Using Schur—Weyl duality and Yamanouchi words to compute SU(n) Wigner coefficients more efficiently

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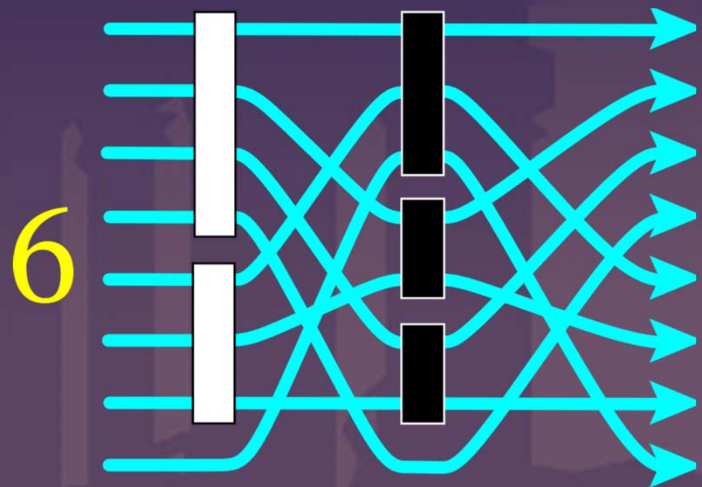
Introduction

Schur—Weyl Duality



- An irrep of SU(n) may be constructed from the tensor product $V^{\otimes n}$ of the defining (fundamental) irrep V
- It corresponds to a Young diagram = a partition of n
- It is projected from $V^{\otimes n}$ by a Young projector
- Hermitian projectors are even better, of course

Projector Orthogonality

- von Neumann observed that Young projectors project onto disjoint invariant subspaces

A. Young, *Proc. London Math. Soc. (1)*, **33**, p. 97 (1900); (1), **34**, p. 361 (1902)
G. Frobenius, *Sitzungsber. Preuss. Akad.*, p. 328 (1903)

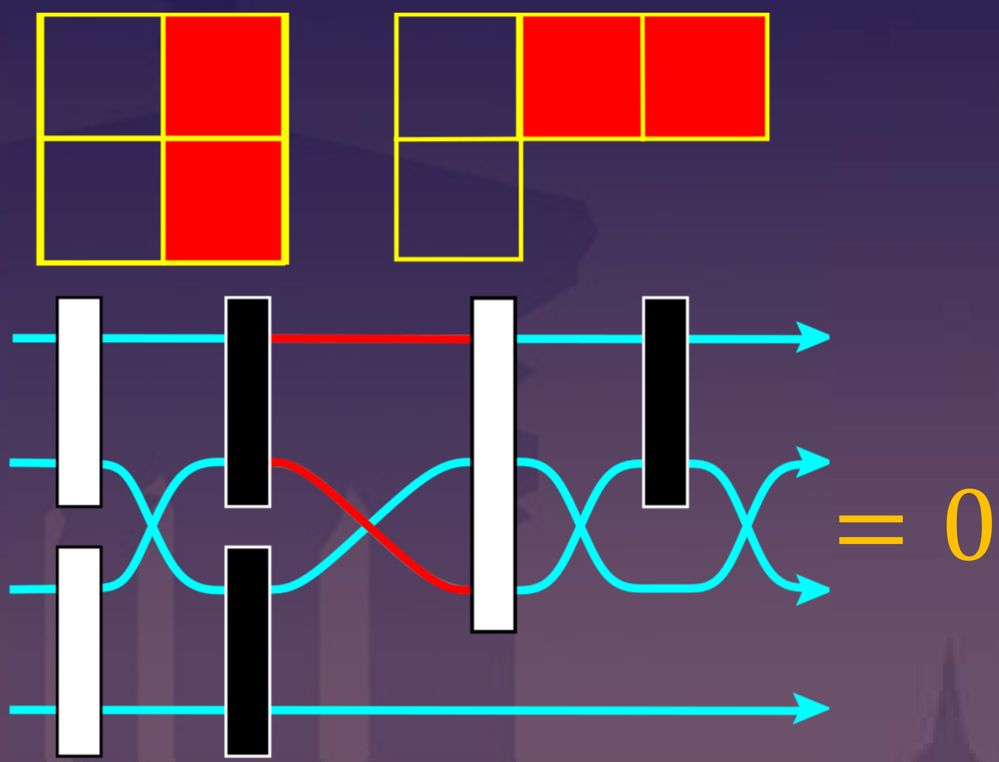
  – *These "symmetries" were first invented by A. Young. In studying their properties we shall follow G. Frobenius or rather the simplified arrangement of Frobenius' proofs due to J. v. Neumann.*

