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Quantum field theory in strong external fields at high densities and temperatures. I.

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Spinor matter in extremal conditions:

- hot and dense
- in strong magnetic field

Physical systems in:

- relativistic heavy-ion collisions
- compact astrophysical objects (neutron stars and magnetars)
- the early universe
- novel materials (the Dirac and Weyl semimetals) Cd_3As_2, Na_3Bi, K_3Bi, Rb_3Bi, $TaAs$, $BaAuBi$, $BaCuBi$, $BaAgBi$, Bi_2Se_3, $TlBiSe_2$, ...

Operator of quantized spinor field in a static background

$$\hat{\Psi}(\mathbf{x},t) = \sum_{E_k>0} e^{-\mathrm{i}E_k t} \langle \mathbf{x}|k\rangle \, \hat{a}_k + \sum_{E_k<0} e^{-\mathrm{i}E_k t} \langle \mathbf{x}|k\rangle \, \hat{b}_k^\dagger, \qquad (1)$$

where $\hat{a}_k^\dagger$ and $\hat{a}_k$ ($\hat{b}_k^\dagger$ and $\hat{b}_k$) are the spinor particle (antiparticle) creation and destruction operators satisfying anticommutation relations,

$$\left[\hat{a}_k,\, \hat{a}_{k'}^\dagger\right]_+ = \left[\hat{b}_k,\, \hat{b}_{k'}^\dagger\right]_+ = \left\langle k|k'\right\rangle, \qquad (2)$$

and $\langle \mathbf{x}|k\rangle$ is the solution to the stationary Dirac equation,

$$H \langle \mathbf{x}|k\rangle = E_k \langle \mathbf{x}|k\rangle, \qquad (3)$$

H is the Dirac hamiltonian, k is the set of parameters (quantum numbers) specifying a one-particle state, E_k is the energy of the state, and the vacuum is defined as

$$\hat{a}_k \, |\mathrm{vac}\rangle = \hat{b}_k \, |\mathrm{vac}\rangle = 0. \qquad (4)$$

Operators of dynamical variables (physical observables) in second-quantized theory are defined as bilinears of the fermion field operator (1).

Fermion number operator

$$\hat{N} = \frac{1}{2}\int \mathrm{d}^d x(\hat{\Psi}^\dagger\hat{\Psi} - \hat{\Psi}^T\hat{\Psi}^{\dagger T}) = \sum_k \left[\hat{a}_k^\dagger\hat{a}_k - \hat{b}_k^\dagger\hat{b}_k - \frac{1}{2}\mathrm{sgn}(E_k)\right], \tag{5}$$

where $\mathrm{sgn}(\pm u) = \pm 1$ at $u > 0$.

Energy (temporal component of the energy-momentum vector) operator,

$$\hat{P}^0 = \frac{1}{2}\int \mathrm{d}^d x(\hat{\Psi}^\dagger H\hat{\Psi} - \hat{\Psi}^T H^T\hat{\Psi}^{\dagger T}) = \sum_k |E_k| \left(\hat{a}_k^\dagger\hat{a}_k + \hat{b}_k^\dagger\hat{b}_k - \frac{1}{2}\right). \tag{6}$$

Partition function of the grand canonical ensemble:

$$Z(\beta,\mu) = \mathrm{Sp}\exp\left[-\beta(\hat{P}^0 - \mu\hat{N})\right], \quad \beta = (k_B T)^{-1}. \qquad (7)$$

Averages over the grand canonical ensemble:

$$\left\langle \hat{N} \right\rangle_{\beta,\mu} = Z^{-1}(\beta,\mu)\,\mathrm{Sp}\,\hat{N}\exp\left[-\beta(\hat{P}^0 - \mu\hat{N})\right] \qquad (8)$$

and

$$\left\langle \hat{P}^0 \right\rangle_{T,\mu} = Z^{-1}(\beta,\mu)\,\mathrm{Sp}\,\hat{P}^0\exp\left[-\beta(\hat{P}^0 - \mu\hat{N})\right]. \qquad (9)$$

Sp is the sum over all possible occupation numbers

Fock space in second-quantized theory: $|\{i_k\}\rangle$

$$\hat{n}_k = \hat{a}_k^\dagger \hat{a}_k \qquad E_k > 0 \qquad \text{(particles)}$$

$$\hat{n}_k = \hat{b}_k^\dagger b_k \qquad E_k < 0 \qquad \text{(antiparticles)}$$

$$\hat{n}_l \, |\{i_k\}\rangle = i_l \, |\{i_k\}\rangle \quad l \in \{k\} \quad i_k = 0,\, 1$$

$$\hat{N}|\{i_k\}\rangle = \left[\sum_{l,\, E_l > 0} \left(\hat{n}_l - \frac{1}{2}\right)\right] |\{i_k\}\rangle - \left[\sum_{l,\, E_l < 0} \left(\hat{n}_l - \frac{1}{2}\right)\right] |\{i_k\}\rangle$$

$$\hat{P}^0|\{i_k\}\rangle = \left[\sum_{l} |E_l| \left(\hat{n}_l - \frac{1}{2}\right)\right] |\{i_k\}\rangle$$

$\mu = 0$

$$\left\langle \hat{n}_l - \frac{1}{2} \right\rangle_\beta = \frac{\sum_{0,1} \sum_{k \neq l} \langle \ \{i_k\} | \ e^{-\beta \hat{P}^0} \ | \ \{i_k\} \rangle \left\langle i_l \left| \left(\hat{n}_l - \tfrac{1}{2}\right) e^{-\beta \hat{P}^0} \right| i_l \right\rangle}{\sum_{0,1} \sum_{k \neq l} \langle \ \{i_k\} | \ e^{-\beta \hat{P}^0} \ | \ \{i_k\} \rangle \left\langle i_l \left| e^{-\beta \hat{P}^0} \right| i_l \right\rangle}$$

$$= \frac{-\frac{1}{2} e^{\frac{1}{2}\beta |E_l|} + \frac{1}{2} e^{-\frac{1}{2}\beta |E_l|}}{e^{\frac{1}{2}\beta |E_l|} + e^{-\frac{1}{2}\beta |E_l|}} = -\frac{1}{2} \tanh\left(\frac{1}{2}\beta |E_l|\right)$$

$$\left\langle \hat{N} \right\rangle_\beta = \sum_{l,\, E_l > 0} \left\langle \hat{n}_l - \frac{1}{2} \right\rangle_\beta - \sum_{l,\, E_l < 0} \left\langle \hat{n}_l - \frac{1}{2} \right\rangle_\beta$$

