Color-kinematics duality, double copy and the unitarity method for higher-derivative QCD and quadratic gravity

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VII NATIONAL WORKSHOP ON QUANTUM FIELD THEORY

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Outline:

- Lee-Wick theories and Quadratic Gravity
- Color-kinematics Duality in Higher-Derivative QCD
- Double Copy for quadratic gravity amplitudes
- Loops and Generalized Unitarity
- Outlook

Quadratic gravity: An overview

- Early explorers: Stelle, Fradkin-Tsetlyn, Adler, Zee, Smilga, Tomboulis, Antoniadis, Hasslacher-Mottola, Lee-Wick, Coleman, Boulware-Gross...
- Current explorers: Einhorn-Jones, Salvio-Strumia, Holdom-Ren, Donoghue-Menezes, Mannheim, Anselmi, Odintsov, Shapiro, Accioly, Narain-Anishetty...
- Related work: Lu-Perkins-Pope-Stelle, 't Hooft, Grinstein-O'Connell-Wise...

See also Ronaldo Thibes's talk

- Action ($\kappa^2 = 32\pi G$):

$$S_{\rm quad} = \int d^4x\sqrt{-g}\left[\frac{2}{\kappa^2}R + \frac{1}{6f_0^2}R^2 - \frac{1}{\xi^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right)\right]$$

- Spin-two part of the propagator: (Parametrization: $g_{\mu\nu} = \eta_{\mu\lambda}(e^h)^\lambda{}_\nu = \eta_{\mu\nu} + h_{\mu\nu} + \frac{1}{2}h_{\mu\lambda}h^\lambda{}_\nu + \ldots$)

Donoghue and GM, PRD 97, 126005 (2018)

$$\begin{aligned} iD_{\mu\nu\alpha\beta} &= i\mathcal{P}^{(2)}_{\mu\nu\alpha\beta}D_2(q) \\ D_2^{-1}(q) &= \frac{q^2 + i\epsilon}{\tilde{\kappa}^2(q)} - \frac{q^4}{2\xi^2(\mu)} - \frac{q^4 N_{\rm eff}}{640\pi^2}\ln\left(\frac{-q^2 - i\epsilon}{\mu^2}\right) - \frac{q^4 N_q}{1280\pi^2}\ln\left[\frac{(q^2)^2}{\mu^4}\right] \end{aligned}$$

Here $N_q = 199/3$ and $N_{\rm eff}$, is a number that depends on the number of light degrees of freedom with the usual couplings to gravity, $N_{\rm eff} = N_V + \frac{1}{4}N_F + \frac{1}{6}N_S + 21/6$. With the Standard Model fields plus gravity, $N_{\rm eff} = 325/12$.

Lee-Wick theories

- In theories with fundamental curvature-squared terms, the graviton propagator will be quartic in the momentum. This is generally considered to be problematic. With a quartic propagator in free field theory one expects negative norm ghost states, using for example ($\mu^2 > 0$)

$$\frac{1}{q^2 - \frac{q^4}{\mu^2}} = \frac{1}{q^2} - \frac{1}{q^2 - \mu^2}$$

- This is also the case of the so-called Lee-Wick theories (e.g., a higher-derivative QED). Interactions in such theories make the heavy state unstable, with a width which can be calculated in perturbation theory. This feature is a crucial modification as it *removes the ghost from the asymptotic spectrum.*

- Past experience with Lee-Wick theories indicates that they can be stable and unitary, although causality does seem to be violated on microscopic scales of order the width of the resonance.

- The massive states for the Lee-Wick QED model must be heavier than energies probed by the LHC. The associated micro-causality violation would then be associated with a time scale of $\sim\sim 10^{-25}$ seconds. In the gravitational case, the micro-causality violation would be proportional to the Planck time, 10^{-43} seconds.

Ghost resonances

- The theories which we are studying have propagators of the form

$$iD(q) = \frac{i}{q^2 + i\epsilon - \frac{q^4}{M^2} + \Sigma(q)} \quad .$$

The pole at $q^2 = 0$ is the stable particle of the theory.

- At one-loop order, the self-energy typically has the form

$$\Sigma(q) = -\frac{\gamma}{\pi} \log\left(\frac{-q^2 - i\epsilon}{\mu^2}\right) = \left[-\frac{\gamma}{\pi} \log\left(\frac{|q^2|}{\mu^2}\right) + i\gamma\theta(q^2)\right]$$

for some calculable quantity γ with dimensions of mass squared.

- A massive resonance for timelike values of q^2. Expanding near that resonance:

$$iD(q)\bigg|_{q^2 \sim m^2} \sim \frac{-i}{q^2 - m^2 - i\gamma} \quad .$$

Merlin modes!

Observe the minus sign in the numerator – a ghost-like resonance.

Formal discussion of unitarity

- Unitarity:

$$\langle f|T|i\rangle - \langle f|T^\dagger|i\rangle = i\sum_j \langle f|T^\dagger|j\rangle\langle j|T|i\rangle$$

- In processes that involve loop diagrams, the sum over real intermediate states can by accomplished by the Cutkosky cutting rules.
- What we usually do: look first at the free field theory to identify the free particles.
- Turn on interactions: some of these particles become unstable and no longer appear as the asymptotic states of the theory. *The free field limit has mislead us.*
- Should one include such unstable particles in the sums over states required for unitarity? Veltman says *no*!
- Veltman: unitarity is indeed satisfied by the inclusion of *only* the asymptotically stable states. Cuts are not to be taken through the unstable particles, and unstable particles are not to be included in unitarity sums.

UNITARITY AND CAUSALITY IN A RENORMALIZABLE FIELD THEORY WITH UNSTABLE PARTICLES
M. VELTMAN *)

- *However*, in the narrow-width approximation, the off-resonance production becomes small and only resonance production is important. In this limit a cut taken through the unstable particle with its width set to zero reproduces the same result as a cut through the decay products.

- Unitarity works with the stable particles as external states in the unitarity sum.
- The ghost resonance does not occur as an external state.
- Normal resonances and ghost resonances can be described in the same propagator using the coupling to the stable states described by the same $\Sigma(q)$.
- Veltman's work: normal resonances satisfy unitarity to all orders. Hence any discontinuity calculated with normal resonances in the intermediate states, can be converted into a discontinuity with ghost resonances by using:

$$iD(q) = \frac{i}{q^2 - m^2 + \Sigma(q) - q^4/\Lambda^2}$$

- If the normal resonance satisfies the unitarity relation, the ghost resonance will also!

Donoghue and GM, PRD 100, 105006 (2019)

Causality in higher-derivative theories

Donoghue and GM, PRL 123, 171601 (2019) (Editor's suggestion)

- Causality in quantum field theory is defined by the vanishing of field commutators for space-like separations.
- However, this does not imply a direction for causal effects. Hidden in our conventions for quantization is a connection to the definition of an arrow of causality.
- Mixing quantization conventions within the same theory, we get a violation of microcausality. In such a theory with mixed conventions the dominant definition of the arrow of causality is determined by the stable states.
- In some quantum gravity theories, such as quadratic gravity and possibly asymptotic safety, such a mixed causality condition occurs.

Modern Amplitudes Program

- Scattering experiments are fundamental for our understanding of nature. The Standard Model of particle physics was constructed upon scattering experiments.
- The primary observable associated with particle scattering experiments is the scattering cross-section. It incorporates the probability of a given process to take place as a function of the energy and momentum of the particles involved.
- Interpretation of data from scattering experiments relies on predictions of theoretical calculations of scattering cross-sections – quantum field theory (QFT). This is the mathematical language for describing elementary particles and their interactions.
- In quantum field theory, the differential cross-section is proportional to the norm-squared of the *scattering amplitude*.

Modern goals

- Early explorers: Bern, Dixon, Kosower, Dunbar, Chalmers, Morgan, Mahlon, Berends, Giele, Parke, Taylor, Mangano, Kawai, Lewellen, Tye, Zhu, Goebel, Halzen, Leveille, Kleiss, Stirling, Kuijf, … There was a lot of particle phenomenology in the old days.

- Modern on-shell amplitudes program : Development of more efficient ways to calculate scattering amplitudes (recursion relations for tree-level amplitudes).

- Mathematical structure of on-shell amplitudes. Recent ideas comprise representations of amplitudes in terms of contour integrals in Grassmannian spaces (spaces of k-planes in n-dimensional space) and geometrizations such as polytopes, amplituhedrons, associahedrons, etc. The amplitude is related to a volume form for a geometric object in some abstract mathematical space.

- Amplitude bootstrap: Traditionally one starts with a Lagrangian, writes down the Feynman rules, and use them to calculate the amplitudes. A new approach is to consider the physical observables – the amplitudes – as the starting point, impose constraints on particle spectrum and symmetries on the amplitudes and implement tests of mathematical consistency.

Color-kinematics, BCJ relations and double copy

Bern, Carrasco, Chiodaroli, Johansson and Roiban, 2019

- Gravity amplitudes are notoriously complicated!