Hermann Weyl, *The Classical Groups*, p. 129

  – *I am indebted to a conversation with J. von Neumann for the simplified proofs…*

B. L. van der Waerden, *Algebra*, vol. II, 5th ed., p. 93

von Neumann's Argument



- Consider the product of Young projectors corresponding to a pair of different diagrams with the same number of boxes
- There must be a pair of boxes in the Young diagram that are in the same row in one diagram and in the same column in the other
- There must be two lines corresponding to this pair that connect a symmetrizer to an antisymmetrizer
- Hence the product vanishes

Projector Normalization

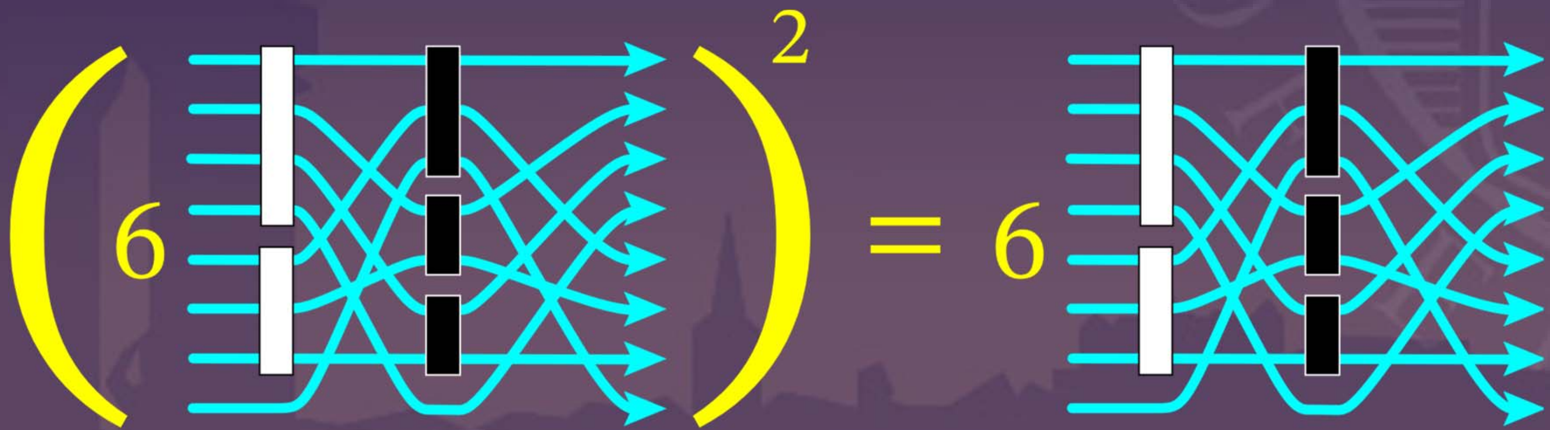
- The <u>hook number</u> is $h =$

6	4	3	1
4	2	1	
1			

$= 3^2 2^6$

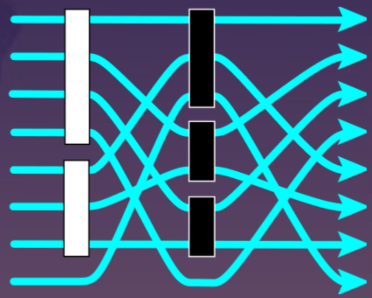
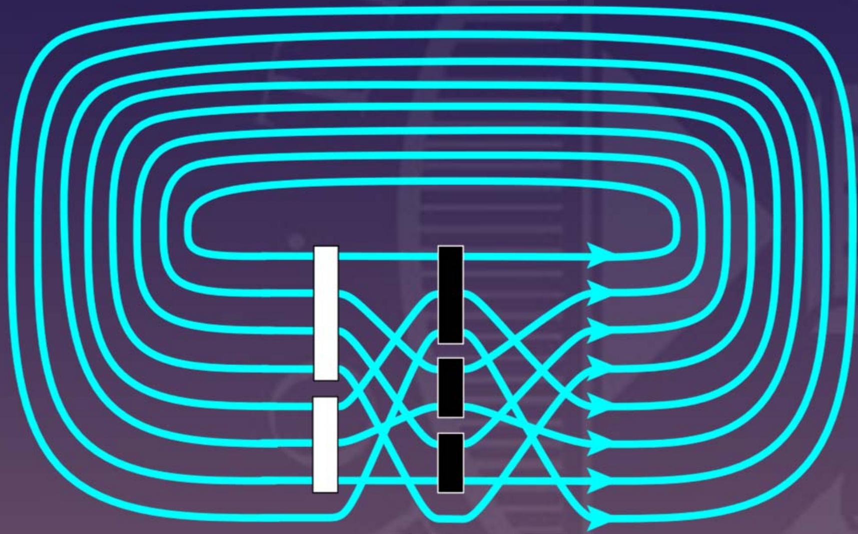
- Idempotent normalization

$$\frac{\left(\prod_i r_i!\right)\left(\prod_j c_j!\right)}{h} = \frac{(4!\,3!\,1!)(3!\,2!\,2!\,1!)}{h} = \frac{4!\,3!^2\,2!^2}{h} = \frac{3^3 2^7}{3^2 2^6} = 6$$



