

Towards hidden symmetries in gauge theories

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The non-abelian generalization of the gauge symmetry proposed by C. N. Yang and R. Mills in 1954 was done à la Maxwell, i.e., in terms of a set of partial differential equations. However, the integral formulation counterpart of this generalization was not known until quite recently.

The critical problem in constructing the integral Yang-Mills equations is the need for a consistent definition of the flux of the non-abelian electric and magnetic fields with which we can build a relationship with the dynamically conserved charges in such a way that these charges are invariant under gauge transformations. Indeed, the naive definition of the flux of the non-abelian fields $\Phi(F) = \int_{\Sigma} F_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau$ is strongly dependent of the gauge choice since under a local gauge transformation $g(x)$, $F_{\mu\nu}(x) \rightarrow g(x)F_{\mu\nu}(x)g^{-1}(x)$ and therefore, the flux through a closed surface cannot be directly associated to gauge-invariant charges inside.

The problem of finding the gauge-invariant charges in non-abelian gauge theories is therefore linked to the problem of formulating the integral version of the Yang-Mills equations.

By scanning the $3 + 1$ dimensional Minkowski space-time with closed 2-dimensional surfaces based at a reference point x_R , which are in turn scanned by a family of homotopically equivalent loops based at x_R , it can be shown that the flux of the “conjugate field-strength” $F_{\mu\nu}^W(x) = W^{-1}F_{\mu\nu}(x)W$ through that closed surface, with W being the holonomy defined along a loop from x_R to x , will transform, under a local gauge transformation $g(x)$, as $\Phi \rightarrow g(x_R)\Phi g(x_R)^{-1}$, i.e., bringing the gauge group element to that defined at the reference point.

A relation between the flux of the conjugate field through the closed surface $\partial\Omega$ and quantities evaluated inside the volume Ω can be established and expanding this construction for the dual field strength $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\rho}F^{\sigma\rho}$, with the use of the (differential) Yang-Mills equations

$$\begin{aligned} D_\mu F^{\mu\nu} &= J^\nu \\ D_\mu \tilde{F}^{\mu\nu} &= J_\nu \end{aligned}$$

with $D_\mu \star = \partial_\mu \star + ie[A_\mu, \star]$ the covariant derivative, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu]$ the field strength and $J_{e,m}^\mu$ the electric and magnetic currents, we obtain their integral formulation:

$$\begin{aligned} \oint_{\partial\Omega} W^{-1} F_{\mu\nu} W \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau &= \oint_{\partial\Omega} \epsilon_{\lambda\mu\nu} \frac{\partial x^\lambda}{\partial \sigma} \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \zeta} J^\zeta d\sigma d\tau \\ &+ \oint_{\Omega} \int_0^1 d\sigma [F_{\mu\nu} W(\sigma) W^{-1}(\sigma) - F_{\alpha\beta} W(\sigma) W^{-1}(\sigma)] \frac{\partial x^\alpha}{\partial \sigma} \frac{\partial x^\beta}{\partial \tau} \frac{\partial x^\nu}{\partial \zeta} d\sigma d\tau \\ &- \frac{\partial x^\alpha}{\partial \sigma} \frac{\partial x^\beta}{\partial \tau} \frac{\partial x^\nu}{\partial \zeta} \frac{\partial x^\lambda}{\partial \sigma} \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \zeta} J^\zeta d\sigma d\tau \\ \oint_{\partial\Omega} W^{-1} \tilde{F}_{\mu\nu} W \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau &= \oint_{\partial\Omega} \epsilon_{\lambda\mu\nu} \frac{\partial x^\lambda}{\partial \sigma} \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \zeta} J^\zeta d\sigma d\tau \\ &+ \oint_{\Omega} \int_0^1 d\sigma [\tilde{F}_{\mu\nu} W(\sigma) W^{-1}(\sigma) - \tilde{F}_{\alpha\beta} W(\sigma) W^{-1}(\sigma)] \frac{\partial x^\alpha}{\partial \sigma} \frac{\partial x^\beta}{\partial \tau} \frac{\partial x^\nu}{\partial \zeta} d\sigma d\tau \\ &- \frac{\partial x^\alpha}{\partial \sigma} \frac{\partial x^\beta}{\partial \tau} \frac{\partial x^\nu}{\partial \zeta} \frac{\partial x^\lambda}{\partial \sigma} \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \zeta} J^\zeta d\sigma d\tau \end{aligned}$$

