

Towards hidden symmetries in gauge theories

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The non-abelian generalization of the gauge symmetry proposed by C. N. Yang and R. Mills in 1954 was done à la Maxwell, i.e., in terms of a set of partial differential equations. However, the integral formulation counterpart of this generalization was not known until quite recently.

The critical problem in constructing the integral Yang-Mills equations is the need for a consistent definition of the flux of the non-abelian electric and magnetic fields with which we can build a relationship with the dynamically conserved charges in such a way that these charges are invariant under gauge transformations. Indeed, the naive definition of the flux of the non-abelian fields $\Phi(F) = \int_{\Sigma} F_{\mu\nu} \frac{\partial x^{\mu}}{\partial \sigma} \frac{\partial x^{\nu}}{\partial \tau} d\sigma d\tau$ is strongly dependent of the gauge choice since under a local gauge transformation $g(x)$, $F_{\mu\nu}(x) \rightarrow g(x)F_{\mu\nu}(x)g^{-1}(x)$ and therefore, the flux through a closed surface cannot be directly associated to gauge-invariant charges inside.

The problem of finding the gauge-invariant charges in non-abelian gauge theories is therefore linked to the problem of formulating the integral version of the Yang-Mills equations.

By scanning the $3 + 1$ dimensional Minkowski space-time with closed 2-dimensional surfaces based at a reference point x_R , which are in turn scanned by a family of homotopically equivalent loops based at x_R , it can be shown that the flux of the “conjugate field-strength” $F_{\mu\nu}^W(x) = W^{-1}F_{\mu\nu}(x)W$ through that closed surface, with W being the holonomy defined along a loop from x_R to x , will transform, under a local gauge transformation $g(x)$, as $\Phi \rightarrow g(x_R)\Phi g(x_R)^{-1}$, i.e., bringing the gauge group element to that defined at the reference point.

A relation between the flux of the conjugate field through the closed surface $\partial\Omega$ and quantities evaluated inside the volume Ω can be established and expanding this construction for the dual field strength $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\sigma\rho}$, with the use of the (differential) Yang-Mills equations

$$\begin{aligned} D_{\mu} F^{\mu\nu} &= J^{\nu} \\ D_{\mu} \tilde{F}^{\mu\nu} &= J_{\nu} \end{aligned}$$

with $D_{\mu} \star = \partial_{\mu} \star + ie[A_{\mu}, \star]$ the covariant derivative, $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + ie[A_{\mu}, A_{\nu}]$ the field strength and $J_{e,m}^{\mu}$ the electric and magnetic currents, we obtain their integral formulation:

$$\begin{aligned} &\int_{\partial\Omega} W^{-1} F_{\mu\nu} W \frac{\partial x^{\mu}}{\partial \sigma} \frac{\partial x^{\nu}}{\partial \tau} d\sigma d\tau = \int_{\Omega} \epsilon_{\lambda\mu\nu\gamma} W^{-1} J^{\gamma} W \frac{\partial x^{\lambda}}{\partial \zeta} \frac{\partial x^{\mu}}{\partial \sigma} \frac{\partial x^{\nu}}{\partial \tau} d\sigma d\tau \\ &+ \int_{\Omega} \int_0^1 \sigma [F_{\mu\nu}(\sigma), F_{\alpha\beta}(\sigma')] \bigg(\frac{\partial x^{\beta}}{\partial \zeta} \frac{\partial x^{\alpha}}{\partial \sigma'} \frac{\partial x^{\mu}}{\partial \sigma} \frac{\partial x^{\nu}}{\partial \tau} \\ &- \frac{\partial x^{\beta}}{\partial \zeta} \frac{\partial x^{\alpha}}{\partial \sigma'} \frac{\partial x^{\mu}}{\partial \sigma} \frac{\partial x^{\nu}}{\partial \tau} \frac{\partial x^{\lambda}}{\partial \zeta} \frac{\partial x^{\gamma}}{\partial \sigma} \frac{\partial x^{\nu}}{\partial \tau} \bigg) \frac{\partial x^{\lambda}}{\partial \zeta} \frac{\partial x^{\gamma}}{\partial \sigma} d\sigma' d\sigma d\tau d\zeta \end{aligned}$$