SU(n) Dimension

- SU(n) dimension is trace of Young projector

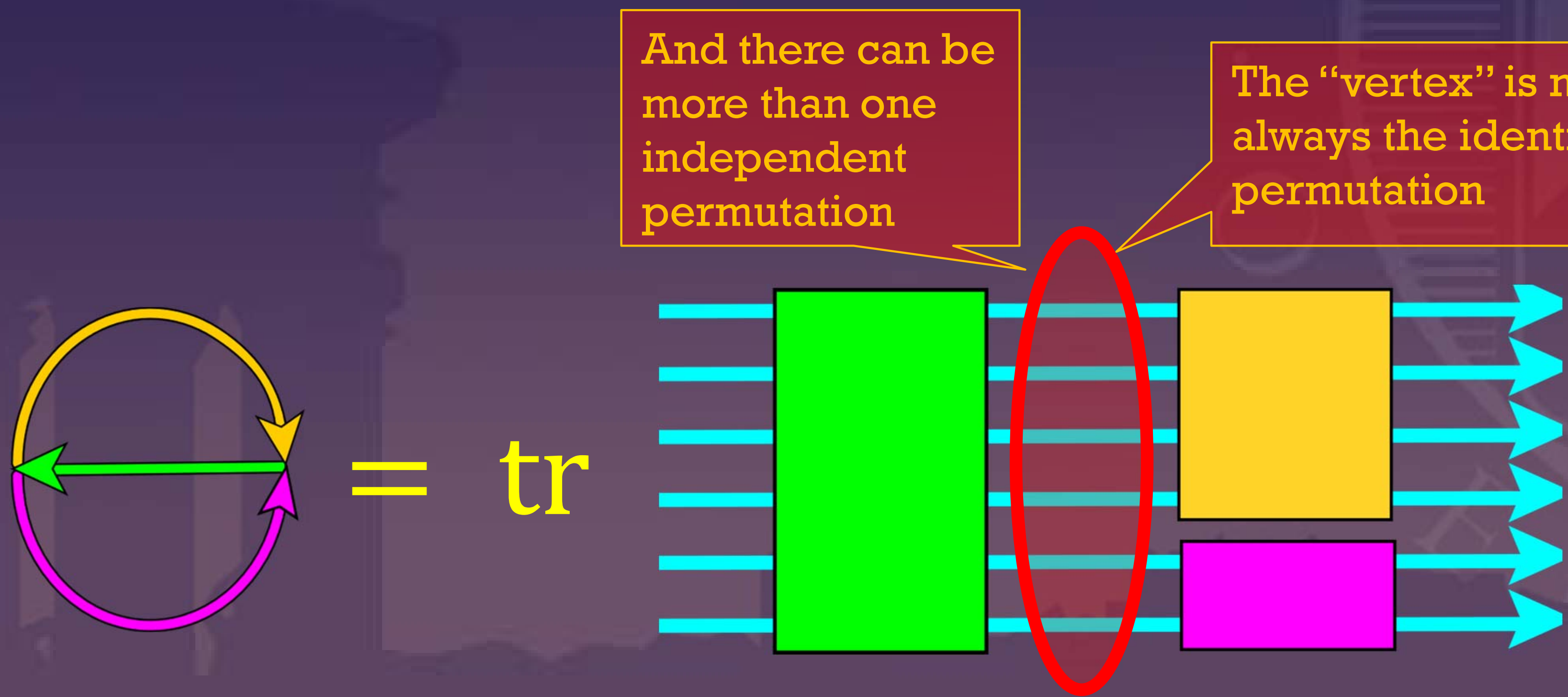
- $d_{SU(n)}$ = tr 6  = 6 

$$= \frac{1}{h}\begin{array}{|c|c|c|c|}\hline n & n+1 & n+2 & n+3 \\ \hline n-1 & n & n+1 & \\ \cline{1-3} n-2 & & & \\ \cline{1-1}\end{array} = \frac{n^2(n^2-1)(n^2-4)(n+3)(n+1)}{3^2 2^6}$$

Clebsch—Gordan Vertex Permutations

CG Vertex Permutations

- Consider the birdtrack expression for a 3*j*
  - We draw the Young projectors as boxes whose colour corresponds to its Young diagram