- In the mid-80s it was realized that tree-level closed string amplitudes can be written as sums of products of tree-level open-string amplitudes, the Kawai-Lewellen-Tye (KLT) relations. In the limit of infinite string tension, this becomes the field theory statement that the graviton tree amplitudes can be obtained as a sum of products of gluon scattering amplitudes:

$$\text{Gravity} = \text{Gauge Theory}^2.$$

Kawai, Lewellen and Tye, 1986

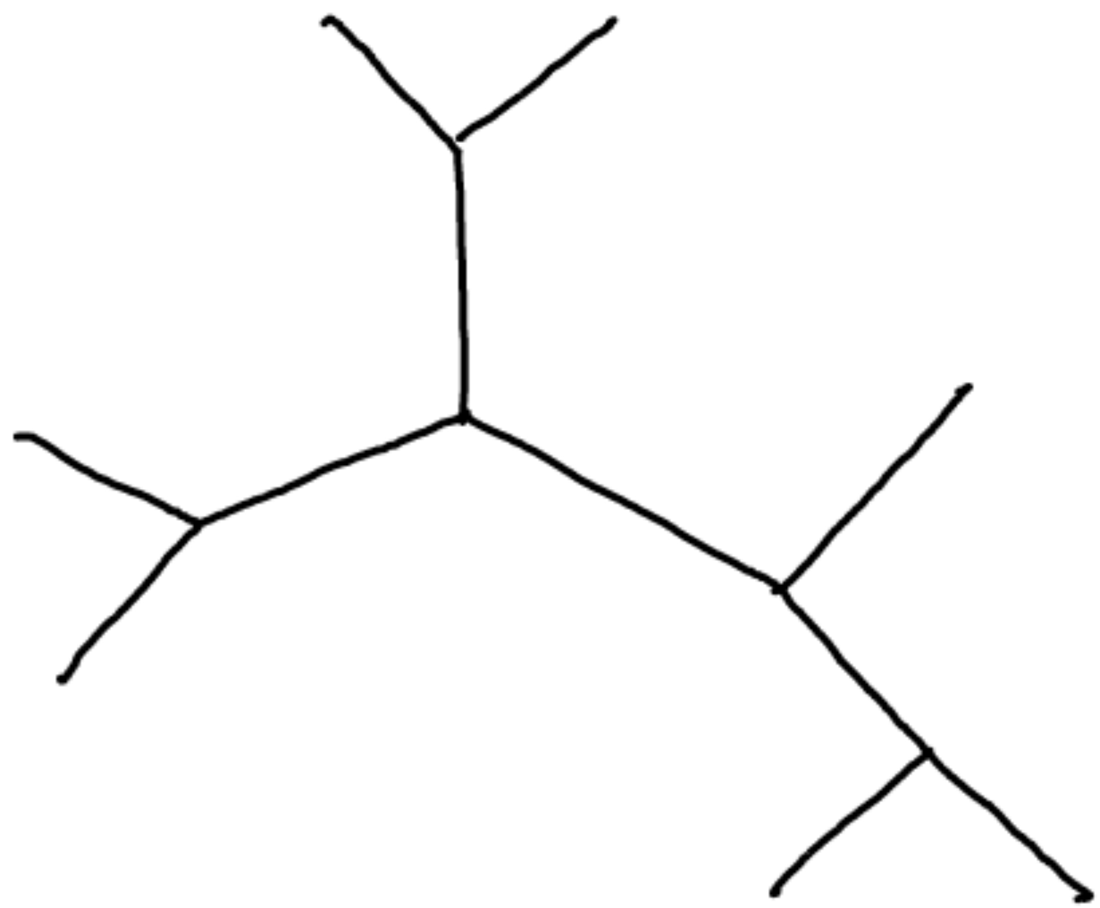
This is the so-called *double copy*.

- Tree-level gauge theory amplitudes of gluon scattering could be written in a form where certain kinematic numerators obey the same Jacobi identities as the algebraic color factors of the non-abelian gauge group of the theory. This is called color-kinematics duality. Moreover, if one replaces the color factors in this representation of the amplitude with the kinematic factors of gauge theory, remarkably the result is the gravity tree amplitude! This is the BCJ (Bern, Carrasco, and Johansson) double copy.

Bern, Carrasco and Johansson, 2008

How does it work?

- The full color-dressed n-point tree amplitude of Yang–Mills theory can be conveniently organized in terms of diagrams with only cubic vertices

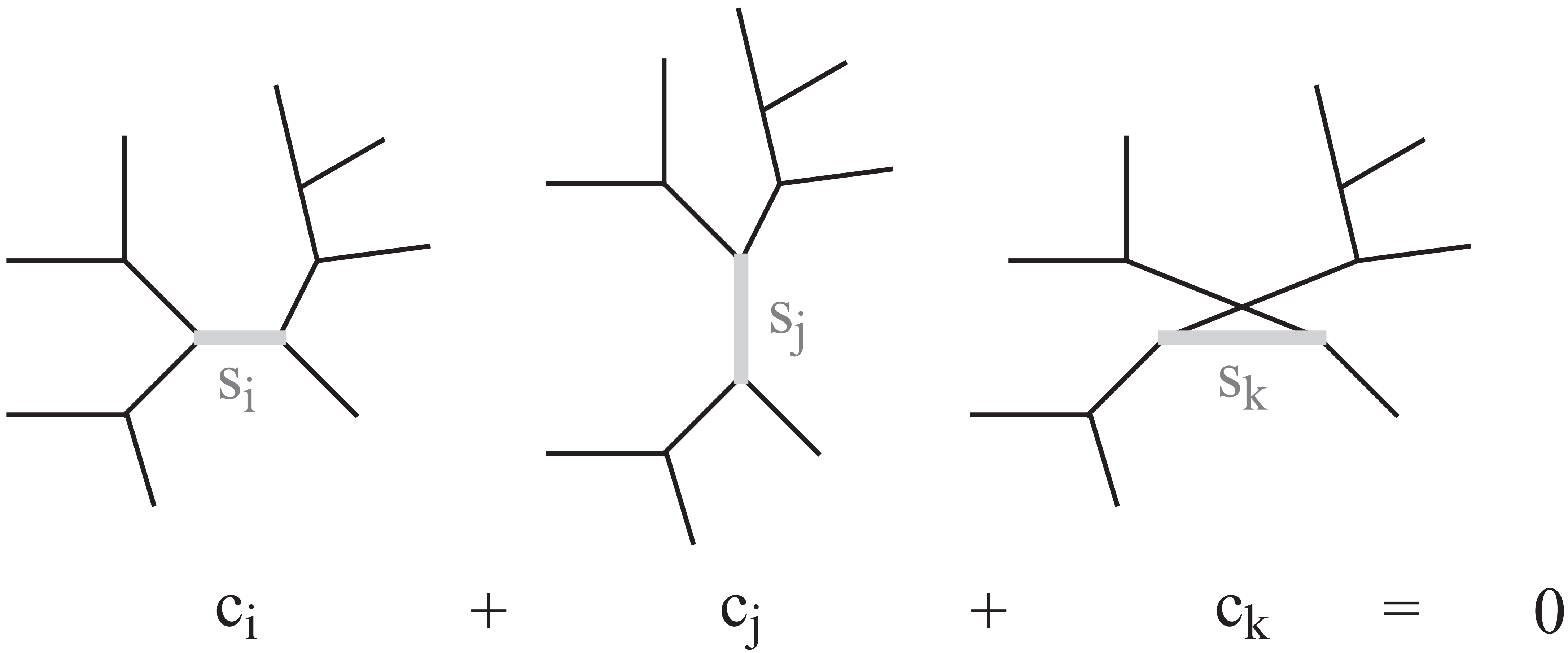


- The tree amplitude is then written as a sum over all distinct trivalent diagrams:

$$A_n^{\text{tree}} = \sum_i \frac{c_i n_i}{\Pi_{\alpha_i} p_{\alpha_i}^2}$$

- In general, for any set of three trivalent diagrams whose color factors are related through a Jacobi identity, $c_i + c_j + c_k = 0$, the following numerator-deformation leaves the amplitude invariant:

$$n_i \to n_i + s_i \Delta \quad n_j \to n_j + s_j \Delta \quad n_k \to n_k + s_k \Delta$$



Elvang and Huang, 2015

Color-kinematics duality

- The duality states that scattering amplitudes of Yang–Mills theory, and its supersymmetric extensions, can be given in a representation where the numerators have the same algebraic properties as the corresponding color factors:

$$c_i = -c_j \leftrightarrow n_i = -n_j$$

$$c_i + c_j + c_k = 0 \leftrightarrow n_i + n_j + n_k = 0$$

- The duality does not state that the numerator factors have to be local; they are allowed to have poles.

Gauge redundancy of numerators!

