XIV-th International School-Conference
Actual Problems of Microworld Physics,
August 12-24, 2018, Hrodna, Belarus

Quantum field theory in strong external fields at high densities and temperatures. II.

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Spinor matter in extremal conditions:

- hot and dense
- in strong magnetic field

Physical systems in:

- relativistic heavy-ion collisions
- compact astrophysical objects (neutron stars and magnetars)
- the early universe
- novel materials (the Dirac and Weyl semimetals) Cd_3As_2, Na_3Bi, K_3Bi, Rb_3Bi, $TaAs$, $BaAuBi$, $BaCuBi$, $BaAgBi$, Bi_2Se_3, $TlBiSe_2$, ...

Ultrarelativistic (chiral) effects

$$|eB| >> m^2, \quad T >> m, \quad \mu >> m.$$

Partition function

$$Z(T,\mu) = \mathrm{Sp}\exp\left[-(\hat{P}^0 - \mu\hat{N})/T\right]. \tag{1}$$

Average of operator $\hat{U}$ over the grand canonical ensemble

$$\left\langle \hat{U} \right\rangle_{T,\mu} = Z^{-1}(T,\mu)\,\mathrm{Sp}\,\hat{U}\exp\left[-(\hat{P}^0 - \mu\hat{N})/T\right]. \tag{2}$$

$$\hat{U} = \frac{1}{2}\left(\hat{\Psi}^\dagger \Upsilon \hat{\Psi} - \hat{\Psi}^T \Upsilon^T \hat{\Psi}^{\dagger T}\right), \tag{3}$$

where Υ is an element of the Dirac-Clifford algebra. We obtain

$$\left\langle \hat{U} \right\rangle_{T,\mu} = -\frac{1}{2}\mathrm{tr}\left\langle \mathbf{x} | \Upsilon \tanh[(H - \mu I)(2T)^{-1}] | \mathbf{x} \right\rangle. \tag{4}$$

Vector current density

$$\mathbf{J} = \left\langle \hat{U} \right\rangle_{T,\mu}\Big|_{\Upsilon=\gamma^0\boldsymbol{\gamma}}. \tag{5}$$

Axial current density

$$\mathbf{J}^5 = \left\langle \hat{U} \right\rangle_{T,\mu}\Big|_{\Upsilon=\gamma^0\boldsymbol{\gamma}\gamma^5}. \tag{6}$$

Axial charge density

$$J^{05} = \left\langle \hat{U} \right\rangle_{T,\mu}\Big|_{\Upsilon=\gamma^5}, \tag{7}$$

where $\gamma^5 = -\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$.

Owing to the presence of chiral symmetry,

$$[H, \gamma^5]_- = 0, \tag{8}$$

one can define axial charge operator

$$\hat{N}^5 = \frac{1}{2} \int\limits_{\Omega} \mathrm{d}^3 r (\hat{\Psi}^\dagger \gamma^5 \hat{\Psi} - \hat{\Psi}^T \gamma^{5T} \hat{\Psi}^{\dagger T}) \tag{9}$$

and modified partition function

$$\tilde{Z}(T, \mu_5) = \mathrm{Sp} \exp\left[-(\hat{P}^0 - \mu_5 \hat{N}^5)/T \right], \tag{10}$$

where μ_5 is the chiral chemical potential.

Average of operator $\hat{U}$ over the modified grand canonical ensemble

$$\left\langle \hat{U} \right\rangle_{T, \mu_5} = \tilde{Z}^{-1}(T, \mu_5)\, \mathrm{Sp}\, \hat{U} \exp\left[-(\hat{P}^0 - \mu_5 \hat{N}^5)/T \right]. \tag{11}$$

$$\left\langle \hat{U} \right\rangle_{T,\mu_5} = -\frac{1}{2}\mathrm{tr}\left\langle \mathbf{x} | \Upsilon \tanh[(H - \mu_5\gamma^5)(2T)^{-1}] | \mathbf{x} \right\rangle . \qquad (12)$$

Vector current density

$$\mathbf{J} = \left\langle \hat{U} \right\rangle_{T,\mu_5} \Big|_{\Upsilon=\gamma^0\boldsymbol{\gamma}} . \qquad (13)$$

Axial current density

$$\mathbf{J}^5 = \left\langle \hat{U} \right\rangle_{T,\mu_5} \Big|_{\Upsilon=\gamma^0\boldsymbol{\gamma}\gamma^5} . \qquad (14)$$

Axial charge density

$$J^{05} = \left\langle \hat{U} \right\rangle_{T,\mu_5} \Big|_{\Upsilon=\gamma^5} . \qquad (15)$$

chiral magnetic effect
(A. Vilenkin, 1980; K. Fukushima, D. E. Kharzeev, and H. J. Warringa, 2008):

$$\mathbf{J} = -\frac{e\mathbf{B}}{2\pi^2}\mu_5$$

chiral separation effect
(M. A. Metlitski and A. R. Zhitnitsky, 2005):

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in unbounded (infinite) medium

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Role of boundaries?

Outline

- Confining boundary condition for quantized spinor matter.
- Impact of magnetized matter on the Casimir effect.
- Hot dense magnetized matter in particle and astroparticle physics.

Confining boundary condition

A quest for boundary conditions ensuring the confinement of the quantized spinor matter was initiated in the context of a model description of hadrons as composite systems with their internal structure being associated with quark-gluon constituents (A.Chodos, R.L.Jaffe, K.Johnson, C.B.Thorn and V.Weisskopf, 1974). If an hadron is an extended object occupying spatial region Ω bounded by surface $\partial\Omega$, then the condition that the quark matter field be confined inside the hadron is formulated as

$$\boldsymbol{n} \cdot \boldsymbol{J}(\mathbf{r})|_{\mathbf{r}\in\partial\Omega} = 0,$$

where $\boldsymbol{n}$ is the unit normal to the boundary surface, and $\boldsymbol{J}(\mathbf{r}) = \psi^\dagger(\mathbf{r})\boldsymbol{\alpha}\psi(\mathbf{r})$ with $\psi(\mathbf{r})$ ($\mathbf{r} \in \Omega$) being the quark matter field (α^1, α^2, α^3 and β are the generating elements of the Dirac-Clifford algebra); an appropriate condition is also formulated for the gluon matter field.