$$= -\frac{1}{2} \sum_k \tanh\left(\frac{1}{2}\beta E_k\right)$$

H is the hamiltonian in first-quantized theory

$$H\langle x|k\rangle = E_k \langle x|k\rangle$$

orthonormality: $\int d^d x \langle k|x\rangle \langle x|k'\rangle = \langle k|k'\rangle$
completeness: $\sum_k \langle x|k\rangle \langle k|x'\rangle = \langle x|x'\rangle I$
H is self-adjoint, then $f(H)$ can be defined,

$$\mathrm{Tr}\ f(H) \equiv \int d^d x\ \mathrm{tr}\langle x|f(H)|x\rangle = \sum_k \sum_{k'} \int d^d x\ \mathrm{tr}\langle x|k\rangle$$

$$\times \left\langle k|f(H)|k'\right\rangle \left\langle k'|x\right\rangle = \sum_k \sum_{k'} \int d^d x\ \mathrm{tr}\left\langle k'|x\right\rangle \langle x|k\rangle \left\langle k|f(H)|k'\right\rangle$$

$$= \sum_k \sum_{k'} \left\langle k'|k\right\rangle \left\langle k|f(H)|k'\right\rangle = \sum_k \langle k|f(H)|k\rangle = \sum_k f(E_k)$$

$$\left\langle \hat{N} \right\rangle_{\beta,\mu} = \frac{\mathrm{Sp}\ \hat{N}\ \exp\left[-\beta(\hat{P}^0 - \mu\hat{N})\right]}{\mathrm{Sp}\ \exp\left[-\beta(\hat{P}^0 - \mu\hat{N})\right]} = -\frac{1}{2}\mathrm{Tr}\ \tanh\left[\frac{1}{2}\beta(H - \mu I)\right]$$

and

$$\left\langle \hat{P}^0 \right\rangle_{\beta,\mu} = \frac{\mathrm{Sp}\ \hat{P}^0\ \exp\left[-\beta(\hat{P}^0 - \mu\hat{N})\right]}{\mathrm{Sp}\ \exp\left[-\beta(\hat{P}^0 - \mu\hat{N})\right]} = -\frac{1}{2}\mathrm{Tr}\ H\ \tanh\left[\frac{1}{2}\beta(H - \mu I)\right]$$

Fermi-Dirac distribution:

$$\langle \hat{n}_l \rangle_{\beta,\mu} = \{\exp[\beta(E_l - \mu))] + 1\}^{-1}\,, \quad E_l > 0$$

and

$$\langle \hat{n}_l \rangle_{\beta,\mu} = \{\exp[\beta(-E_l + \mu)] + 1\}^{-1}\,, \quad E_l < 0.$$

One can define densities of the averages, for instance:

$$\left\langle \hat{N} \right\rangle_{\beta,\mu} = \int d^d x\ \rho_N(x;\ \beta, \mu)$$

$$\rho_N(x;\ \beta, \mu) = -\frac{1}{2}\mathrm{tr}\left\langle x \left| \tanh\left[\frac{1}{2}\beta(H - \mu I)\right] \right| x \right\rangle$$

Matsubara formalism

Causal Green function:

$$G(t-t',\, x, x') = i \left\langle \mathrm{vac} \left| T\hat{\Psi}(t,x)\bar{\hat{\Psi}}(t',x') \right| \mathrm{vac} \right\rangle$$

$$(-i\gamma^{\nu} D_{\nu} + m)\, G(t-t',\, x, x') = \delta(t-t')\delta^{d}(x-x')$$

$$D_{\nu} = \partial_{\nu} + ieA_{\nu}, \quad -i\gamma^{v} D_{v} + m = \gamma^{0}(-i\partial_{0} + H)$$

$H = -i\vec{\alpha}\cdot\vec{D} + eA_0 + \gamma^0 m$ is the Dirac hamiltonian; $\vec{\alpha} = \gamma^0\vec{\gamma}$.
Transition to imaginary time: $t = -i\tau$

$$0 < \tau < \beta, \quad \beta = (k_B T)^{-1}$$

$$\gamma^{0}(\partial_{\tau} + H)G\left(-i(\tau-\tau'), x, x'\right) = i\Delta(\tau-\tau')\delta^{d}(x-x')$$

Chemical potential μ: $A_0(x,t) = \tilde{A}_0(x,t) - \mu$

$$H_\mu \equiv H - \mu I, \quad \gamma^0(\partial_\tau + H_\mu) G\Big(-i(\tau-\tau'), x, x'\Big) = i\Delta(\tau-\tau')\delta^d(x-x')$$

antiperiodicity: $\Delta(\tau+\beta) = -\Delta(\tau)$

$$G\Big(-i(\tau+\beta-\tau'), x, x'\Big) = -G\Big(-i(\tau-\tau'), x, x'\Big)$$

$$G\Big(-i(\tau-\tau'),\, x, x'\Big) = \frac{i}{\beta}\sum_{n\in\mathbb{Z}} e^{-i\omega_n(\tau-\tau')}\left\langle x \left| \left[\gamma^0\left(H_\mu - i\omega_n\right)\right]^{-1} \right| x'\right\rangle$$

$$\Delta(\tau) = \frac{1}{\beta}\sum_{n\in\mathbb{Z}} e^{i\omega_n\tau}, \quad \omega_n = \frac{2\pi}{\beta}(n+\frac{1}{2})$$

$$\left\langle T\hat{\Psi}(-i\tau, x)\bar{\hat{\Psi}}(-i\tau', x')\right\rangle_{\beta,\mu} = -iG\Big(-i(\tau-\tau'),\, x, x'\Big)$$

$$= \frac{1}{\beta}\sum_{n\in\mathbb{Z}} e^{-i\omega_n(\tau-\tau')}\left\langle x \left| \left[\gamma^0\left(H_\mu - i\omega_n\right)\right]^{-1} \right| x'\right\rangle$$

