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Quasipotential equations solutions for three dimensional harmonic oscillator in the relativistic configuration representation

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The actual problems of microworld physics, Grodno, Belarus, 12-24, August, 2018

Plan

- Quasipotential equations
- Transformation of integral equations to the Sturm-Liouville problem
- The methods for approximate solution of the Logunov-Tavkhelidze equation
- Numerical solution of quasipotential integral equations in the RCR
- Analysis of results

Quasipotential equations

- Quasipotential equations in the momentum representation (MR):

$$\psi_{(j)}(E_q,p) = -\frac{2m}{\pi} G_{(j)}(E_q,p) \int_0^\infty \frac{dk}{E_k} V(p,k)\psi_{(j)}(E_q,k) \qquad E_k = \sqrt{k^2+m^2}$$

j=1 (j=3) – is the Logunov-Tavkhelidze equation (modified)
j=2 (j=4) – is the Kadyshevsky equation (modified)

$2E_q \geq 2m$ - is the energy of two-particle system

m - is the mass of each particle

The Green functions:

$$G_{(1)}(E_q,p) = \frac{1}{E_p^2 - E_q^2 - i0} \qquad G_{(2)}(E_q,p) = \frac{1}{2E_p(E_p - E_q - i0)}$$

$$G_{(3)}(E_q,p) = \frac{1}{(E_p^2 - E_q^2 - i0)}\frac{E_p}{m} \qquad G_{(4)}(E_q,p) = \frac{1}{2(E_p - E_q - i0)}\frac{1}{m}$$

$$E_p = \sqrt{p^2+m^2}$$

$V(p,k)$ - potential

Quasipotential equations in the relativistic configurational representation (RCR):

$$\psi_{(j)}(r) = \int_0^\infty dr' G_{(j)}(\chi_q, r, r') V(r') \psi_{(j)}(r')$$

r - is the module of radius-vector in the RCR

$\chi_q \geq 0$ - is the parameter related to energy as $2E_q = 2m\,\mathrm{ch}\,\chi_q$

The Green functions in the RCR:

$$G_{(j)}(\chi_q, r, r') = G_{(j)}(\chi_q, r - r') - G_{(j)}(\chi_q, r + r')$$

$$G_{(1)}(\chi_q, r) = \frac{-i}{m\,\mathrm{sh}\,2\chi_q} \frac{\mathrm{sh}\left[\left(\pi/2 + i\chi_q\right) m r\right]}{\mathrm{sh}\left[\pi m r/2\right]} \qquad G_{(2)}(\chi_q, r) = \frac{(4m\,\mathrm{ch}\,\chi_q)^{-1}}{\mathrm{ch}\left[\pi m r/2\right]} - \frac{i}{m\,\mathrm{sh}\,2\chi_q} \frac{\mathrm{sh}\left[\left(\pi + i\chi_q\right) m r\right]}{\mathrm{sh}\left[\pi m r\right]}$$

$$G_{(3)}(\chi_q, r) = \frac{-i}{2m\,\mathrm{sh}\,\chi_q} \frac{\mathrm{ch}\left[\left(\pi/2 + i\chi_q\right) m r\right]}{\mathrm{ch}\left[\pi m r/2\right]} \qquad G_{(4)}(\chi_q, r) = \frac{-i}{2m\,\mathrm{sh}\,\chi_q} \frac{\mathrm{sh}\left[\left(\pi + i\chi_q\right) m r\right]}{\mathrm{sh}\left[\pi m r\right]}$$

$V(r)$ - potential in the RCR

The relationship between the values in the MR and in the RCR:

The values in the RCR and in the MR are interrelated by means of the Shapiro integral transformation, which in the spherically symmetric case is the Fourier transformation

Wave functions:

$$\psi_{(j)}(r) = \frac{2m}{\pi}\int_0^\infty d\chi \sin(\chi m r)\psi_{(j)}(m\,\mathrm{ch}\,\chi_q, m\,\mathrm{sh}\,\chi)$$

