Towards the Yang-Mills ensemble

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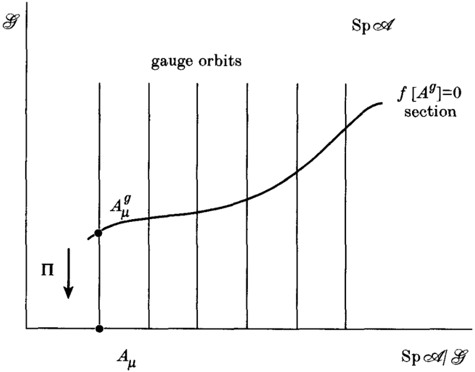
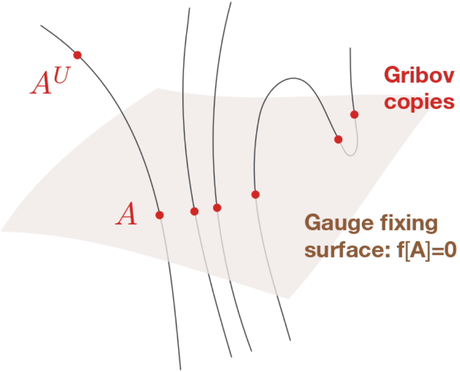
Quantization à la Faddeev-Popov

Choice of a gauge fixing condition in the space for all field configurations: In geometrical terms, such a choice is equivalent to define a global section.



The Gribov problem

FP procedure does not fix the gauge globally, but just locally (if we are restricted to some neighborhood of the cross-point, the multiply solutions are hiding, as occurs when we see uniquely to the perturbative region of the QCD)

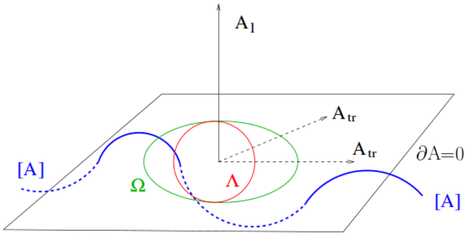


The Singer's obstruction...

- After FP, path integral still contains redundant d.o.f associated with gauge fields obeying $g(A) = 0$ and related by nontrivial gauge transformations. Such spurious configurations are typically called Gribov copies.
- The existence of a global section on The Principal Bundle SpA would imply the triviality of the bundle. This last relation is impossible to fulfill because of the nonvanishing of some of the homotopy groups of $SU(N)$.
- Singer's (no-go) theorem: Because of a topological obstruction, there is no condition $g(A) = 0$ that can globally fix the gauge on the whole configuration space $\{A_\mu\}$ ["Some remarks on the Gribov ambiguity". Comm. in Math. Phys. 60 (1): 7–12].

Gribov-Zwanziger(-Sorella) solution

Gribov (Ω)and Fundamental modular (Λ) regions for minimal Landau gauge: Orbits $[A]$ cross the hyperplane $\partial A = 0$ of physical configurations in multiple times, but only once the fundamental modular region $\Lambda \subset \Omega$ (absolute minimum of the Hilbert norm). Copies may appear in the Ω/Λ (relative minimums of the norm).



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Wait! What did Singer say?

I. Singer, "Some remarks on the Gribov ambiguity". Comm. in Math. Phys. 60 (1), 7–12:

> Since $\mathfrak{N}_{\mathscr{B}}$ and $\hat{\mathfrak{N}}_{\mathscr{B}}$ are paracompact, one way to avoid the difficulty posed by the Gribov ambiguity is to use a locally finite covering $\{\mathscr{V}\}$ and a subordinate partition of unity $\{f_{\mathscr{V}}\}$. For $\mathfrak{N}_{\mathscr{B}}$, the generalized Coloumb gauges $\mathscr{S}_A$ are cross sections locally. Use the Fadeev-Popov trick locally and weight by $f_{\mathscr{V}}$. On reducible connections $A \in \mathfrak{A}$, the $\mathscr{S}_A$ are only slices and intersect nearby orbits in a compact submanifold, an orbit of the stability group $\mathfrak{G}_A$. This is not a serious difficulty for one can still find a local cross section. This partition of unity argument would be useful if the covering used could be made explicit.

Singer's theorem does not rule out the possibility of covering $Sp\mathcal{A}$ with local regions $\mathcal{V}_\alpha$, each one having its own well-defined local section f_α, for $\mathcal{A}_\mu \in \mathcal{V}_\alpha$.