$T = 0,\ \mu = 0$

Identity:

$$T\hat{\Psi}(t,x)\bar{\hat{\Psi}}(t',x') = \frac{1}{2}\left[\hat{\Psi}(t,x)\bar{\hat{\Psi}}(t',x') - \bar{\hat{\Psi}}^T(t',x')\hat{\Psi}^T(t,x)\right]$$

$$+\frac{1}{2}\mathrm{sgn}(t-t')\left[\hat{\Psi}(t,x)\bar{\hat{\Psi}}(t',x') + \bar{\hat{\Psi}}^T(t',x')\hat{\Psi}^T(t,x)\right]$$

its consequence:

$$\hat{\Psi}^\dagger(t,x)\hat{\Psi}(t,x') - \hat{\Psi}^T(t,x')\hat{\Psi}^{\dagger T}(t,x)$$

$$= \lim_{t-t'\to 0_+} T\bar{\hat{\Psi}}(t,x)\gamma^0\hat{\Psi}(t',x') + \lim_{t-t'\to 0_-} T\bar{\hat{\Psi}}(t,x)\gamma^0\hat{\Psi}(t',x')$$

and $\left\langle \mathrm{vac}\left|\hat{N}\right|\mathrm{vac}\right\rangle$

$$= \frac{1}{2}\int d^dx \left\langle \mathrm{vac}\left|\left[\hat{\Psi}^\dagger(t,x)\hat{\Psi}(t,x) - \hat{\Psi}^T(t,x)\hat{\Psi}^{\dagger T}(t,x)\right]\right|\mathrm{vac}\right\rangle$$

$$= -\frac{1}{2}\left(\lim_{t-t'\to 0_+} + \lim_{t-t'\to 0_-}\right)\int d^dx\ \mathrm{tr}\ \gamma^0\left\langle \mathrm{vac}\left|T\hat{\Psi}(t',x)\bar{\hat{\Psi}}(t,x)\right|\mathrm{vac}\right\rangle$$

$T > 0, \mu \neq 0$

$$\left\langle \hat{N} \right\rangle_{\beta,\mu} = -\frac{1}{2}\left(\lim_{\tau-\tau'\to 0_+} + \lim_{\tau-\tau'\to 0_-} \right) \int d^d x \, \mathrm{tr}\, \gamma^0 \left\langle T\hat{\Psi}(-i\tau')\hat{\bar{\Psi}}(-i\tau \right\rangle_{\beta,\mu}$$

$$= -\frac{1}{\beta} \int d^d x \ \mathrm{tr} \sum_{n\in Z} \gamma^0 \left\langle x \left| \left[\gamma^0 \left(H_\mu - i\omega_n \right) \right]^{-1} \right| x \right\rangle$$

$$= -\frac{1}{\beta} \mathrm{Tr} \sum_{n\in Z} (H_\mu - i\omega_n)^{-1} = -\frac{1}{\beta} \mathrm{Tr} \sum_{n\geq 0} \left[(H_\mu - i\omega_n)^{-1} + (H_\mu + i\omega_n)^{-1} \right]$$

$$= -\frac{2}{\beta} \mathrm{Tr} \sum_{n\geq 0} \frac{H_\mu}{H_\mu^2 + \omega_n^2} = -\frac{1}{2} \mathrm{Tr} \ \tanh\left(\frac{1}{2} H_\mu \right),$$

where the use is made of relation

$$\sum_{n\geq 0} \frac{y}{y^2 + (2n+1)^2} = \frac{\pi}{4} \tanh(\frac{1}{2}\pi y).$$

Taking limit:

$$\left\langle \hat{N} \right\rangle_{\infty,0} = -\frac{1}{2} \mathrm{Tr} \ \mathrm{sgn}(H) = \left\langle \mathrm{vac} \left| \hat{N} \right| \mathrm{vac} \right\rangle$$

Polarization tensor

$$\Pi^{\nu\nu'}(x,y) = \frac{\delta^2\Gamma(A)}{\delta A_{\nu'}(y)\delta A_\nu(x)}$$

$$= i\left\langle \mathrm{vac}\left| Tj^\nu(x)j^{\nu'}(y)\right| \mathrm{vac}\right\rangle - i\left\langle \mathrm{vac}\left|j^\nu(x)\right| \mathrm{vac}\right\rangle \left\langle \mathrm{vac}\left|j^{\nu'}(y)\right| \mathrm{vac}\right\rangle$$

where

$$e^{i\Gamma(A)} = \int d\bar\psi d\psi\ \exp\left\{ i\int d^{d+1}x\ \bar\psi\left[i\gamma^\nu\left(\partial_\nu + ieA_\nu\right) - m\right]\psi\right\}$$

$$j^\nu(x) = e\bar\psi(x)\gamma^\nu\psi(x)$$

$$\Pi^{\nu\nu'}(x,y) = -ie^2\mathrm{tr}\gamma^\nu\left\langle \mathrm{vac}\left| T\hat\Psi(x)\bar{\hat\Psi}(y)\right| \mathrm{vac}\right\rangle \gamma^{\nu'}\left\langle \mathrm{vac}\left| T\hat\Psi(y)\bar\Psi(x)\right| \mathrm{vac}\right\rangle$$

Transition to imaginary time: $x^0 = -i\tau \quad y^0 = -i\tau'$

$$\left\langle T\hat{\Psi}(-i\tau,\vec{x})\bar{\hat{\Psi}}(-i\tau',\vec{y})\right\rangle_{\beta,\mu} = \frac{1}{\beta}\sum_{n\in Z} e^{-i\omega_n(\tau-\tau')}$$

$$\times\left\langle \vec{x}\left|\left[\gamma^0(H_\mu - i\omega_n)\right]^{-1}\right|\vec{y}\right\rangle$$

$$H_\mu = -i\vec{\alpha}(\vec{\partial} - ie\vec{A} + \gamma^0 m - \mu, \quad \omega_n = \frac{2\pi}{\beta}(n+\frac{1}{2})$$

$$\Pi^{jk}(\vec{x},i\tau;\,\vec{y},-i\tau') = -\frac{ie^2}{\beta}\sum_{n,n'\in Z}\exp\left[-i(\omega_n-\omega_{n'})(\tau-\tau')\right]$$

$$\times\mathrm{tr}\;\alpha^j\left\langle\vec{x}\left|(H_\mu - i\omega_n)^{-1}\right|\vec{y}\right\rangle\alpha^k\left\langle\vec{y}\left|(H_\mu - i\omega_{n'})^{-1}\right|\vec{x}\right\rangle$$