$\chi \geq 0$ - is the rapidity related to momentum as $p = m\,\mathrm{sh}\,\chi$

Green functions:

$$G_{(j)}(\chi_q, r, r') = \frac{-2m}{\pi}\int_0^\infty d\chi \sin(\chi m r) G_{(j)}(m\,\mathrm{ch}\,\chi_q, m\,\mathrm{sh}\,\chi)\sin(\chi m r')$$

Potential:

$$V(p,k) = \int_0^\infty dr \sin(\chi m r) V(r) \sin(\chi' m r)$$

The non-relativistic limit of the above formulas and equations leads to the corresponding formulas and equations of the non-relativistic quantum theory.

Transformation of integral equations to the Sturm-Liouville problem

The harmonic oscillator type potential in the RCR :

$$V(r)=\omega^2 r^2$$

The potential in the momentum representation:

$$V(p,k)=-\frac{\pi\omega^2}{2m^3}\left(\sqrt{m^2+p^2}\,\frac{d}{dp}\right)^2\sqrt{m^2+k^2}\,\delta(p-k)=-\frac{\pi\omega^2}{2m^3}\frac{d^2}{d\chi^2}\delta(\chi-\chi')$$

The substitution of the potential in the integral equation in the MR leads to the differential equation (DE):

$$\frac{d^2}{d\chi^2}\psi_{(j)}(\chi_q,\chi)=\frac{m^2}{\omega^2}G_{(j)}^{-1}(m\,\mathrm{ch}\,\chi_q,m\,\mathrm{ch}\,\chi)\psi_{(j)}(\chi_q,\chi)$$

The boundary conditions:

$$\psi(\chi_q,0)=0 \qquad \psi(\chi_q,\chi)\Big|_{\chi\to\infty}\cong 0$$

The Sturm-Liouville problem

The methods for approximate solving of the Logunov-Tavkhelidze equation

The Sturm-Liouville problem for the Logunov-Tavkhelidze equation:

$$\frac{d^2}{d\chi^2}\psi(\chi_q,\chi)=\frac{m^4}{\omega^2}(\mathrm{ch}^2\chi-\mathrm{ch}^2\chi_q)\psi(\chi_q,\chi) \qquad \chi\geq 0$$

$$\psi(\chi_q,0)=0 \qquad \psi(\chi_q,\chi)\big|_{\chi\to\infty}\cong 0$$

The DE solution can be expressed through the modified Mathieu functions. However, the study of such solutions is a cumbersome problem. Let us consider the approximate analytical solution of the Sturm-Liouville problem.

Reduction of the equation to the modified Bessel equation

Let us replace the variable $z=\omega^{-1}m^2\exp(\chi)/2$

$$\left[\left(z\frac{d}{dz}\right)^2-z^2+\frac{m^4}{2\omega^2}\mathrm{ch}\,2\chi_q\right]\psi(\chi_q,z)=\frac{m^8\omega^{-4}}{16z^2}\psi(\chi_q,z) \qquad z\geq\omega^{-1}m^2/2$$

$$\psi(\chi_q,\omega^{-1}m^2/2)=0 \qquad \psi(\chi_q,z)\big|_{z\to\infty}\cong 0$$

In the presented DE, we omit the right-hand side. The modified Bessel functions satisfy the equation obtained in this way.

The second of the boundary conditions holds for the Macdonald function

$K_{i\nu}(z)$, где $\nu=m^2/\omega\sqrt{(1/2)\,\mathrm{ch}\,2\chi_q}$

Taking into account the first boundary condition leads to a transcendental equation for the quantity ν

$$K_{i\nu}\left(\omega^{-1}m^2/2\right)=0$$

which is the condition for quantization of energy.