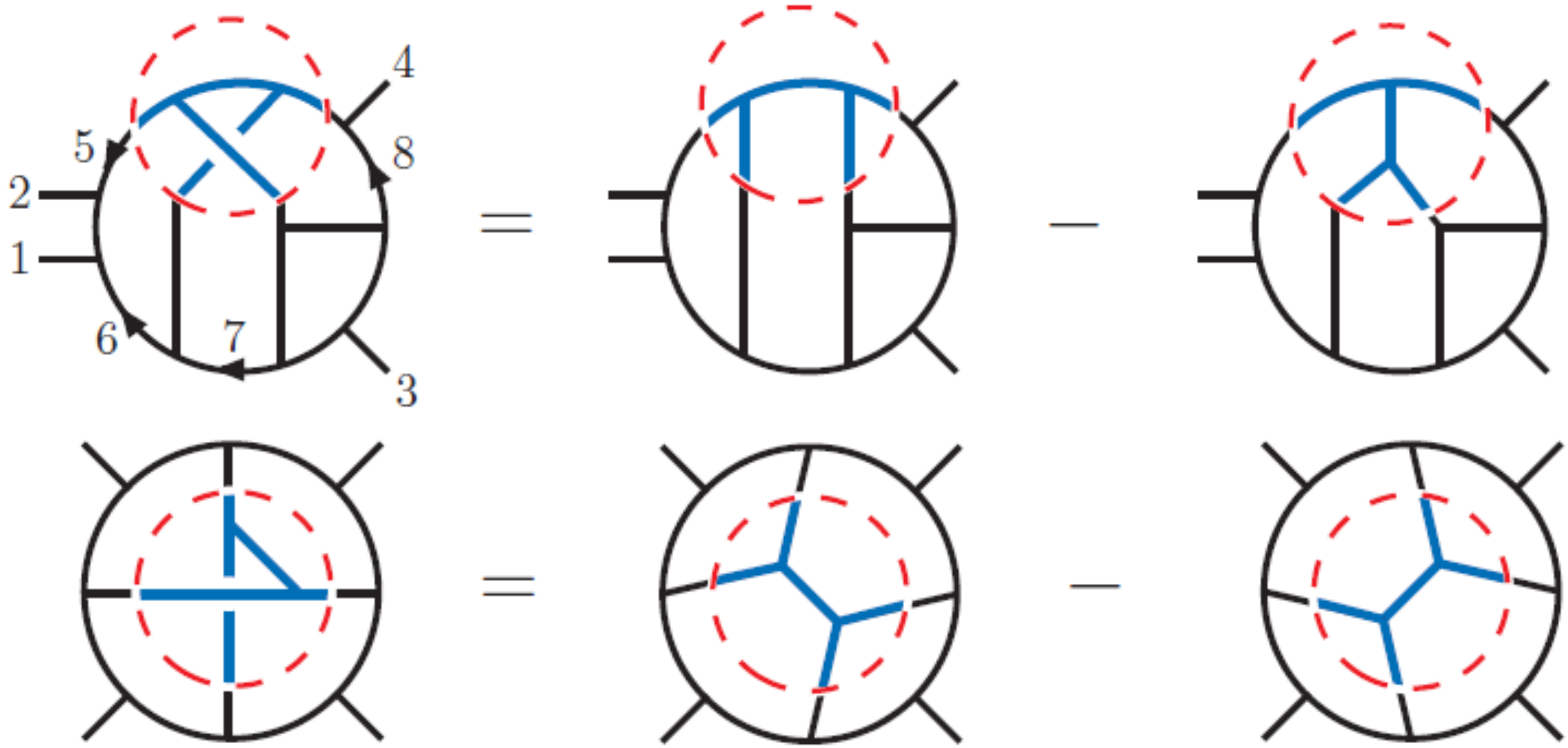
- Under color-kinematics duality, one can derive relations among color-ordered amplitudes, such as

$$tA_4^{\text{tree}}[1324] = sA_4^{\text{tree}}[1234].$$

This is an example of the BCJ relations.

Bern, Carrasco and Johansson, 2008

Non-planar contributions can be derived from planar terms!



Zvi Bern, Cargese 2010

Gluon scattering in QCD

Five gluon tree level scattering with Feynman diagrams:

Result of a brute force calculation:

$k_1 \cdot k_4\, \varepsilon_2 \cdot k_1\, \varepsilon_1 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5$

Picture from Zvi Bern

Higher-derivative Yang-Mills theory

See also: Grinstein, O'Connell and Wise, 2008

GM, arXiv:2112.00978 [hep-th]

- We consider that only the gauge sector displays a higher-derivative contribution. The Lagrangian reads:

$$M^2\mathcal{L} = -\frac{M^2}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D_\mu F^{a\mu\nu}D_\lambda F^{a\lambda}_{\ \ \nu}.$$

- 3-particle amplitudes involving only physical gluons will not display contributions coming from higher-order derivative terms:

$$A_3^{(4)}[1^{h_1}, 2^{h_2}, 3^{h_3}] = M^2 A_3^{(2)}[1^{h_1}, 2^{h_2}, 3^{h_3}].$$

  Using the BCFW recursion relations, one can show that this result carries out to an arbitrary number of gluons.

- Amplitudes involving a single Merlin particle vanish:

$$A_{n+1}^{(4)}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}, k^{IJ}) = 0.$$

First obtained by H. Johansson, G. Mogull and F. Teng (2018).

3-point Amplitudes with Merlins

- Propagator:

$$D^{ab}_{\mu\nu}(p) = -\frac{\delta^{ab}}{p^2}\left(\eta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right) + \frac{\delta^{ab}}{p^2 - M^2 - iM\Gamma}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{M^2}\right)$$

- 3-particle amplitudes with two Merlins:

$$\begin{aligned} A_3[1^+, \mathbf{2}, \mathbf{3}] &= \sqrt{2}\frac{\langle r|\mathbf{3}|1]}{\langle r1\rangle}\langle \mathbf{32}\rangle^2 \\ A_3[1^-, \mathbf{2}, \mathbf{3}] &= \sqrt{2}\frac{[r|\mathbf{3}|1\rangle}{[1r]}[\mathbf{32}]^2. \end{aligned}$$

- 3 Merlins:

$$A_3[\mathbf{1}, \mathbf{2}, \mathbf{3}] = 2\sqrt{2}\Big([\mathbf{12}][\mathbf{13}]\langle \mathbf{23}\rangle + [\mathbf{12}]\langle \mathbf{13}\rangle[\mathbf{23}] + \langle \mathbf{12}\rangle[\mathbf{13}][\mathbf{23}]\Big).$$

See also: Durieux, Kitahara, Shadmi and Weiss, 2020

Tree-level Compton Amplitudes: gluon-Merlin scattering

- Amplitudes involving gluons and Merlins:

$$A_4[\mathbf{2}, 1^+, 4^+, \mathbf{3}] = 2M^4 \frac{[14]}{\langle 14 \rangle} \frac{\langle \mathbf{32} \rangle^2}{s_{12} - M^2}$$

$$A_4[\mathbf{2}, 1^-, 4^+, \mathbf{3}] = 2M^4 \frac{1}{s_{14}(s_{12} - M^2)} \Big([4\mathbf{3}] \langle 1\mathbf{2} \rangle + \langle 1\mathbf{3} \rangle [4\mathbf{2}] \Big)^2$$

- The color-ordered amplitudes in this case obey the standard BCJ relations. For instance:

$$(s_{12} - M^2) A_4[\mathbf{2}, 1^+, 4^+, \mathbf{3}] = (s_{13} - M^2) A_4[\mathbf{2}, 4^+, 1^+, \mathbf{3}].$$

See also: H. Johansson and A. Ochirov, 2019

Tree-level Compton Amplitudes: scalar-Merlin scattering

- Amplitude involving massive scalars and Merlins:

$$A_4[\ell_A, \mathbf{1}, \mathbf{2}, \ell_B] = 2\frac{\langle \mathbf{1}|\ell_A|\mathbf{1}]\langle \mathbf{2}|\ell_B|\mathbf{2}]}{M^2}\frac{1}{s-m^2} - \frac{2}{M^2}\left[\langle \mathbf{12}\rangle[\mathbf{21}](\mathbf{1}-\mathbf{2})\cdot\ell_A - \left(\langle \mathbf{1}|\ell_A|\mathbf{1}]\langle \mathbf{2}|\mathbf{1}|\mathbf{2}] - \langle \mathbf{2}|\ell_A|\mathbf{2}]\langle \mathbf{1}|\mathbf{2}|\mathbf{1}]\right)\right]\left(\frac{1}{t} - \frac{2}{t-M^2}\right)$$

- In order to discuss CK duality, we need to work with the full amplitude. Numerators:

$$A(1,2,3,4) = \sum_k \frac{c_k n_k^{(0)}}{s_k}$$

$$n_s = \frac{2}{M^2}\langle \mathbf{1}|\ell_A|\mathbf{1}]\langle \mathbf{2}|\ell_B|\mathbf{2}] + \frac{\langle \mathbf{12}\rangle[\mathbf{21}]}{M^2}[2(\mathbf{1}\cdot\ell_A)+M^2]$$

$$n_t = \frac{2}{M^2}\left(\langle \mathbf{12}\rangle[\mathbf{21}](\mathbf{1}-\mathbf{2})\cdot\ell_A + \langle \mathbf{1}|\ell_A|\mathbf{1}]\langle \mathbf{2}|\ell_B|\mathbf{2}] - \langle \mathbf{2}|\ell_A|\mathbf{2}]\langle \mathbf{1}|\ell_B|\mathbf{1}]\right)$$

$$n_u = \frac{2}{M^2}\langle \mathbf{1}|\ell_B|\mathbf{1}]\langle \mathbf{2}|\ell_A|\mathbf{2}] + \frac{\langle \mathbf{12}\rangle[\mathbf{21}]}{M^2}[2(\mathbf{2}\cdot\ell_A)+M^2].$$

Unique, gauge-invariant numerators!

- Observe that

$$c_s - c_u = c_t \leftrightarrow n_s - n_u = n_t.$$

- Amplitudes do not obey standard BCJ relations!

Tree-level Compton Amplitudes: fermion-Merlin scattering

- Amplitude involving massless fermions and Merlins:

$$
\begin{aligned}
A_4[3^{-1/2}, \mathbf{1}, \mathbf{2}, 4^{+1/2}] &= 2\frac{\langle 3\mathbf{1}\rangle [\mathbf{1}|(\mathbf{1}+3)|\mathbf{2}\rangle [\mathbf{2}4]}{M^2}\frac{1}{s} \\
&+ \frac{1}{M^2}\Big[\langle \mathbf{12}\rangle [\mathbf{21}] \langle 3|(\mathbf{1}-\mathbf{2})|4] - 2\Big(\langle \mathbf{1}3\rangle [4\mathbf{1}] \langle \mathbf{2}|\mathbf{1}|\mathbf{2}] - \langle \mathbf{2}3\rangle [4\mathbf{2}] \langle \mathbf{1}|\mathbf{2}|\mathbf{1}]\Big)\Big] \\
&\times \left(\frac{1}{t} - \frac{2}{t-M^2}\right)
\end{aligned}
$$

- Numerators for the full amplitude:

$$
\begin{aligned}
n_s &= \frac{2}{M^2}\langle 3\mathbf{1}\rangle [\mathbf{1}|(\mathbf{1}+3)|\mathbf{2}\rangle [\mathbf{2}4] \\
n_t &= -\frac{1}{M^2}\Big[\langle \mathbf{12}\rangle [\mathbf{21}] \langle 3|(\mathbf{1}-\mathbf{2})|4] + 2\Big(\langle 3\mathbf{1}\rangle [\mathbf{1}4] \langle \mathbf{2}|(3-4)|\mathbf{2}] - \langle 3\mathbf{2}\rangle [\mathbf{2}4] \langle \mathbf{1}|(3-4)|\mathbf{1}]\Big)\Big] \\
n_u &= \frac{2}{M^2}\langle 3\mathbf{2}\rangle [\mathbf{2}|(\mathbf{2}+3)|\mathbf{1}\rangle [\mathbf{1}4].
\end{aligned}
$$

Unique, gauge-invariant numerators!

