Some Higher-Derivative Theories in Practical Terms

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Summary

1 - Historical Aspects

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R. Courant and D. Hilbert, Methoden der Mathematischen Physik I (Julius Springer, 1924);

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B. Podolsky, “A Generalized Electrodynamics Part I-Non-Quantum,” Phys. Rev. **62**, 68 (1942);

B. Podolsky and C. Kikuchi, “A Generalized Electrodynamics Part II-Quantum,” Phys. Rev. **65**, 228-235 (1944);

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Pais-Uhlenbeck Oscillator

Pais and Uhlenbeck, "On Field theories with nonlocalized action," Phys. Rev. **79**, 145 (1950).

$$\frac{d^4q}{dt^4} + (\omega_1^2 + \omega_2^2)\frac{d^2q}{dt^2} + \omega_1^2\omega_2^2 q = 0 \tag{1}$$

$$L(q, \dot{q}, \ddot{q}) = \frac{\gamma}{2}[\ddot{q}^2 - ({\omega_1}^2 + {\omega_2}^2)\dot{q}^2 + {\omega_1}^2{\omega_2}^2 q^2] \tag{2}$$

$$(\partial_0^2 - \nabla^2)(\partial_0^2 - \nabla^2 + M^2)\phi(t, \mathbf{x}) = 0\,, \tag{3}$$

$$\phi(t, \mathbf{x}) = q(t)e^{i\mathbf{k}\cdot\mathbf{x}} \tag{4}$$

$$\omega_1^2 + \omega_2^2 = 2\mathbf{k}^2 + M^2 \quad \text{and} \quad \omega_1^2\omega_2^2 = \mathbf{k}^2(\mathbf{k}^2 + M^2) \tag{5}$$

2 - Generalizations of the Klein-Gordon Equation

Since Fock, Gordon, Klein and Schrödinger, the KG equation is a deep fundamental relation in QFT, concerning all elementary particles.
In modern notation, the KG equation may be written as

$$(\Box + m^2)\phi = 0\,, \tag{6}$$

or simply $K_m\phi = 0$ with

$$K_m \equiv \Box + m^2 \tag{7}$$

denoting the *Klein-Gordon operator*.

As a second-order partial differential equation, it is in the essence of the mathematician/theoretical physicist to investigate consistent extensions and generalizations of the Klein-Gordon equation (6).

2 - Generalizations of the Klein-Gordon Equation

C. G. Bollini and J. J. Giambiagi, "Generalized Klein-Gordon Equation in d-dimensions From Supersymmetry," Phys. Rev. D **32**, 3316 (1985).

D. G. Barci, M. C. Rocca and C. G. Bollini, "Quantization of a fourth order wave equation," Nuovo Cim. A **103**, 597 (1990).

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A. Cherman, L. G. Ferreira Filho, L. L. Santos Guedes and J. A. Helayël-Neto, "Explicit classical solutions and comments on Higher-Derivative Klein-Gordon equation in (1+1)-D," Rev. Mex. Fis. **58**, 384 (2012).

Y. W. Kim, Y. S. Myung and Y. J. Park, "BRST quantization of a sixth-order derivative scalar field theory," Mod. Phys. Lett. A **28**, 1350182 (2013).

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Mod. Phys. Lett. A **36**, no.28, 2150205 (2021)

The d'Alembertian $\Box \equiv \partial_\mu \partial^\mu$ is a regular local covariant 2nd-order differential operator, which can act recursively.
Introduce a length-dimensional multiplicative factor $a > 0$ and, for $n \in \mathbb{N}$, define

$$\mathcal{L}_n \equiv -\frac{a^{2(n-1)}}{2n!}\phi \Box^n \phi \tag{8}$$

Next, consider the $2N$-th order Lagrangian density

$$\mathcal{L}^{(2N)} \equiv \sum_{n=0}^{n=N} \mathcal{L}_n = -\frac{1}{2}\phi \left(\sum_{n=0}^{N} \frac{a^{2(n-1)}\Box^n}{n!} \right) \phi \,, \tag{9}$$

for a fixed $N \in \mathbb{N}$.
It is then natural to investigate the behavior of (9) in the limit of arbitrarily large N, for which we define further the complete Lagrangian density

$$\mathcal{L}_\phi \equiv -\frac{1}{2a^2}\phi e^{a^2 \Box} \phi \,. \tag{10}$$

Integrating in space-time, we may define the natural actions

$$S^{(2N)} = -\frac{1}{2}\int d^D x\, \phi(x) \sum_{n=0}^{N} a^{2(n-1)} \frac{\Box^n \phi(x)}{n!} \tag{11}$$