The approximate solution of the Logunov-Tavkhelidze equation in the MR:

$$\psi(\chi_q^{(n)},\chi)=C_n K_{i\nu_n}\left(\omega^{-1}m^2\exp(\chi)/2\right)$$

$n=1,2,3,\ldots$ - is the state number of the relativistic harmonic oscillator

C_n - is the normalization constant

ν_n - is the root of the transcendental equation, connected with the energy of the relativistic harmonic oscillator by the formula

$$2E_q^{(n)}=\sqrt{2m^2+\left(2\nu_n\omega/m\right)^2}$$

The approximate wave function in the RCR:

$$\psi_n(r)=\frac{C_n}{4i}\left\{\frac{1}{2}\sum_{s=\pm1}s\left(\frac{4\omega}{m^2}\right)^{ismr}\Gamma\left(\frac{ismr-i\nu_n}{2}\right)\Gamma\left(\frac{ismr+i\nu_n}{2}\right)+\right.$$

$$+\left(\frac{4\omega}{m^2}\right)^{i\nu_n}\Gamma(i\nu_n)\sum_{s=\pm1}\frac{1}{is\nu_n-imr}\,{}_1F_2\left(\frac{ismr-i\nu_n}{2};1-i\nu_n,1+\frac{ismr-i\nu_n}{2};\frac{m^4}{16\omega^2}\right)-$$

$$\left.-\left(\frac{4\omega}{m^2}\right)^{-i\nu_n}\Gamma(-i\nu_n)\sum_{s=\pm1}\frac{1}{is\nu_n+imr}\,{}_1F_2\left(\frac{ismr+i\nu_n}{2};1+i\nu_n,1+\frac{ismr+i\nu_n}{2};\frac{m^4}{16\omega^2}\right)\right\}$$

$\Gamma(z)$ – is the gamma function, ${}_1F_2(a;b,c;z)$ – is the generalized hypergeometric series

To find the constants C_n we use the normalization conditions for the wave functions :

<u>in the MR</u>

$$m\int_0^\infty d\chi\,\psi^2(\chi_q^{(n)},\chi)=1$$

<u>in the RCR</u>

$$\int_0^\infty dr\,\psi_n^2(r)=1$$

The advantages of the solving method:

- the possibility of finding the analytical solution: wave functions in the MR and in the RCR

Disadvantages of the solving method:

- the absence of a non-relativistic limit of the results obtained

Solution by the Galerkin method

Performing a change of variable $p = m \operatorname{sh} \chi$, we represent the Sturm-Liouville problem in the form:

$$\frac{\omega^2}{m^2}\left(\sqrt{m^2+p^2}\,\frac{d}{dp}\right)^2 \psi(\chi_q, p) = \left(p^2 - m^2 \operatorname{sh}^2 \chi_q\right)\psi(\chi_q, p)$$

$$\psi(\chi_q, 0) = 0 \qquad \psi(\chi_q, p)\big|_{p\to\infty} \cong 0$$

We represent the unknown wave function as the sum:

$$\psi(\chi_q, p) = \sum_{s=0}^{N} C_s \varphi_s(\chi_q, p)$$

C_s – is the unknown coefficients

$\varphi_s(\chi_q, p)$ - is the exact solutions of the Sturm-Liouville problem for the equation

$$\omega^2 \frac{d^2}{dp^2}\varphi_s(\chi_q, p) = \left(p^2 - \lambda_s\right)\varphi_s(\chi_q, p)$$

The number of terms N in the sum depends on the required accuracy of the solution obtained.