- Observe that

$$c_s - c_u = c_t \leftrightarrow n_s - n_u = n_t.$$

- Amplitudes do not obey standard BCJ relations!

This is all so nice, but…what about gravity?

- Color-kinematics and BCJ relations allows us to write gravity amplitudes as

$$M_n^{\text{tree}} = \sum_i \frac{n_i^2}{\Pi_{\alpha_i} p_{\alpha_i}^2}$$

It can be generalized in terms of the product of two possibly distinct Yang–Mills numerators:

$$M_n^{\text{tree}} = \sum_i \frac{n_i \widetilde{n}_i}{\Pi_{\alpha_i} p_{\alpha_i}^2}$$

BCJ double copy!

- The general idea is to start from the above formula, use color-kinematics duality and BCJ relations to show that we obtain the KLT relations, such as

$$\begin{aligned} M_4^{\text{tree}}(1,2,3,4) &= -is_{12} A_4^{\text{tree}}(1,2,3,4) \widetilde{A}_4^{\text{tree}}(1,2,4,3) \\ M_5^{\text{tree}}(1,2,3,4,5) &= is_{12}s_{34} A_5^{\text{tree}}(1,2,3,4,5) \widetilde{A}_5^{\text{tree}}(2,1,4,3,5) \\ &+ is_{13}s_{24} A_5^{\text{tree}}(1,3,2,4,5) \widetilde{A}_5^{\text{tree}}(3,1,4,2,5) \end{aligned}$$

Bern, Carrasco and Johansson, 2008, 2010

Bjerrum-Bohr, Donoghue and Vanhove, 2014

So indeed gravity = (gauge theory)^2!

- This is the famous double-copy formula. Only one copy of the numerators needs to satisfy the duality, not both.

- Assuming that there exists a duality-satisfying set of local numerators for the Yang– Mills tree amplitude, one can rigorously prove that the doubling-relation produces the correct gravity tree amplitude for any n. The proof is established inductively by showing that the difference between this equation and the gravity amplitude obtained from BCFW recursion vanishes if one assumes the doubling-relation to hold for all lower-point amplitudes.

Bern, Dennen, Huang and Kiermaier, 2010

Elvang and Huang, 2015

3-point graviton amplitudes:

$$M_3(1^{--},2^{--},3^{++}) \;=\; \frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 32\rangle^2} = A_3[1^-,2^-,3^+]^2$$

$$M_3(1^{++},2^{++},3^{--}) \;=\; \frac{[12]^6}{[13]^2[32]^2} = A_3[1^+,2^+,3^-]^2.$$

Caveats…

But see: Borsten, Jurco, Kim, Macrelli, Saemann, Wolf, arXiv:2102.11390 [hep-th]

Unlike at tree-level, there is currently not a formal proof of the existence of duality-satisfying numerators at loop level. Furthermore, the map does not produce a pure Einstein gravity. Typically the double copy of two vector fields contains more than just the graviton. For example, a gluon in four dimension has two helicities ± 1, so the square has four states: the ± 2 correspond to graviton, and the zero-helicity states which are identified as the dilaton scalar ϕ and a Kalb-Ramond two-form $B_{\mu\nu}$. This is a consequence of the fact that low-energy string theory produces

$$\begin{aligned} \mathcal{L} &= \frac{2}{\kappa^2}\Big(-\frac{2(D-26)}{3\alpha'}e^{4\phi/(D-2)} + R \\ &- \frac{1}{12}e^{-8\phi/(D-2)}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \frac{4}{D-2}\partial_\mu\phi\partial^\mu\phi + \mathcal{O}(\alpha')\Big) \end{aligned}$$

where $H_{\mu\nu\lambda}$ is the fully antisymmetric field strength for $B_{\mu\nu}$. The projection to pure gravity, which eliminates the dilaton and axion, can be performed in any case: When using the four-dimensional helicity method, we simply correlate the helicities of the two copies of gauge theory amplitudes. Moreover, we can simply apply the projection to graviton on each of the massless cut legs, which are external lines for tree-level blobs in a generalized unitarity cut.

In summary:

$$\text{4d axion-dilaton gravity} = \text{YM} \otimes \text{YM}$$

and similarly

$$\mathcal{N} = 8 \text{ SUGRA} = (\mathcal{N} = 4 \text{ SYM}) \otimes (\mathcal{N} = 4 \text{ SYM})$$

Quadratic gravity amplitudes

See also: Bob Holdom, 2021

GM, arXiv:2112.00978 [hep-th]

- Work in the Einstein frame (Einstein-Weyl theory):

$$S = \int d^4x \sqrt{-g} \left[\frac{2}{\kappa^2} R - \frac{1}{2\xi^2} C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} \right]$$

where $\kappa^2 = 32\pi G$.

- Employ the map:

YM: spontaneously broken!

$$\text{(Higher-derivative YM)} \otimes \text{YM} = \text{Weyl-Einstein}$$

See also: H. Johansson and J. Nohle, 2017; H. Johansson, G. Mogull and F. Teng, 2018

- Spectrum: Besides the graviton, dilaton and axion, we also have additional five physical degrees of freedom associated with a gravitational Merlin, three states associated with a Merlin 2-form field and a Merlin scalar!

- Projection to pure gravity: Simply correlate the helicities in the two gauge-theory copies. This works in a similar fashion for quadratic gravity: In order to work with only gravitational Merlins, we take the symmetric tensor product of gauge-theory Merlins.

3-point amplitudes and propagator

- Propagator:

$$D_{\mu\nu\rho\sigma}(p) = \frac{1}{2p^2}\Big(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-\eta_{\mu\nu}\eta_{\rho\sigma}\Big)-\frac{1}{2(p^2-M^2-iM\Gamma)}\Big(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-\frac{2}{3}\eta_{\mu\nu}\eta_{\rho\sigma}\Big).$$

- 3-particle amplitudes involving only physical gravitons do not display contributions coming from higher-order derivative terms:

$$M_3^{(4)}[1^{h_1},2^{h_2},3^{h_3}] = M^2 M_3^{(2)}[1^{h_1},2^{h_2},3^{h_3}]$$

and this result generalizes to an arbitrary number of gravitons by using BCFW recursion relations.

- Amplitudes involving a single gravitational Merlin particle vanishes:

$$M_{n+1}^{(4)}(1^{h_1},2^{h_2},\ldots,n^{h_n},\mathbf{k}) = 0$$

- 3-particle amplitude involving two gravitational Merlin particles:

$$\begin{aligned} M_3(1^{++},\mathbf{2},\mathbf{3}) &= iA_3^{\text{tree, HD}}[1^+,\mathbf{2},\mathbf{3}]A_3^{\text{tree,YM}}[1^+,\mathbf{2},\mathbf{3}] = -2i\frac{\langle r|\mathbf{3}|1]^2}{M^4\langle r1\rangle^2}\langle\mathbf{32}\rangle^4 \\ M_3(1^{--},\mathbf{2},\mathbf{3}) &= iA_3^{\text{tree, HD}}[1^-,\mathbf{2},\mathbf{3}]A_3^{\text{tree,YM}}[1^-,\mathbf{2},\mathbf{3}] = -2i\frac{[r|\mathbf{3}|1\rangle^2}{M^4[1r]^2}[\mathbf{32}]^4. \end{aligned}$$

Tree-level Compton Amplitudes: graviton-Merlin scattering

- Amplitudes involving gravitons and Merlins:

$$\begin{aligned}
M_4^{\text{tree}}(\mathbf{2}, 1^{++}, 4^{++}, \mathbf{3}) &= -is_{23} A_4^{\text{tree, HD}}[\mathbf{2}, 1^+, 4^+, \mathbf{3}] A_4^{\text{tree, YM}}[\mathbf{2}, 4^+, 1^+, \mathbf{3}] \\
&= 4i \frac{[14]^4}{s_{23}} \frac{\langle \mathbf{32} \rangle^4}{(s_{12} - M^2)(s_{13} - M^2)} \\
M_4^{\text{tree}}(\mathbf{2}, 1^{--}, 4^{++}, \mathbf{3}) &= -is_{23} A_4^{\text{tree, HD}}[\mathbf{2}, 1^-, 4^+, \mathbf{3}] A_4^{\text{tree, YM}}[\mathbf{2}, 4^+, 1^-, \mathbf{3}] \\
&= 4i \frac{1}{s_{23}(s_{12} - M^2)(s_{13} - M^2)} \Big([4\mathbf{3}] \langle 1\mathbf{2} \rangle + \langle 1\mathbf{3} \rangle [4\mathbf{2}] \Big)^4
\end{aligned}$$