The concept of confined matter fields is quite familiar in the context of condensed matter physics: collective excitations (e.g., spin waves and phonons) exist only inside material objects and do not spread outside. Moreover, in the context of quantum electrodynamics, if one is interested in the effect of a classical background magnetic field on the vacuum of the quantized electron-positron matter, then the latter should be considered as confined to the spatial region between the sources of the magnetic field, as long as collective quasielectronic excitations inside a magnetized material differ from electronic excitations in the vacuum. It should be noted in this respect that the study of the effect of the background electromagnetic field on the vacuum of quantized charged matter has begun already eight decades ago (W.Heisenberg and H.Euler, 1936; V.S.Weisskopf, 1936).

However, the case of a background field filling the whole (infinite) space is hard to be regarded as realistic. The case of both the background and quantized fields confined to a bounded spatial region with boundaries serving as sources of the background field looks much more physically plausible, it can even be regarded as realizable in laboratory. Moreover, there is no way to detect the energy density that is induced in the vacuum in the first case, whereas the pressure from the vacuum onto the boundaries, resulting in the second case, is in principle detectable.

In view of the above, an issue of a choice of boundary conditions for the quantized matter fields gains a crucial significance, and condition for the current should be resolved to take the form of a boundary condition that is linear in $\psi(\mathbf{r})$. An immediate way of such a resolution is known as the MIT bag boundary condition (K.Johnson, 1975),

$$(I + \mathrm{i}\beta \boldsymbol{n} \cdot \boldsymbol{\alpha})\psi(\mathbf{r})|_{\mathbf{r}\in\partial\Omega} = 0,$$

but it is needless to say that this way is not a unique one.

The most general boundary condition is provided by the condition of the self-adjointness of the differential operator of one-particle energy in first-quantized theory (Dirac hamiltonian operator in the case of relativistic spinor matter). The self-adjointness of operators of physical observables is required by general principles of comprehensibility and mathematical consistency, see, e.g.,

J.von Neumann, *Mathematische Grundlagen der Quantummechanik* (Springer, Berlin, 1932).

To put it simply, a multiple action is well defined for a self-adjoint operator only, allowing for the construction of functions of the operator, such as resolvent, evolution, heat kernel and zeta-function operators, with further implications upon second quantization.

QUANTUM THEORY

- physical observables $\Longrightarrow$ operators
- physical states $\Longrightarrow$ functions

Stability of quantum systems: real values of observables

$$\left\{ \begin{array}{l} \textit{Boundary conditions for functions of states?} \\ \\ \textit{Real values of observales?} \end{array} \right\}$$

QUANTUM THEORY

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NEW PARAMETERS?

Self-adjointness

Let us consider differential (unbounded in general) operator H and scalar products

$$(\tilde{\chi}, H\chi) = \int_{\Omega} \mathrm{d}v\, \tilde{\chi}^\dagger H\chi, \qquad (H^\dagger\tilde{\chi}, \chi) = \int_{\Omega} \mathrm{d}v\, (H^\dagger\tilde{\chi})^\dagger\chi,$$

and get, using integration by parts,

$$(\tilde{\chi}, H\chi) = (H^\dagger\tilde{\chi}, \chi) - \mathrm{i}\int_{\partial\Omega} \mathrm{d}\boldsymbol{s}\cdot\mathbf{J}[\tilde{\chi}, \chi],$$

where Ω is a spatial region with boundary $\partial\Omega$.
Operator H is Hermitian (symmetric),

$$(\tilde{\chi}, H\chi) = (H^\dagger\tilde{\chi}, \chi),$$

if

$$\int_{\partial\Omega} \mathrm{d}\boldsymbol{\sigma}\cdot\mathbf{J}[\tilde{\chi}, \chi] = 0 \qquad \text{or} \qquad \mathbf{n}\cdot\mathbf{J}[\tilde{\chi}, \chi]|_{\mathbf{r}\in\partial\Omega} = 0.$$

The latter condition can be satisfied in various ways by imposing different boundary conditions for χ and $\tilde{\chi}$. However, among the whole variety, there may exist a possibility that a boundary condition for $\tilde{\chi}$ is the same as that for χ; then the domain of definition of $H^\dagger$ (set of functions $\tilde{\chi}$) coincides with that of H (set of functions χ),

$$\chi \in D(H), \quad \tilde{\chi} \in D(H^\dagger), \quad D(H^\dagger) \equiv D(H),$$

and operator H is called self-adjoint. The action of a self-adjoint operator results in functions belonging to its domain of definition only, and, therefore, a multiple action and functions of such an operator (for instance, the evolution and resolvent operators) can be consistently defined.

Weyl – von Neumann theory of deficiency indices (n, n')

- 1. $n = n'$: self-adjoint extension

$$D(H^\dagger) \equiv D(H), \qquad n^2 \text{ parameters}$$

- 2. $n = 0, n' = 0$: essential self-adjointness

$$D(H^\dagger) \equiv D(H),$$

- 3. $n \neq n'$: non-self-adjointness

$$D(H^\dagger) \supset D(H)$$

Applicable to effectively one-dimensional systems with point-like boundaries

Dirac fermions in $2+1$-dimensional space-time

Dirac Hamiltonian operator is

$$H_{(1/2)} = H^{\dagger}_{(1/2)} = -\mathrm{i}\alpha^x \nabla_x + -\mathrm{i}\alpha^y \nabla_y + \beta m, \;\; \boldsymbol{\nabla} = \boldsymbol{\partial} - \mathrm{i}e\mathbf{A} + \frac{\mathrm{i}}{2}\boldsymbol{\omega}.$$

