# Towards hidden symmetries in gauge theories

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FUNDAÇÃO DE AMPARO À PESQUISA E INOVAÇÃO DO ESPÍRITO SANTO

# Inspirations from flatland

- Hidden symmetries play a crucial role in $1+1$ dimensional integrable field theories.

## Zero-curvature representation of the equations of motion

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0$$

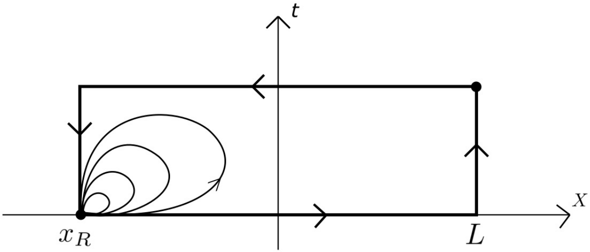
- Flatness of $A = A_\mu^a T_a dx^\mu$ leads to conserved charges.

## The holonomy (parallel transport operator)

$$\frac{dW}{d\sigma} + A_\mu \frac{dx^\mu}{d\sigma} W = 0 \qquad W = P\, e^{-\int_\gamma A_\mu \frac{dx^\mu}{d\sigma} d\sigma} W_\circ$$

- Variations of the path $x^\mu \to x^\mu + \delta x^\mu$ will change the holonomy as

$$\delta W = W \int_0^{2\pi} d\sigma \; W^{-1} F_{\mu\nu} W \frac{dx^\mu}{d\sigma} \delta x^\nu = 0 \Rightarrow W[\gamma] = 1$$




$$W[\gamma_t] = W[\gamma_L] \cdot W[\gamma_0] \cdot W^{-1}[\gamma_{x_R}]$$

- With boundary conditions, $A_t|_{x=x_R} = A_t|_{x=L}$, $U(t) = P\; e^{-\int_0^t dt A_t|_{x=L}}$
- The holonomy restricted to space undergoes a isospectral evolution

$$W[\gamma_t] = U(t) \cdot W[\gamma_0] \cdot U^{-1}(t) \qquad W[\gamma_t] = P\; e^{-\int_{x_R}^{L} dx A_x|_{t=0}}$$

- The eigenvalues of the holonomy are conserved charges.
- The zero curvature equation is gauge invariant

**The gauge transformation is a symmetry**

$$A_\mu \to g A_\mu g^{-1} - \partial_\mu g\; g^{-1} \qquad F_{\mu\nu} \to g F_{\mu\nu} g^{-1}$$

- These conserved charges are gauge invariant.

# From integrable to integral equations

- Could we use a similar construction in gauge theories?
- But why? There is a problem concerning the charges of Yang-Mills theories.
- Yang-Mills equations were constructed *à la* Maxwell, for non-abelian gauge groups
  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu], \quad \widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho},$
  $D_\mu \star = \partial_\mu \star + ie[A_\mu, \star]$

**Yang-Mills equations**

$$D_\mu F^{\mu\nu} = J^\nu \qquad D_\mu \widetilde{F}^{\mu\nu} = 0$$

- The local conservation of non-abelian charges is not trivial:

$$\mathcal{J}^\nu = J^\nu - ie[A_\mu, F^{\mu\nu}] \qquad \partial_\mu \mathcal{J}^\mu = 0$$

- A key point in Maxwell electrodynamics is the Stokes theorem.

**The Stokes theorem for a 2-form $B = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu$**

$$\oint_{\partial\Omega} B = \int_\Omega dB$$

- The integral equations are a consequence of the Stokes theorem and the Maxwell differential equations

$$B_{\mu\nu} = \alpha f_{\mu\nu} + \beta \widetilde{f}_{\mu\nu} \quad dB = \frac{1}{3!}\left(\alpha \partial_\mu \widetilde{f}^{\mu\nu} + \beta \partial_\mu f^{\mu\nu}\right) \epsilon_{\nu\rho\sigma\lambda} dx^\rho \wedge dx^\sigma \wedge dx^\lambda$$

$$\begin{aligned} \oint_{\partial\Omega} f_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau &= 0 \\ \oint_{\partial\Omega} \widetilde{f}_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau &= \frac{4\pi}{c} \int_\Omega \epsilon_{\lambda\rho\gamma\mu} j^\mu \frac{\partial x^\lambda}{\partial \sigma} \frac{\partial x^\rho}{\partial \tau} \frac{\partial x^\gamma}{\partial \zeta} d\sigma d\tau d\zeta \end{aligned}$$