See also: H. Johansson and A. Ochirov, 2019

Triple graviton vertex, calculated from quadratic gravity (Jordan frame)

$$
\begin{aligned}
\mathcal{T}^{\mu\nu}{}_{\alpha\beta\gamma\delta}(k,q) &= -\frac{i}{2\kappa^2}\mathcal{E}^{\mu\nu}{}_{\alpha\beta\gamma\delta}(k,q) - \frac{i}{12 f_0^2}\mathcal{F}^{\mu\nu}{}_{\alpha\beta\gamma\delta}(k,q) - \frac{i}{2\xi^2}\mathcal{W}^{\mu\nu}{}_{\alpha\beta\gamma\delta}(k,q) \\
&+ \frac{i}{16}\left(\frac{3}{2f_0^2+\xi^2}+\frac{1}{\xi^2}\right)\Big[\eta_{\alpha\beta}\eta_{\gamma\delta}(k^\mu p^\nu + k^\nu p^\mu)(k^2+p^2) - 2(I^{\mu\nu}{}_{\alpha\beta}\eta_{\gamma\delta}p^4 + I^{\mu\nu}{}_{\gamma\delta}\eta_{\alpha\beta}k^4) \\
&+ \eta^{\mu\nu}\eta_{\alpha\beta}\eta_{\gamma\delta}\left(-q_\kappa(k^2p^\kappa + p^2k^\kappa) + k^2p^2\right)\Big]
\end{aligned} \tag{63}
$$

$$
\begin{aligned}
\mathcal{E}^{\mu\nu}{}_{\alpha\beta\gamma\delta}(k,q) &= \left(I_{\alpha\beta\gamma\delta} - \frac{1}{2}\eta_{\gamma\delta}\eta_{\alpha\beta}\right)\left(k^\mu k^\nu + p^\mu p^\nu + q^\mu q^\nu - \frac{3}{2}\eta^{\mu\nu}q^2\right) \\
&+ 2q_\lambda q_\sigma\left(I_{\alpha\beta}{}^{\lambda\sigma}I_{\gamma\delta}{}^{\mu\nu} + I_{\gamma\delta}{}^{\lambda\sigma}I_{\alpha\beta}{}^{\mu\nu} - I_{\alpha\beta}{}^{\lambda\mu}I_{\gamma\delta}{}^{\sigma\nu} - I_{\gamma\delta}{}^{\lambda\mu}I_{\alpha\beta}{}^{\sigma\nu}\right) \\
&+ q_\lambda q^\mu\left(\eta_{\alpha\beta}I_{\gamma\delta}{}^{\lambda\nu} + \eta_{\gamma\delta}I_{\alpha\beta}{}^{\lambda\nu}\right) + q_\lambda q^\nu\left(\eta_{\alpha\beta}I_{\gamma\delta}{}^{\lambda\mu} + \eta_{\gamma\delta}I_{\alpha\beta}{}^{\lambda\mu}\right) \\
&- q^2\left(\eta_{\alpha\beta}I_{\gamma\delta}{}^{\mu\nu} + \eta_{\gamma\delta}I_{\alpha\beta}{}^{\mu\nu}\right) - \eta^{\mu\nu}q^\lambda q^\sigma\left(\eta_{\alpha\beta}I_{\gamma\delta\lambda\sigma} + \eta_{\gamma\delta}I_{\alpha\beta\lambda\sigma}\right) \\
&- 2q^\lambda\left(I_{\alpha\beta\lambda\sigma}I_{\gamma\delta}{}^{\sigma\nu}p^\mu + I_{\alpha\beta\lambda\sigma}I_{\gamma\delta}{}^{\sigma\mu}p^\nu + I_{\gamma\delta\lambda\sigma}I_{\alpha\beta}{}^{\sigma\nu}k^\mu + I_{\gamma\delta\lambda\sigma}I_{\alpha\beta}{}^{\sigma\mu}k^\nu\right) \\
&+ q^2\left(I_{\alpha\beta}{}^{\sigma\mu}I_{\gamma\delta\sigma}{}^{\nu} + I_{\gamma\delta}{}^{\sigma\mu}I_{\alpha\beta\sigma}{}^{\nu}\right) + \eta^{\mu\nu}q^\lambda q_\sigma\left(I_{\alpha\beta\lambda\rho}I_{\gamma\delta}{}^{\rho\sigma} + I_{\gamma\delta\lambda\rho}I_{\alpha\beta}{}^{\rho\sigma}\right) \\
&+ (k^2+p^2)\left(I_{\alpha\beta}{}^{\sigma\mu}I_{\gamma\delta\sigma}{}^{\nu} + I_{\alpha\beta}{}^{\sigma\nu}I_{\gamma\delta\sigma}{}^{\mu} - \frac{1}{2}\eta^{\mu\nu}\left(I_{\alpha\beta\gamma\delta} - \frac{1}{2}\eta_{\gamma\delta}\eta_{\alpha\beta}\right)\right) - \left(k^2\eta_{\alpha\beta}I_{\gamma\delta}{}^{\mu\nu} + p^2\eta_{\gamma\delta}I_{\alpha\beta}{}^{\mu\nu}\right) \\
&- \frac{1}{2}\eta^{\mu\nu}\eta_{\alpha\beta}\eta_{\gamma\delta}k_\lambda p^\lambda + \eta^{\mu\nu}\eta_{\alpha\beta}I^{\lambda\kappa\gamma\delta}k_\lambda p_\kappa + \eta^{\mu\nu}\eta_{\gamma\delta}I^{\lambda\kappa\alpha\beta}p_\lambda k_\kappa + 2(I^{\mu\nu}{}_{\alpha\beta}p_\gamma p_\delta + I^{\mu\nu}{}_{\gamma\delta}k_\alpha k_\beta) \\
&+ 2(I^{\mu\lambda}{}_{\alpha\beta}p^\nu p_\lambda + I^{\nu\lambda}{}_{\alpha\beta}p^\mu p_\lambda)\eta_{\gamma\delta} + 2(I^{\mu\lambda}{}_{\gamma\delta}k^\nu k_\lambda + I^{\nu\lambda}{}_{\gamma\delta}k^\mu k_\lambda)\eta_{\alpha\beta} + q_\rho(I^{\mu\nu}{}_{\alpha\beta}\eta_{\gamma\delta}p^\rho + I^{\mu\nu}{}_{\gamma\delta}\eta_{\alpha\beta}k^\rho) \\
&- q_\sigma\Big[(I^{\sigma\mu}{}_{\alpha\beta}p^\nu + I^{\sigma\nu}{}_{\alpha\beta}p^\mu + \eta^{\mu\nu}I^{\lambda\sigma}{}_{\alpha\beta}p_\lambda)\eta_{\gamma\delta} + (I^{\sigma\mu}{}_{\gamma\delta}k^\nu + I^{\sigma\nu}{}_{\gamma\delta}k^\mu + \eta^{\mu\nu}I^{\lambda\sigma}{}_{\gamma\delta}k_\lambda)\eta_{\alpha\beta}\Big] \\
&- \eta^{\mu\nu}\left(I_{\lambda\kappa\alpha\beta}I^{\lambda\sigma}{}_{\gamma\delta}k^\kappa p_\sigma + I_{\lambda\kappa\gamma\delta}I^{\lambda\sigma}{}_{\alpha\beta}p^\kappa k_\sigma\right) - I^{\mu\lambda}{}_{\alpha\beta}I^{\nu\kappa}{}_{\gamma\delta}p_\lambda p_\kappa - I^{\mu\lambda}{}_{\gamma\delta}I^{\nu\kappa}{}_{\alpha\beta}k_\lambda k_\kappa - I^{\nu\lambda}{}_{\alpha\beta}I^{\mu\kappa}{}_{\gamma\delta}p_\lambda p_\kappa - I^{\nu\lambda}{}_{\gamma\delta}I^{\mu\kappa}{}_{\alpha\beta}k_\lambda k_\kappa \\
&- 2\left(I^{\mu\lambda}{}_{\alpha\beta}p^\nu p^\kappa + I^{\nu\lambda}{}_{\alpha\beta}p^\mu p^\kappa\right)I_{\lambda\kappa\gamma\delta} - 2\left(I^{\mu\lambda}{}_{\gamma\delta}k^\nu k^\kappa + I^{\nu\lambda}{}_{\gamma\delta}k^\mu k^\kappa\right)I_{\lambda\kappa\alpha\beta} \\
&+ q_\lambda\Big(I^{\nu\lambda}{}_{\alpha\beta}I^{\mu\kappa}{}_{\gamma\delta}p_\kappa + I^{\mu\lambda}{}_{\alpha\beta}I^{\nu\kappa}{}_{\gamma\delta}p_\kappa - 2I^{\mu\nu}{}_{\alpha\beta}I^{\lambda\kappa}{}_{\gamma\delta}p_\kappa + 2\eta^{\mu\nu}I^{\sigma\lambda}{}_{\alpha\beta}I_{\sigma\kappa\gamma\delta}p^\kappa \\
&+ I^{\nu\lambda}{}_{\gamma\delta}I^{\mu\kappa}{}_{\alpha\beta}k_\kappa + I^{\mu\lambda}{}_{\gamma\delta}I^{\nu\kappa}{}_{\alpha\beta}k_\kappa - 2I^{\mu\nu}{}_{\gamma\delta}I^{\lambda\kappa}{}_{\alpha\beta}k_\kappa + 2\eta^{\mu\nu}I^{\sigma\lambda}{}_{\gamma\delta}I_{\sigma\kappa\alpha\beta}k^\kappa\Big)
\end{aligned} \tag{64}
$$