In order to obtain the conserved charges, we consider the generalization of the holonomy operator by assigning to each loop parameterized by τ , scanning a closed 2-dimensional surface with base-point at x_R , the quantity $\mathcal{B} = \oint_{\gamma} W^{-1} B_{\mu\nu} W \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma$ and define the 2-holonomy by the differential equation

$$\begin{aligned} \frac{dV}{d\tau} + ieV\mathcal{B} &= 0, \\ \end{aligned}$$

whose solution is the ordered series

$$\begin{aligned} & \begin{aligned} & \text{\begin{equation}} \\ & V[\partial \Omega] = P_{-2}; e^{-i \oint W^{-1} B_{\mu\nu}} W \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau. \\ & \text{\end{equation}} \end{aligned} \end{aligned}$$

This same operator can be obtained if we consider the 2-dimensional surface where it is calculated to be the result of continuous deformations from an infinitesimal surface at x_R . This leads to a definition of the 2-holonomy as the ordered series

$$\begin{aligned} & \begin{aligned} & \text{\begin{equation}} \\ & V[\Omega] = P_{-3}; e^{i \int_0^{2\pi} \text{tr} \{ A(\zeta) d\zeta \}}; V_{-1} \\ & \text{\end{equation}} \end{aligned} \end{aligned}$$

with

$$\begin{aligned} & \begin{aligned} & \text{\begin{eqnarray}} \\ & \text{\mathcal{A}} = \int_{\Sigma} V W^{-1} \left(D_{\lambda} B_{\mu\nu} + D_{\mu} B_{\nu\lambda} + D_{\nu} B_{\lambda\mu} \right) W V^{-1} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} \frac{\partial x^\lambda}{\partial \zeta} d\sigma d\tau d\zeta \\ & + i \int_{\Sigma} V \int_0^{2\pi} \text{tr} \left\{ F_{\mu\nu} W(\sigma) B^{\mu\nu}(\sigma) \right\} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} \frac{\partial x^\lambda}{\partial \zeta} \frac{\partial x^\alpha}{\partial \sigma} \frac{\partial x^\beta}{\partial \tau} - \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} \frac{\partial x^\lambda}{\partial \zeta} \frac{\partial x^\alpha}{\partial \sigma} \frac{\partial x^\beta}{\partial \tau} \frac{\partial x^\gamma}{\partial \zeta} \right) V^{-1} d\sigma d\tau d\zeta \\ & \text{\end{eqnarray}} \end{aligned} \end{aligned}$$

where $\mathcal{F}_{\mu\nu} = F_{\mu\nu} - B_{\mu\nu}$.

The fact that the operator V can be calculated in these two different but equivalent approaches lead us to the identity

$$\begin{aligned} & \begin{aligned} & \text{\begin{equation}} \\ & P_{-3}; e^{i \int_0^{2\pi} \text{tr} \{ A(\zeta) d\zeta \}} = P_{-2}; e^{-i \oint W^{-1} B_{\mu\nu}} W \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau. \\ & \text{\end{equation}} \end{aligned} \end{aligned}$$

For $B_{\mu\nu} = \alpha F_{\mu\nu} + \beta \tilde{F}_{\mu\nu}$, the above equation, which is the non-abelian Stokes theorem, leads to the integral Yang-Mills equations.

Two given closed surfaces in space-time can be regarded as points in the loop space $L^2\Omega$ and the volume between them will define a path in this space.

A consequence of the integral Yang-Mills equations is that the operator $V[\Omega]$ is path-independent in $L^2\Omega$, i.e., it does not change under a reparameterization of the volume enclosed by $\partial\Omega$.

By appropriately splitting space-time into space and time one can then show that V evolves from a $t = 0$ volume Ω_0 to a $t > 0$ volume Ω_t as

$$\begin{aligned} & \begin{aligned} & \text{\begin{equation}} \\ & V[\Omega_t] = U V[\Omega_0] U^{-1}, \\ & \text{\end{equation}} \end{aligned} \end{aligned}$$

i.e., it undergoes a unitary transformation, thus preserving its eigenvalues which can be identified with the conserved charges.

The integral Yang-Mills equations can be regarded as a zero-curvature equation in the loop space $L^2\Omega$ and the conserved charges are a consequence of the hidden gauge symmetry there.

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