- *To appear soon: Zaché, Victor and L.,G.*

# Yang-Mills integral equations

- Can we relate charge and flux in Yang-Mills theories? For $B_{\mu\nu}$ either $F_{\mu\nu}$ or $\widetilde{F}_{\mu\nu}$,

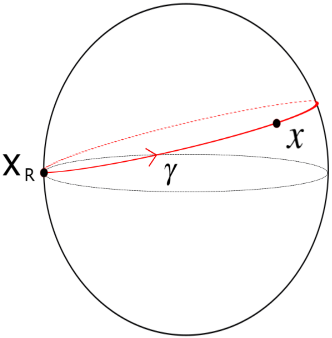
$$\Phi = \oint_{\partial\Omega} B_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau$$

- We expect charges to be gauge invariant and under a gauge transformation,

$$\Phi \rightarrow \oint_{\partial\Omega} g(x) B_{\mu\nu}(x) g^{-1}(x) \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau.$$

- So, we have a problem... but we can make things better!
- First thing to learn: a fancy way to define points in space-time.

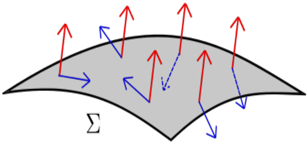
- Introduce the "conjugate field": $B^W_{\mu\nu} = W\ B_{\mu\nu}\ W^{-1}$.

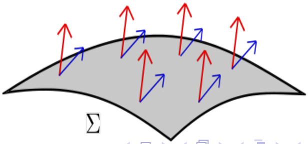



Under a gauge transformation

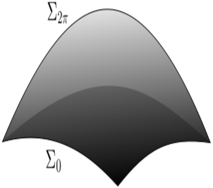
$$
\begin{aligned}
B_{\mu\nu}(x) &\rightarrow g(x)B_{\mu\nu}(x)g^{-1}(x) \\
W[\gamma] &\rightarrow g(x)\ W[\gamma]\ g^{-1}(x_R) \\
B^W_{\mu\nu}(x) &\rightarrow g(x_R)B^W_{\mu\nu}(x)g^{-1}(x_R).
\end{aligned}
$$

$$\Phi \rightarrow g(x_R)\ \Phi\ g^{-1}(x_R)$$



# A derivation of Stokes theorem for the conjugate field




$$\Phi(B^W_{\mu\nu}, \Sigma_0) = \int_{\Sigma_0} W^{-1} B_{\mu\nu} W \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau.$$

$$\frac{d\Phi}{d\zeta} = \int_{\Sigma} W^{-1}(D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu}) W \frac{\partial x^\lambda}{\partial \zeta} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau$$
$$+ \quad \int_{\Sigma} \int_0^{\sigma} \left( [B^W_{\mu\nu}(\sigma), F^W_{\alpha\beta}(\sigma')] S^{\beta\nu} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\alpha}{\partial \sigma'} d\sigma d\tau \right) d\sigma'.$$

$$S^{\beta\nu} \equiv \left( \frac{\partial x^\beta}{\partial \zeta}(\sigma') \frac{\partial x^\nu}{\partial \tau}(\sigma) - \frac{\partial x^\beta}{\partial \tau}(\sigma') \frac{\partial x^\nu}{\partial \zeta}(\sigma) \right)$$