Defining a scalar product as $(\tilde{\chi}, \chi) = \int\limits_{\Omega} \mathrm{d}^2 r \, \tilde{\chi}^{\dagger} \chi$,

$$(\tilde{\chi}, H_{(1/2)}\chi) = (H^{\dagger}_{(1/2)}\tilde{\chi}, \chi) - \mathrm{i} \int\limits_{\partial\Omega} \mathrm{d}\boldsymbol{\sigma} \cdot \mathbf{J}[\tilde{\chi}, \chi],$$

where

$$\mathbf{J}[\tilde{\chi}, \chi] = \tilde{\chi}^{\dagger} \boldsymbol{\alpha} \chi.$$

Disconnected boundary

$$\partial\Omega = \partial\Omega^{(+)} \bigcup \partial\Omega^{(-)} : \;\; x = a, \, x = b,$$

then

$$\tilde{\chi}^{\dagger} \alpha^x \chi \,|_{x=a} = \tilde{\chi}^{\dagger} \alpha^x \chi \,|_{x=b} .$$

Impenetrability:

$$\tilde{\chi}^{\dagger} \alpha^x \chi \,|_{x=a} = 0, \;\; \tilde{\chi}^{\dagger} \alpha^x \chi \,|_{x=b} = 0.$$

The problem of self-adjointness of operator $H_{(1/2)}$ is resolved by imposing the same boundary condition for χ and $\tilde{\chi}$ in the form

$$\chi|_{\mathbf{r}\in\partial\Omega} = K\chi|_{\mathbf{r}\in\partial\Omega}, \quad \tilde{\chi}|_{\mathbf{r}\in\partial\Omega} = K\tilde{\chi}|_{\mathbf{r}\in\partial\Omega},$$

where K is a Hermitian matrix (element of the Clifford algebra) which is determined by two conditions:

$$K^2 = I$$

and
either $[K, \alpha^x]_- = 0$, or $[K, \alpha^x]_+ = 0$.
Linearly independent elements of the Clifford algenbra are

$$I, \quad \alpha^x, \quad \beta, \quad \mathrm{i}\alpha^x\beta.$$

Either $K = c_1 I + c_2\alpha^x$, then $K^2 \neq 1$.
Or $K = c_1\beta + c_2\mathrm{i}\alpha^x\beta$, then $K^2 = 1$, if $c_1^2 + c_2^2 = 1$.

Two-parametric boundary condition:

$$\left(I - \mathrm{i}\beta\alpha^x e^{-\mathrm{i}\theta\alpha^x}\right)\chi\,|_{x=a} = 0,\ \left(I - \mathrm{i}\beta\alpha^x e^{-\mathrm{i}\theta\alpha^x}\right)\tilde{\chi}\,|_{x=a} = 0$$

$$(c_1 = \sin\theta,\ c_2 = \cos\theta),$$

$$\left(I - \mathrm{i}\beta\alpha^x e^{\mathrm{i}\tilde{\theta}\alpha^x}\right)\chi\,|_{x=b} = 0,\ \left(I - \mathrm{i}\beta\alpha^x e^{\mathrm{i}\tilde{\theta}\alpha^x}\right)\tilde{\chi}\,|_{x=b} = 0$$

$$(c_1 = \sin\tilde{\theta},\ c_2 = \cos\tilde{\theta}).$$

It should be emphasized that parameters θ and $\tilde{\theta}$ are in general dependent on y. Thus, the “number” of self-adjoint extension parameters is infinite, moreover, it is not countable but is of power of a continuum. This distinguishes the case of an extended boundary from the case of an excluded point (contact interaction) when the number of self-adjoint extension parameters is finite, being equal to n^2 for the deficiency index equal to $\{n, n\}$.

If $b \to \infty$, the restriction at $x = b$ disappears
$(\lim_{x\to\infty}\chi = 0,\ \lim_{x\to\infty}\tilde{\chi} = 0)$.

Operator of quantized spinor field in a static background field

$$\hat{\Psi}(t,\mathbf{x}) = \sum\!\!\!\!\!\!\!\int_{E_k>0} e^{-\mathrm{i}E_k t} \langle \mathbf{x}|k\rangle\, \hat{a}_k + \sum\!\!\!\!\!\!\!\int_{E_k<0} e^{-\mathrm{i}E_k t} \langle \mathbf{x}|k\rangle\, \hat{b}_k^\dagger$$

$$\left[\hat{a}_k,\, \hat{a}_{k'}^\dagger\right]_+ = \left[\hat{b}_k,\, \hat{b}_{k'}^\dagger\right]_+ = \left\langle k|k'\right\rangle, \qquad \hat{a}_k\,|\mathrm{vac}\rangle = \hat{b}_k\,|\mathrm{vac}\rangle = 0,$$

$$H_{(1/2)}\,\langle \mathbf{x}|k\rangle = E_k\,\langle \mathbf{x}|k\rangle\,.$$

Temporal component of the energy-momentum tensor

$$\hat{T}^{00} = \frac{\mathrm{i}}{4}[\hat{\Psi}^\dagger(\partial_0\hat{\Psi}) - (\partial_0\hat{\Psi}^T)\hat{\Psi}^{\dagger T} - (\partial_0\hat{\Psi}^\dagger)\hat{\Psi} + \hat{\Psi}^T(\partial_0\hat{\Psi}^{\dagger T})]$$

Vacuum energy density

$$\varepsilon = \langle\mathrm{vac}|\,\hat{T}^{00}|\mathrm{vac}\rangle = -\frac{1}{2}\sum\!\!\!\!\!\!\!\int_k |E_k|\,\langle k|\mathbf{x}\rangle\,\langle \mathbf{x}|k\rangle\,.$$