Momentum representation

$$\int_0^\beta d\tau\int_0^\beta d\tau'\, e^{i\Omega_m\tau - i\Omega_{m'}\tau'}\Pi^{jk} = i\beta^2\delta_{mm'}\tilde{\Pi}(\Omega_m;\,\vec{x},\vec{y})$$

where $\Omega_m = \frac{2\pi}{\beta}m, \quad m\in Z$

$$\tilde{\Pi}^{jk}(\Omega_m;\ \vec{x},\vec{y}) = -\frac{e^2}{\beta^2}\sum_{n\in Z}\mathrm{tr}\,\alpha^j \left\langle \vec{x}\left|\left(H_\mu - i\omega_n - \frac{i}{2}\Omega_m\right)^{-1}\right|\vec{y}\right\rangle$$

$$\times\alpha^k \left\langle \vec{y}\left|\left(H_\mu - i\omega_n + \frac{i}{2}\Omega_m\right)^{-1}\right|\vec{x}\right\rangle$$

Translational invariance: $\tilde{\Pi}^{jk}(\Omega_m;\ \vec{x},\vec{y}) = \tilde{\Pi}^{jk}(\Omega_m;\ \vec{x}-\vec{y})$

$$\int d^d x\, d^d y\, e^{-i\vec{q}(\vec{x}-\vec{y})}\tilde{\Pi}^{jk}(\Omega_m;\ \vec{x}-\vec{y}) = (2\pi)^d \pi^{jk}(\Omega_m,\vec{q})\delta^d(\vec{q}-\vec{q}')$$

where

$$\pi^{jk}(\Omega_m,\vec{q}) = -\frac{e^2}{\beta^2}\int d^d x\ e^{-i\vec{q}\vec{x}}\sum_{n\in Z}\mathrm{tr}$$

$$\times\alpha^j \left\langle \vec{x}\left|\left(H_\mu - i\omega_n - \frac{i}{2}\Omega_m\right)^{-1}\right|\vec{0}\right\rangle \alpha^k \left\langle \vec{0}\left|\left(H_\mu - i\omega_n + \frac{i}{2}\Omega_m\right)^{-1}\right|\vec{x}\right\rangle$$

Polarization tensor at $\vec{q} = \vec{0}$ and $\Omega_m \neq 0$:

$$\pi^{0k}(\Omega_m, \vec{0})$$

$$= -\frac{e^2}{\beta^2} \sum_{n \in Z} \mathrm{tr}\ \alpha^k \left\langle \vec{0} \left| (H_\mu - i\omega_n + \frac{i}{2}\Omega_m)^{-1} (H_\mu - i\omega_n - \frac{i}{2}\Omega_m)^{-1} \right| \vec{0} \right\rangle = 0$$

where the use is made of formulas

$$\int d^d x \mid x \rangle \langle x \mid = 1, \quad \sum_{n \in Z} \frac{1}{n + ix} \frac{1}{n + iy} = \frac{\pi}{x - y} [\mathrm{cth}(\pi x) - \mathrm{cth}(\pi y)]$$

also $\pi^{00}(\Omega_m, \vec{0}) = 0$
This is consistent with gauge invariance

$$q_\nu \pi^{\nu\nu'} = 0 \quad \Rightarrow \quad \Omega_m \pi^{0k} = \Omega_m \pi^{00} = 0$$

Polarization tensor at $\vec{q} = \vec{0}$ and $\Omega_m = 0$:

$$\pi^{0k}(0,\vec{0}) = -\frac{e^2}{\beta^2}\sum_{n\in Z} \mathrm{tr}\ \alpha^k \left\langle \vec{0} \left| (H_\mu - i\omega_n)^{-2} \right| \vec{0} \right\rangle$$

$$= \frac{e^2}{4}\mathrm{tr}\ \alpha^k \left\langle \vec{0} \left| \mathrm{sech}^2\left(\frac{1}{2}\beta H_\mu\right) \right| \vec{0} \right\rangle$$

$$\pi^{00}(0,\vec{0}) = \frac{e^2}{4}\mathrm{tr} \left\langle \vec{0} \left| \mathrm{sech}^2\left(\frac{1}{2}\beta H_\mu\right) \right| \vec{0} \right\rangle$$

Quadratic fluctuation of the charge:

$$V\pi^{00}(0,\vec{0}) = \frac{e^2}{4}\mathrm{Tr}\ \mathrm{sech}^2\left[\frac{1}{2}\beta\left(H - \mu I\right)\right]$$

Correlation of the charge with the current:

$$V\pi^{0k}(0,\vec{0}) = \frac{e^2}{4}\mathrm{Tr}\ \alpha^k \mathrm{sech}^2\left[\frac{1}{2}\beta\left(H - \mu I\right)\right]$$

Physical observables

	first-quantized theory	second-quantized theory
charge	eI	$\hat{Q} = \frac{e}{2}\int d^d x \left(\hat{\Psi}^\dagger \hat{\Psi} - \hat{\Psi}^T \hat{\Psi}^{\dagger T}\right)$
energy	H	$\hat{P}^0 = \frac{1}{2}\int d^d x \left(\hat{\Psi}^\dagger H \hat{\Psi} - \hat{\Psi}^T H^T \hat{\Psi}^{\dagger T}\right)$
conserved observable	J	$\hat{M} = \frac{1}{2}\int d^d x \left(\hat{\Psi}^\dagger J \hat{\Psi} - \hat{\Psi}^T J^T \hat{\Psi}^{\dagger T}\right)$
nonconserved observable	Υ	$\hat{U} = \frac{1}{2}\int d^d x \left(\hat{\Psi}^\dagger \Upsilon \hat{\Psi} - \hat{\Psi}^T \Upsilon^T \hat{\Psi}^{\dagger T}\right)$

Figure:

Partition function: $Z(\beta, \mu_e) = Sp\, e^{-\beta(\hat{P}^0 - \mu_e \hat{Q})}$
Thermodynamic potential: $\Omega(\beta, \mu_e) = -\frac{1}{\beta} \ln Z(\beta, \mu_e)$
Average of the charge:

$$\left\langle \hat{Q} \right\rangle_{\beta,\mu_e} = -\frac{\partial}{\partial \mu_e} \Omega(\beta, \mu_e) = Z^{-1}(\beta, \mu_e) Sp\ \hat{Q}\ e^{-\beta(\hat{P}^0 - \mu_e \hat{Q})}$$

$$= -\frac{e}{2} \mathrm{Tr}\ \tanh\left[\frac{1}{2}\beta(H - \mu_e eI)\right]$$

Quadratic fluctuation

$$\left\langle \hat{Q}^2 \right\rangle_{\beta,\mu_e} - \left(\left\langle \hat{Q} \right\rangle_{\beta,\mu_e}\right)^2 = -\frac{1}{\beta} \frac{\partial^2}{\partial^2 \mu_e} \Omega\left(\beta, \mu_e\right)$$

$$= \frac{e^2}{4} \mathrm{Tr}\ \mathrm{sech}^2\left[\frac{1}{2}\beta\left(H - \mu_e eI\right)\right]$$

