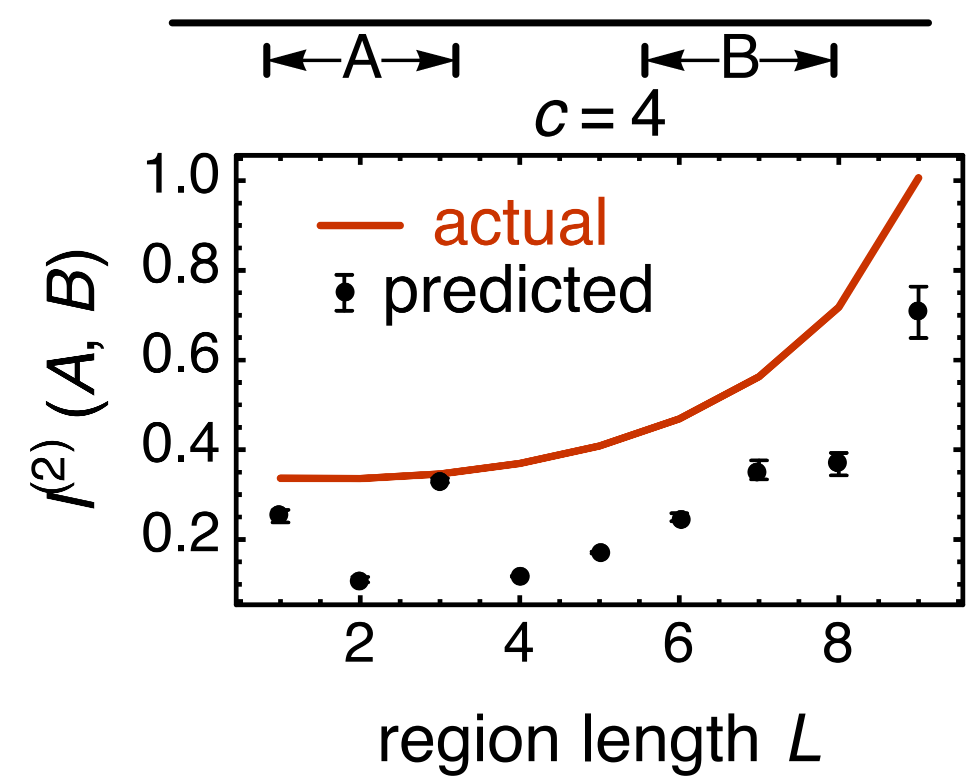
Mutual Information

- Adjacent regions: mutual information still fits well, because there is still a geometric interpretation



- Deep network at work!

- Separated regions: can not provide sufficient mutual information, need additional matter on top of the background geometry…



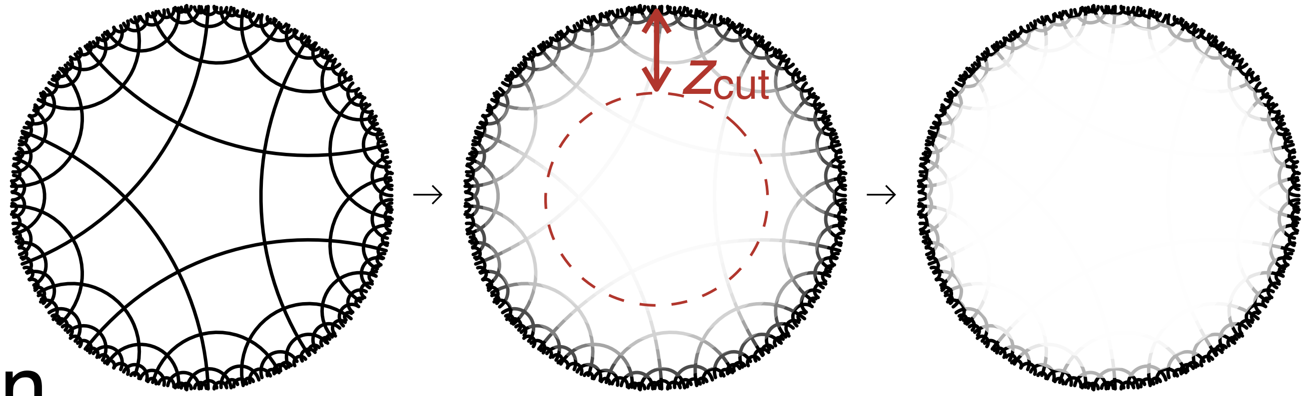
- Learnt the classical bulk geometry, nothing more