Self-adjointness of the Dirac operator

$$H_{(1/2)} = H^{\dagger}_{(1/2)} = -\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m$$

$$(\tilde{\chi}, H_{(1/2)}\chi) = (H^{\dagger}_{(1/2)}\tilde{\chi}, \chi) - \mathrm{i}\int\limits_{\partial\Omega} \mathrm{d}\boldsymbol{\sigma}\cdot\mathbf{J}[\tilde{\chi}, \chi],$$

where

$$\mathbf{J}[\tilde{\chi}, \chi] = \tilde{\chi}^{\dagger}\boldsymbol{\alpha}\chi.$$

$$\mathbf{n}\cdot\mathbf{J}[\tilde{\chi}, \chi]|_{\mathbf{r}\in\partial\Omega} = 0.$$

$$\Downarrow\Downarrow$$

$$\chi|_{\mathbf{r}\in\partial\Omega} = K\chi|_{\mathbf{r}\in\partial\Omega}, \quad \tilde{\chi}|_{\mathbf{r}\in\partial\Omega} = K\tilde{\chi}|_{\mathbf{r}\in\partial\Omega},$$

where K is a Hermitian matrix (element of the Clifford algebra) which is determined by two conditions:

$$K^2 = I$$

and

$$[\boldsymbol{n}\cdot\boldsymbol{\alpha}, K]_+ = 0.$$

It should be noted that, in addition, the following combination of χ and $\tilde{\chi}$ is also vanishing at the boundary:

$$\tilde{\chi}^{\dagger}(\boldsymbol{n}\cdot\boldsymbol{\alpha})K\chi|_{\mathbf{r}\in\partial\Omega} = \tilde{\chi}^{\dagger}K(\boldsymbol{n}\cdot\boldsymbol{\alpha})\chi|_{\mathbf{r}\in\partial\Omega} = 0.$$

16 linear independent elements of the Clifford algebra in $3+1$-dimensional space-time, and the explicit form is

$$K = \left[\beta e^{i\varphi\gamma^5}\cos\theta + (\alpha^1\cos\varsigma + \alpha^2\sin\varsigma)\sin\theta\right] e^{i\tilde{\varphi}\boldsymbol{n}\cdot\boldsymbol{\alpha}},$$

where $\gamma^5 = i\alpha^1\alpha^2\alpha^3$, matrices α^1 and α^2 are chosen to obey condition

$$[\alpha^1,\, \boldsymbol{n}\cdot\boldsymbol{\alpha}]_+ = [\alpha^2,\, \boldsymbol{n}\cdot\boldsymbol{\alpha}]_+ = [\alpha^1,\, \alpha^2]_+ = 0,$$

and the boundary parameters are chosen to vary as

$$-\frac{\pi}{2} < \varphi \le \frac{\pi}{2}, \quad -\frac{\pi}{2} \le \tilde{\varphi} < \frac{\pi}{2}, \quad 0 \le \theta < \pi, \quad 0 \le \varsigma < 2\pi.$$

The MIT bag boundary condition (K.Johnson, 1975),

$$(I + i\beta\boldsymbol{n}\cdot\boldsymbol{\alpha})\chi(\mathbf{r})\,|_{\mathbf{r}\in\partial\Omega} = 0,$$

is obtained at $\varphi = \theta = 0$, $\tilde{\varphi} = -\pi/2$.

In the case of a disconnected boundary consisting of two simply-connected components, $\partial\Omega = \partial\Omega^{(+)} \bigcup \partial\Omega^{(-)}$, there are in general 8 boundary parameters: φ_+, $\tilde{\varphi}_+$, θ_+ and ς_+ corresponding to $\partial\Omega^{(+)}$ and φ_-, $\tilde{\varphi}_-$, θ_- and ς_- corresponding to $\partial\Omega^{(-)}$. If spatial region Ω has the form of a slab bounded by surfaces, $\partial\Omega^{(+)}$ and $\partial\Omega^{(-)}$, separated by distance a, then the boundary condition takes form

$$\left(I - K^{(\pm)}\right) \chi(\mathbf{r}) \mid_{z=\pm a/2} = 0, \tag{16}$$

where

$$K^{(\pm)} = \left[\beta \mathrm{e}^{\mathrm{i}\varphi_\pm \gamma^5} \cos\theta_\pm + (\alpha^1 \cos\varsigma_\pm + \alpha^2 \sin\varsigma_\pm) \sin\theta_\pm\right] \mathrm{e}^{\pm\mathrm{i}\tilde{\varphi}_\pm \alpha^z}, \tag{17}$$

coordinates $\mathbf{r} = (x,\, y,\, z)$ are chosen in such a way that x and y are tangential to the boundary, while z is normal to it, and the position of $\partial\Omega^{(\pm)}$ is identified with $z = \pm a/2$.

The confinement of matter inside the slab means that the vector bilinear, $\chi^\dagger(\mathbf{r})\alpha^z\chi(\mathbf{r})$, vanishes at the slab boundaries,

$$\chi^\dagger(\mathbf{r})\alpha^z\chi(\mathbf{r})\,|_{z=\pm a/2} = 0, \qquad (18)$$

and this is ensured by condition (16). As to the axial bilinear, $\chi^\dagger(\mathbf{r})\alpha^z\gamma^5\chi(\mathbf{r})$, it vanishes at the slab boundaries,

$$\chi^\dagger(\mathbf{r})\alpha^z\gamma^5\chi(\mathbf{r})\,|_{z=\pm a/2} = 0, \qquad (19)$$

in the case of $\theta_+ = \theta_- = \pi/2$ only, that is due to relation

$$[K^{(\pm)}\,|_{\theta_\pm=\pi/2}\,, \gamma^5]_- = 0. \qquad (20)$$

However, there is a symmetry with respect to rotations around a normal to the slab, and the cases differing by values of ς_+ and ς_- are physically indistinguishable, since they are related by such a rotation. The only way to avoid the unphysical degeneracy of boundary conditions with different values of ς_+ and ς_- is to fix $\theta_+ = \theta_- = 0$. Then $\chi^\dagger(\mathbf{r})\alpha^z\gamma^5\chi(\mathbf{r})$ is nonvanishing at the slab boundaries, and the boundary condition takes form

$$\left\{I - \beta \exp\left[\mathrm{i}\left(\varphi_\pm\gamma^5 \pm \tilde{\varphi}_\pm\alpha^z\right)\right]\right\}\chi(\mathbf{r})\,|_{z=\pm a/2} = 0. \qquad (21)$$

Condition (21) determines the spectrum of the wave number vector in the z-direction, k_l. The requirement that this spectrum be real and unambiquous yields constraint

$$\varphi_+ = \varphi_- = \varphi, \quad \tilde{\varphi}_+ = \tilde{\varphi}_- = \tilde{\varphi}; \qquad (22)$$

then the k_l-spectrum is determined implicitly from relation

$$k_l \sin\tilde{\varphi}\cos(k_l a) + (E_{\ldots l}\cos\tilde{\varphi} - m\cos\varphi)\sin(k_l a) = 0, \qquad (23)$$

where $E_{\ldots l}$ is the energy of the one-particle state.