Local domains

If usual Fadeev-Popov procedure could be separately implemented on each domain ϑ_α, then:

$$Z_{\mathrm{YM}} = \sum_\alpha Z_{(\alpha)} \quad , \quad \langle O \rangle_{\mathrm{YM}} = \sum_\alpha \frac{Z_{(\alpha)}}{Z_{\mathrm{YM}}} \langle O \rangle_{(\alpha)}$$

$$Z_{(\alpha)} = \int_{\vartheta_\alpha} [DA_\mu]\, e^{-S_{\mathrm{YM}}[A]} \; , \; \langle O \rangle_{(\alpha)} = \frac{1}{Z_{(\alpha)}} \int_{\vartheta_\alpha} [DA_\mu]\, e^{-S_{\mathrm{YM}}[A]} O[A]$$

How to implement this construction? Two ideas:

- Center dominance: according to which center vortices, and nothing else, are responsible forthe non-perturbative YM features.
- In Laplacian Center Gauge (LCG), center vortices and monopoles appear together as local gauge defects.

Topological excitations

- How to isolate a subset of collective d.o.f. which would be responsible for the non-perturbative features of SU(N) YM?
- Following lattice: non-perturbative effects $\equiv$ topological excitations.
- Topological excitations are related to non-trivial homotopy groups (for SU(2):
  1. $\Pi_3(SU(2)) = \mathbf{Z}$: instantons.
  2. $\Pi_2(SU(2)/U(1)) = \mathbf{Z}$: Abelian monopoles.
  3. $\Pi_1(SU(2)/\mathbf{Z}_2) = \mathbf{Z}_2$: center vortices (non-contractible Wilson loops in the adjoint).
- Topology is a feature of continuum field theory, so in lattice some kind of interpolation is required.
- Any topological excitation can be exposed as a local gauge-fixing singularity: MAG for AM and MCG for CV.

Some remarks about LCG

- In lattice, LCG is used to detect center vortices by looking at the lowest eigenfuntions of the adjoint covariant Laplacian.
- The idea is to bring each link element (in lattice) as close as possible to a center element.
- The eigenvalues of the adjoint Laplacian operator are real and gauge invariant, and gauge transformations rotate the eigenvectors in color space.
- Then, we can fix the gauge by requiring a conventional arbitrary orientation for the eigenvectors.
- In [Oxman et al., "Detecting topological sectors in continuum YM theory and the fate of BRST symmetry"; PRD , 92:125025, 2015] was proposed a LCG in the continuum.

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The YM quantization on the $\mathcal{V}(S_0)$-sectors

We considered a gauge invariant auxiliary action $S_{\rm aux}[A,\psi]$ ($\psi = (\psi_1, \dots, \psi_{N_{\rm f}})$, adjoint scalar fields ψ_I, $I = 1, \dots, N_{\rm f}$. Then, the gauge field A_μ is correlated with the solution $\psi = \psi(A)$ to the set of classical eom

$$\frac{\delta S_{\rm aux}}{\delta \psi_I} = 0 \quad , \quad \psi_I \in \mathfrak{su}(N) \quad , \quad I = 1, \dots, N_{\rm f}$$

When an orbit of A_μ is followed, an orbit in the auxiliary space $\{\psi\}$ is described, with components

$$\psi_I(A^U) = U\psi_I(A)\,U^{-1} \quad , \quad A^U_\mu = U A_\mu U^{-1} + iU\partial_\mu U^{-1} \,.$$

and a polar decomposition of the tuple ψ is introduced:

$$\psi_1 = Sq_1S^{-1},\ \dots\ ,\ \psi_{N_{\rm f}} = Sq_{N_{\rm f}}S^{-1}\ ,$$

Topological defects

- When moving along the orbit of A_μ, $q(A)$ stays invariant while the phase describes an orbit $S(A^U) = US(A)$.
- A_μ, $S(A)$ will generally contain defects, which cannot be removed by means of the regular U-mappings associated with gauge transformations.
- That is, it is not possible to define a "unitary" gauge $S(A) = I$ on $\{A\}$. What can be done is to define regions $\mathscr{V}(S_0) \subset \{A_\mu\}$ formed by gauge fields that can be gauge-transformed to $S(A^U) = S_0$, where U is regular and S_0 is a reference (class representative), characterized by a given distribution of defects:

$$f_{S_0}(A) = 0 \quad , A_\mu \in \mathscr{V}(S_0)$$

$$f_{S_0}(A) = f(S_0^{-1}\psi_1(A)\, S_0, \; \dots \; , S_0^{-1}\psi_{N_\mathrm{f}}(A)\, S_0)$$