Conservation of observables

first-quantized theory $\Rightarrow$ second-quantized theory

$[J, H]_- = 0 \quad \Rightarrow \quad \hat{M}$ is diagonal along with $\hat{P}^0$ and $\hat{Q}$

Generalized partition function:

$Z(\beta, \mu_e, \mu_J) = e^{-\beta\Omega(\beta,\mu_e,\mu_J)} = Sp\, e^{-\beta(\hat{P}^0 - \mu_e \hat{Q} - \mu_J \hat{M})}$

Average of the conserved observable:

$$\left\langle \hat{M} \right\rangle_{\beta,\mu_e} = -\frac{\partial}{\partial \mu_J}\Omega(\beta, \mu_e, \mu_J)\mid_{\mu_J=0} = Z^{-1}(\beta, \mu_e) Sp \hat{M} e^{-\beta(\hat{P}^0 - \mu_e \hat{Q})}$$

$$= -\frac{1}{2}\mathrm{Tr}\, J \tanh\left[\frac{1}{2}\beta(H - \mu_e e I)\right]$$

Quadratic fluctuation:

$$\left\langle \hat{M}^2 \right\rangle_{\beta,\mu_e} - \left(\left\langle \hat{M} \right\rangle_{\beta,\mu_e}\right)^2 = -\frac{1}{\beta}\frac{\partial^2}{\partial^2 \mu_J}\Omega(\beta, \mu_e, \mu_J)\mid_{\mu_J=0}$$

$$= \frac{1}{4}\mathrm{Tr}\, J^2 \,\mathrm{sech}^2\left[\frac{1}{2}\beta(H - \mu_e e I)\right]$$

The partition function of the fermionic system is presented as the functional integral over the Grassman fields

$$\exp[-\beta\Omega(\beta,\mu_e,\mu_J)] = \int d\psi^\dagger d\psi\, e^{-S},$$

where

$$S = \int_0^\beta d\tau \int d^d x\, \psi^+(\partial_\tau - \mu_e e I - \mu_J J + H)\psi$$

is the Euclidean action, τ is the imaginary time. The integral is of the Gauss type and can be immediately computed

$$\exp[-\beta\Omega(\beta,\mu_e,\mu_J)] = \det(\partial_\tau + H_\mu), \quad H_\mu = H - \mu_e e I - \mu_J J.$$

Hence the thermodynamic potential is given by expression

$$\Omega(\beta,\mu_e,\mu_J) = -\frac{1}{\beta}\ln\det(\partial_\tau + H_\mu) =$$

$$= -\frac{1}{\beta}\int_0^\beta d\tau \int d^d x\, \mathrm{tr}\langle \mathbf{x},\tau|\ln(\partial_\tau + H_\mu)|\mathbf{x},\tau\rangle.$$

In the case of a static background, operators H and J are τ-independent, and the integration over τ is performed by using the antiperiodicity boundary condition at the ends of the imaginary time interval:

$$\Omega(\beta, \mu_e, \mu_J) = -\frac{1}{\beta} \sum_{n \in Z} \int d^d x \, \mathrm{tr} \langle \mathbf{x} | \ln(H_\mu - i\omega_n) | \mathbf{x} \rangle,$$

where $\omega_n = \frac{2\pi}{\beta}(n + \frac{1}{2})$. Using the notation of functional trace, one gets further

$$\Omega(\beta, \mu_e, \mu_J) = -\frac{1}{\beta} \sum_{n \in Z} \mathrm{Tr} \ln(H_\mu - i\omega_n) = -\frac{1}{\beta} \mathrm{Tr} \ln \prod_{n=0}^{\infty} (H_\mu^2 + \omega_n^2)$$

$$= -\frac{1}{\beta} \left\{ \mathrm{Tr} \ln \prod_{n=0}^{\infty} \left[\left(\frac{H_\mu}{\omega_n} \right)^2 + 1 \right] + \mathrm{Tr} \ln \prod_{n=0}^{\infty} \omega_n^2 \right\}.$$

The second term in the figure brackets is dropped as an irrelevant infinite constant, and the infinite product in the first term is computed with the use of relation

$$\cosh(\frac{\pi a}{2}) = \prod_{n=0}^{\infty} \left[1 + a^2(2n+1)^{-2}\right].$$

As a result we obtain

$$\Omega(\beta, \mu_e, \mu_J) = -\frac{1}{\beta}\mathrm{Tr}\ln\cosh\left(\frac{1}{2}\beta H_\mu\right)$$

$$= -\frac{1}{\beta}\mathrm{Tr}\ln\{1 - \exp\left[-\beta\left(H - \mu_e eI - \mu_J J\right)\mathrm{sgn}\left(H\right)\right]\}$$

Average of the nonconserved observable:

$$\left\langle \hat{U} \right\rangle_{\beta,\mu_e} = Z^{-1}(\beta,\mu_e) Sp\, \hat{U}\, e^{-\beta(\hat{P}^0 - \mu_e \hat{Q})}$$

$$= -\frac{1}{2} \mathrm{Tr}\, \Upsilon\ \tanh \left[\frac{1}{2} \beta (H - \mu_e e I) \right]$$

Correlation:

$$\left\langle \hat{U}\hat{M} \right\rangle_{\beta,\mu_e} - \left\langle \hat{U} \right\rangle_{\beta,\mu_e} \left\langle \hat{M} \right\rangle_{\beta,\mu_e} = \frac{1}{4} \mathrm{Tr}\, \Upsilon J\, \mathrm{sech}^2 \left[\frac{1}{2} \beta (H - \mu_e e I) \right]$$