Background: uniform magnetic field orthogonal to the boundary

$$\boldsymbol{B} = (0, 0, B), \qquad \boldsymbol{A} = (-yB, 0, 0)$$

$$\partial\Omega : \quad \partial\Omega^{(+)} \oplus \partial\Omega^{(-)}$$

$$\mathbf{r} = (x, y, z) \qquad \partial\Omega^{(+)} : \quad z = a/2; \qquad \partial\Omega^{(-)} : \quad z = -a/2$$

.

$$\boldsymbol{\nabla}\hat{\Psi} = (\boldsymbol{\partial} - \mathrm{i}e\mathbf{A})\hat{\Psi}, \quad \boldsymbol{\nabla}\hat{\Psi}^\dagger = (\boldsymbol{\partial} + \mathrm{i}e\mathbf{A})\hat{\Psi}^\dagger, \quad \mathbf{B} = \boldsymbol{\partial} \times \mathbf{A}$$

One-particle energy spectrum (Landau levels):

$$\omega_{snl} = \sqrt{|eB|(2n + 1 - 2s) + k_l^2 + m^2},$$

$$s = 0, 1/2, \quad n = 0, 1, 2, ...,$$

Vacuum energy per unit area of the boundary surface

$$\frac{E_{(s)}}{S} = \frac{|eB|}{2\pi}(1-4s)\sum_{l}\sum_{n=0}^{\infty}(1+2s-2s\delta_{n0})\omega_{snl}$$

Use of generalizations of the Abel-Plana summation formula ields

$$\frac{E_{(s)}}{S} = a\,\varepsilon_{(s)}^{\infty} + \Omega_{(s)}(a) + \tilde{\Omega}_{(s)},$$

where

$$\varepsilon_{(s)}^{\infty} = \frac{|eB|}{(2\pi)^2}(1-4s)\int\limits_{-\infty}^{\infty}\mathrm{d}k\sum_{n=0}^{\infty}(1+2s-2s\delta_{n0})\omega_{snk}.$$

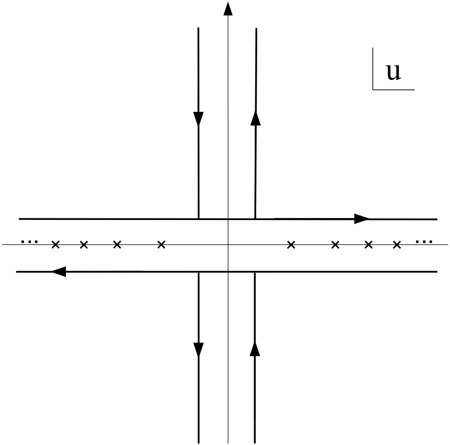

Figure: Contours $C_{=}$, $C_{\sqcap}$ and $C_{\sqcup}$ on the complex u-plane; the positions of poles of function $G_{+}(u) + G_{-}(u)$ are indicated by crosses.

Regularization & renormalization: $\varepsilon^{\infty}_{(s)} \to \varepsilon^{\infty}_{(s)\mathrm{ren}}$

$$\varepsilon^{\infty}_{(s)\mathrm{ren}} = \frac{e^2 B^2}{(4\pi)^2} \int\limits_0^{\infty} \frac{\mathrm{d}\eta}{\eta} \exp\left(-\frac{m^2 \eta}{|eB|}\right) \left[\frac{4s\cosh\eta - 1 + 2s}{\eta \sinh\eta} + (1-6s)\frac{1}{\eta^2} - \frac{1}{6}(1+6s)\right]$$

V.S.Weisskopf, *Kong. Dans. Vid. Selsk. Mat-Fys. Medd.* **14**, 6 (1936).
W.Heisenberg and H.Euler, *Z. Phys.* **98**, 714 (1936).

Regularization & renormalization: $\frac{E_{(s)}}{S} \to \frac{E_{(s)\mathrm{ren}}}{S}$

$$\frac{E_{(s)\mathrm{ren}}}{S} = a\,\varepsilon^{\infty}_{(s)\mathrm{ren}} + \Omega_{(s)}(a) + \tilde{\Omega}_{(s)}$$

Casimir force (or pressure)

$$F_{(s)} \equiv -\frac{\partial}{\partial a}\frac{E_{(s)\mathrm{ren}}}{S} = -\varepsilon^{\infty}_{(s)\mathrm{ren}} + \Delta_{(s)}(a),$$

where

$$\Delta_{(s)}(a) \equiv -\frac{\partial}{\partial a}\Omega_{(s)}(a)$$

$$= -\frac{|eB|}{\pi^2}\sum_{n=0}^{\infty}(1+2s-2s\delta_{n0})\int\limits_{M_{sn}}^{\infty} \mathrm{d}\kappa\, \Upsilon_{(s)}(\kappa)\kappa^{2-4s}(\kappa^2 - M_{sn}^2)^{2s-1/2},$$

$$M_{sn} = \sqrt{|eB|(2n+1-2s)+m^2},$$

$$\Upsilon_{(0)}(\kappa) = \frac{1}{2}\,\frac{\cos\rho - \mathrm{e}^{-\kappa a}}{\cosh(\kappa a) - \cos\rho},$$

