Instantons in Quantum Mechanics and Quantum Field Theory

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INTRODUCTION

Instantons are nontrivial solutions of **Euclidean** equations of motion with finite action

Instantons in classical theory

- Instantons are nontrivial solutions of classical Euclidean equations of motion with finite action

- Usually Euclidean equations of motion are obtained by Wick rotation or $t \to -i\tau$

Instantons in classical theory

- QM can be considered as QFT in 0+1 dimension

- Instantons in (Euclidean) D dimensions space can be considered as **static** solitons in (Minkovski) D+1 space-time; (Euclidean) action has sense of energy

Instantons in Quantum Mechanics

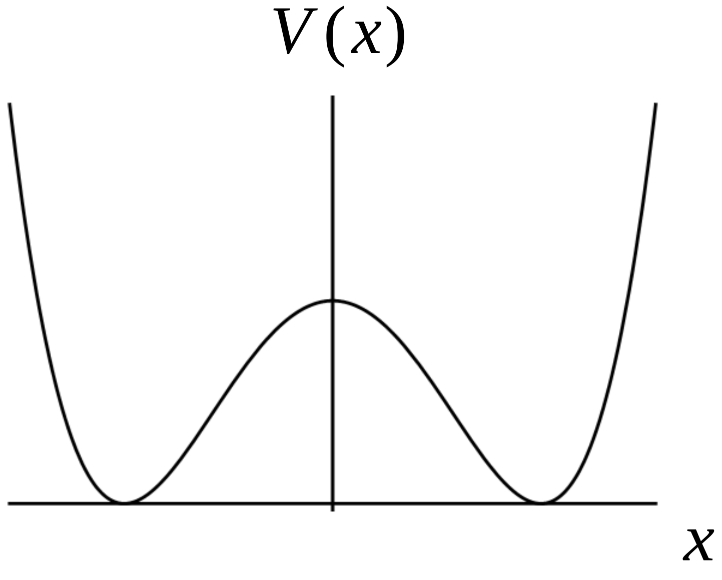
Quantum mechanical instantons

The simplest instantons arise in the one-dimensional problem of a particle in a potential, such as

- Double-well potential
- Periodical potential

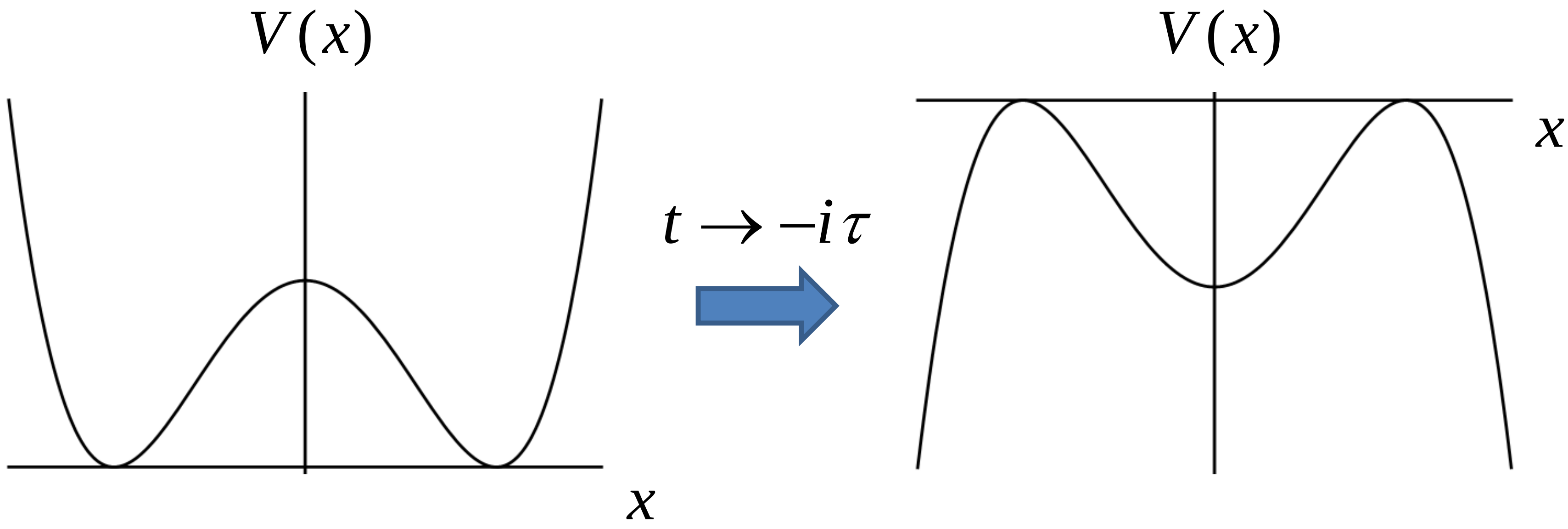
Quantum mechanical instantons. Double-well potential

$$L = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - V(x), \quad V(x) = \lambda(x^2 - \rho^2)^2$$



Quantum mechanical instantons. Double-well potential

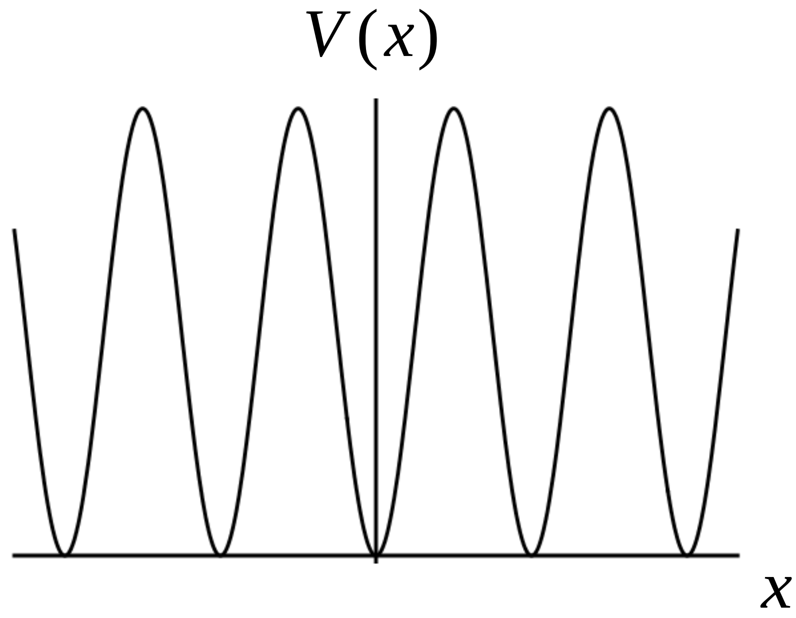
$$L = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - V(x), \quad V(x) = \lambda (x^2 - \rho^2)^2$$



Quantum mechanical instantons.

Periodical potential

$$L = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \lambda\left(1 - \cos(\rho x)\right)$$