$$
\begin{aligned}
\mathcal{F}^{\mu\nu}{}_{\alpha\beta\gamma\delta}(k,q) &= 2\Big[\eta^{\mu\nu}q_\lambda q_\kappa + I^{\mu\nu}{}_{\lambda\kappa}q^2 - q^\sigma\left(I^{\mu}{}_{\sigma\lambda\kappa}q^\nu + I^{\nu}{}_{\sigma\lambda\kappa}q^\mu\right)\Big]\Big[I^{\lambda\kappa}{}_{\alpha\beta}\left(p^2\eta_{\gamma\delta} - p_\gamma p_\delta\right) + I^{\lambda\kappa}{}_{\gamma\delta}\left(k^2\eta_{\alpha\beta} - k_\alpha k_\beta\right)\Big] \\
&- 2\left(\eta^{\mu\nu}q^2 - q^\mu q^\nu\right)\left\{\bar{\mathcal{R}}_{\alpha\beta\gamma\delta} + \frac{3}{2}\left[I_{\lambda\kappa\alpha\beta}I^{\kappa\sigma}{}_{\gamma\delta}(k^\lambda p_\sigma + p^\lambda p_\sigma) + I_{\lambda\kappa\gamma\delta}I^{\kappa\sigma}{}_{\alpha\beta}(p^\lambda k_\sigma + k^\lambda k_\sigma)\right]\right\}
\end{aligned} \tag{65}
$$

$$
\begin{aligned}
\mathcal{W}^{\mu\nu}{}_{\alpha\beta\gamma\delta}(k,q) &= \Big[\eta^{\mu\nu}q_\lambda q_\kappa + I^{\mu\nu}{}_{\lambda\kappa}q^2 - q^\sigma\left(I^{\mu}{}_{\sigma\lambda\kappa}q^\nu + I^{\nu}{}_{\sigma\lambda\kappa}q^\mu\right)\Big]\bar{\mathcal{R}}^{\lambda\kappa}{}_{\alpha\beta\gamma\delta}(k,q) \\
&- \frac{2}{3}\left(\eta^{\mu\nu}q^2 - q^\mu q^\nu\right)\bar{\mathcal{R}}_{\alpha\beta\gamma\delta}(k,q) - 2\mathcal{P}^{\mu\nu\sigma\tau}I_\sigma{}^{\lambda}{}_{\gamma\delta}I_{\tau\lambda\alpha\beta}p^2k^2 \\
&+ \left(\mathcal{P}^{\mu\nu}{}_{\sigma\lambda} - \frac{1}{4}\eta^{\mu\nu}\eta_{\sigma\lambda}\right)q^\sigma\left(p^\lambda k^2 + k^\lambda p^2\right)I_{\alpha\beta\gamma\delta} - \left(p^\mu p^\nu k^2 + k^\mu k^\nu p^2\right)I_{\alpha\beta\gamma\delta} \\
&- \frac{1}{2}q^2\Big[I_\lambda{}^{\nu}{}_{\gamma\delta}I^{\lambda\mu}{}_{\alpha\beta}k^2 + I_\lambda{}^{\nu}{}_{\alpha\beta}I^{\lambda\mu}{}_{\gamma\delta}p^2 + (\mu\leftrightarrow\nu)\Big] \\
&+ q_\rho\Big[k^2\Big(I_\lambda{}^{\nu}{}_{\gamma\delta}I^{\lambda\mu}{}_{\alpha\beta}p^\rho + p^\mu(I^{\nu\lambda}{}_{\gamma\delta}I^{\rho}{}_{\lambda\alpha\beta} - I^{\rho\lambda}{}_{\gamma\delta}I^{\nu}{}_{\lambda\alpha\beta}) + \frac{1}{2}q^\mu\left(I^{\rho\lambda}{}_{\gamma\delta}I^{\nu}{}_{\lambda\alpha\beta} - I^{\nu\lambda}{}_{\gamma\delta}I^{\rho}{}_{\lambda\alpha\beta}\right)\Big) \\
&+ p^2\Big(I_\lambda{}^{\nu}{}_{\alpha\beta}I^{\lambda\mu}{}_{\gamma\delta}k^\rho + k^\mu(I^{\nu\lambda}{}_{\alpha\beta}I^{\rho}{}_{\lambda\gamma\delta} - I^{\rho\lambda}{}_{\alpha\beta}I^{\nu}{}_{\lambda\gamma\delta}) + \frac{1}{2}q^\mu\left(I^{\rho\lambda}{}_{\alpha\beta}I^{\nu}{}_{\lambda\gamma\delta} - I^{\nu\lambda}{}_{\alpha\beta}I^{\rho}{}_{\lambda\gamma\delta}\right)\Big) + (\mu\leftrightarrow\nu)\Big] \\
&+ q_\lambda q_\sigma\Big[k^2\Big(I^{\lambda\nu}{}_{\gamma\delta}I^{\mu\sigma}{}_{\alpha\beta} - \frac{1}{2}\left(I^{\mu\nu}{}_{\gamma\delta}I^{\sigma\lambda}{}_{\alpha\beta} + I^{\mu\nu}{}_{\alpha\beta}I^{\lambda\sigma}{}_{\gamma\delta}\right)\Big) + p^2\Big(I^{\lambda\nu}{}_{\alpha\beta}I^{\mu\sigma}{}_{\gamma\delta} - \frac{1}{2}\left(I^{\mu\nu}{}_{\alpha\beta}I^{\sigma\lambda}{}_{\gamma\delta} + I^{\mu\nu}{}_{\gamma\delta}I^{\lambda\sigma}{}_{\alpha\beta}\right)\Big) \\
&+ \frac{1}{2}q^\sigma\Big(k_\kappa\left(I^{\mu\nu}{}_{\gamma\delta}I^{\kappa\lambda}{}_{\alpha\beta} - I^{\lambda\nu}{}_{\gamma\delta}I^{\mu\kappa}{}_{\alpha\beta}\right) + p_\kappa\left(I^{\mu\nu}{}_{\alpha\beta}I^{\kappa\lambda}{}_{\gamma\delta} - I^{\lambda\nu}{}_{\alpha\beta}I^{\mu\kappa}{}_{\gamma\delta}\right)\Big) \\
&+ \frac{1}{2}q^\nu\Big(k_\kappa\left(I^{\lambda\sigma}{}_{\gamma\delta}I^{\mu\kappa}{}_{\alpha\beta} - I^{\mu\lambda}{}_{\gamma\delta}I^{\kappa\sigma}{}_{\alpha\beta}\right) + p_\kappa\left(I^{\lambda\sigma}{}_{\alpha\beta}I^{\mu\kappa}{}_{\gamma\delta} - I^{\mu\lambda}{}_{\alpha\beta}I^{\kappa\sigma}{}_{\gamma\delta}\right)\Big) + (\mu\leftrightarrow\nu)\Big] \\
&+ \frac{1}{2}\left(I^{\sigma\kappa\mu\nu}q^\tau q^\lambda + I^{\lambda\tau\mu\nu}q^\kappa q^\sigma - I^{\lambda\kappa\mu\nu}q^\tau q^\sigma - I^{\sigma\tau\mu\nu}q^\kappa q^\lambda\right)\Big(I_{\sigma\tau\alpha\beta}\eta_{\gamma\delta}k_\kappa p_\lambda + I_{\sigma\tau\gamma\delta}\eta_{\alpha\beta}p_\kappa k_\lambda \\
&+ I_{\sigma\tau\alpha\beta}\eta_{\gamma\delta}p_\kappa p_\lambda + I_{\sigma\tau\gamma\delta}\eta_{\alpha\beta}k_\kappa k_\lambda\Big)
\end{aligned} \tag{66}
$$