$$\Upsilon_{(1/2)}(\kappa) = \frac{\left[(2\kappa a - 1)\left(\kappa^2 - m^2\cos^2\varphi\right) - 2\kappa m\cos\varphi\right]\mathrm{e}^{2\kappa a}}{\left[(\kappa + m\cos\varphi)\,\mathrm{e}^{2\kappa a} + \kappa - m\cos\varphi\right]^2}$$

$$-\frac{(\kappa - m\cos\varphi)^2}{\left[(\kappa + m\cos\varphi)\,\mathrm{e}^{2\kappa a} + \kappa - m\cos\varphi\right]^2} \qquad (\tilde{\varphi} = -\pi/2).$$

$$F_{(s)} = -\varepsilon^{\infty}_{(s)\mathrm{ren}} + \Delta_{(s)}(a),$$

$-\varepsilon^{\infty}_{(s)\mathrm{ren}}$ is positive

In the case of a weak magnetic field, $|B| \ll m^2|e|^{-1}$, one has

$$-\varepsilon^{\infty}_{(s)\mathrm{ren}} = \frac{1}{360\pi^2}\left[1 - \frac{9}{8}\left(\frac{1}{2} - s\right)\right]\left(\frac{eB}{m}\right)^4 .$$

Note that the critical value is the lowest one, $B_{\mathrm{crit}} = m^2|e|^{-1} = 4.41 \times 10^{13}\,\mathrm{Gauss}$, for the case of quantized electron-positron matter.

In the case of a strong magnetic field, $|B| \gg m^2|e|^{-1}$, one has

$$-\varepsilon^{\infty}_{(s)\mathrm{ren}} = \frac{1}{24\pi^2}\left[1 - \frac{3}{2}\left(\frac{1}{2} - s\right)\right] e^2 B^2 \ln\frac{2|eB|}{m^2}.$$

$\Delta_{(1/2)}(a)$ at $|B| \ll m^2|e|^{-1}$ takes the forms in the limits of large and small distances between the plates

$$\Delta_{(1/2)}(a) = \left\{ \begin{array}{ll} -\frac{3}{16\pi^{3/2}}\frac{m^{3/2}}{a^{5/2}}\mathrm{e}^{-2ma}[1 + O(\frac{1}{ma})], & \varphi = 0 \\ \\ -\frac{\tan^2(\varphi/2)}{2\pi^{3/2}}\frac{m^{5/2}}{a^{3/2}}\mathrm{e}^{-2ma}[1 + O(\frac{1}{ma})], & \varphi \neq 0 \end{array} \right\},$$

$$|eB| \ll m^2, \qquad ma \gg 1$$

and

$$\Delta_{(1/2)}(a) = -\frac{7}{8}\frac{\pi^2}{120}\frac{1}{a^4}, \quad |eB| \ll m^2, \qquad ma \ll 1.$$

$\Delta_{(1/2)}(a)$ at $|B| \gg m^2|e|^{-1}$ takes the forms in the limits of large and small distances between the plates

$$\Delta_{(1/2)}(a) = \left\{ \begin{array}{ll} -\frac{|eB|}{16\pi^{3/2}} \frac{m^{1/2}}{a^{3/2}} \mathrm{e}^{-2ma}[1 + O(\frac{1}{ma})], & \varphi = 0 \\ \\ -\frac{|eB| \tanh^2(\varphi/2)}{2\pi^{3/2}} \frac{m^{3/2}}{a^{1/2}} \mathrm{e}^{-2ma}[1 + O(\frac{1}{ma})], & \varphi \neq 0 \end{array} \right\},$$

$$\sqrt{|eB|}a \gg ma \gg 1$$

and

$$\Delta_{(1/2)}(a) = -\frac{|eB|}{48a^2}, \quad ma \ll 1, \ \sqrt{|eB|}a \gg 1.$$

$$m^{-1} = 3.86 \times 10^{-13}\,\mathrm{m}, \quad \mathrm{a} > 10^{-8}\,\mathrm{m}$$

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Conclusion 1

- The pressure from the vacuum of confined charged massive matter in the background of magnetic field is repulsive and independent of the choice of a boundary condition, as well as of the distance between the plates.

Slab of spinor matter in extremal conditions

$$|eB| >> m^2, \quad T >> m, \quad \mu >> m. \tag{24}$$

The Dirac Hamiltonian takes form ($m = 0$)

$$H = -\mathrm{i}\gamma^0 \boldsymbol{\gamma} \cdot (\boldsymbol{\partial} - \mathrm{i}e\mathbf{A}) \tag{25}$$

and the one-particle energy spectrum is

$$E_{nl} = \pm\omega_{nl}, \quad \omega_{nl} = \sqrt{2n|eB| + k_l^2}, \quad n = 0, 1, 2, \ldots, \tag{26}$$

where B is the value of the magnetic field strength, $\mathbf{B} = \boldsymbol{\partial} \times \mathbf{A}$, n labels the Landau levels, and k_l is the value of the wave number vector along the magnetic field; the set of the k_l values is determined by condition

$$k_l \sin\tilde{\varphi} \cos(k_l a) + E_{nl} \cos\tilde{\varphi} \sin(k_l a) = 0, \tag{27}$$

depending on one parameter only, although the boundary condition depends on two parameters:

$$\left\{ I - \gamma^0 \exp\left[\mathrm{i}\left(\varphi\gamma^5 \pm \tilde{\varphi}\gamma^0\gamma^z\right)\right]\right\} \chi(\mathbf{r})|_{z=\pm a/2} = 0. \tag{28}$$

Chiral effects

$$\mathbf{J} = J^{05} = 0. \quad J^{x5} = J^{y5} = 0. \tag{29}$$

As to the component of the axial current density, which is along the magnetic field, only the lowest Landau level ($n = 0$) contributes to it. The spectrum of the wave number vector along the magnetic field is determined from (30) at $n = 0$, i.e.