The functions $\varphi_s(p)$ have the form analogous to the wave functions of a non-relativistic harmonic oscillator:

$$\varphi_s(p) = \frac{1/\sqrt{\omega}}{\left[2^{2s}(2s+1)!\sqrt{\pi}\right]^{1/2}} \exp\left(-\frac{1}{2\omega}p^2\right) H_{2s+1}\left(p/\sqrt{\omega}\right)$$

$H_n(x)$ - is the Hermite polynomials

The corresponding eigenvalues are defined as $\lambda_s = \omega(3+4s)$

After substituting the sum into the equation, we obtain the following equality:

$$\left(\frac{\omega}{m}\right)^2 \sum_{s=0}^{N} C_s \left(p\frac{d}{dp}\right)^2 \varphi_s(p) - \sum_{s=0}^{N} C_s \lambda_s \varphi_s(p) = -q^2 \sum_{s=0}^{N} C_s \varphi_s(p)$$

Multiplying the resulting equality by the function $\varphi_n(p)$, and integrating the resulting equality from zero to infinity, we obtain the linear system of $\mathrm{N}+1$ equations

$$MC = E_q^2 C$$

C – is the vector composed of unknown coefficients

M – is the five-diagonal matrix whose elements have the form

$$M_{ns} = [\lambda_n + 1]\delta_{n,s} + \left(\frac{\omega}{m}\right)^2 \left[a_n^2 \delta_{ns} + b_s \delta_{n+1s} - b_n \delta_{ns+1} - c_s \delta_{n+2s} - c_n \delta_{ns+2}\right]$$

δ_{ns} - are the elements of the identity matrix

$$a_n^2 = \frac{1}{4}\left(8n^2 + 12n + 5\right) \qquad b_n = \frac{1}{2}\sqrt{2n(2n+1)} \qquad c_n = \frac{1}{4}\sqrt{(2n-2)(2n-1)2n(2n+1)}$$

We used recurrence relations for the Hermite polynomials when calculating the matrix elements

$$2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x) \qquad \frac{d}{dx}H_n(x) = 2nH_{n-1}(x)$$

The advantages of the solving method:

- the possibility to calculate quickly a large number of energy values simultaneously

Disadvantages of the solving method:

- the need for cumbersome preliminary analytical calculations

Numerical solving quasipotential integral equations in the RCR

The solution was found by the method that we used to study resonant states earlier on the basis covariant two-particle integral equations in the RCR

Using quadrature formulas we replace integrals in the equations by the sums. As the results we obtain homogeneously systems of linear algebraic equations

$$M\psi = 0 \qquad\qquad M_{nm} = \delta_{nm} - W_m G_l^{(j)}(\chi_q, r_n, r_m) V(r_m)$$

W_n, r_n – are the coefficients and nodes of the quadrature formula

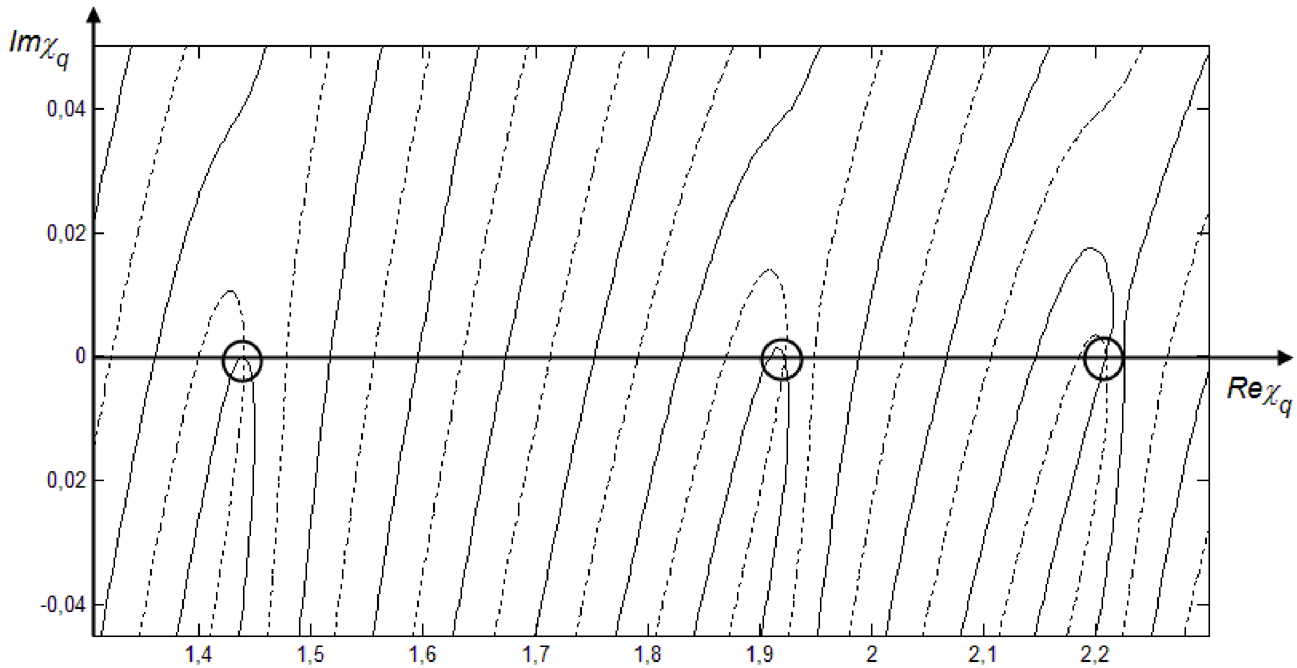
ψ – is the vector of wave functions in the nodes