and

$$S_\phi = -\frac{1}{2a^2}\int d^D x \phi(x) e^{a^2\Box}\phi(x)\,. \tag{12}$$

The field equation associated to (11) reads

$$\sum_{n=0}^{N} \frac{a^{2(n-1)}\Box^n}{n!}\phi = 0\,, \tag{13}$$

while in the limit of arbitralily large N, corresponding to (12), we have

$$e^{a^2\Box}\phi = 0\,. \tag{14}$$

It can be shown that, for even N or in the infinity limit, equations (13) or (14) do not have nontrivial real solutions while, for odd N, equation (13) has exactly one nontrivial positive real classical solution.

For odd N, equation (13) has exactly one positive real classical solution given by

$$\phi_{(N)}(x) = \int \frac{d^{D-1}\mathbf{p}}{E_N(\mathbf{p}^2)} \left\{ \varphi_N(\mathbf{p}) e^{-i(E_N(\mathbf{p}^2)\,t - \mathbf{p}\cdot\mathbf{x})} + \varphi_N^*(\mathbf{p}) e^{i(E_N(\mathbf{p}^2)\,t - \mathbf{p}\cdot\mathbf{x})} \right\},$$

with

$$E_N(\mathbf{p}^2) \equiv \sqrt{\mathbf{p}^2 - q_N/a^2}\,, \tag{15}$$

where q_N represents the dimensionless real root of the algebraic equation

$$f_N(q) = 0 \tag{16}$$

with $f_N(q)$ defined as the N-th order polynomial in the dimensionless real variable q given by

$$f_N(q) \equiv \sum_{n=0}^{N} \frac{(-1)^n N!}{(N-n)!} q^{N-n}\,. \tag{17}$$

In terms of a given external current $J(x)$, we may write the functional generator associated to action (11) as

$$Z^{(2N)}[J] = \mathcal{N} \int [d\phi] \exp\left\{ iS^{(2N)} + i \int d^D x \, J(x)\phi(x) \right\}, \tag{18}$$

with

$$\mathcal{N}^{-1} \equiv \int [d\phi] \exp\left\{ iS^{(2N)} \right\}. \tag{19}$$

The propagator for the scalar field $\phi(x)$ can be immediately computed as

$$D^{(2N)} = \frac{-ia^2}{\sum_{n=0}^{N} (-1)^n \frac{\left(a^2 p^2\right)^n}{n!}} \tag{20}$$

and has a real pole for odd N at $p^2 = q_N/a^2$ with q_N denoting the only real solution to the polynomial equation (16).

More details can be found in

Mod. Phys. Lett. A **36**, no.28, 2150205 (2021).

3 - Bopp-Podolsky and Lee-Wick Theories

$$\mathcal{L}_B = -\frac{1}{4}\left[F_{\mu\nu}F^{\mu\nu} - a^2\partial_\rho F^{\mu\nu}\partial^\rho F_{\mu\nu}\right] \quad \text{Bopp (1940)}$$

$$\mathcal{L}_P = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\rho F_{\mu\rho} \quad \text{Podolsky (1942)}$$

$$\frac{1}{q^2} - \frac{1}{q^2 - m^2} \quad \text{Lee-Wick (1969)}$$

$$\mathcal{L} = -\frac{1}{4}\left(G_{\mu\nu}^2 + F_{\mu\nu}^2\right) - \frac{1}{2}(m_B B_\mu)^2 \quad \text{Lee-Wick (1970)}$$

$$D_{\mu\nu} = \frac{-i}{k^2}\left(\frac{m_B^2}{k^2 + m_B^2}\right)\delta_{\mu\nu} + (\cdots k_\mu k_\nu)$$

$$\mathcal{L}_{LW} = -\frac{1}{4}\left[F_{\mu\nu}F^{\mu\nu} + a^2F^{\mu\nu}\Box F_{\mu\nu}\right] \quad \text{The Lee-Wick Lagrangian}$$

A generalized electrodynamics

Generalized Coulomb Potential (electrostacitcs) $V(r) = \dfrac{1 - e^{-r/a}}{r}$

Generalized Maxwell equations

$$(1 + a^2\Box)\nabla \cdot \mathbf{E} = j^0 \qquad \nabla \cdot \mathbf{B} = 0\,, \tag{21}$$

$$(1 + a^2\Box)(\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t}) = \mathbf{j} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0\,. \tag{22}$$