$$k_l^{(\pm)} = (l\pi \mp \tilde{\varphi})/a, \quad l \in \mathbb{Z}, \quad k_l^{(\pm)} > 0, \tag{30}$$

where the upper (lower) sign corresponds to $E_{0l} > 0$ ($E_{0l} < 0$) and $\mathbb{Z}$ is the set of integer numbers. Hence, the z-component of the axial current density is

$$J^{z5} = \frac{eB}{4\pi a}\left\{\sum_{k_l^{(+)}>0} \tanh[(k_l^{(+)} - \mu)(2T)^{-1}] \right.$$
$$\left. - \sum_{k_l^{(-)}>0} \tanh[(k_l^{(-)} + \mu)(2T)^{-1}] \right\}, \tag{31}$$

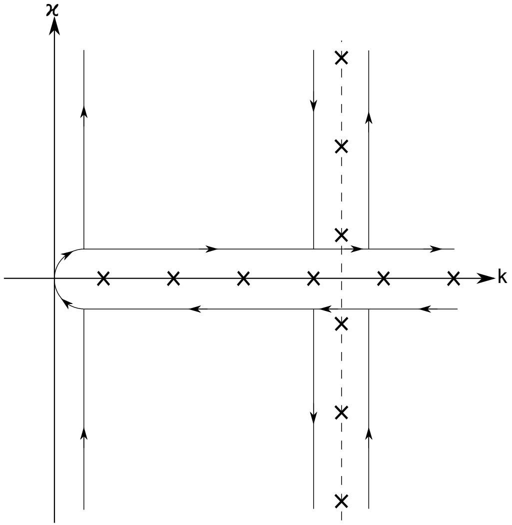


Figure: Contour $C_{\subset}$ enclosing the positive real semiaxis can be continuously deformed into a contour consisting of vertical lines on the complex ω-plane; positions of simple poles of the integrand are indicated by crosses.

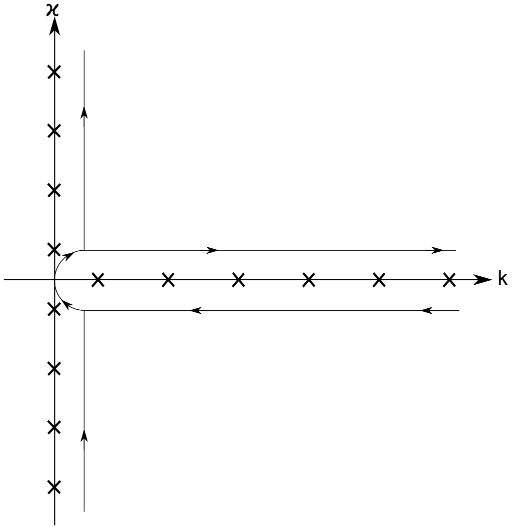


Figure: Poles on the complex plane are on the imaginary axis. Contour enclosing infinite number of poles on the positive real semiaxis is deformed into a contour which is infinitesimally close from the right to the imaginary axis.

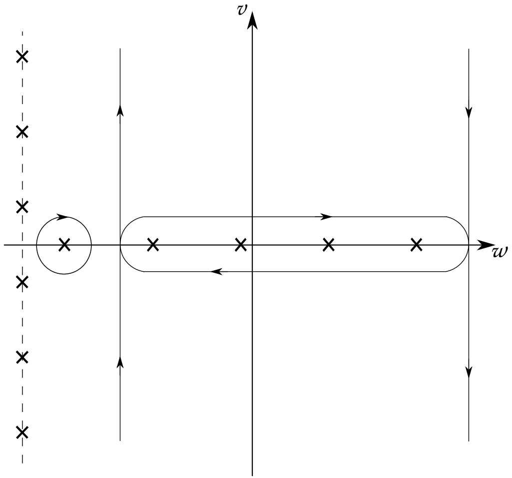


Figure: Finite number of poles on the real axis at $-s < w < s$ is enclosed by two separate contours. Poles on the complex plane are on a vertical axis at $w = -s$. The right closed contour is deformed into a contour consisting of two vertical lines at $w = -s + \pi$ and $w = s$.

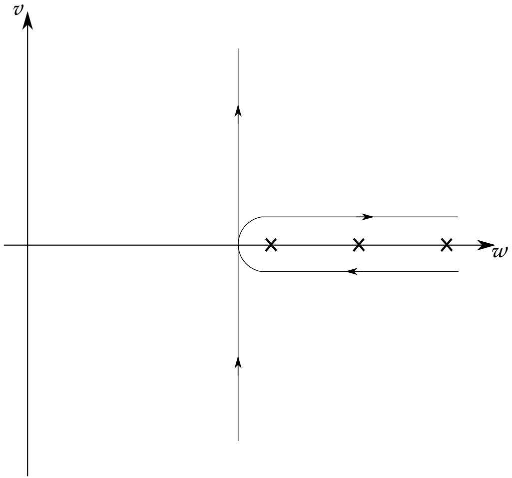


Figure: Contour enclosing infinite number of poles on the real axis at $s < w < \infty$ is deformed into a contour consisting of a vertical line at $w = s$.

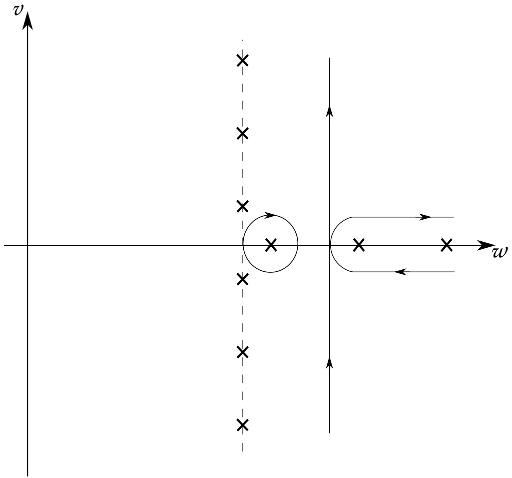


Figure: Infinite number of poles on the real axis at $s < w < \infty$ is enclosed by two separate contours, the right one is deformed into a contour consisting of a vertical line at $w = s + \pi$. Poles on the complex plane are on a vertical line at $w = s$.