Topological defects II

- As this is a local condition in the configuration space $\{A\}$, it is possibile to have no copies in this setting.
- Any pair of different class representatives S_0, S_0' s.t. $S_0' \neq U S_0$ (for regular U) so that a gauge field A_μ cannot be in different regions.
- As all the gauge fields belong to some region, the above procedure gives a partition of $\{A_\mu\}$: $\vartheta_\alpha \to \mathscr{V}(S_0)$.
- The labels correspond to oriented and nonoriented center vortices with nonabelian d.o.f., where the nonoriented component is generated by monopoles.
- YM field averages involve an ensemble integration over topological defects (sector labels) with a weight $Z_{(S_0)}/Z_{\mathrm{YM}}$ that is in principle calculable.

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A well-defined gauge-fixing condition

- Appropriate regularity and boundary conditions on the auxiliary fields so as to have a unique solution $\psi(A)$.
- The auxilary-field content and the auxiliary action must be such that $\psi(A)$ is injective on any gauge orbit:

$$\psi(A^U) = \psi(A) \Rightarrow U \in Z(N)$$

  where $Z(N)$ is the center group of $SU(N)$ (fields transform homogeneously under the gauge group).
- A univocally defined polar decomposition of $\psi(A)$, inducing the partition $\mathscr{V}(S_0)$, and a local condition on $\mathscr{V}(S_0)$ with no copies:

$$f_{S_0}(\psi(A)) = 0\,, f_{S_0}(\psi(A^U)) = 0 \Rightarrow U = \mathbb{I}$$

Introducing the auxiliary fields in PI: SSB

$$1 = \int [D\psi] \det \left(\frac{\delta^2 S_{\text{aux}}}{\delta\psi_I \delta\psi_J} \right) \delta \left(\frac{\delta S_{\text{aux}}}{\delta\psi_I} \right)$$

Given A_μ, to correctly implement this identity:

- δ-functional must have a unique zero, and the quadratic operator in the determinant must be positive definite.
- This is nothing but the uniqueness requirement
- Positivity of the quadratic form is related to solutions $\psi(A)$ with minimum auxiliary action (SSB).

Introducing the auxiliary fields in PI: gauge-fixing

To represent the YM quantities in terms of a partition in the local sectors $\mathscr{V}(S_0)$:

$$1 = \sum_{S_0} 1_{S_0} \, , 1_{S_0} = \int [DU] \, \delta(f_S(\psi)) \, \det(J(\psi)) \, , S = U S_0$$

- $J(q)$ is the FP op. associated to the g.-f. condition.
- The characteristic function 1_{S_0} is nontrivial on fields of the form $\psi = q^S$, $f(q) = 0$.
- As ψ is single-valued, when we get close to the defects of S_0, the fields accompanying Lie algebra components rotated by S_0 must tend to zero.
- When restricted to $\mathscr{V}(S_0)$, 1_{S_0} should be a unique U that solves $f_S(\psi) = 0$.
- This is expected to be addressed by the consideration of the $SU(N) \to Z(N)$ SSB pattern and a good definition of polar decomposition with a univocally defined phase (and modulus)

Localization of the new measure

We initially choose

$$\frac{\delta S_{\text{aux}}}{\delta \psi_I^a} = 0 \; , S_{\text{aux}} = \int_x \left(\frac{1}{2} D_\mu^{ab} \psi_I^b D_\mu^{ac} \psi_I^c + V_{\text{aux}} \right)$$

with the auxiliary potential $V_{\text{aux}}(\psi)$ to be the most general one constructed in terms of antisymmetric structure constants, and containg up to quartic terms:

$$V_{\text{aux}}(\psi) = \frac{\mu^2}{2} \psi_I^a \psi_I^a + \frac{\kappa}{3} f^{abc} f_{IJK} \psi_I^a \psi_J^b \psi_K^c + \frac{\lambda}{4} \gamma_{IJKL}^{abcd} \psi_I^a \psi_J^b \psi_K^c \psi_L^d$$

In order for the procedure to be well-defined, a potential with minima displaying $SU(N) \to Z(N)$ is essential. This pattern, which can be easily accommodated by $N^2 - 1$ flavors, produces a strong correlation between $\mathcal{A}_\mu$ and the local phase $S[\mathcal{A}]$.