Correlations in Matsubara formalism

Functional integral

$$e^{i\Gamma[\{g_j\}]} = \int d\bar{\psi} d\psi \exp\left[i \left(S + \int dt\, d^d x \sum_j g_j \bar{\psi}\gamma^0 \Upsilon_j \psi \right) \right]$$

where $S = \int dt\, d^d x\, \bar{\psi}\gamma^0 (i\partial_t - H_\mu)\psi$
Transition to imaginary time $t = -i\tau$ with $0 < \tau < \beta$

$$e^{-\tilde{\Gamma}[\{g_j\}]} = \int d\psi^\dagger d\psi \exp\left[- \left(\tilde{S} - \int d\tau d^d x \sum_j g_j \psi^\dagger \Upsilon_j \psi \right) \right]$$

where $\tilde{S} = \int d\tau d^d x\, \psi^\dagger (\partial_\tau + H_\mu)\psi$

$$\left\langle T \hat{\Psi}^\dagger \Upsilon_k \hat{\Psi} \right\rangle_{\beta,\mu} = -\frac{\delta \tilde{\Gamma}[\{g_j\}]}{\delta g_k} \Big|_{\{g_j\}=0} = -\frac{\int d\psi^\dagger d\psi\, \psi^\dagger \Upsilon_k \psi\, \exp(-\tilde{S})}{\int d\psi^\dagger d\psi\, \exp(-\tilde{S})}$$

$$\hat{U}_k = \frac{1}{2}\left(\lim_{\tau-\tau'\to 0_+} + \lim_{\tau-\tau'\to 0_-}\right)\int d^d x\, T\hat{\Psi}^\dagger(-i\tau, x)\Upsilon_k\hat{\Psi}(-i\tau', x)$$

Generalized polarization tensor

$$\Pi^{jk}(x,\tau;y,\tau') = -\left.\frac{\delta^2\,\tilde{\Gamma}[\{g\}]}{\delta g_k(y,\tau')\delta g_j(x,\tau)}\right|_{\{g\}=0}$$

$$= -\mathrm{tr}\,\Upsilon_j\left\langle T\hat{\Psi}(x,\tau)\hat{\Psi}^\dagger(y,\tau')\right\rangle_{\beta,\tau}\Upsilon_k\left\langle T\hat{\Psi}(y,\tau')\hat{\Psi}^\dagger(x,\tau)\right\rangle_{\beta,\tau}$$

where

$$\left\langle T\hat{\Psi}(x,\tau)\hat{\Psi}^\dagger(y,\tau')\right\rangle_{\beta,\tau} = \left\langle x,\tau\left|(\partial_\tau + H_\mu)^{-1}\right|y,\tau'\right\rangle$$

$$= \frac{1}{\beta}\sum_{n\in Z} e^{-i\omega_n(\tau-\tau')}\left\langle x\left|(H - i\omega_n)^{-1}\right|y\right\rangle$$

$$\omega_n = \frac{2\pi}{\beta}(n + \frac{1}{2})$$

Fourier transformation:

$$\int_0^\beta d\tau \int_0^\beta d\tau' \, e^{i\Omega_m \tau - i\Omega_{m'}\tau'} \int d^d x \int d^d y \, e^{-iqx + iq'y} \, \Pi^{jk}(x,\tau; y,\tau')$$

$$= \beta^2 \delta_{mm'} \pi^{jk}(\Omega_m; q, q'), \quad \Omega_m = \frac{2\pi}{\beta} m, \quad \omega_n = \frac{2\pi}{\beta}(n + \frac{1}{2})$$

where, in the case of $q = q' = 0$

$$\pi^{jk}(\Omega_m; 0, 0) = -\frac{1}{\beta^2} \mathrm{Tr} \sum_{n \in Z} \Upsilon_j (H_\mu - i\omega_n - \frac{i}{2}\Omega_m)^{-1} \Upsilon_k (H_\mu - i\omega_n + \frac{i}{2}\Omega_m)^{-1}$$

Quadratic fluctuation:

$$\left\langle \hat{U}_k^2 \right\rangle_{\beta,\mu} - \left(\left\langle \hat{U}_k^2 \right\rangle_{\beta,\mu} \right)^2 = \pi^{kk}(0;\ 0,0)$$

$$= -\frac{1}{\beta^2} \mathrm{Tr} \sum_{n \in Z} \Upsilon_k (H_\mu - i\omega_n)^{-1} \Upsilon_k (H_\mu - i\omega_n)^{-1}$$

Correlation:

$$\left\langle \hat{U}_j \hat{U}_k \right\rangle_{\beta,\mu} - \left\langle \hat{U}_j \right\rangle_{\beta,\mu} \left\langle \hat{U}_k \right\rangle_{\beta,\mu} = \pi^{jk}(0;\ 0,0)$$

$$= -\frac{1}{\beta^2} \mathrm{Tr} \sum_{n \in Z} \Upsilon_j (H_\mu - i\omega_n)^{-1} \Upsilon_k (H_\mu - i\omega_n)^{-1}$$

Correlation with conserved observable:

$$\left\langle \hat{U}_k \hat{M} \right\rangle_{\beta,\mu} - \left\langle \hat{U}_k \right\rangle_{\beta,\mu} \left\langle \hat{M} \right\rangle_{\beta,\mu} = -\frac{1}{\beta^2} \mathrm{Tr} \sum_{n \in Z} \Upsilon_k J (H_\mu - i\omega_n)^{-2}$$

$$= \frac{1}{4} \mathrm{Tr}\, \Upsilon_k\, J\, \mathrm{sech}^2 \left(\frac{1}{2} \beta H_\mu \right)$$

Further thermodynamic characteristics

Entropy:

$$S = -\frac{\partial}{\partial T}\Omega(\beta,\mu) = \frac{1}{T}\left[\left\langle \hat{P}^0 \right\rangle_{\beta,\mu} - \mu \left\langle \hat{N} \right\rangle_{\beta,\mu} - \Omega(\beta,\mu)\right]$$

Heat capacity:

$$C_v = T\left[\frac{\partial S}{\partial T} - \left(\frac{\partial}{\partial T}\left\langle \hat{N} \right\rangle_{\beta,\mu}\right)^2 \left(\frac{\partial}{\partial \mu}\left\langle \hat{N} \right\rangle_{\beta,\mu}\right)^{-1}\right]$$

$$= \frac{1}{k_{\mathrm{B}} T^2}\left[\left\langle \left(\hat{P}^0\right)^2 \right\rangle_{\beta,\mu} - \left(\left\langle \hat{P}^0 \right\rangle_{\beta,\mu}\right)^2 - \frac{\left(\left\langle \hat{P}^0 \hat{N} \right\rangle_{\beta,\mu} - \left\langle \hat{P}^0 \right\rangle_{\beta,\mu}\left\langle \hat{N} \right\rangle_{\beta,\mu}\right)^2}{\left\langle \hat{N}^2 \right\rangle_{\beta,\mu} - \left(\left\langle \hat{N} \right\rangle_{\beta,\mu}\right)^2}\right]$$