The condition for existence of nontrivial system solution

$f(\chi_q) = \det M = 0$ - the energy quantization conditions

It is advisable to represent roots of equation graphically on the complex plane χ_q .

In the case of the potential under consideration the roots are located on the real axis.



The advantages of the solving method:

- the possibility to apply this method to solve various equations with a wide class of potentials, in the case of bound states and resonant states

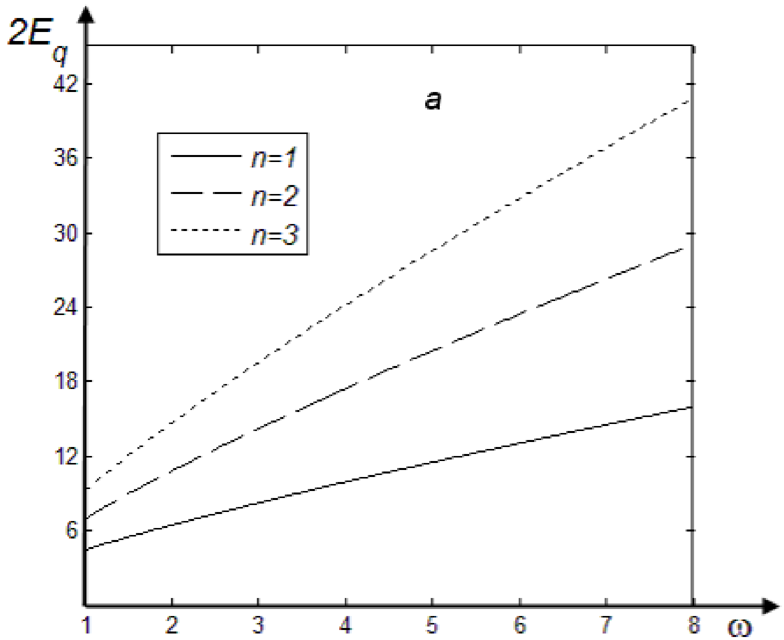
Disadvantages of the solving method:

- low computing speed

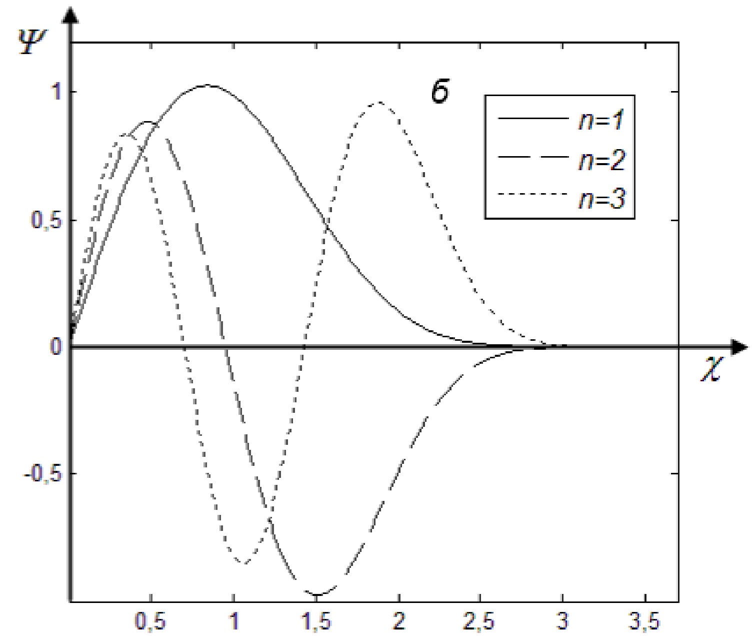
Energy eigenvalues of the relativistic harmonic oscillator

State number n	Numerical solution of DE in the MR by Numerov's method	Solution of the modified Bessel equation in the MR	Solution by Galerkin's method in the MR	Solution of integral equation in the RCR
ω=0,5				
1	3,3266856	3,2827875	3,3266856	3,3266856
2	4,7745776	4,7499059	4,7745776	4,7745776
3	6,0647951	6,0475481	6,0647951	6,0647951
4	7,2602146	7,2469130	7,2602147	7,2602147
5	8,3899913	8,3791426	8,3899916	8.3899916
ω=1				
1	4,4575153	4,4340641	4,4575153	4,4575153
2	6,9820604	6,9688641	6,9820604	6,9820604
3	9,2137508	9,2045646	9,2137508	9,2137508
4	11,2795927	11,2725341	11,2795928	11,2795928
5	13,2337174	13.2279780	13,2337177	13,2337177
ω=4				
1	9,9176430	9,9125166	9,9176452	9,9176430
2	17,4372346	17,4338824	17,4372462	17,4372346
3	24,1520149	24,1496003	24,1521891	24,1520149
4	30,4214462	30,4195702	30,4220768	30,4214463
5	36,3935674	36,3920360	36,3983593	36,3935676
ω=10				
1	18,7458897	18,7441887	18,7470688	18,7458718
2	34,3783383	34,3770590	34,3833417	34,3783383
3	48,5435979	48,5426370	48,5820152	48,5435979
4	61,8714963	61,8707376	61,9781868	61,8714966
5	74,6343046	74,6336807	75,0707392	74,6343053

- energy levels are not equidistant;
- the accuracy of the solutions by the method of reduction to the modified Bessel equation is <u>improved</u> with increasing of coupling constant ω;
- the accuracy of the solutions by the Galerkin's method is <u>worsens</u> with increasing of coupling constant ω



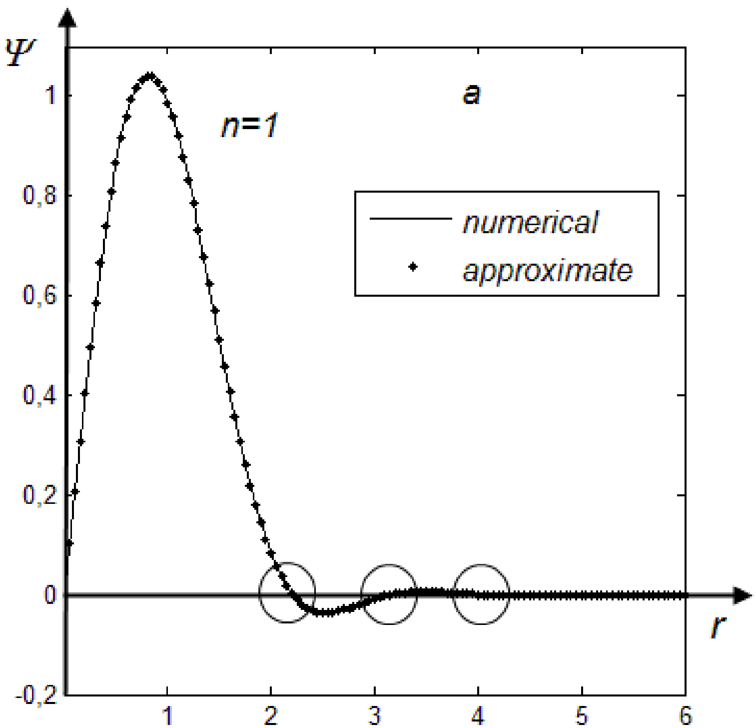
The dependence of energy on the value of coupling constant at $m=1$