$$\begin{aligned} &\int_{\partial\Omega} W^{-1} \tilde{F}^{\mu\nu} W \frac{\partial x^{\mu}}{\partial \sigma} \frac{\partial x^{\nu}}{\partial \tau} d\sigma d\tau = \int_{\Omega} \epsilon_{\lambda\mu\nu\gamma} W^{-1} J^{\gamma} W \frac{\partial x^{\lambda}}{\partial \zeta} \frac{\partial x^{\mu}}{\partial \sigma} \frac{\partial x^{\nu}}{\partial \tau} d\sigma d\tau \\ &+ \int_{\Omega} \int_0^1 \sigma [\tilde{F}^{\mu\nu}(\sigma), \tilde{F}^{\alpha\beta}(\sigma')] \bigg(\frac{\partial x^{\beta}}{\partial \zeta} \frac{\partial x^{\alpha}}{\partial \sigma'} \frac{\partial x^{\mu}}{\partial \sigma} \frac{\partial x^{\nu}}{\partial \tau} \\ &- \frac{\partial x^{\beta}}{\partial \zeta} \frac{\partial x^{\alpha}}{\partial \sigma'} \frac{\partial x^{\mu}}{\partial \sigma} \frac{\partial x^{\nu}}{\partial \tau} \frac{\partial x^{\lambda}}{\partial \zeta} \frac{\partial x^{\gamma}}{\partial \sigma} \frac{\partial x^{\nu}}{\partial \tau} \bigg) \frac{\partial x^{\lambda}}{\partial \zeta} \frac{\partial x^{\gamma}}{\partial \sigma} d\sigma' d\sigma d\tau d\zeta \end{aligned}$$

In order to obtain the conserved charges, we consider the generalization of the holonomy operator by assigning to each loop parameterized by τ , scanning a closed 2-dimensional surface with base-point at x_R , the quantity $\mathcal{B} = \oint_{\gamma} W^{-1} B_{\mu\nu} W \frac{\partial x^{\mu}}{\partial \sigma} \frac{\partial x^{\nu}}{\partial \tau} d\sigma$ and define the 2-holonomy by the differential equation

$$\begin{aligned} \frac{dV}{d\tau} + ieV\mathcal{B} &= 0, \end{aligned}$$

whose solution is the ordered series

$$V[\partial\Omega] = P_{-2}; e^{-ie \oint W^{-1}} B_{\mu\nu} W \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau.$$

This same operator can be obtained if we consider the 2-dimensional surface where it is calculated to be the result of continuous deformations from an infinitesimal surface at x_R . This leads to a definition of the 2-holonomy as the ordered series

$$V[\Omega] = P_{-3}; e^{ie \int_0^{2\pi} \mathcal{A}(\zeta) d\zeta}; V_{-1}$$

with

$$\begin{aligned} \mathcal{A} &= \int_{\Sigma} V W^{-1} \left(D_\lambda B_{\mu\nu} + D_\nu B_{\lambda\mu} + D_\nu B_{\lambda\mu} \right) W \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau \\ &+ ie \int_{\Sigma} V \int_0^{2\pi} \mathcal{F} \left(W(\sigma), B W(\sigma) \right) \left(\frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} \frac{\partial x^\alpha}{\partial \zeta} \frac{\partial x^\beta}{\partial \tau} - \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} \frac{\partial x^\alpha}{\partial \tau} \frac{\partial x^\beta}{\partial \zeta} \right) V^{-1} d\sigma d\tau \end{aligned}$$

where $\mathcal{F}_{\mu\nu} = F_{\mu\nu} - B_{\mu\nu}$.

The fact that the operator V can be calculated in these two different but equivalent approaches lead us to the identity

$$P_{-3}; e^{ie \int_0^{2\pi} \mathcal{A}(\zeta) d\zeta} = P_{-2}; e^{-ie \oint W^{-1}} B_{\mu\nu} W \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau.$$

For $B_{\mu\nu} = \alpha F_{\mu\nu} + \beta \tilde{F}_{\mu\nu}$, the above equation, which is the non-abelian Stokes theorem, leads to the integral Yang-Mills equations.

Two given closed surfaces in space-time can be regarded as points in the loop space $L^2\Omega$ and the volume between them will define a path in this space.

A consequence of the integral Yang-Mills equations is that the operator $V[\Omega]$ is path-independent in $L^2\Omega$, i.e., it does not change under a reparameterization of the volume enclosed by $\partial\Omega$.

By appropriately splitting space-time into space and time one can then show that V evolves from a $t = 0$ volume Ω_0 to a $t > 0$ volume Ω_t as

$$V[\Omega_t] = UV[\Omega_0]U^{-1},$$

i.e., it undergoes a unitary transformation, thus preserving its eigenvalues which can be identified with the conserved charges.

The integral Yang-Mills equations can be regarded as a zero-curvature equation in the loop space $L^2\Omega$ and the conserved charges are a consequence of the hidden gauge symmetry there.

Author: Prof. LUCHINI, Gabriel (Universidade Federal do Espírito Santo)

Co-author: Prof. FERREIRA, Luiz Agostinho (Instituto de Física de São Carlos / USP)

Presenter: Prof. LUCHINI, Gabriel (Universidade Federal do Espírito Santo)