Local characteristics (densities)

Conserved observables

$\hat{M}_k(x) = \frac{1}{2}\left[\hat{\Psi}^\dagger(x) J_k \hat{\Psi} - \hat{\Psi}^T(x) J_k^T \hat{\Psi}^{\dagger\, T}(x)\right]$

Average:

$$\left\langle \hat{M}_k(x) \right\rangle_{\beta,\mu} = -\frac{1}{2}\mathrm{tr}\left\langle x \left| J_k \tanh(\frac{1}{2}\beta H_\mu) \right| x \right\rangle$$

Quadratic fluctuation:

$$\left\langle \hat{M}_k^2(x) \right\rangle_{\beta,\mu} - \left(\left\langle \hat{M}_k(x) \right\rangle_{\beta,\mu} \right)^2 = \frac{1}{4}\mathrm{tr}\left\langle x \left| J_k^2 \mathrm{sech}^2\left(\frac{1}{2}\beta H_\mu\right) \right| x \right\rangle$$

Correlation:

$$\left\langle \hat{M}_k(x)\hat{M}_{k'}(x) \right\rangle_{\beta,\mu} - \left\langle \hat{M}_k(x) \right\rangle_{\beta,\mu} \left\langle \hat{M}_{k'}(x) \right\rangle_{\beta,\mu}$$

$$= \frac{1}{4}\mathrm{tr}\left\langle x \left| J_k J_{k'} \mathrm{sech}^2\left(\frac{1}{2}\beta H_\mu\right) \right| x \right\rangle$$

Nonconserved observables

$\hat{U}_k(x) = \frac{1}{2}\left[\hat{\Psi}^\dagger(x)\Upsilon_k\hat{\Psi} - \hat{\Psi}^T(x)\Upsilon_k^T\hat{\Psi}^{\dagger T}(x)\right]$

Average:

$$\left\langle \hat{U}_k(x) \right\rangle_{\beta,\mu} = -\frac{1}{2}\mathrm{tr}\left\langle x \left| \Upsilon_k \tanh\left(\frac{1}{2}\beta H_\mu\right) \right| x \right\rangle$$

Quadratic fluctuation:

$$\left\langle \hat{U}_k^2(x) \right\rangle_{\beta,\mu} - \left(\left\langle \hat{U}_k(x) \right\rangle_{\beta,\mu}\right)^2$$

$$= -\frac{1}{\beta^2}\sum_{n\in Z}\mathrm{tr}\left\langle x \left| \Upsilon_k (H_\mu - i\omega_n)^{-1}\Upsilon_k (H_\mu - i\omega_n)^{-1} \right| x \right\rangle$$

Correlation of two nonconserved observables:

$$\left\langle \hat{U}_k(x)\hat{U}_{k'}(x)\right\rangle_{\beta,\mu} - \left\langle \hat{U}_k(x)\right\rangle_{\beta,\mu}\left\langle \hat{U}_{k'}(x)\right\rangle_{\beta,\mu}$$

$$= -\frac{1}{\beta^2}\sum_{n\in Z}\mathrm{tr}\left\langle x\left|\Upsilon_k(H_\mu - i\omega_n)^{-1}\Upsilon_{k'}(H_\mu - i\omega_n)^{-1}\right| x\right\rangle$$

Correlation of nonconserved and conserved observables:

$$\left\langle \hat{U}_k(x)\hat{M}_{k'}(x)\right\rangle_{\beta,\mu} - \left\langle \hat{U}_k(x)\right\rangle_{\beta,\mu}\left\langle \hat{M}_{k'}(x)\right\rangle_{\beta,\mu}$$

$$= \frac{1}{4}\mathrm{tr}\left\langle x\left|\Upsilon_k J_{k'}\mathrm{sech}^2\left(\frac{1}{2}\beta H_\mu\right)\right| x\right\rangle$$

Resolvent and spectral density

$(H-\omega)^{-1}$ where ω is a complex parameter

If H is self-adjoint operator, than
resolvent is defined at $Im\,\omega \neq 0$

Spectral density of the self-adjoint operator:

$$\tau(E) = \pm\frac{1}{\pi} Im\,\mathrm{Tr}(H - E \mp i0)^{-1} = \mathrm{Tr}\,\delta(H-E)$$

$$\mathrm{Tr}\,f(H) = \int_{-\infty}^{\infty} dE\,\tau(E)\,f(E) = \int_{C} \frac{d\omega}{2\pi i}\mathrm{Tr}(H-\omega)^{-1} f(\omega)$$

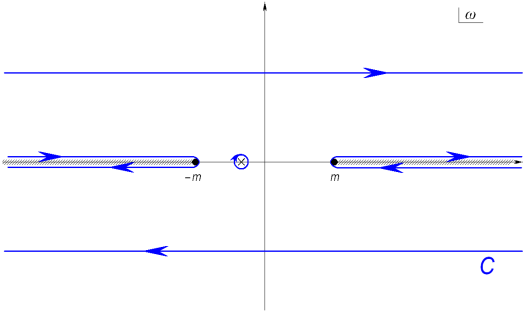

Figure: Singularities of the resolvent on the complex ω-plane.

Ideal nonrelativistic Fermi gas

spectral density:

$$\tau(E) = \frac{gV}{(4\pi)^{d/2}} \left(\frac{2m}{\hbar^2}\right)^{d/2} \frac{E^{\frac{d}{2}-1}}{\Gamma(d/2)}, \qquad E = \frac{\hbar^2 k^2}{2m}$$

thermodynamic potential:

$$\Omega(\beta, \mu) = -\frac{1}{\beta} \int_0^\infty dE\, \tau(E) \ln\left[1 + e^{-\beta(E-\mu)}\right]$$

average particle number:

$$\left\langle \hat{N} \right\rangle_{\beta,\mu} = \int_0^\infty \frac{dE\, \tau(E)}{\exp[\beta(E-\mu)] + 1}, \quad \int_0^\infty dE \tau(E) \ldots = \frac{gV}{(2\pi)^d} \int d^d k \ldots$$

quadratic fluctuation of particle number:

$$\left\langle \hat{N}^2 \right\rangle_{\beta,\mu} - \left(\left\langle \hat{N} \right\rangle_{\beta,\mu}\right)^2 = \frac{1}{4} \int_0^\infty \frac{dE\, \tau(E)}{\cosh^2[\frac{1}{2}\beta(E-\mu)]}$$