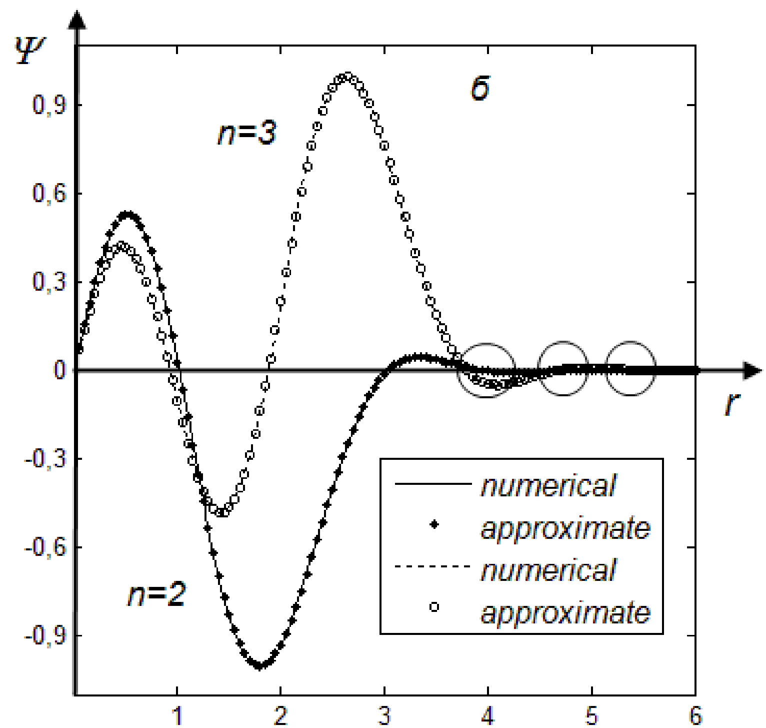
The wave functions in the momentum representation at $m=1$ $\omega=1$

- the dependence of energy on the value of coupling constant ω in this interval is almost linear;
- the graphics of the approach wave function are indistinguishable visually from numerical ones for the indicated quantities m и ω;
- the number of wave functions zeros in the MR is equal to state number of relativistic harmonic oscillator

The wave functions in the RCR at $m=1$, $\omega=5$



The wave functions of ground state



The wave functions of second and third states

- the graphics of the approach wave function are indistinguishable visually from numerical ones for the indicated quantities m и ω;
- the wave functions in the RCR have additional zeros in comparison with the wave functions in the MR and the wave functions of non-relativistic harmonic oscillator

Conclusions and results

- the solutions of the quasipotential equations for harmonic oscillator are found in the spherically symmetric case;
- the Logunov-Tavkhelidze equation in the momentum representation was transformed to the Sturm-Liouville problem. The approximate analytical and numerical solutions of this problem were found;
- the obtained wave functions in the RCR have additional zeros in comparison with corresponding wave functions in the MR and the wave functions of non-relativistic harmonic oscillator. The number of zeros for the fixed quantum state depends on value of coupling constant of relativistic harmonic oscillator.

Thank you for your attention!