Generalized Poisson Equation (GPE)

$$(1 - a^2\nabla^2)\nabla^2\phi = -4\pi\rho \tag{23}$$

The general solution of the GPE (23) as well as some interesting particular solutions can be found in reference

C. R. Ji, A. T. Suzuki, J. H. O. Sales and RT, Eur. Phys. J. C **79**, no.10, 871 (2019).

Generalized Scalar Electrodynamics

Interaction with a charged bosonic field

$$\mathcal{L}_{int} = ieA_\mu \left[\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right] + e^2A^2|\phi|^2 \,, \tag{24}$$

with $|\phi|^2 \equiv \phi^*\phi$. The dynamics is given by the Lagrangians

$$\mathcal{L}_\phi = \partial_\mu\phi^*\partial^\mu\phi - m^2|\phi|^2 \qquad \text{and} \tag{25}$$

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\rho F_{\mu\rho} \;\; \text{with} \;\; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{26}$$

By demanding stationarity of the total gauge invariant action

$$S = \int d^4x \, \left[\mathcal{L}_\phi + \mathcal{L}_A + \mathcal{L}_{int}\right] \tag{27}$$

with respect to arbitrary variations of ϕ and A_μ we obtain the field equations:

Generalized Scalar Electrodynamics

Field equations

$$(\Box + m^2)\phi = -ieA_\mu \partial^\mu \phi - ie\partial^\mu(\phi A_\mu) + e^2 A^2 \phi \,, \tag{28}$$

$$(\Box + m^2)\phi^* = ieA_\mu \partial^\mu \phi^* + ie\partial^\mu(\phi^* A_\mu) + e^2 A^2 \phi^* \tag{29}$$

$$(1 + a^2\Box)\partial_\nu F^{\mu\nu} = ie(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) + 2e^2 A^\mu |\phi|^2 \tag{30}$$

4 - Gauge-Fixing and Propagators

C. R. Ji, A. T. Suzuki, J. H. O. Sales and RT, Eur. Phys. J. C **79**, no.10, 871 (2019).
I. G. Oliveira, J. H. Sales and RT, Eur. Phys. J. Plus **135**, no.9, 713 (2020).

Considering the Landau gauge

$$\mathcal{L}_L = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \frac{a^2}{2\xi}(\partial_\lambda \partial_\mu A^\mu)(\partial^\lambda \partial_\nu A^\nu) \qquad (31)$$

we can invert the corresponding gauge field kinetic term and obtain

$$P_{\mu\nu}(k) = \frac{-i}{(1-a^2k^2)k^2}\left[\eta_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2}\right] . \qquad (32)$$

$$\begin{aligned} Z[j^\mu] &= N \int DA_\mu DC D\bar{C} DB \exp \{ iS_0 \\ &+ i \int d^4x \left[\bar{C}(1+a^2\Box)\Box C + B(1+a^2\Box)\partial^\mu A_\mu \right. \\ &\left. - \frac{a^2\xi}{2}\partial_\mu B \partial^\mu B + \frac{\xi B^2}{2} - j^\mu A_\mu \right] \} . \end{aligned} \qquad (33)$$

Axial Gauge in the Light-Front

Concerning the axial gauge in the Light-Front

$$\mathcal{L}_A = -\frac{1}{2\alpha}(n_a A^a)^2 + \frac{a^2}{2\alpha}(n_a \partial_c A^a)(n_b \partial^c A^b)\,, \tag{34}$$

we have the propagator

$$P_{ab}(k) = \frac{-i}{k^2(1-a^2k^2)}\left[\eta_{ab} + \frac{(\alpha k^2 + n^2)}{(n\cdot k)^2}k_a k_b - \frac{1}{(n\cdot k)}(k_a n_b + k_a n_b)\right]\,.$$

For the usual light-front gauge we choose a light-like direction n, with $n^2 = 0$, and consider the limit $\alpha \to 0$. In this case

$$P_{ab} = \frac{-i}{k^2(1-a^2k^2)}\left[\eta_{ab} - \frac{1}{(n\cdot k)}(k_a n_b + k_b n_a)\right]\,. \tag{35}$$