The action

- A tuple $q_I = S^{-1}\psi_I S$ is called the modulus of ψ_I, if $S \in SU(N)$ minimizes $(q_I - u_I)^2$, where $u_I \in \mathfrak{su}(N)$ is a reference tuple of linearly independent vectors.
- Then, if we perform an infinitesimal rotation of q_I with generator X, we get $(q_I - u_I, [q_I, X]) = 0$ for every $X \in \mathfrak{su}(N)$.
- Polar decomposition and the modulus condition: $\psi_I = S q_I S^{-1}\,, [u_I, q_I] = 0$

The full Yang Mills action in this gauge is then given by

$$\begin{aligned}\Sigma \quad = \quad & S_{\rm YM} + \int_x \Big\{ \left(D^{ab}_\mu \bar{c}^b_I\right) D^{ac}_\mu c^c_I + \left(D^{ab}_\mu b^b_I\right) D^{ac}_\mu \zeta^c_I + \kappa f_{IJK} f^{abc} \left(b^a_I \zeta^b_J \zeta^c_K - 2\bar{c}^a_I \zeta^b_K c^c_J\right) \\ & + \lambda \gamma^{abcd}_{IJKL} (b^a_I \zeta^b_J \zeta^c_K \zeta^d_L + 3\bar{c}^a_I c^b_J \zeta^c_K \zeta^d_L) + \mu^2 \left(\bar{c}^a_I c^a_I + b^a_I \zeta^b_I\right) + \\ & + i f^{abc} b^a \eta^b_I \zeta^c_I + f^{ecd} f^{eba} \bar{c}^a \eta^b_I \zeta^c_I c^d + i f^{abc} \bar{c}^a \eta^b_I c^c_I \Big\}\end{aligned}$$

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BRST

The action is invariant under the following BRST transformations

$$
\begin{aligned}
sA^a_\mu &= \frac{i}{g} D^{ab}_\mu c^b \,, & sc &= -\frac{i}{2} f^{abc} c^b c^c \\
s\bar{c}^a &= -b^a \,, & sb^a &= 0 \\
s\zeta^a_I &= i f^{abc} \zeta^b_I c^c + c^a_I \,, & s\bar{c}^a_I &= -i f^{abc} \bar{c}^b_I c^c - b^a_I \\
sb_I &= i f^{abc} b^b_I c^c \,, & sc^a_I &= -i f^{abc} c^b_I c^c
\end{aligned}
$$

Conveniently, the gauge fixing terms can be written as a BRST-exact term, so that the action is equivalent to

$$
\begin{aligned}
S &= S_{\mathrm{YM}} - s \int_x \Big[D^{ab}_\mu \bar{c}^b_I D^{ac}_\mu \zeta^c_I + \bar{c}^a_I \Big(\mu^2 \zeta_I + \kappa f^{IJK} f^{abc} \zeta^b_J \zeta^c_K \Big) \\
&\quad + \gamma^{abcd}_{IJKL} \lambda \bar{c}^a_I \zeta^b_J \zeta^c_K \zeta^d_L + i f^{abc} \bar{c}^a \eta^b_I \zeta^c_I \Big] \,.
\end{aligned}
$$

Flavor charge

Ghost number equation and global flavor symmetry:

$$\mathcal{N}_{gh}\Sigma = 0 \quad , \mathcal{Q}\Sigma = 0$$

$$\begin{aligned}
\mathcal{N}_{gh} &\equiv \int_x \left(c^a_I \frac{\delta}{\delta c^a_I} - \bar{c}^a_I \frac{\delta}{\delta \bar{c}^a_I} + c^a \frac{\delta}{\delta c^a} - \bar{c}^a \frac{\delta}{\delta \bar{c}^a} + .. \right) \\
\mathcal{Q} &\equiv \int_x \left(q^a_I \frac{\delta}{\delta q^a_I} - b^a_I \frac{\delta}{\delta b^a_I} - \bar{c}^a_I \frac{\delta}{\delta \bar{c}^a_I} + c^a_I \frac{\delta}{\delta c^a_I} - u^a_I \frac{\delta}{\delta u^a_I} - \kappa \frac{\delta}{\delta \kappa} - 2\lambda \frac{\delta}{\delta \lambda} + ... \right)
\end{aligned}$$

This symmetry can be used to define new conserved quantum number in the auxiliary flavor sector, the $\mathcal{Q}$-charge. Thus, this symmetry forbids combinations of composite fields with nonvanishing $\mathcal{Q}$-charge.