Relation

$$\sum_{n\in\mathbb{Z}} \frac{y \sin x}{\cos x + \cosh[(2n+1)\pi y]}$$

$$= \frac{1}{\pi} \int_0^\infty \mathrm{d}\eta \, \frac{\sin x \sinh(2\pi/y)}{(\cos x + \cosh \eta)[\cosh(2\pi/y) + \cos(\eta/y)]}$$

$$- \frac{2 \sinh\{2[\arctan\left(\tan \frac{x}{2}\right)]/y\}}{\cosh(\pi/y) + \cosh\{2[\arctan\left(\tan \frac{x}{2}\right)]/y\}}$$

may have diverse applications in fermion field theory at finite temperature and chemical potential.

Limiting case:

$$\lim_{y\to 0} \sum_{n\in\mathbb{Z}} \frac{y \sin x}{\cos x + \cosh[(2n+1)\pi y]} = \frac{x}{\pi}, \quad -\pi < x < \pi.$$

$$J^{z5} = -\frac{eB}{2\pi a}\left\{\mathrm{sgn}(\mu) F\left(|\mu| a + \mathrm{sgn}(\mu)\left[\tilde{\varphi} - \mathrm{sgn}(\tilde{\varphi})\pi/2\right]; Ta\right) - \frac{1}{\pi}\left[\tilde{\varphi} - \mathrm{sgn}(\tilde{\varphi})\pi/2\right]\right\}, \quad (32)$$

where

$$F(s;t) = \frac{s}{\pi} - \frac{1}{\pi}\int_0^\infty \mathrm{d}v \frac{\sin(2s)\sinh(\pi/t)}{[\cos(2s) + \cosh(2v)][\cosh(\pi/t) + \cos(v/t)]} + \frac{\sinh\{[\arctan(\tan s)]/t\}}{\cosh[\pi/(2t)] + \cosh\{[\arctan(\tan s)]/t\}}. \quad (33)$$

In the case of a magnetic field filling the whole (infinite) space we obtain the known resut:

$$\lim_{a\to\infty} J^{z5} = -\frac{eB}{2\pi^2}\mu. \quad (34)$$

Asymptotics at small and large temperatures:

$$\lim_{T\to 0} J^{z5} = -\frac{eB}{2\pi a}\left[\mathrm{sgn}(\mu)\left[\!\left[\frac{|\mu|a + \mathrm{sgn}(\mu)\tilde{\varphi}}{\pi} + \Theta(-\mu\tilde{\varphi})\right]\!\right] - \frac{\tilde{\varphi}}{\pi} + \frac{1}{2}\mathrm{sgn}(\tilde{\varphi})\right] \tag{35}$$

and

$$\lim_{T\to\infty} J^{z5} = -\frac{eB}{2\pi^2}\mu; \tag{36}$$

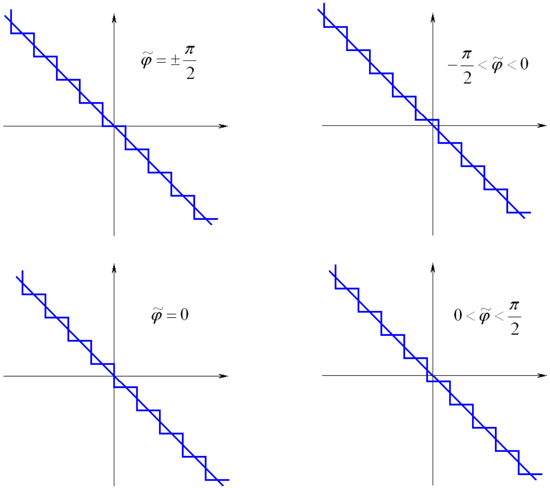
here $[\![u]\!]$ denotes the integer part of quantity u, and $\Theta(u) = \frac{1}{2}[1 + \mathrm{sgn}(u)]$ is the step function.

The chiral separation effect can be nonvanishing even at zero chemical potential:

$$J^{z5}|_{\mu=0} = -\frac{eB}{2\pi a}\left\{F(\tilde{\varphi} - \mathrm{sgn}(\tilde{\varphi})\pi/2; Ta) - \frac{1}{\pi}\left[\tilde{\varphi} - \mathrm{sgn}(\tilde{\varphi})\pi/2\right]\right\}; \tag{37}$$

the latter vanishes in the limit of infinite temperature,

$$\lim_{T\to\infty} J^{z5}|_{\mu=0} = 0. \tag{38}$$



$\widetilde{\varphi} = \pm\frac{\pi}{2}$
$-\frac{\pi}{2} < \widetilde{\varphi} < 0$
$\widetilde{\varphi} = 0$
$0 < \widetilde{\varphi} < \frac{\pi}{2}$

Figure:

The trivial boundary condition, $\tilde{\varphi} = -\pi/2$, gives spectrum $k_l = (l + \frac{1}{2})\frac{\pi}{a} \quad (l = 0, 1, 2, \ldots)$, and the axial current density at zero temperature for this case was obtained earlier (E. V. Gorbar et al, 2015)

$$J^{z5} \mid_{T=0,\ \tilde{\varphi}=-\pi/2} = -\frac{eB}{2\pi a}\mathrm{sgn}(\mu)[\![|\mu|a/\pi + 1/2]\!]. \qquad (39)$$

The "bosonic-type" spectrum, $k_l = l\frac{\pi}{a}$ $(l = 0, 1, 2, \ldots)$, is given by $\tilde{\varphi} = 0$; note that the axial current density is continuous at this point:

$$\lim_{\tilde{\varphi}\to 0_+} J^{z5} - \lim_{\tilde{\varphi}\to 0_-} J^{z5} = 0. \qquad (40)$$

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Conclusion 2

The significant role of boundaries for chiral effects in hot dense magnetized spinor matter:

- dependence on both temperature and chemical potential,
- dependence on the boundary parameter,
- the boundary condition can serve as a source that is additional to the spinor matter density.

Thank you for your attention!