Light-Front Gauges

In order to obtain the doubly transverse three-term propagator we may use the mixed gauge-fixing (A. T. Suzuki and J. H. O. Sales, Nucl. Phys. A **725**, 2003)

$$\mathcal{L}_3 = -\frac{1}{\beta}(n \cdot A)(\partial \cdot A) + \frac{a^2}{\beta}(n_a \partial_c A^a)(\partial_b \partial^c A^b)\,, \tag{36}$$

$$\begin{aligned} P_{ab}(k) &= \frac{-i}{k^2(1-a^2k^2)}\left[\eta_{ab} + \frac{\beta^2 k^2 + n^2}{(n\cdot k)^2 - n^2k^2}k_a k_b - \frac{n\cdot k + i\beta k^2}{(n\cdot k)^2 - n^2k^2}k_a n_b + \right. \\ &\left. - \frac{n\cdot k + i\beta k^2}{(n\cdot k)^2 - n^2k^2}k_b n_a + \frac{k^2}{(n\cdot k)^2 - n^2k^2}n_a n_b\right] \end{aligned} \tag{37}$$

Light-Front Gauges

Going back to the particular light-front gauge case where $n^2 = 0$ and taking the limit $\beta \to 0$ we get

$$P_{ab}(k) = \frac{-1}{k^2(1-a^2k^2)}\left[\eta_{ab} - \frac{1}{(n\cdot k)}(k_a n_b + k_b n_a) + \frac{k^2}{(n\cdot k)^2} n_a n_b\right]$$

which is the corresponding three-term generalized photon propagator in the light-front gauge for the BP model.

As can be directly checked, the three-term propagator satisfies

$$k^a P_{ab} = 0 \tag{38}$$

and

$$n^a P_{ab} = 0 \tag{39}$$

being in this sense doubly transverse.

5 - Reduction of Order

$$L(q,\dot{q},\ddot{q}) = \frac{\gamma}{2}[\ddot{q}^2 - ({\omega_1}^2 + {\omega_2}^2)\dot{q}^2 + {\omega_1}^2{\omega_2}^2 q^2] \tag{40}$$

$$L_c(q,x,\lambda,\dot{q},\dot{x},\dot{\lambda}) = \frac{\gamma}{2}[\dot{x}^2 - ({\omega_1}^2 + {\omega_2}^2)x^2 + {\omega_1}^2{\omega_2}^2 q^2] + \lambda(\dot{q} - x)\,, \tag{41}$$

P. D. Mannheim and A. Davidson, Phys. Rev. A **71**, 042110 (2005).

$$H_T = \frac{p_x^2}{2\gamma} + \frac{\gamma}{2}(\omega_1^2 + \omega_2^2)x^2 - \frac{\gamma}{2}\omega_1^2\omega_2^2 q^2 + p_q x + \gamma\omega_1^2\omega_2^2 q p_\lambda\,. \tag{42}$$

$$\chi_1 \equiv p_q - \lambda\,, \quad \text{and} \quad \chi_2 \equiv p_\lambda\,, \tag{43}$$

5 - Reduction of Order

Back to Bopp-Podolsky, we may try the same

$$\mathcal{L}_{int} + \mathcal{L}_{\phi} = ieA_{\mu}\left[\phi\partial^{\mu}\phi^{*} - \phi^{*}\partial^{\mu}\phi\right] + e^{2}A^{2}|\phi|^{2} + \partial_{\mu}\phi^{*}\partial^{\mu}\phi - m^{2}|\phi|^{2} \tag{44}$$

$$\mathcal{L}_{A} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^{2}}{2}\partial_{\nu}F^{\mu\nu}\partial^{\rho}F_{\mu\rho} \tag{45}$$

Reducing the derivatives order, we decouple the massive and massless modes

$$\mathcal{L}_{AB} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{a^{2}}{2}B_{\mu}B^{\mu} + a^{2}\partial_{\mu}B_{\nu}F^{\mu\nu}\,, \tag{46}$$

and obtain the equivalent reduced-order model

$$S_{red} = \int d^{4}x\left\{\mathcal{L}_{\phi} + \mathcal{L}_{AB} + \mathcal{L}_{int}\right\} \tag{47}$$

I. G. Oliveira, J. H. Sales and RT, Eur. Phys. J. Plus **135**, no.9, 713 (2020).

5 - Reduction of Order

Field equations

$$(\Box + m^2)\phi = -ieA_\mu\partial^\mu\phi - ie\partial^\mu(\phi A_\mu) + e^2A^2\phi\,, \tag{48}$$

$$(\partial^\mu\partial^\nu - \Box\eta^{\mu\nu})A_\nu = B^\mu\,, \tag{49}$$

$$(\partial^\mu\partial^\nu - \Box\eta^{\mu\nu})(A_\nu - a^2B_\nu) = ie\,[\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi] + 2e^2A^\mu|\phi|^2\,. \tag{50}$$

The auxiliary vector field B_μ takes the mass from A_μ which becomes now massless. The gauge invariance is preserved.