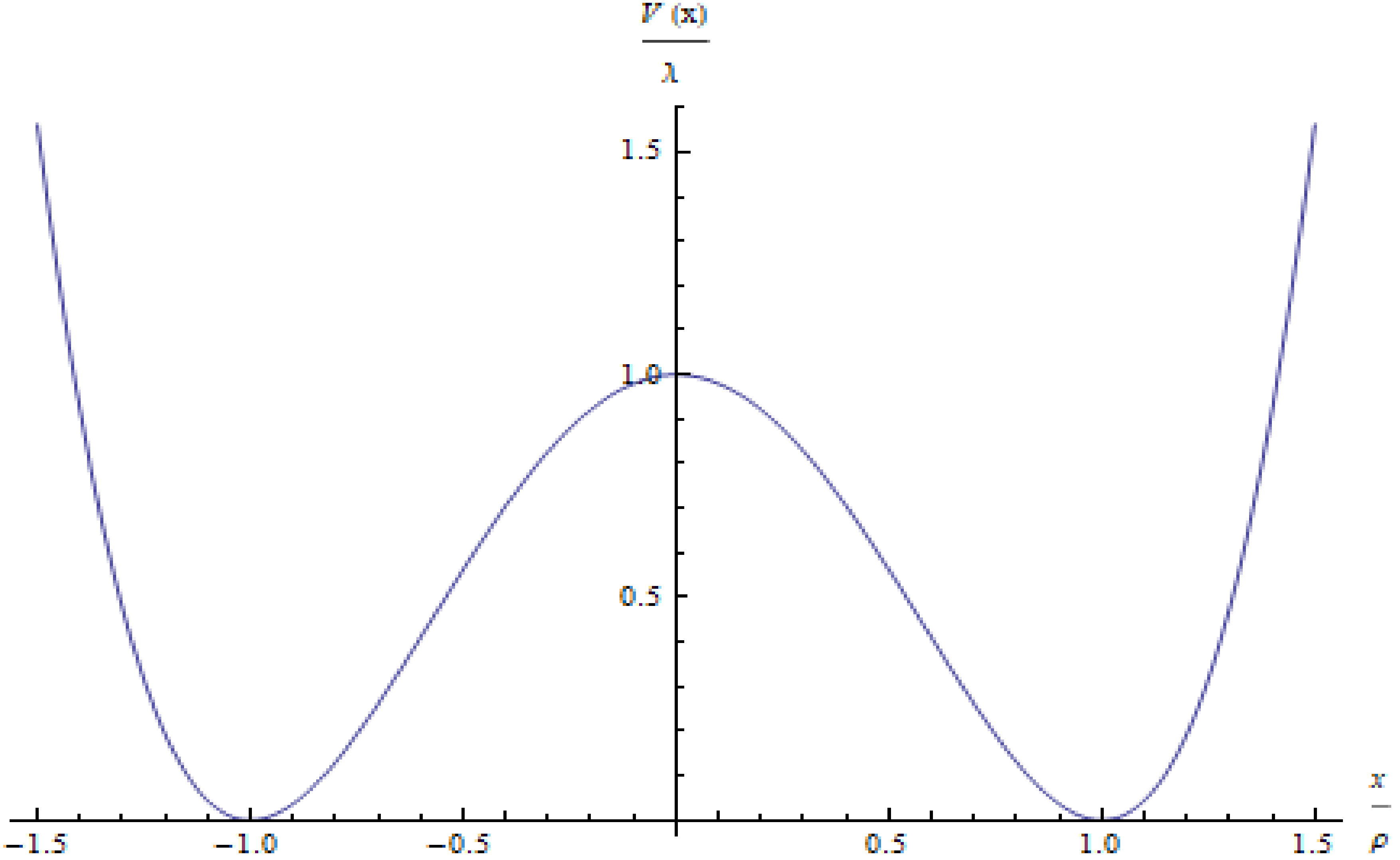


Particle in a ring

$$L = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - V(x) \qquad x \in [0, l] \qquad V(x + l) = V(x)$$