with

$$
\begin{aligned}
\bar{\mathcal{R}}^{\lambda\kappa}{}_{\alpha\beta\gamma\delta}(k,q) &= -I_{\alpha\beta\gamma\delta}q^\lambda q^\kappa + q^\sigma\Big[I_{\sigma\tau\gamma\delta}\left(\frac{1}{2}I^{\tau\kappa}{}_{\alpha\beta}k^\lambda - I^{\lambda\kappa}{}_{\alpha\beta}k^\tau\right) + I_{\sigma\tau\alpha\beta}\left(\frac{1}{2}I^{\tau\kappa}{}_{\gamma\delta}p^\lambda - I^{\lambda\kappa}{}_{\gamma\delta}p^\tau\right)\Big] \\
&+ \left(I^{\tau\lambda}{}_{\gamma\delta}p_\sigma + I^{\tau}{}_{\sigma\gamma\delta}p^\lambda - I^{\lambda}{}_{\sigma\gamma\delta}p^\tau\right)\left(I^{\sigma\kappa}{}_{\alpha\beta}k_\tau + I^{\sigma}{}_{\tau\alpha\beta}k^\kappa - I_\tau{}^{\kappa}{}_{\alpha\beta}k^\sigma\right) + k^2I^{\kappa}{}_{\tau\gamma\delta}I^{\lambda\tau}{}_{\alpha\beta} + p^2I^{\kappa}{}_{\tau\alpha\beta}I^{\lambda\tau}{}_{\gamma\delta} \\
&- 2k_\sigma k_\tau\left(\frac{1}{3}I^{\lambda\kappa}{}_{\gamma\delta}I^{\sigma\tau}{}_{\alpha\beta} + \frac{1}{2}I^{\kappa\sigma}{}_{\gamma\delta}I^{\tau\lambda}{}_{\alpha\beta}\right) - 2p_\sigma p_\tau\left(\frac{1}{3}I^{\lambda\kappa}{}_{\alpha\beta}I^{\sigma\tau}{}_{\gamma\delta} + \frac{1}{2}I^{\kappa\sigma}{}_{\alpha\beta}I^{\tau\lambda}{}_{\gamma\delta}\right) \\
&- q^\lambda\left(I^{\kappa\sigma}{}_{\gamma\delta}I_{\tau\sigma\alpha\beta}k^\tau + I^{\kappa\sigma}{}_{\alpha\beta}I_{\tau\sigma\gamma\delta}p^\tau\right) \\
&+ \frac{1}{2}\left(k^\kappa k^\lambda\eta_{\alpha\beta} - k^\kappa k_\rho I^{\rho\lambda}{}_{\alpha\beta} - k^\lambda k_\rho I^{\rho\kappa}{}_{\alpha\beta} + k^2I^{\kappa\lambda}{}_{\alpha\beta}\right)\eta_{\gamma\delta} + \frac{1}{2}\left(p^\kappa p^\lambda\eta_{\gamma\delta} - p^\kappa p_\rho I^{\rho\lambda}{}_{\gamma\delta} - p^\lambda p_\rho I^{\rho\kappa}{}_{\gamma\delta} + p^2I^{\kappa\lambda}{}_{\gamma\delta}\right)\eta_{\alpha\beta} \\
&- \frac{1}{2}(k^\kappa I^{\lambda}{}_{\rho\alpha\beta} + p^\lambda I^{\kappa}{}_{\rho\alpha\beta} - k_\rho I^{\kappa\lambda}{}_{\alpha\beta})p^\rho\eta_{\gamma\delta} - \frac{1}{2}(p^\kappa I^{\lambda}{}_{\rho\gamma\delta} + p^\lambda I^{\kappa}{}_{\rho\gamma\delta} - p_\rho I^{\kappa\lambda}{}_{\gamma\delta})k^\rho\eta_{\alpha\beta} \\
&+ \frac{2}{3}I^{\kappa\lambda}{}_{\alpha\beta}p^2\eta_{\gamma\delta} + \frac{2}{3}I^{\kappa\lambda}{}_{\gamma\delta}k^2\eta_{\alpha\beta} + I_{\alpha\beta\rho}{}^{(\lambda}p^{\kappa)}p^\rho\eta_{\gamma\delta} + I_{\gamma\delta\rho}{}^{(\lambda}k^{\kappa)}k^\rho\eta_{\alpha\beta} \\
&+ \frac{1}{4}\Big[\eta_{\alpha\beta}\eta_{\gamma\delta}(k^\lambda p^\kappa + p^\lambda k^\kappa) - 2(I_{\alpha\beta\rho}{}^{(\kappa}p^{\lambda)}k^\rho\eta_{\gamma\delta} + I_{\alpha\beta\rho}{}^{(\kappa}p^{\lambda)}p^\rho\eta_{\gamma\delta} + I_{\gamma\delta\rho}{}^{(\kappa}k^{\lambda)}p^\rho\eta_{\alpha\beta} + I_{\gamma\delta\rho}{}^{(\kappa}k^{\lambda)}k^\rho\eta_{\alpha\beta})\Big]
\end{aligned} \tag{67}
$$

$$
\begin{aligned}
\bar{\mathcal{R}}_{\alpha\beta\gamma\delta}(k,q) &= -q^2I_{\alpha\beta\gamma\delta} + \left(I^{\lambda}{}_{\tau\gamma\delta}p_\sigma + I^{\lambda}{}_{\sigma\gamma\delta}p_\tau - I_{\tau\sigma\gamma\delta}p^\lambda\right)\left(I_\lambda{}^{\tau}{}_{\alpha\beta}k^\sigma + I^{\tau\sigma}{}_{\alpha\beta}k_\lambda - I_\lambda{}^{\sigma}{}_{\alpha\beta}k^\tau\right) \\
&+ I^{\lambda}{}_{\sigma\gamma\delta}k_\tau\left(I^{\tau}{}_{\lambda\alpha\beta}k^\sigma + I^{\tau\sigma}{}_{\alpha\beta}k_\lambda - I^{\sigma}{}_{\lambda\alpha\beta}k^\tau\right) + I^{\lambda}{}_{\sigma\alpha\beta}p_\tau\left(I^{\tau}{}_{\lambda\gamma\delta}p^\sigma + I^{\tau\sigma}{}_{\gamma\delta}p_\lambda - I^{\sigma}{}_{\lambda\gamma\delta}p^\tau\right) \\
&+ \frac{1}{2}q^\lambda\left(I_{\lambda\sigma\gamma\delta}I^{\kappa\sigma}{}_{\alpha\beta}k_\kappa + I_{\lambda\sigma\alpha\beta}I^{\kappa\sigma}{}_{\gamma\delta}p_\kappa\right) \\
&+ \eta_{\alpha\beta}\eta_{\gamma\delta}k^\lambda p_\lambda - 2\left[I^{\kappa}{}_{\lambda\alpha\beta}\eta_{\gamma\delta}(k_\kappa p^\lambda + p_\kappa p^\lambda) + I^{\kappa}{}_{\lambda\gamma\delta}\eta_{\alpha\beta}(p_\kappa k^\lambda + k_\kappa k^\lambda)\right] \\
&+ \eta_{\alpha\beta}\left(p^2\eta_{\gamma\delta} - p^\lambda p^\kappa I_{\lambda\kappa\gamma\delta}\right) + \eta_{\gamma\delta}\left(k^2\eta_{\alpha\beta} - k^\lambda k^\kappa I_{\lambda\kappa\alpha\beta}\right).
\end{aligned} \tag{68}
$$

Tree-level Compton Amplitudes: scalar-Merlin scattering

- To evaluate Compton scattering amplitude involving two gravitational Merlin particles and two matter particles, we apply the BCJ double-copy prescription:

$$M(1_s, 2, 3, 4_s) = i \sum_k \frac{n_k^{(s_1)} \tilde{n}_k^{(s_2)}}{s_k}$$

  where $s = s_1 + s_2$, $2, 3$ are graviton or Merlin particles, $\tilde{n}_k$ are numerators belonging to the spontaneously broken gauge theory of the double copy described earlier and s_k are inverse propagators (they could be massive).

- For the scalar case:

$$\begin{aligned} M_4(\ell_A, \mathbf{1}, \mathbf{2}, \ell_B) &= \left[\frac{2}{M^2}\langle \mathbf{1}|\ell_A|\mathbf{1}]\langle \mathbf{2}|\ell_B|\mathbf{2}] + \frac{\langle \mathbf{12}\rangle[\mathbf{21}]}{M^2}(2(\mathbf{1}\cdot\ell_A) + M^2)\right]^2 \frac{i}{s - m^2} \\ &+ \left[\frac{2}{M^2}\langle \mathbf{1}|\ell_B|\mathbf{1}]\langle \mathbf{2}|\ell_A|\mathbf{2}] + \frac{\langle \mathbf{12}\rangle[\mathbf{21}]}{M^2}(2(\mathbf{2}\cdot\ell_A) + M^2)\right]^2 \frac{i}{u - m^2} \\ &- \frac{4i}{M^4}\Big(\langle \mathbf{12}\rangle[\mathbf{21}](\mathbf{1}-\mathbf{2})\cdot\ell_A + \langle \mathbf{1}|\ell_A|\mathbf{1}]\langle \mathbf{2}|\ell_B|\mathbf{2}] - \langle \mathbf{2}|\ell_A|\mathbf{2}]\langle \mathbf{1}|\ell_B|\mathbf{1}]\Big)^2 \left(\frac{1}{t} - \frac{2}{t - M^2}\right) \end{aligned}$$

Loops and Generalized Unitarity

Bern, Dixon, Dunbar and Kosower, 1994, 1995

- The strategy of calculating gluon amplitudes of standard Yang-Mills theory from Grassmann integrations of $\mathcal{N} = 4$ SYM does not work at loop level. Now the gluon amplitudes differ in both theories. However, the tree-level method will still be useful since we are going to use tree-level amplitudes to reconstruct loop-level amplitudes. This is the so-called generalized unitarity method

- The knowledge of tree amplitudes can be recycled into information about loop integrands. The operation of taking loop propagators on-shell is called a unitarity cut. It originates from the unitarity constraint of the S-matrix. To see how, recall the unitarity demands that generalized optical theorem holds, that is, for an arbitrary process $a \to b$ one has that

$$i\mathcal{A}(a \to b) - i\mathcal{A}^*(b \to a) = -\sum_f \int d\Pi_f \mathcal{A}^*(b \to f)\mathcal{A}(a \to f)(2\pi)^4\delta^4(a-f)$$

  and there is an overall delta function assuring energy-momentum conservation.

- If we examine this constraint order by order in perturbation theory, it tells us that the imaginary part of scattering amplitudes at a given order is related to the product of lower-order results. Using the cutting rules, it is possible to associated to each cut a one-particle final state; hence, for two-particle cuts one associates a two-particle final state, for three-particle cuts one associated a three-particle state, and so on.

- For one-loop process, one finds a product of two tree amplitudes. This product involves the sum over all possible on-shell states that can cross the cut. *Only states from the physical spectrum of the theory are included in this sum.* In a unitarity cut, we restrict the loop-momenta to be on-shell and only physical modes are included in the two on-shell amplitudes .

- The cutting rules also include integrals of any remaining freedom in the loop-momentum after imposing the so-called cut constraints and momentum conservation. For instance, for one-loop $2 \to 2$ processes, the integral over all allowed kinematic constraints takes the form

$$\int d^4\ell \delta_+(\ell^2)\delta_+((\ell - p_1 - p_2)^2).$$

Carrasco and Johansson, 2011

- The imaginary part of the amplitude probes its branch-cut structure, hence the unitarity cut allows us to relate the pole structure of the integrand with the branch-cut structure of the loop-integral. One can reconstruct the integrand by analyzing different sets of unitarity cuts. The unitarity cuts can also involve more than two cut lines. An N-line cut simply means that N internal line are taken on-shell. Reconstruction of the full loop amplitude from systematic application of unitarity cuts is called the generalized unitarity method. To identify contributions to the amplitude, one computes generalized unitarity cuts on the level of the integrand, expressing them as a product of on-shell subamplitudes

$$\sum_{\text{states}} A_{(1)} A_{(2)} A_{(3)} \cdots A_{(m)}.$$

  The cuts can be computed directly from the theory by feeding in the corresponding subamplitudes and summing over the intermediate states.