Renormalizability

Counter-term $\Sigma^{\text{c.t.}}$ can be decomposed as

$$\Sigma^{\text{c.t.}} = \Delta + \mathcal{B}_{\Sigma}\Delta^{(-1)}$$

$\mathcal{B}_{\Sigma}\Delta^{(-1)}$ corresponds to the trivial solution (exact part of the cohomology of the BRST operator) and Δ identifies the nontrivial solution (cohomology of $\mathcal{B}_{\Sigma} \rightarrow \Delta \neq \mathcal{B}_{\Sigma}\tilde{\Delta}$)

- Terms containing the parameters of the model are forbidden in Δ due to their BRST doublet structure
- Linear and quadratic terms in A_{μ} mixed with other fields vanish because the BRST.
- Any other possibility results in a combination of flavored fields with mass dimension 4 and vanishing ghost number and $\mathcal{Q}$-charge, all of them being BRST-exact forms.
- Thus, we conclude that the nontrivial cohomology of the present model is the usual cohomology corresponding to the YM ($\Delta = a_0 S_{\text{YM}}$)

Including center vortices

- In order to account for a sector containing vortexes, the components of ζ_I that rotate under S_0 must vanish at Ω.
- To implement this type of boundary condition: $\delta-$functional *à la* FP in the PI, and exponentiate it using auxiliary fields that only exist in Ω:

$$\prod_\alpha \delta_\Omega(\zeta_I^\alpha)\delta_\Omega(\zeta_I^{\bar\alpha}) = \int [D\lambda D\bar\lambda]\, e^{i\int dx\; J_\Omega(x)\left(\lambda_I^\alpha(x)\zeta_I^\alpha(x)+\lambda_I^{\bar\alpha}(x)\zeta_I^{\bar\alpha}(x)\right)}$$

$$J_\Omega(x) = \int d\sigma_1 d\sigma_2 \sqrt{g(\sigma_1,\sigma_2)}\,\delta(x-x(\sigma_1,\sigma_2))$$

- $x(\sigma_1,\sigma_2)$ is a parametrization of Ω, λ_I^α and $\bar\lambda_I^{\bar\alpha}$ are auxiliary fields, and g is the determinant of the world-sheet metric.

Renormalizability, again

- These boundary terms break the WI, which would allow too many new counter-terms in the renormalizability analysis.
- To circumvent this problem we to promote J_Ω to a generic Schwinger source $J^a(x)$ so the color-flavor symmetries can be restored.
- At the end, J^a must collapse to its physical value J_Ω.
- The method is equivalent to embed the physical theory into a larger one and then, eventually, contracting down to the physical theory.
- It turns out that the same set of Ward identities can be accommodated for the action containing boundary terms.

The *frontier* charge

New WI in the boundary sector.

$$\mathcal{F}\Sigma = \lambda^a_I \frac{\delta\Sigma}{\delta\lambda^a_I} + \xi^a_I \frac{\delta\Sigma}{\delta\xi^a_I} - J^a \frac{\delta\Sigma}{\delta J^a} - n^{ab}_{IJ} \frac{\delta\Sigma}{\delta n^{ab}_{IJ}} - m^{ab}_{IJ} \frac{\delta\Sigma}{\delta m^{ab}_{IJ}} = 0$$

Due to this Ward identity, boundary variables cannot enter in counter-term.

No new divergences, no new counter-terms!

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Infinitesimal injectivity of $\psi(A)$

- Injectivity is related to the positivity of the operator introduced in gauge-fixing and to the absence of nontrivial gauge transformations that leave invariant the auxiliary fields.
- For non-trivial gauge transformations, this is related to the existence of zero-modes for ψ.
- We conclude that a lack of infinitesimal injectivity would be associated to configurations that satisfy $\det \Psi = 0$.
- We show that this property holds infinitesimally for typical configurations of the vortex-free sector and for the simplest example in the one-vortex sector.

Vortex-free sector

- For the vortex-free sector, in the limit of large v, we expect that $\Psi = v\mathbb{I} + \epsilon$, where ϵ is a small matrix.
- Defining $b(\epsilon) = \det(v\mathbb{I} + \epsilon)$, we must show that $b(\epsilon) \neq 0$ for small ϵ.
- By expanding it, we may write

$$b(\epsilon) \approx b(0) + \frac{\partial g}{\partial \epsilon^a}\epsilon^a$$

- Since $b(0) = \det v\mathbb{I} = v^{N^2-1}$ is a finite (and large) value, we may conclude that the only solution in this regime is $X = 0$ (trivial gauge transf.).
- Hence, on the vortex-free sector, injectivity is ensured.