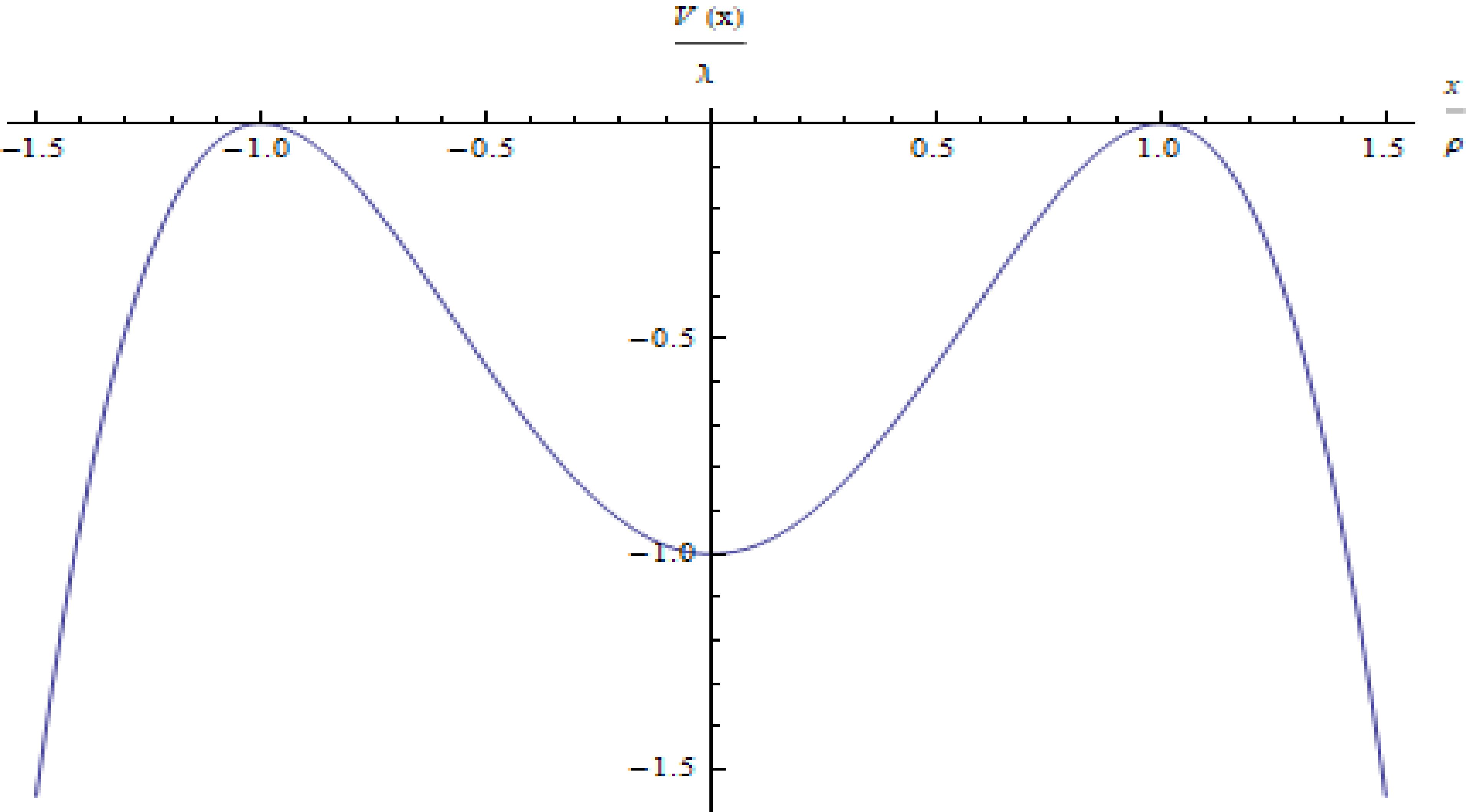
Quantum mechanical example

$$V(x) = \lambda(x^2 - \rho^2)^2$$



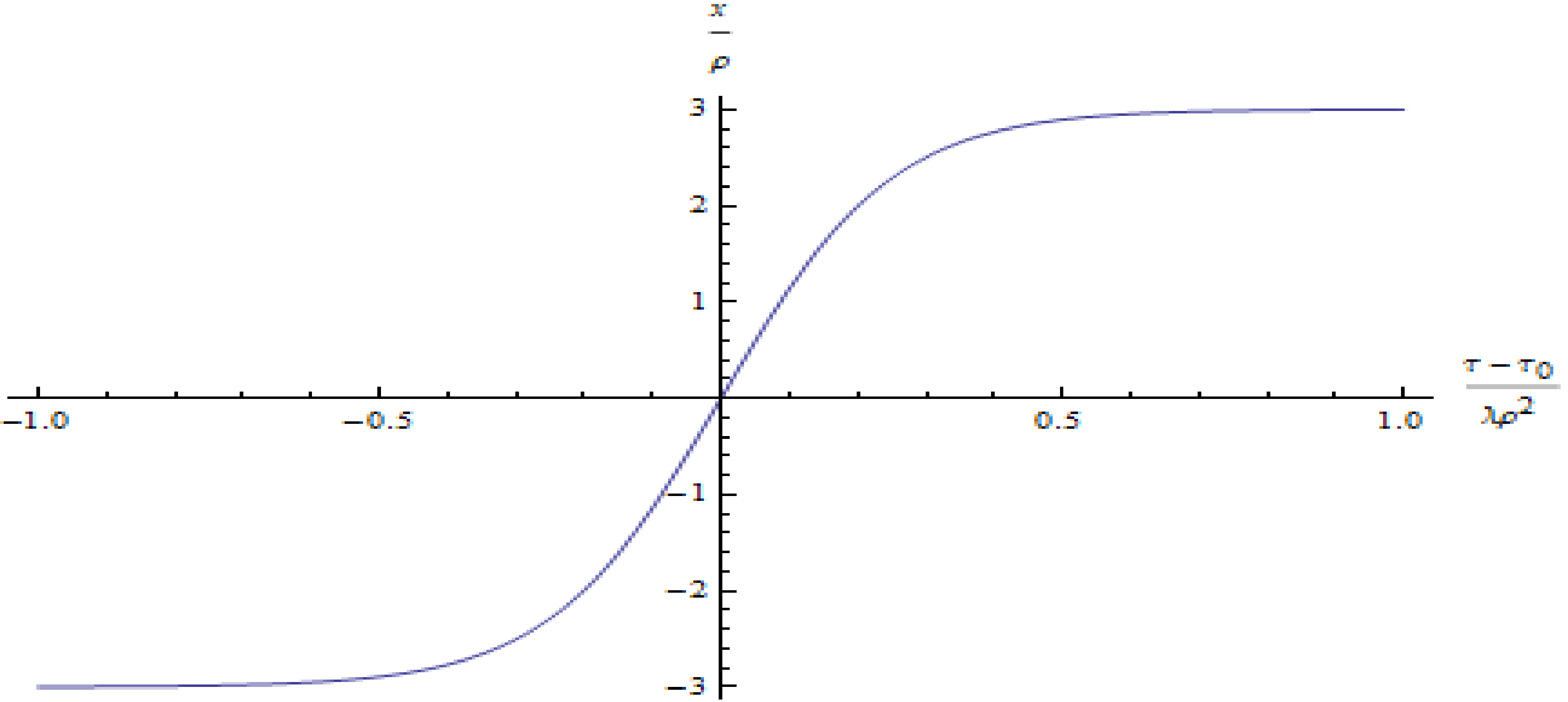
After Wick rotation we get inverted potential

$$t \to -i\tau \qquad V_E(x) = -V(x)$$



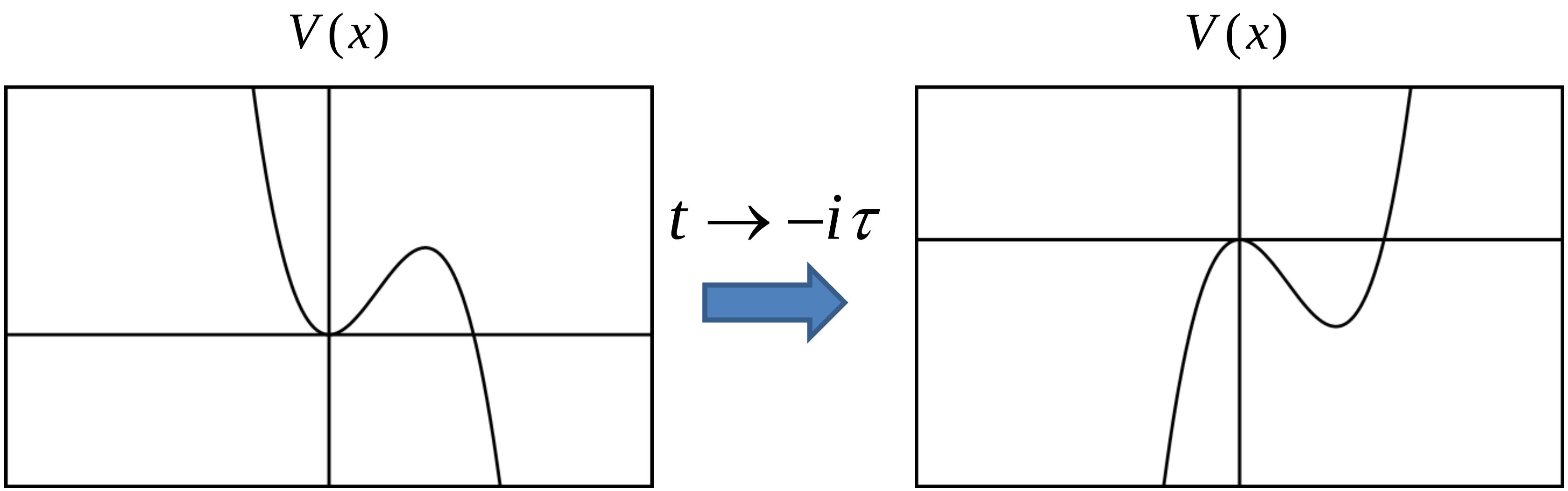
In this model, there is exact analytical Euclidean solution of the following form

$$x^{inst}(\tau) = \rho \tanh \frac{8\lambda\rho^2(\tau - \tau_0)}{2}$$



Quantum mechanical instantons.

Decay of a metastable state

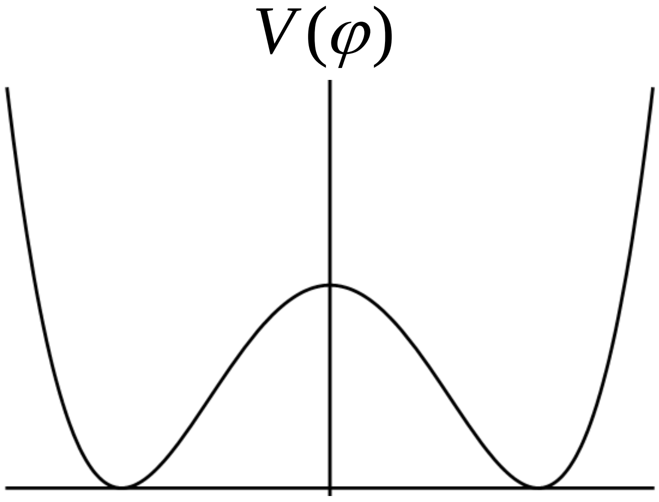


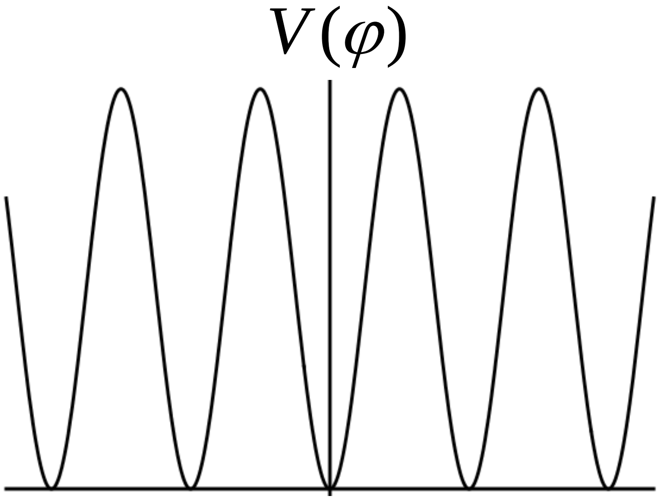
Instantons in quantum field theory

Scalar field theories with (classically) degenerate vacuum

$$L = \frac{1}{2}\partial_\mu \varphi \, \partial_\mu \varphi - V(\varphi)$$

$$V(\varphi) = \lambda(\varphi^2 - \rho^2)^2 \qquad V(\varphi) = \lambda\left(1 - \cos(\rho\varphi)\right)$$