- The unitarity method instructs us to reconstruct amplitudes using the information from the set of all generalized unitarity cuts of a theory.

- The method of generalized unitarity consists in finding an integrand that reproduces all the unitarity cuts, including of course all the maximal cuts.

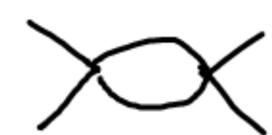
Carrasco and Johansson, 2011 Bern and Huang, 2011 Britto, 2011

- The information from unitarity cuts can be utilized most efficiently if we know a priori a complete basis of integrals that can appear in the scattering amplitudes. In D dimensions all one-loop amplitudes can be written as a sum of m-gon one-loop scalar integrals I_m for $m = 1, 2, 3, \ldots, D$:

$$A_n^{1-\text{loop}} = \sum_i C_D^{(i)} I_{D;n}^{(i)} + \sum_j C_{D-1}^{(j)} I_{D-1;n}^{(j)} + \cdots + \sum_k C_2^{(k)} I_{2;n}^{(k)} + \sum_l C_1^{(l)} I_{1;n}^{(l)} + \mathcal{R}$$

  where $\mathcal{R}$ denotes terms that are rational in the external variables and $C_D^{(i)}$ are kinematic-dependent coefficients related to the tree-level amplitudes for the m-gon scalar integrals $I_m^{(i)}$.

- The $D = 4$ one-loop integral reduction to box, triangle, bubble and tadpole scalar integrals. The latter are related to the coefficients $C_1^{(l)}$; such integrals vanish in dimensional regularization when only massless particles circulate in the loop.

$A^{1\text{-loop}}_{n, D=4} = C_\Box$  $+ \; C_\Delta$  $+ \; C_\infty$ 

$+$ possible tadpoles $+$ rational terms

Loop amplitudes involving unstable particles

- In a theory with unstable particles (of any kind) unitarity is satisfied by the inclusion of only the asymptotically stable states. This means that cuts should not be taken through the unstable particles, and unstable particles are not to be included in unitarity sums.

UNITARITY AND CAUSALITY IN A RENORMALIZABLE FIELD THEORY WITH UNSTABLE PARTICLES
M. VELTMAN *)

- In an one-loop analysis, four-dimensional amplitudes can be written as a linear combination of scalar boxes, scalar triangles, scalar bubbles and scalar tadpoles, with rational coefficients:

$$A^{1-\text{loop}} = \sum_{n=1}^{4} \sum_{\mathbf{K}} c_n(\mathbf{K}) \mathbf{I_n}(\mathbf{K})$$

where K_i are sums of external momenta and I_n are scalar integrals. The coefficients c_n are calculated using generalized cuts.

- For instance, the quadruple cut picks up a contribution from exactly one box integral, namely the one with momenta K_1, K_2, K_3, K_4 at the corners. Therefore, the cut expansion collapses to a single term

$$\Delta_4 A^{1-\text{loop}} = c_4(K_1, K_2, K_3, K_4) \Delta_4 I_4(K_1, K_2, K_3, K_4).$$

- Hence in principle the quadruple cut of the scalar box integral would suffice to calculate the box coefficient.

- However, for unstable particle this procedure does not quite determine such a coefficient. Using a modified Lehmann representation for the cut propagator, one obtains the general expression

$$\begin{aligned}\Delta_4 A^{1-\text{loop}} &= \left(\prod_{I=1}^{4}\int_0^\infty ds_i\,\frac{\widetilde{\rho}(s_i)}{\pi}\right)\int d^4\ell\,\delta(\ell_1^2(\ell)-s_1)\theta(\ell_1^0)\delta(\ell_2^2(\ell)-s_2)\theta(\ell_2^0)\delta(\ell_3^2(\ell)-s_3)\\ &\times\quad \theta(\ell_3^0)\delta(\ell_4^2(\ell)-s_4)\theta(\ell_4^0)A_1^{\text{tree}}(\ell)A_2^{\text{tree}}(\ell)A_3^{\text{tree}}(\ell)A_4^{\text{tree}}(\ell).\end{aligned}$$

- This means that we were not able to put the internal momenta on-shell; this will only happen if the spectral function has a contribution from one-particle states, which is not the case for unstable particles. So when cutting an internal line corresponding to an unstable particle, the result we obtain is not a contribution to the scattering amplitude. Hence in order to implement the generalized unitarity method to a theory containing unstable particles, we must consider the inclusion of only cuts from stable states in unitarity sums.

- The question lies on whether external momentum configurations of an amplitude allows the unstable propagator to become resonant. At least in the Complex-Mass Scheme, this can be thoroughly answered: The cut of an unstable propagator off resonance gives a contribution of higher contribution in perturbation theory. Such cuts can be ignored: In this sense, they give a vanishing result. On the other hand, for a resonant unstable propagator, one can show that the cut of this propagator proceeds through the cut of only stable particles, preserving unitarity in Veltman's sense. At leading order, the cut of the bare unstable propagator equals the cut through its first loop correction – through stable particle propagators.

How to implement the unitarity method in the presence of unstable particles

GM, arXiv:2111.11570 [hep-th]

- In order to implement the technique in a straightforward way, one must ensure that external momentum configurations of an amplitude allows the unstable propagator to become resonant. In this case the cut unstable propagator will have the correct cut structure to guarantee that unitarity is satisfied.

- On the other hand, there is also other situation that the method can be applied without further issues: In the narrow-width approximation! Near the resonance, we can treat the resonant particle as being on-shell. This means that in this limit a cut taken through the unstable particle with its width set to zero reproduces the same result as a cut through the decay products.

- In other words, for unstable particles the present practice of the unitarity method is valid if the assumption of a resonant unstable propagator is warrant. This can happen depending on external momentum configurations or else one should verify whether the narrow-width approximation holds in the particular case under studied.

Beware...

- Suppose we wish to study a particular process $a + b \to c + d$ which takes place exclusively through loops of unstable particles and let us assume that we are off resonance. Using the reasoning above, the cuts of internal unstable propagators will produce a vanishing contribution, resulting in a vanishing amplitude if we use the current practice of the unitarity method to reconstruct it. This is obviously an unsatisfactory answer.

- There are ways to circumvent this issue. For instance, one can eliminate the unstable fields in the Lagrangian. So we are able to reformulate the theory in terms of the stable particles only.

- But this will actually introduce non-local vertices in our description. There is one constraint that we should impose in this situation. In order to preserve unitarity, the only acceptable poles in amplitudes are the ones that come from propagators. Since non-local vertices may generate unphysical poles that would not correspond to an exchange of a physical particle, we must impose that such poles have zero residue. Or we must claim that the residues of all such spurious poles must cancel among the diagrams to give zero.

A Standard Model example: $\gamma - \gamma$ scattering via W loops

GM, arXiv:2111.11570 [hep-th]

- Consider the scattering $\gamma^+\gamma^+ \to \gamma^+\gamma^+$ proceeding via W-boson loops. At one-loop, this is finite (no bubbles).

- The unitarity method produces:

$$A_4^{1-\text{loop}} = -16e^4M^4 \frac{s_{12}s_{23}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} I_4(p_1, p_2, p_3, p_4) + \mathcal{R}.$$

- Could we have considered a non-local description? Yes! For instance, consider the non-local scalar QED:

$$\mathcal{L}_{\text{NL}} = \phi^*(x)\Sigma(x-y)U(x,y)\phi(y)$$

For the same process $\gamma^+\gamma^+ \to \gamma^+\gamma^+$, now via scalar loops, one finds that

$$A_4^{1-\text{loop}} = -32e^4m^4[F(m^2)]^4 \frac{s_{12}s_{23}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} I_4(p_1, p_2, p_3, p_4) + \mathcal{R}$$

$F(p'^2) = \frac{\partial\Sigma(p'^2)}{\partial p'^2}$.

Color-ordered one-loop amplitude associated with the process $g^+g^+ \to g^+g^+$

GM, arXiv:2112.00978 [hep-th]

Unitarity method produces

$$A_4^{1-\text{loop}}(1^+,2^+,3^+,4^+) = -\frac{2i}{(4\pi)^{2-\epsilon}} \frac{s_{12}s_{23}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\Big[I_4[\mu^4] + 8\mathcal{I}_4[(M^2+\mu^2)^2]\Big]$$

One-loop amplitude for the graviton scattering process $h^{++}h^{++} \to h^{++}h^{++}$

Unitarity method produces

$$\begin{aligned} M_4^{1-\text{loop}}(1^{++},2^{++},3^{++},4^{++}) \quad &= \quad \frac{2i}{(4\pi)^{2-\epsilon}}\left[\frac{s_{12}s_{23}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right]^2 \Big[I_{1234}[\mu^8] + I_{1243}[\mu^8] + I_{1324}[\mu^8] \\ &+ \quad 128\Big(\mathcal{I}_{1234}[(M^2+\mu^2)^4] + \mathcal{I}_{1243}[(M^2+\mu^2)^4] + \mathcal{I}_{1324}[(M^2+\mu^2)^4]\Big)\Big] \end{aligned}$$

Outlook

Modern Scattering Amplitudes Program is increasingly relevant to diverse areas across Physics

Essential directions:

- New computational tools
- Sophisticated mathematical structure (positive geometry)
- Amplitude Bootstrap
- Double copy
- Gravitational-wave Physics and related subjects