## The Stokes theorem

$$\int_{\partial\Omega} W^{-1} B_{\mu\nu} W \frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau} d\sigma d\tau = \int_0^{2\pi} \mathcal{K}(\zeta) d\zeta$$

$$\begin{aligned}\mathcal{K}(\zeta) \equiv & \int_\Sigma W^{-1}(D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu}) W \frac{\partial x^\lambda}{\partial\zeta}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau} d\sigma d\tau \\ + & \int_\Sigma \int_0^\sigma \left( [B^W_{\mu\nu}(\sigma), F^W_{\alpha\beta}(\sigma')] S^{\beta\nu} \frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\alpha}{\partial\sigma'} \right) d\sigma' d\sigma d\tau.\end{aligned}$$

- Taking $B_{\mu\nu} = \alpha F_{\mu\nu} + \beta \widetilde{F}_{\mu\nu}$ and using the Yang-Mills differential equations we make this theorem an integral physical law.

$$\oint_{\partial\Omega} W^{-1} F_{\mu\nu} W \frac{\partial x^{\mu}}{\partial\sigma}\frac{\partial x^{\nu}}{\partial\tau} d\sigma d\tau = \int_{\Omega} \epsilon_{\lambda\mu\nu\gamma} W^{-1} J_m^{\gamma} W \frac{\partial x^{\lambda}}{\partial\zeta}\frac{\partial x^{\mu}}{\partial\sigma}\frac{\partial x^{\nu}}{\partial\tau} d\sigma d\tau d\zeta$$
$$+ \int_{\Omega}\int_0^{\sigma} [F_{\mu\nu}^W(\sigma), F_{\alpha\beta}^W(\sigma')] S^{\beta\nu} \frac{\partial x^{\alpha}}{\partial\sigma'}\frac{\partial x^{\mu}}{\partial\sigma} d\sigma' d\sigma d\tau d\zeta,$$

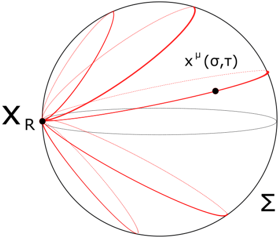
and

$$\oint_{\partial\Omega} W^{-1} \tilde{F}_{\mu\nu} W \frac{\partial x^{\mu}}{\partial\sigma}\frac{\partial x^{\nu}}{\partial\tau} d\sigma d\tau = \int_{\Omega} \epsilon_{\lambda\mu\nu\gamma} W^{-1} J_e^{\gamma} W \frac{\partial x^{\lambda}}{\partial\zeta}\frac{\partial x^{\mu}}{\partial\sigma}\frac{\partial x^{\nu}}{\partial\tau} d\sigma d\tau d\zeta$$
$$+ \int_{\Omega}\int_0^{\sigma} [\tilde{F}_{\mu\nu}^W(\sigma), F_{\alpha\beta}^W(\sigma')] S^{\beta\nu} \frac{\partial x^{\alpha}}{\partial\sigma'}\frac{\partial x^{\mu}}{\partial\sigma} d\sigma' d\sigma d\tau d\zeta.$$

# The conserved charges

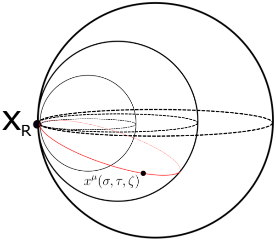
- We define a "new" holonomy such that the dynamical (integral) equations imply its path independence.
- Introduce the 2-holonomy on a closed surface $\partial\Omega$ scanned by loops labelled by $\tau \in [0, 2\pi]$:

$$\frac{dV}{d\tau} + ieV\mathcal{B} = 0, \qquad \mathcal{B} = \oint_\gamma W^{-1} B_{\mu\nu} W \frac{\partial x^\mu}{\partial \sigma}\frac{\partial x^\nu}{\partial \tau} d\sigma$$




$$V[\partial\Omega] = V_\circ \, P_2 \, e^{-ie\int_{\partial\Omega} W^{-1} B_{\mu\nu} W \frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau} d\sigma d\tau}$$

Consider $\partial\Omega$ to be obtained from $\Sigma_0$. $V$ changes from $V[\Sigma_0]$ as




$$\frac{dV}{d\zeta} - ie\mathcal{A}V = 0 \qquad V[\Omega] = P_3\; e^{ie\int_0^{2\pi}\mathcal{A}(\zeta)d\zeta}\; V_\circ$$