Scalar field theories with (classically) degenerate vacuum

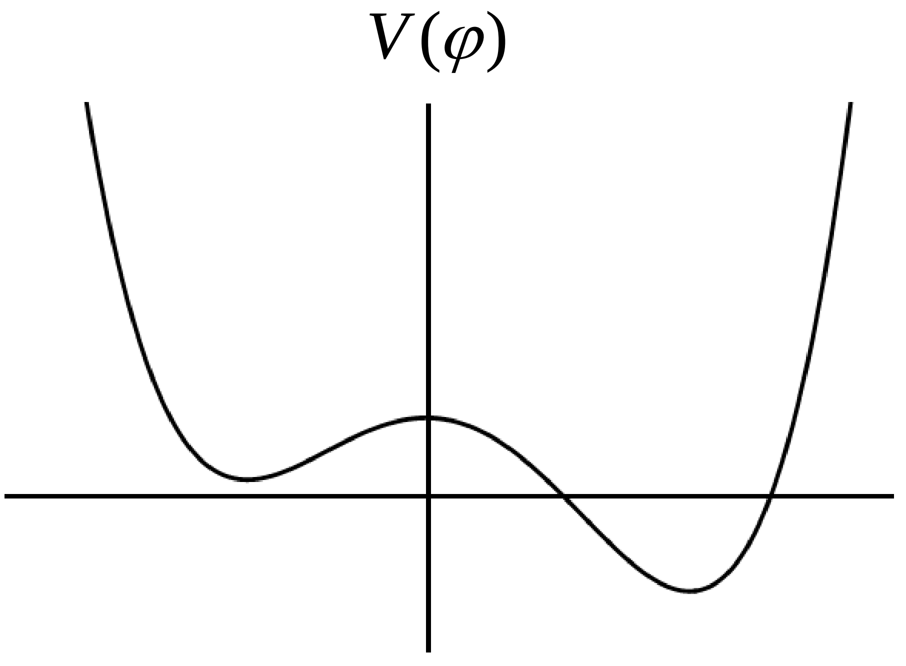
By Derrick's theorem, **there are no instantons in these theories**.

Physically this prohibition is due to the fact that vacuum tunneling transitions are impossible because of the infinite magnitude of the energy barrier between neighboring vacuums (since considered spatial region is infinite).

[G.H. Derrick, J. Math. Phys. 5, 1252 (1964)

R. Hobart. Proc. Phys. Soc. 82, 201 (1963)]

False vacuum decay



Tunneling can be described by using classical solutions ("bounces"),

$$\Gamma \propto e^{-2S_E[\varphi_b(x,\tau)]}$$

Instantons in quantum field theory

Gauge theories

2-dimensional Abelian Higgs model

$$L = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2} D_\mu \varphi \, D_\mu \varphi - \lambda(\varphi\varphi^* - \rho^2), \qquad D_\mu = \partial_\mu - ieA_\mu$$

Varieties of vacua (pure gauge)

$$A_x = \frac{1}{e}\frac{d\alpha(x)}{dx}, \quad \varphi = \rho \; e^{i\alpha(x)}, \quad A_0 = 0.$$

splits in the quantum case into a discrete number of classes. They are bounded by Nielsen-Olesen vortices.

BPST-instanton

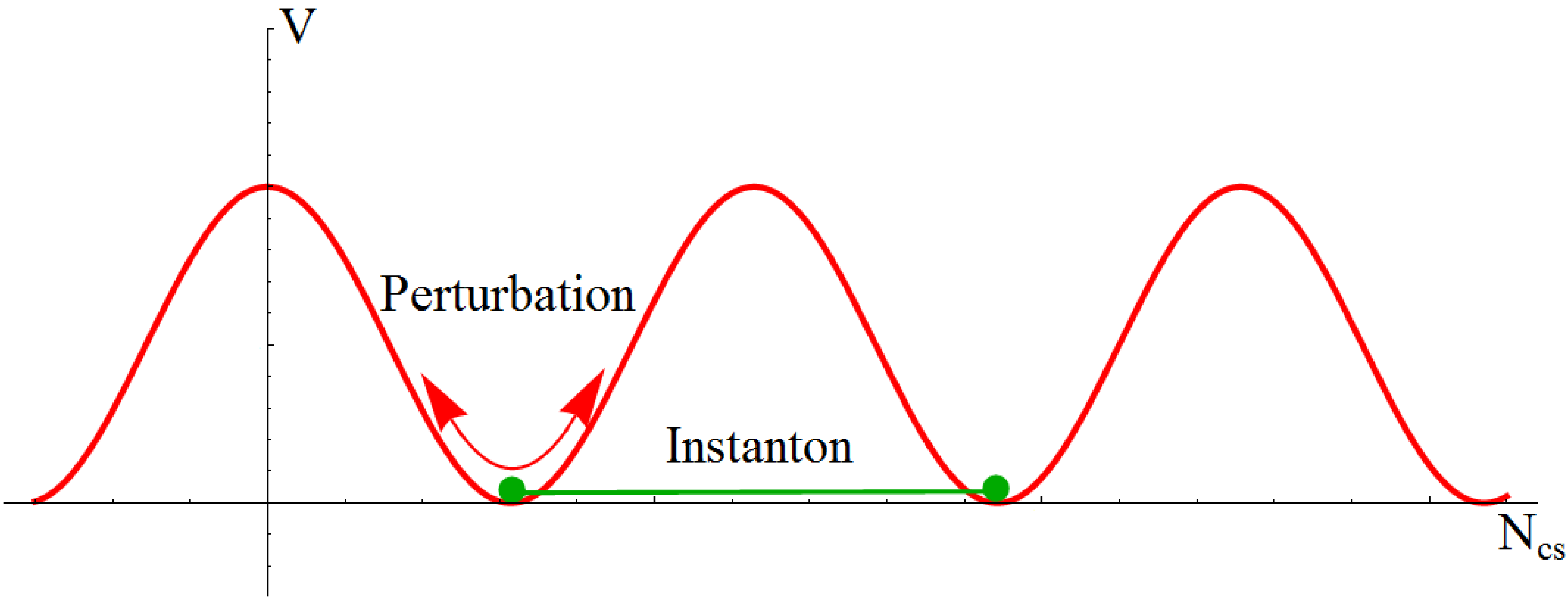
Instantons were found in 1975 [1]:

$$A_{\mu}^{a\,(ins)} = \frac{2}{g}\eta_{a\mu\nu}\frac{(x - x_0)_\nu}{(x - x_0)^2 + \rho^2}$$

$$S_E = \frac{2\pi}{\alpha_s}\left|Q\right|,$$

[A. Belavin, A. Polyakov, A. Schwarz, and Yu. Tyupkin, Phys. Lett. 59B, 85 (1975).]

Vacuum QCD



$$N_{cs} = \frac{g^2}{16\pi^2} \int d^3x \;\; \varepsilon_{ijk} \left(A_i^a \partial_j A_k^a + \frac{g}{3} \varepsilon^{abc} A_i^a A_j^b A_k^c \right)$$

[R. Jackiw, C. Rebbi, Phys. Lett. 37, 172 (1976)]

The experimental status of instantons

Instanton processes are strongly suppressed in physically meaningful gauge theories:

$$e^{-\frac{4\pi}{\alpha}} \approx \begin{cases} 10^{-160}, EWT \\ 10^{-10}, QCD \end{cases}$$

but can become observable at high energies or at high temperatures.

Instantons induce a violation of the baryon and lepton number, what may be bounded with the problem of the asymmetry of matter and antimatter in visible part of the Universe.

Electroweak theory

Instantons induce a violation of the baryon and lepton number, what may be bounded with the problem of the asymmetry of matter and antimatter in visible part of the Universe

Quantum chromodynamics

Instantons cause processes with non-conservation of chirality

Instantons and vacuum structure in quantum chromodynamics

In singular gauge instanton solution is given by

$$A_\mu^a = \frac{2}{g_s} \frac{\bar{\eta}^a{}_{\mu\nu}(x - x_0)_\nu \rho}{(x - x_0)^2((x - x_0)^2 + \rho^2)},$$

where $\bar{\eta}^a{}_{\mu\nu}$ is t'Hooft symbol. This solution have finite action

$$S_E = \frac{2\pi}{\alpha_s}|Q|,$$

where Q is topological charge of instanton.