Canonical Quantization

$$\mathcal{L} = \frac{1}{2} F_{0i} F_{0i} - a^2 (\partial_0 B_i - \partial_i B_0) F_{0i} - \mathcal{H}_{sp} \, , \tag{51}$$

$$\mathcal{H}_{sp} \equiv \frac{1}{4} F_{ij} F_{ij} - a^2 \partial_i B_j F_{ij} + \frac{a^2}{2} B_\mu B^\mu \, . \tag{52}$$

$$H_c = \int d^3x \left[-\frac{\Pi^i \Pi_B^i}{a^2} - \frac{\Pi_B^i \Pi_B^i}{2a^4} + \mathcal{H}_{sp} - A_0 \partial_i \Pi^i - B_0 \partial_i \Pi_B^i \right] \, . \tag{53}$$

Constraints in phase space:

$$\chi_1 = \Pi_B^0 \approx 0 \, , \tag{54}$$

$$\chi_2 = \partial_i \Pi_B^i - a^2 B_0 \approx 0 \, , \tag{55}$$

$$\chi_3 = \Pi^0 \approx 0 \, , \tag{56}$$

$$\chi_4 = \partial_i \Pi^i \approx 0 \, . \tag{57}$$

Constraints χ_1 and χ_2 are second-class while χ_3 and χ_4 are first-class.

Gauge fixing:

$$\chi_5 = A_0\,, \quad \chi_6 = \partial_i A_i\,. \tag{58}$$

Table: Dirac Brackets

	A_j	B_0	B_j	Π^j	Π^j_B
A_i	.	.	.	$(\delta^j_i - \frac{\partial_i\partial_j}{\nabla^2})$	.
B_0	.	.	$-\frac{1}{a^2}\partial_j$	.	.
B_i	.	$-\frac{1}{a^2}\partial_i$	.	.	δ^j_i
Π^i	$(-\delta^i_j + \frac{\partial_i\partial_j}{\nabla^2})$	.	.	.	.
Π^i_B	.	.	$-\delta^i_j$	.	.

$$[A_i(\mathbf{x}), \Pi^j(\mathbf{y})]^* = (\delta^j_i - \frac{\partial_i\partial_j}{\nabla^2})\delta(\mathbf{x}-\mathbf{y}) \qquad [B_i(\mathbf{x}), \Pi^j_B(\mathbf{y})]^* = \delta^j_i\delta(\mathbf{x}-\mathbf{y}) \tag{59}$$

$$[B_i(\mathbf{x}), B_0(\mathbf{y})]^* = -\frac{1}{a^2}\partial_i\delta(\mathbf{x}-\mathbf{y}) \tag{60}$$

More details can be found in

Braz. J. Phys. **47**, no.1, 72-80 (2017).

Conclusion and Final Remarks

- The quest for understanding higher-derivative models in QFT is a longterm one - an important challenging still open problem.
- We have proposed a natural higher-order generalization for the KG equation and investigated its classical solutions.
- We have obtained a natural class of higher-order gauge-fixing terms for Bopp-Podolsky and Lee-Wick like theories.
- The reduction of order technique can be very helpful and handy for higher-derivative theories.
- We have not discussed here the important open interelated issues of unitarity, causality, positiveness and propagating ghost modes.

The main parts of this talk were based on the following four papers:

RT, "Reduced Order Podolsky Model," Braz. J. Phys. **47**, no.1, 72-80 (2017) [arXiv:1606.09319].

C. R. Ji, A. T. Suzuki, J. H. O. Sales and RT, "Pauli–Villars regularization elucidated in Bopp–Podolsky's generalized electrodynamics," Eur. Phys. J. C **79**, no.10, 871 (2019) [arXiv:1902.07632].

I. G. Oliveira, J. H. Sales and RT, "Bopp–Podolsky scalar electrodynamics propagators and energy-momentum tensor in covariant and light-front coordinates," Eur. Phys. J. Plus **135**, no.9, 713 (2020) [arXiv:2008.03735].

RT, "Natural Higher-Derivatives Generalization for the Klein-Gordon Equation," Mod. Phys. Lett. A **36**, No. 28, 2150205 (2021) [arXiv:2011.02567].

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