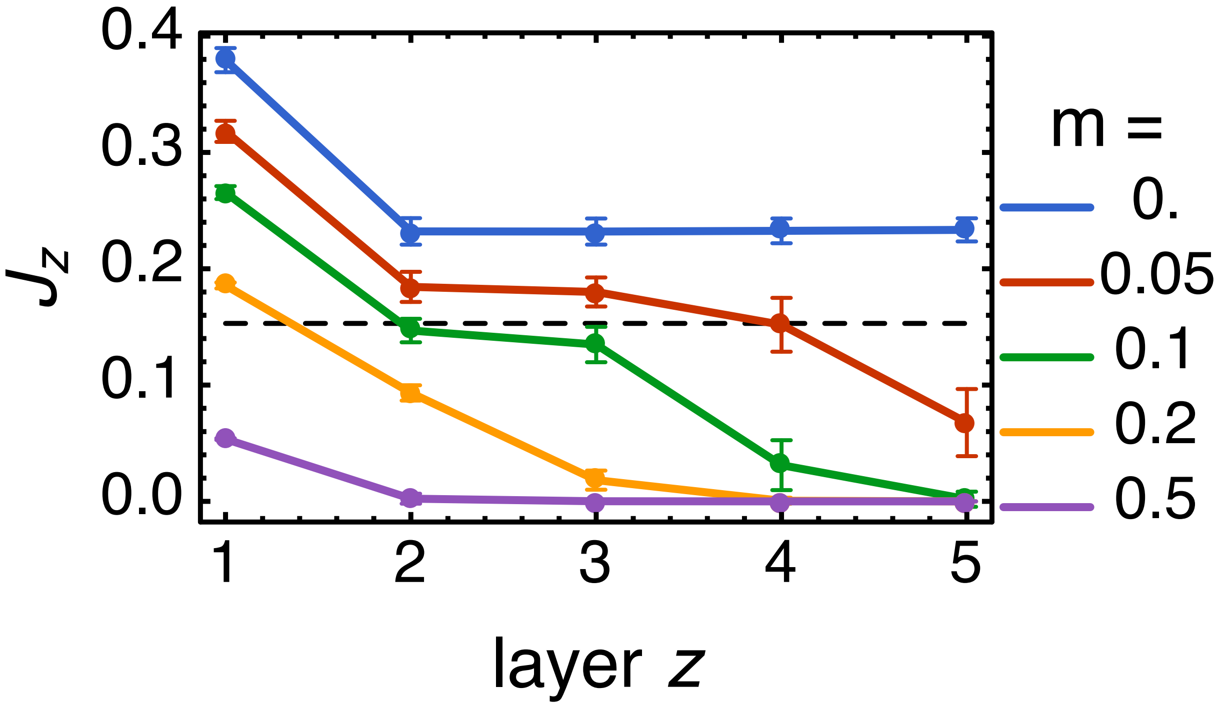
How Deep Could It Be?

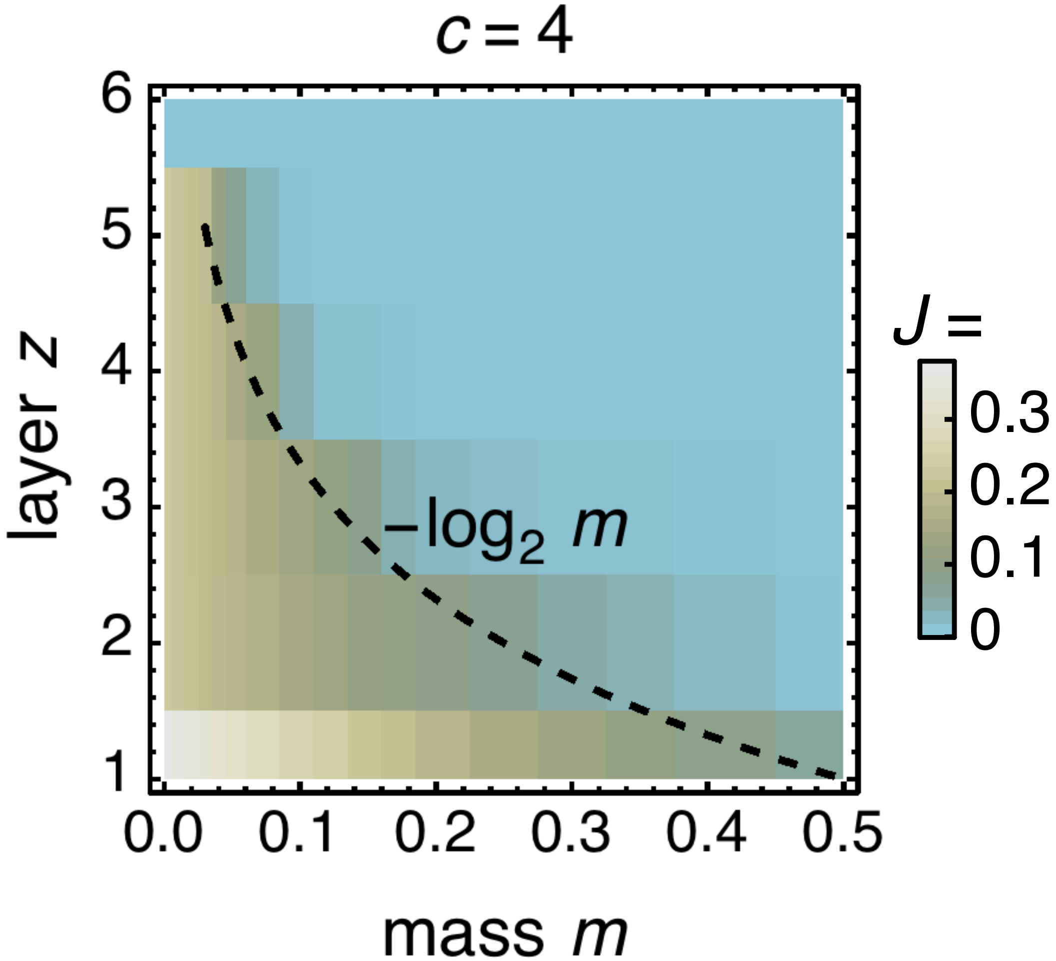
- Turn on the fermion mass *m*, the holographic bulk is cut off at the mass scale (in IR)

$$z_{\text{cut}} \sim \ln \xi = -\ln m$$

  - Deep layers fade away
  - Quantum phase transition of the Majorana chain is signified by the peak of z_{cut}







Summary

- **It from Qubit**: matter/spacetime from quantum information
- Holographic geometry emerges from deep learning the entanglement features in a quantum many-body state.

Entanglement big data inflow from holographic boundary

↓

Entanglement Feature Learning

↓

Emergent geometry develops in the holographic bulk

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Thanks for your attention!