Instantons and vacuum structure in quantum chromodynamics

Instantons leads to a specific multiquark t'Hooft vertex.

It can be described (for $\rho \to 0,\ N_c = N_f = 3$) by effective Lagrangian

$$L_{eff}^{(3)} = \int d\rho \quad n(\rho)\{ \prod_{i=u,d,s} (m_i \rho - \frac{4\pi}{3}\rho^3) +$$

$$+\frac{3}{32}(\frac{4}{3}\pi^2\rho^3)^2[(j_u^a j_d^a - \frac{3}{4} j_{u\mu\nu}^a j_{d\mu\nu}^a)(m_s\rho - \frac{4}{3}\pi^2\rho^3 \bar{q}_{SR} q_{sL}) +$$

$$+\frac{9}{40}(\frac{4}{3}\pi^2\rho^3)^2 d^{abc} j_{u\mu\nu}^a j_{d\mu\nu} j_s^c + 2\,perm.] + \frac{9}{320}(\frac{4}{3}\pi^2\rho^3)^3 d^{abc} j_u^a j_d^b j_s^c +$$

$$+\frac{igf^{abc}}{256}(\frac{4}{3}\pi^2\rho^3)^3 j_{u\mu\nu}^a j_{d\nu\lambda}^b j_{s\lambda\mu}^c + (R \leftrightarrow L)\}$$

Instantons and vacuum structure in quantum chromodynamics

Where

$$q_{R,L} = (1 \pm \gamma_5)/2q(x), \quad j_i^a = \bar{q}_{iR}\lambda^a q_{iL}, \quad j_{i\mu\nu}^a = \bar{q}_{iR}\sigma_{\mu\nu}\lambda^a q_{iL},$$

$n(\rho)$ is instanton density.

For massless quarks and number of flavors $N_f = 2$ structure of effective Lagrangian is significantly reduced:

$$L_{eff}^{(3)} = \int d\rho \quad n(\rho)\left(\frac{4}{3}\pi^2\rho^3\right)^2 \times$$

$$\left\{\bar{u}_R u_L \bar{d}_R d_L \left[1 + \frac{3}{32}\left(1 - \frac{3}{4}\sigma_{\mu\nu}^u \sigma_{\mu\nu}^\delta\right)\lambda_u^a \lambda_d^a\right] + (R \leftrightarrow L)\right\}$$

Instantons and vacuum structure in quantum chromodynamics

Previous Lagrangians are obtained from the consideration of quark scattering by so-called zero mode in the instanton field. The quark zero mode was found by t'Hooft, who showed that the Dirac equation

$$\left(i\partial_\mu + g\frac{\lambda^a}{2} A_\mu^a(x) \right) \Psi_n(x) = \varepsilon_n \Psi_n(x)$$

in instanton field has a solution with zero energy ($\varepsilon_0 = 0$)

$$\Psi_0(x - x_0) = \frac{\rho(1-\gamma_5)}{2\pi((x-x_0)^2 + \rho^2)^{3/2}} \frac{\hat{x}}{\sqrt{x^2}} \varphi$$

with φ is two-component spinor $\varphi_m^a = \varepsilon_m^a / \sqrt{2}$.

Instantons and vacuum structure in quantum chromodynamics

Let us note some features of function $\Psi_0(x - x_0)$:

• instanton zero modes have a certain helicity;

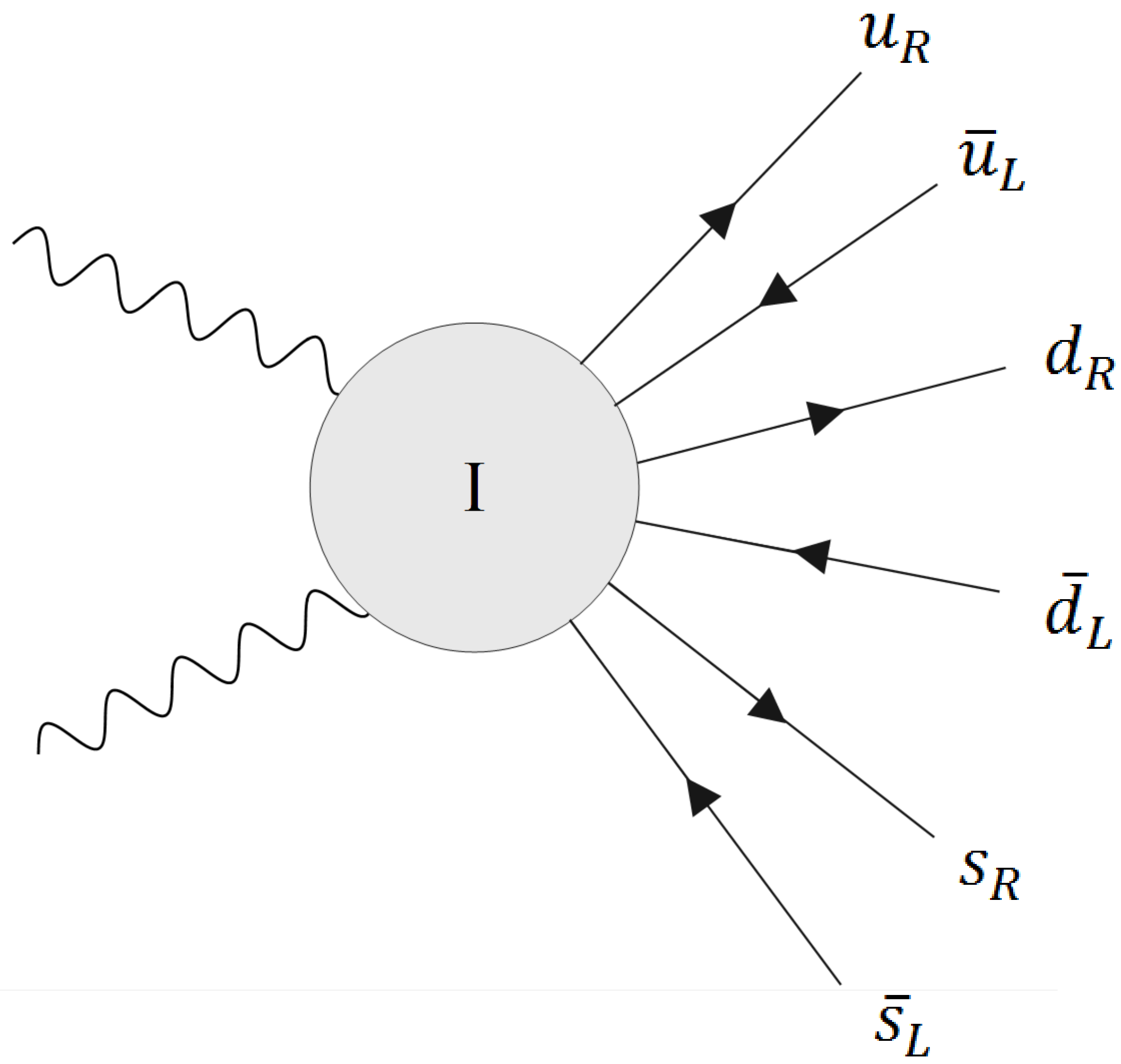
•at zero mode, the values of the color of the quark and its spin are strictly correlated through the spinor, so that their sum is zero.

Consequently,

•only one quark of a certain flavor can be in zero mode

•the helicity of the quark is reversed upon scattering by an instanton.

Chirality violation



Tunneling and confinement in scalar field theories

As was said earlier, there are no instantons in scalar theories.

However, this obstacle can be avoided by considering a system in a limited spatial volume.