Sectors with center-vortices

- Ψ will necessarily be far from the identity near their guiding-centers.
- We may, however, consider a particular example for $SU(2)$. The simplest case is the sector labeled by an antisymmetric vortex with charge k=1.
- Then, as $\beta = \sqrt{2}$, we have $S_0 = e^{i\varphi\sqrt{2}T_1}$, where φ is the angle of cylindrical coordinates.
- For $SU(2)$:

$$\psi_1 = h_1(\rho)T_1 \,, \psi_{\alpha_1} = h(\rho)S_0 T_{\alpha_1} S_0^{-1} \,, \psi_{\bar{\alpha}_1} = h(\rho)S_0 T_{\bar{\alpha}_1} S_0^{-1}$$

- For $\rho \neq 0$, this gives $\xi_1 = \xi_2 = \xi_3 = 0$.
- The only problematic region is the plane $\rho = 0$, which is a region of null measure in R^4.
- The functional $\psi(A)$ is therefore infinitesimally injective in the one-vortex sector for this particular example.

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Equation of copies

Injectivity of $\psi(A)$ does not guarantee that the gauge-fixing is free from copies. We still need to show that, for all sectors S_0,

$$f_{S_0}(\psi(A)) = f_{S_0}(\psi(A^U)) = 0 \to U = \mathbb{I}$$

In matricial form, we may write these conditions as

$$J^{AB}\xi^B = 0 \qquad , \qquad J^{AB} \equiv Tr(M^A M^B Q)$$

Copies are associated with configurations having $\det J = 0$. J is introduced in the Yang-Mills partition function by means of the Fadeev-Popov procedure. It is therefore expected that copies are related to zeros of this determinant.

Copies in the vortex-free sector

In the vortex-free sector, for the general group $SU(N)$, the gauge-fixed functional $q_I(A)$ satisfies

$$q_I(A) \wedge u_I = 0 \,, q_I(A) \to vT_I \,, x \to \infty$$

the boundary condition above will imply (on the limit of large v) that the fields Q are close to $v\,\mathbb{I}$ everywhere, i.e. $q_I^a = \delta_I^a + \epsilon_I^a$:

$$\xi^m(\delta^{m\gamma} + f^{aI\gamma} f^{anm} \epsilon_I^n) = 0$$

This yields a system of $N^2 - 1$ linear equations in the variables ξ^a, with coefficients that will depend on ϵ_I^a, i.e.

$$M(\epsilon)\xi = 0 \,, k(\epsilon) \equiv \det M(\epsilon) = 0 \,, k(\epsilon) \approx k(0) + \frac{\partial k(\epsilon)}{\partial \epsilon_I^a} \epsilon_I^a$$

As $M(0)$ is simply the $1_{(N^2-1)\times(N^2-1)}$, we have $k(0) = 1$. Therefore, , in the large v-limit, there are no Gribov copies for the dominant configurations in the vortex-free sector.

Copies in a general sector

Sector labeled by a vortex along the z axis: $A_\mu = a(\rho)\partial_\mu\varphi\beta\cdot T$.
For $SU(2)$:

$$\psi_I = h_{IJ}S_0T_IS_0^{-1}\,, q_I = h_{IJ}T_J$$

The necessary condition for the existence of copies is

$$2h(h_1+h)^2 = 0$$

Since the profiles $h_1(\rho)$ and $h(\rho)$ are positive for all $\rho > 0$, it is easy to see that this condition is only satisfied at $\rho = 0$, which is a region in R^4 of null measure. The transformations that lead to copies are not continuous, as they should be nontrivial only in this plane. Then, they should not be considered as associated to gauge transformations. This configuration, therefore, does not admit Gribov copies.

More information...

Check it out here

- *Detecting topological sectors in continuum Yang-Mills theory and the fate of BRST symmetry* [arXiv:1509.04728 [hep-th]].
- *Renormalizability of the center-vortex free sector of Yang-Mills theory* [arXiv:2002.02343 [hep-th]].
- *Study of Gribov Copies in the Yang-Mills ensemble* [arXiv:2012.01952 [hep-th]]
- *Renormalizability of a first principles Yang-Mills center-vortex ensemble* [arXiv:2108.11361 [hep-th]]