Ideal relativistic Fermi gas.I.

spectral density:

$$\tau(E) = \frac{gV\,|E|\,\left(E^2 - m^2\right)^{\frac{d}{2}-1}}{(4\pi)^{d/2}\Gamma(d/2)}\Theta\left(E^2 - m^2\right), \quad E = \pm\sqrt{k^2 + m^2}$$

thermodynamic potential:

$$\Omega(\beta,\mu) = -\frac{1}{\beta}\int\limits_{-\infty}^{\infty} dE\,\tau(E)\ln\left\{1 + e^{-\beta[|E| - \mu\,\mathrm{sgn}(E)]}\right\}$$

$$= -\frac{1}{\beta}\int\limits_{m}^{\infty} d\epsilon\,\tau(\epsilon)\,\ln\left\{\left[1 + e^{-\beta(\epsilon-\mu)}\right]\left[1 + e^{-\beta(\epsilon+\mu)}\right]\right\}$$

$$\epsilon = |E|, \qquad 2\int\limits_{m}^{\infty} d\epsilon\,\tau(\epsilon)\ldots = \frac{gV}{(2\pi)^d}\int d^d k\ldots$$

Ideal relativistic Fermi gas.II.

average particle number:

$$\left\langle \hat{N} \right\rangle_{\beta,\mu} = -\frac{1}{2}\int_{-\infty}^{\infty} dE\,\tau(E)\tanh\left[\frac{1}{2}\beta(E-\mu)\right] = \int_{m}^{\infty}\frac{d\epsilon\,\tau(\epsilon)\sinh(\beta\mu)}{\cosh(\beta\epsilon)+\cosh(\beta\mu)}$$

quadratic fluctuation of particle number:

$$\left\langle \hat{N}^2 \right\rangle_{\beta,\mu} - \left(\left\langle \hat{N}^2 \right\rangle_{\beta,\mu}\right)^2 = \frac{1}{4}\int_{-\infty}^{\infty}\frac{dE\,\tau(E)}{\cosh^2\left[\frac{1}{2}\beta(E-\mu)\right]}$$

$$= \int_{m}^{\infty} d\epsilon\,\tau(\epsilon)\frac{\cosh(\beta\epsilon)\cosh(\beta\mu)+1}{[\cosh(\beta\epsilon)+\cosh(\beta\mu)]^2}$$

Note

$$\left\langle \hat{N} \right\rangle_{\beta,0} = 0, \quad \left\langle \hat{N} \right\rangle_{\infty,\mu} = 0, \quad \left\langle \hat{N}^2 \right\rangle_{\infty,\mu} = 0, \quad [\beta\Omega(\beta,\mu)]\,|_{\beta=\infty} = 0$$

Ideal relativistic Fermi gas.III.

average energy:

$$\left\langle \hat{P}^0 \right\rangle_{\beta,\mu} = \int_m^\infty d\epsilon\, \tau(\epsilon)\, \epsilon\, \frac{\cosh(\beta\mu) + 1}{\cosh(\beta\epsilon) + \cosh(\beta\mu)}$$

quadratic fluctuation of energy:

$$\left\langle (\hat{P}^0)^2 \right\rangle_{\beta,\mu} - \left(\left\langle \hat{P}^0 \right\rangle_{\beta,\mu} \right)^2 = \frac{\mu}{\beta} \frac{\partial}{\partial\mu} \left\langle \hat{P}^0 \right\rangle_{\beta,\mu} - \frac{\partial}{\partial\beta} \left\langle \hat{P}^0 \right\rangle_{\beta,\mu}$$

$$= \int_m^\infty d\epsilon\, \tau(\epsilon)\, \epsilon^2\, \frac{\sinh(\beta\epsilon)[\cosh(\beta\mu) + 1]}{[\cosh(\beta\epsilon) + \cosh(\beta\mu)]^2}$$

Ideal relativistic Fermi gas.IV.

correlation with particle number:

$$\left\langle \hat{P}^0 \hat{N} \right\rangle_{\beta,\mu} - \left\langle \hat{P}^0 \right\rangle_{\beta,\mu} \left\langle \hat{N} \right\rangle_{\beta,\mu} = \frac{1}{\beta} \frac{\partial}{\partial \mu} \left\langle \hat{P}^0 \right\rangle_{\beta,\mu}$$

$$= \int_m^\infty d\epsilon \, \tau(\epsilon) \, \epsilon \, \frac{[\cosh(\beta\epsilon) - 1]\sinh(\beta\mu)}{[\cosh(\beta\epsilon) + \cosh(\beta\mu)]^2}$$

Note

$$\left\langle \hat{P}^0 \right\rangle_{\infty,\mu} = 0, \quad \left\langle (\hat{P}^0)^2 \right\rangle_{\infty,\mu} = \left\langle \hat{P}^0 \hat{N} \right\rangle_{\infty,\mu} = 0, \quad \left\langle \hat{P}^0 \hat{N} \right\rangle_{\beta,0} = 0$$

Ideal relativistic Fermi gas.V.

heat capacity:

$$C_V = k_{\mathrm{B}} \frac{gV\beta^{-d}}{(4\pi)^{d/2}\Gamma(d/2)} \left\{ \int_{\beta m}^{\infty} du\, u^3 \left(u^2 - \beta^2 m^2\right)^{\frac{d}{2}-1} \frac{\sinh u (\cosh\beta\mu + 1)}{(\cosh u + \cosh\beta\mu)^2} \right.$$

$$\left. - \frac{\left[\int_{\beta m}^{\infty} du\, u^2 \left(u^2 - \beta^2 m^2\right)^{\frac{d}{2}-1} \frac{(\cosh u - 1)\sinh\beta\mu}{(\cosh u + \cosh\beta\mu)^2} \right]^2}{\int_{\beta m}^{\infty} du\, u \left(u^2 - \beta^2 m^2\right)^{\frac{d}{2}-1} \frac{\cosh u \cosh\beta\mu + 1}{(\cosh u + \cosh\beta\mu)^2}} \right\}$$

$$\mu = 0: \quad C_V = k_{\mathrm{B}} \frac{gV\beta^{-d}}{(4\pi)^{d/2}\Gamma(d/2)} \int_{\beta m}^{\infty} du u^3 \left(u^2 - \beta^2 m^2\right)^{\frac{d}{2}-1} \frac{\sinh(u/2)}{\cosh^3(u/2)}$$

Thank you for your attention!