2-dimensional sine-Gordon model

$$L = \frac{1}{2}\partial_\mu \varphi \, \partial_\mu \varphi - V(\varphi)$$

$$V(\varphi) = \lambda(1 - \cos(\rho\varphi)), \qquad \mu = 0,1, \qquad -l/2 \le x \le l/2$$

The instanton solution for this system is given by

$$\varphi^{inst}(\tau, x) = \pm \frac{4}{\rho} \arctan\left(e^{(\tau - \tau_0)\rho\sqrt{\lambda}}\right)$$

Euclidean action of this solution equal

$$S[\varphi^{inst}(\tau, x)] = \frac{8\sqrt{\lambda} l}{\rho}$$

2-dimensional sine-Gordon model

In the quantum version of the theory, the instantons describe tunnel transitions between classical vacua

$$\varphi_n^{vac}(x,t) = \frac{2\pi n}{\rho}, \quad n = 0, \pm 1, \pm 2, \ldots$$

2-dimensional sine-Gordon model

In order to study the confinement possibility, let us introduce Yukawa interaction

$$L_{\text{int}} = g\bar{\psi}\psi\varphi$$

Potential obtained from this system

$$V(L_1) = const\ L_1 e^{-\frac{8\sqrt{\lambda}}{\rho}L}$$

significantly differs from Yukawa potential

$$V(L_1) = const\ L_1 e^{-\rho\sqrt{\lambda}L_1}$$

2-dimensional double-well potential model

$$L = \frac{1}{2}\partial_\mu \varphi \ \partial_\mu \varphi - V(\varphi)$$

$$V(\varphi) = \lambda(\varphi^2 - \rho^2)^2 \qquad \mu = 0,1 \qquad -l/2 \le x \le l/2$$

The instanton solution is given by

$$\varphi^{inst}(\tau, x) = \pm \tanh\left((\tau - \tau_0)\rho\sqrt{2\lambda}\right)$$

Euclidean action

$$S[\varphi^{inst}(\tau, x)] = \frac{4\sqrt{2\lambda}\rho^3 l}{3}$$

Instantons in QFT

In quantum field theory instantons describe **tunneling** processes between classical degenerated vacua (if formalism of Feynman path integral used)

Instantons in QFT

$$A \sim \int [DA]\, e^{-S_E[A]}$$

In leading approximation

$$A \sim e^{-S_E\left[A^{inst}\right]} \sim e^{-\frac{2\pi}{\alpha_s}} \sim 10^{-10}$$

Relationship between amplitude and energy

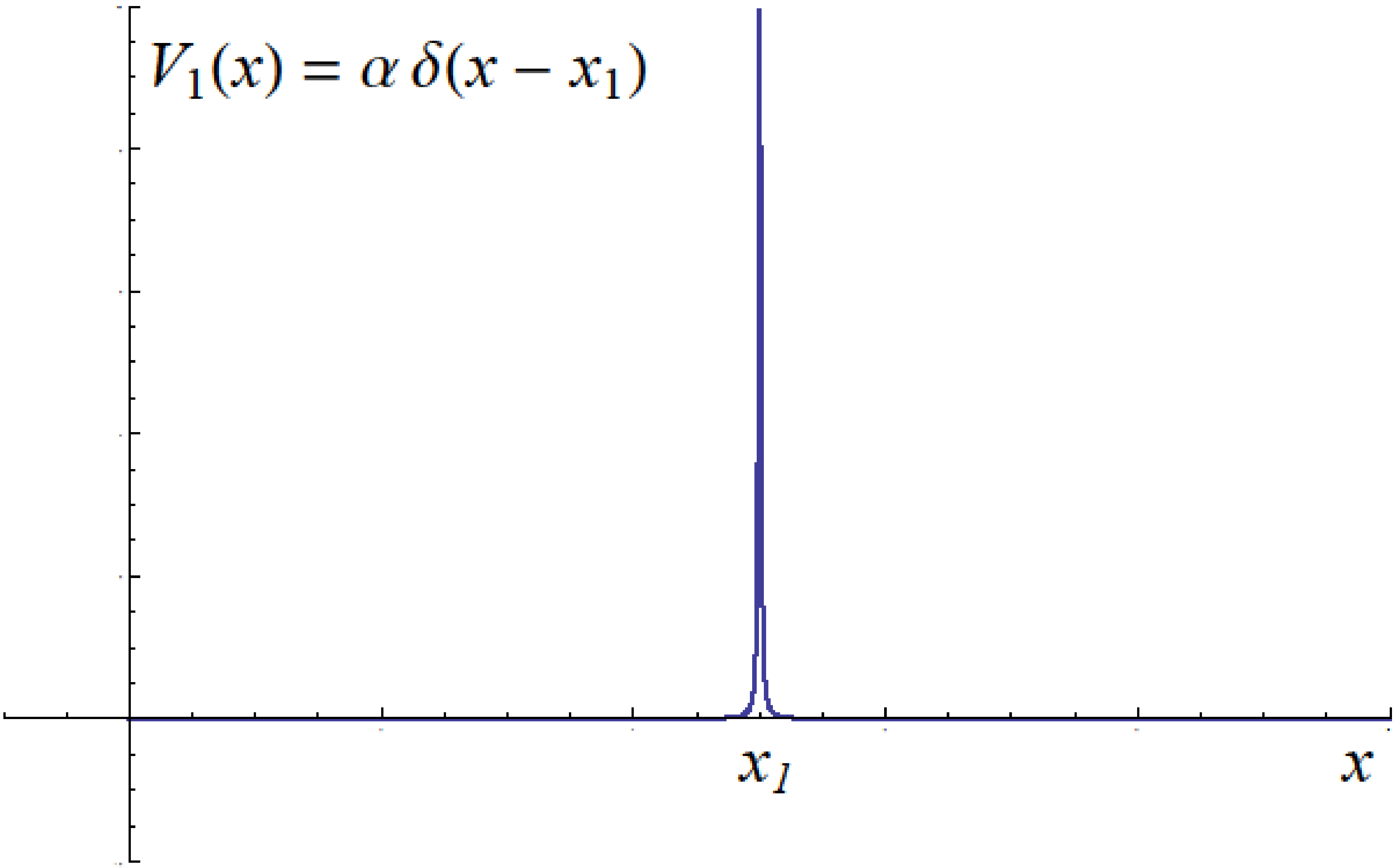
Ringwald showed that the amplitude can exponentially grow with the energy:

$$A(0) \sim e^{-\frac{2\pi}{\alpha_s}}$$

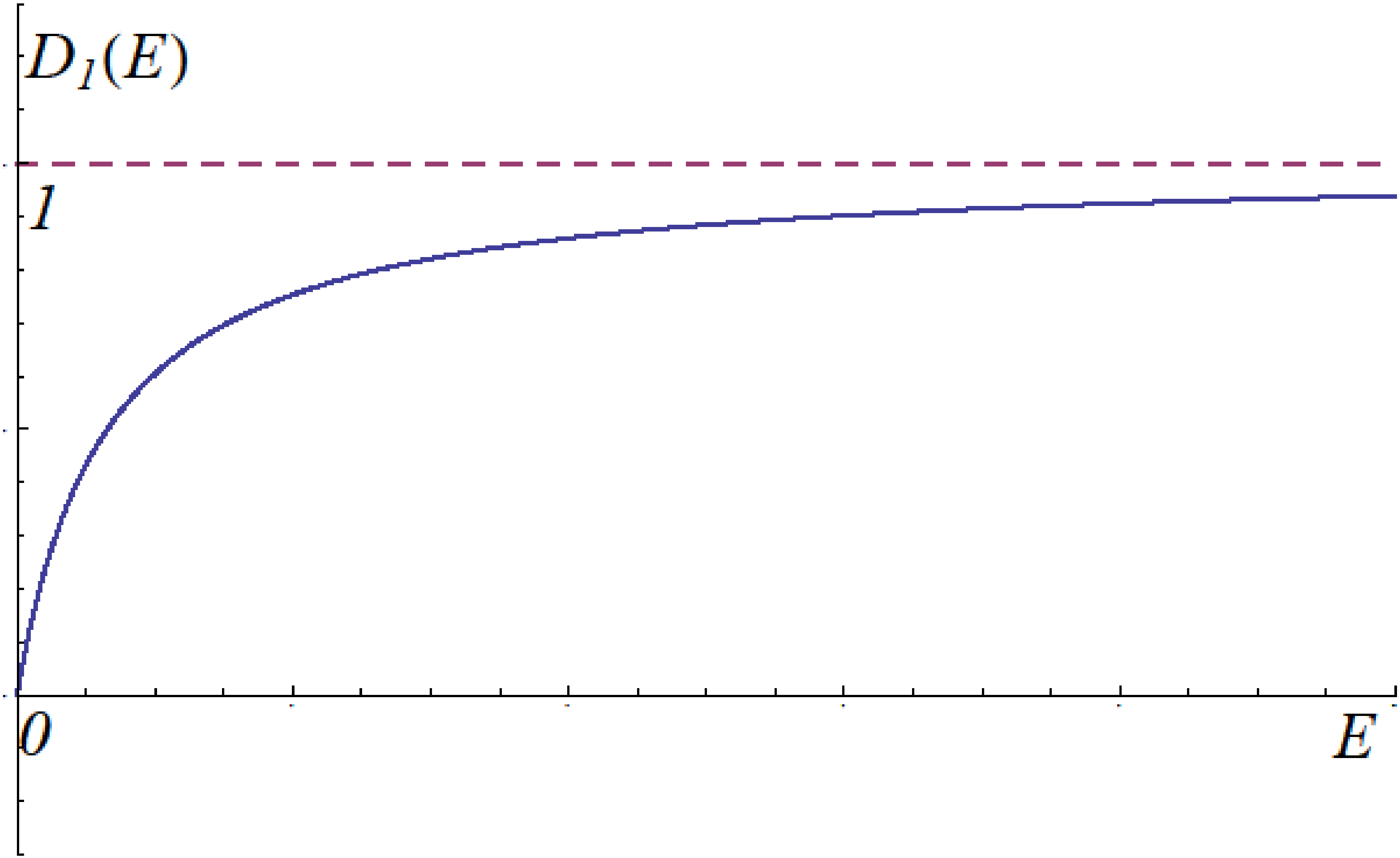
$$A(\varepsilon) \sim e^{\frac{2\pi}{\alpha_s} F_{hG}(\varepsilon)}$$

$$F_{hG}(\varepsilon) = -1 + \frac{9}{8}\varepsilon^{4/3} - \frac{9}{16}\varepsilon^2 + \cdots$$

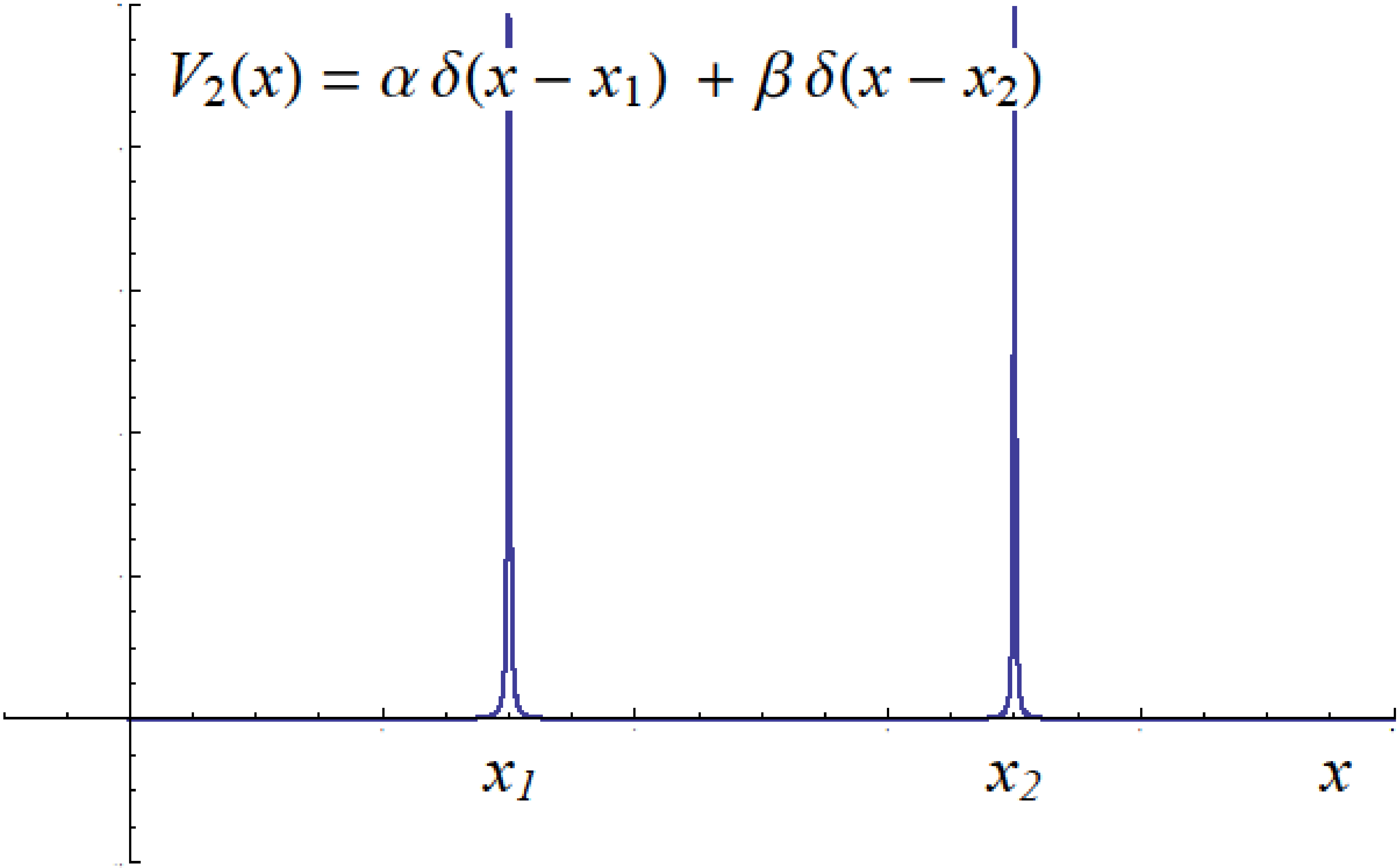
[A. Ringwald, Nucl. Phys. B 330, 1 (1990)]