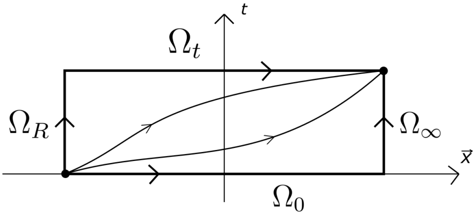
$$\begin{aligned}\mathcal{A}(\zeta) &= \int_\Sigma VW^{-1}\left(D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu}\right) WV^{-1}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}\frac{\partial x^\lambda}{\partial\zeta} d\sigma d\tau \\ &+ ie\int_\Sigma V\int_0^\sigma \left[F^W_{\mu\nu}(\sigma') - B^W_{\mu\nu}(\sigma'), B^W_{\mu\nu}(\sigma)\right] S^{\mu\alpha\beta\nu} V^{-1} d\sigma' d\sigma d\tau\end{aligned}$$

$$S^{\mu\alpha\beta\nu} \equiv \left(\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\zeta}\frac{\partial x^\alpha}{\partial\sigma'}\frac{\partial x^\beta}{\partial\tau} - \frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}\frac{\partial x^\alpha}{\partial\sigma'}\frac{\partial x^\beta}{\partial\zeta}\right)$$

# An important identity

$$P_3 \, e^{ie\int_0^{2\pi}\mathcal{A}(\zeta)d\zeta} = P_2 \, e^{-ie\oint W^{-1}B_{\mu\nu}W\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}d\sigma d\tau}$$

We look at $V[\Omega]$ as an operator in loop space $L^2\Omega$.




The path-independence of $V[\Omega]$ gives

$$V[\Omega_t] = V[\Omega_\infty]V[\Omega_0]V^{-1}[\Omega_R] = U(t)V[\Omega_0]U^{-1}(t).$$

Under a gauge transformation: $V[\Omega_t] \to g(x_R)V[\Omega_t]g^{-1}(x_R)$.
The eigenvalues of $V[\Omega]$ are gauge invariant dynamically conserved charges of Yang-Mills theories.

# The magnetic monopole configuration

Wu-Yang and 't Hooft-Polyakov (at $\|x\| \to \infty$)

$$A_i = -\frac{1}{e}\epsilon_{ija}\frac{x^j}{r^2}T_a, \qquad A_0 = 0, \qquad F_{ij} = \frac{1}{e}\epsilon_{ijk}\frac{\hat{r}^k}{r^2}\hat{r}\cdot\vec{T}.$$

One can show that

$$\frac{d}{d\sigma}\left(W^{-1}\;\hat{r}\cdot\vec{T}\;W\right) = W^{-1}\;D_i\left(\hat{r}\cdot\vec{T}\right)\;W = 0$$

and the flux of the conjugate field through a spatial surface becomes "abelianized":

$$\Phi = \oint_{\partial\Omega} W^{-1}\;F_{ij}\;W\frac{\partial x^i}{\partial\sigma}\frac{\partial x^j}{\partial\tau}d\sigma d\tau = -\frac{1}{e}\oint_{\partial\Omega}\frac{\hat{r}}{r^2}T_R\cdot d\vec{S} = -\frac{4\pi}{e}T_R$$

where $T_R \equiv \left(W^{-1}\;\hat{r}\cdot\vec{T}\;W\right)\Big|_{x_R}$.

The integral equation

$$\oint_{\partial\Omega} W^{-1} F_{ij} W \frac{\partial x^i}{\partial \sigma}\frac{\partial x^j}{\partial \tau} d\sigma d\tau = \int_{\Omega} \epsilon_{kij0} W^{-1} J^0_m W \frac{\partial x^k}{\partial \zeta}\frac{\partial x^i}{\partial \sigma}\frac{\partial x^j}{\partial \tau} d\sigma d\tau d\zeta$$
$$+ \int_{\Omega}\int_0^{\sigma} \cancelto{0}{[F^W_{ij}(\sigma), F^W_{kl}(\sigma')]} S^{lj} \frac{\partial x^k}{\partial \sigma'}\frac{\partial x^i}{\partial \sigma} d\sigma' d\sigma d\tau d\zeta$$