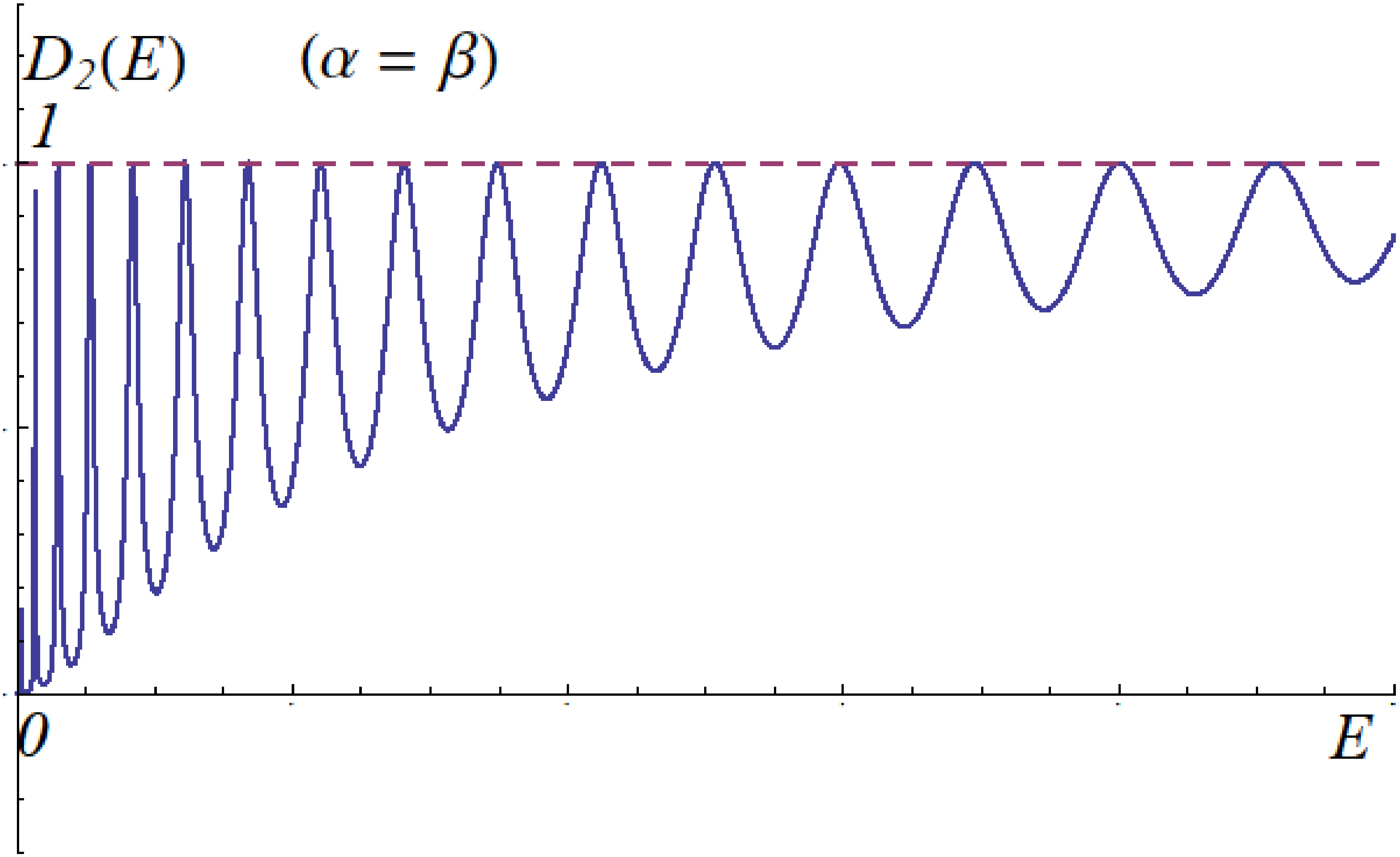


$V_1(x) = \alpha\,\delta(x - x_1)$
x_1
x

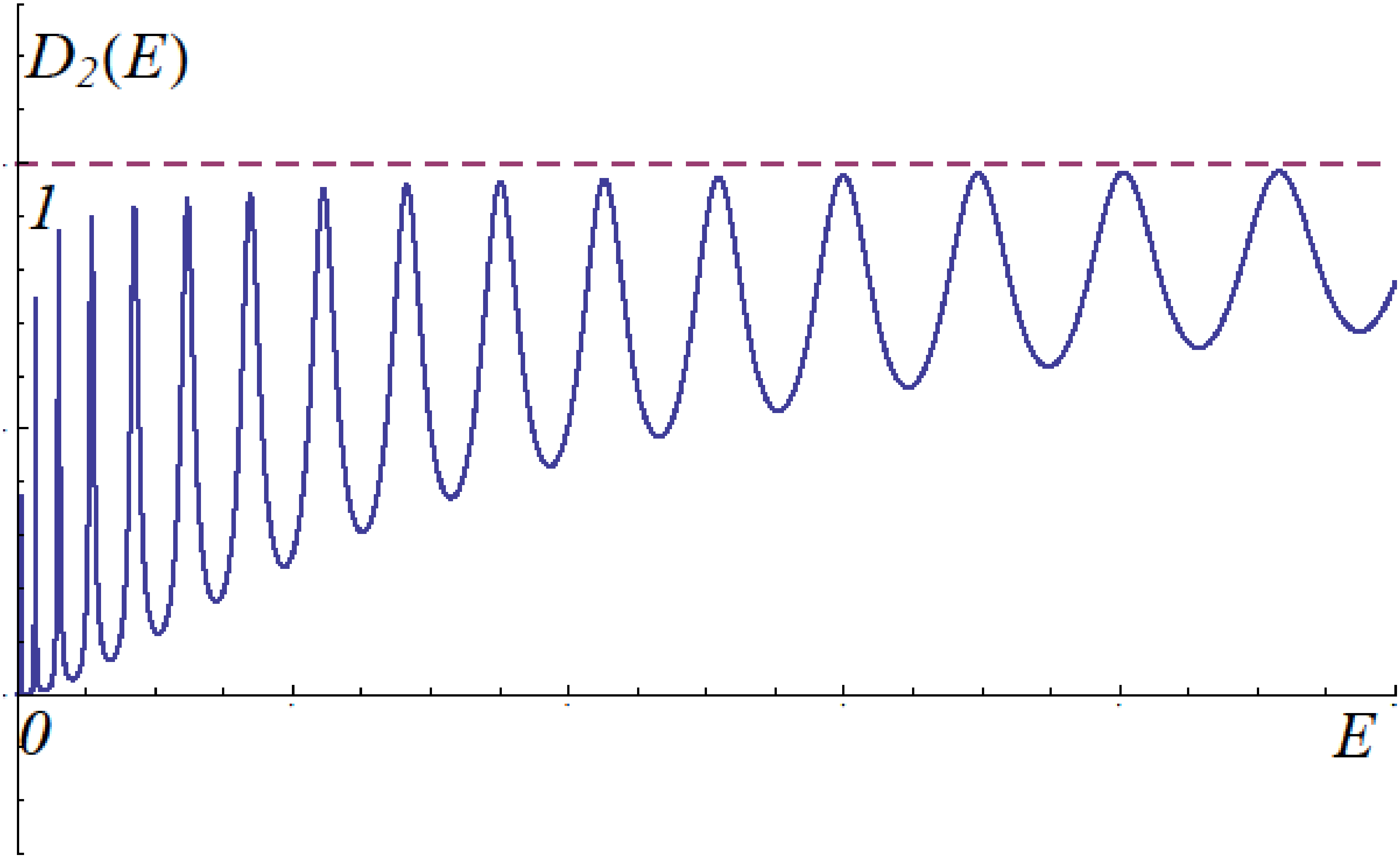


D1(E)
1
0
E


$V_2(x) = \alpha\,\delta(x - x_1) + \beta\,\delta(x - x_2)$
x_1
x_2
x



$D_2(E)$ $(\alpha = \beta)$
1
0
E



$D_2(E)$
1
0
E

✓Resonance tunneling

[V.I. Kuvshinov, A.V. Kuzmin, R.G. Shulyakovsky. Chaos assisted instanton tunnelling in one dimensional perturbed periodic potential. *Phys. Rev.* E67 (2003) 015201-1 - 015201-4]

[Р.Г. Шуляковский. Аналитические инстантонные решения в 2-мерных полевых моделях. *Письма в ЭЧАЯ* №5, 2008, 704-708]

✓ Footprints of QCD-instantons in high energy collisions

[A. Ringwald, Nucl. Phys. B 330, 1 (1990)]

[V.I. Kashkan, V.I. Kuvshinov, R.G. Shulyakovsky. Effect of Hadronization on the Form of Correlation Moments for Instanton Processes and Possibility of Discovering Them Experimentally. Physics of Atomic Nuclei, Vol. 65, No. 5 (2002) 925–928]

✓ Contribution into spin structure function (instanton liquid model)

[T. Schaefe, E. Shuryak. Instantons in QCD. Rev. Mod. Phys. 70:323-426,1998]

[A.E.Dorokhov. Instanton effects in high energy processes. Czech.J.Phys.53:B59-B68, 2003]

[N.I. Kochelev. Instantons and Spin-Flavor effects in Hadron Physics.: arXiv:0809.4773]

✓Instantons at high temperature and high density

[V. A. Rubakov, M. E. Shaposhnikov. Electroweak baryon number non-conservation in the early Universe and in high-energy collisions UFN, 166:5 (1996), 493–537]

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Thank you for attention!