requires a non-vanishing contribution from the conjugate charge density. The Yang-Mills equation ($[A_i, B_i] = 0$ for Wu-Yang configuration)

$$\partial_i B_i = 0$$

can be treated in terms of distributions:

$$\langle T_{B_i}, \mathcal{F}\rangle \equiv \int_{\Omega} B_i \mathcal{F}\, dV, \quad \langle \partial_j T_{B_i}, \mathcal{F}\rangle \equiv -\int_{\Omega} B_i \partial_j \mathcal{F}\, dV,$$

$$\langle \partial_i T_{B_i}, \mathcal{F} \rangle^a = -\frac{1}{e}\mathcal{F}(\vec{0}) \int \hat{r}^a \sin\theta d\theta d\varphi = 0$$

Then, we must find a function $\rho$ such that

$$\int_\Omega d^3x \; \rho \; \mathcal{F} = \int_\Omega d^3x \; \partial_i B_i = 0$$

Any function of the type

$$\rho = C \frac{\delta(r)}{r^2} \hat{\mathrm{r}} \cdot \vec{T}$$

will do but the constant $C$ can only be fixed using the integral equations:

$$\rho = -\frac{1}{e} \frac{\delta(r)}{r^2} \hat{\mathrm{r}} \cdot \vec{T}$$

# The charges of the Wu-Yang monopole

The charge operator reads

$$V = P_3 \; e^{i\alpha e \int_\Omega VW^{-1} J_m^\gamma W V^{-1} \epsilon_{\lambda\mu\nu\gamma} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} \frac{\partial x^\lambda}{\partial \zeta} d\sigma d\tau d\zeta} = e^{i\alpha 4\pi T_R}$$

The charges $g$ of the monopole are then given by the eigenvalues of the flux operator of the conjugate magnetic field

$$Q = \int_{\partial\Omega} W^{-1} \mathrm{B} W \cdot dS = -\frac{4\pi}{e} T_R$$

which in a diagonal basis of the spin $j$ representation of $SU(2)$ are given by

$$g = \frac{4\pi}{e}(j, j-1, \ldots, -j).$$

# The hidden symmetry

- The local Yang-Mills equations imply the zero curvature

$$\delta\mathcal{A} + \mathcal{A}\wedge\mathcal{A} = 0$$

with

$$\begin{aligned}
\mathcal{A} &= \int d\tau \int d\sigma V \Bigg\{ ie\beta \widetilde{J}^{W}_{\mu\nu\lambda} \frac{\partial x^{\mu}}{\partial\sigma}\frac{\partial x^{\nu}}{\partial\tau}\delta x^{\lambda} \\
&+ e^{2}\int_{0}^{\sigma} d\sigma' \Big[ \Big( (\alpha-1)\, F^{W}_{\kappa\rho} + \beta \widetilde{F}^{W}_{\kappa\rho} \Big)(\sigma'), \Big( \alpha F^{W}_{\mu\nu} + \beta \widetilde{F}^{W}_{\mu\nu} \Big)(\sigma) \Big] \\
&\times \frac{\partial x^{\kappa}}{\partial\sigma'}\frac{\partial x^{\mu}}{\partial\sigma} \left( \frac{\partial x^{\rho}}{\partial\tau}(\sigma')\delta x^{\nu}(\sigma) - \frac{\partial x^{\nu}}{\partial\tau}(\sigma)\delta x^{\rho}(\sigma') \right) \Bigg\}
\end{aligned}$$

which is a connection in the loop space $L^{2}\Omega$.

- This is a conservation law.

- There is more to the conserved charge then the matter fields from the current $J^{\mu}$.
- The commutator term in the integral equations is new and it contributes to the flux of the (conjugate) field strength.
- The hidden symmetry leading to the conserved charges is the path independence of the 3-holonomy in the loop space.
- We need to understand what are the gauge transformations in loop space, which are symmetries of the zero-curvature equation.

# Thanks!