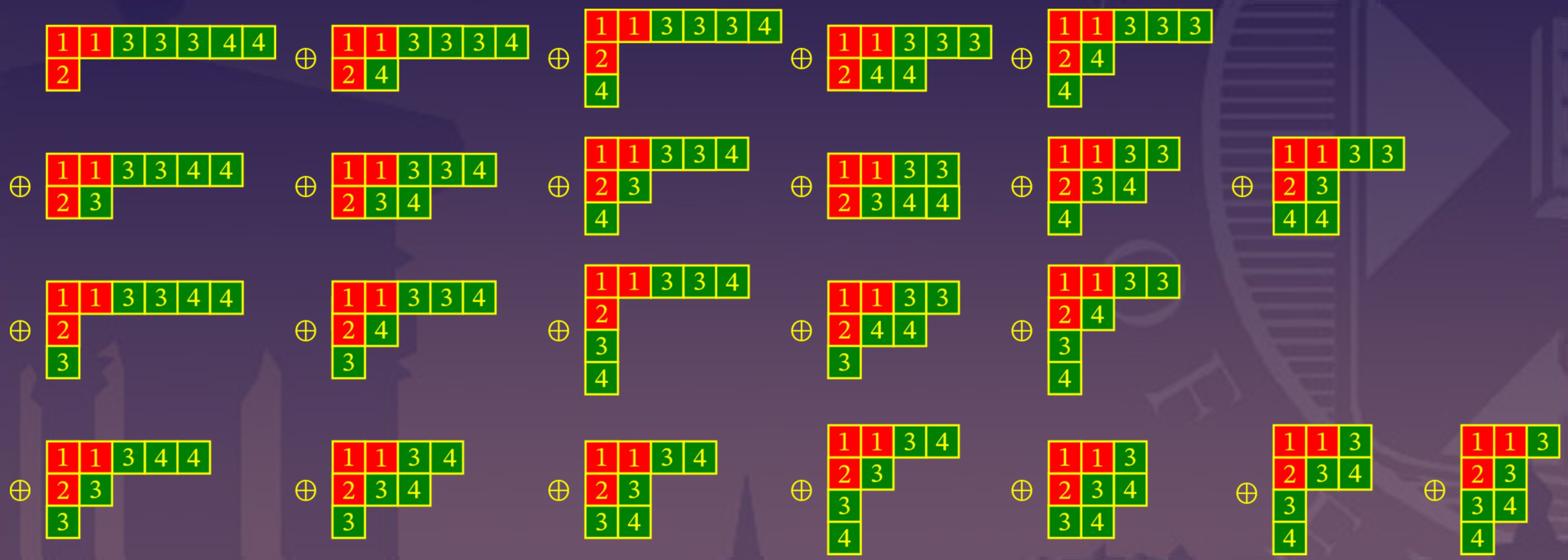
Pieri's Rule

- Which irreps occur in the following tensor product of SU(n) irreps (CG series)?

  - We use <u>Pieri's Rule</u> to add the first (green) row

Pieri's Rule

- We use it again to add the second (green) row



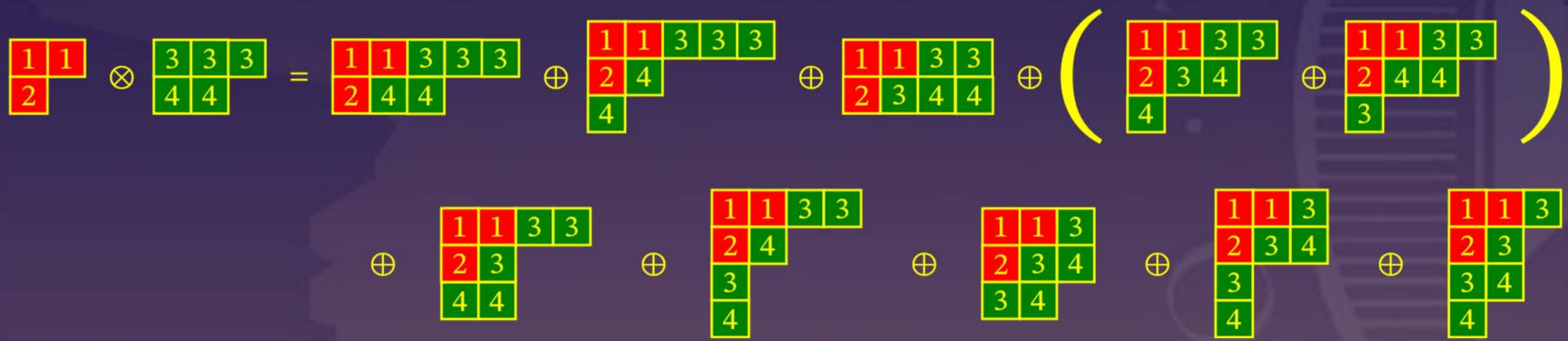
Yamanouchi Words

– We write the reverse lattice words (green numbers top to bottom then right to left) and delete those that are not Yamanouchi (more 4s than 3s at any point from left to right)

~~44333~~, ~~43334~~, ~~43334~~, 33344, 33344,
~~44333~~, ~~43343~~, ~~43334~~, 33443, 33434, 33344,
~~44333~~, ~~43343~~, ~~43334~~, 33443, 33434,
~~44333~~, ~~43433~~, ~~43343~~, ~~43334~~, 34343, 34333, 33434

Littlewood—Richardson Rule

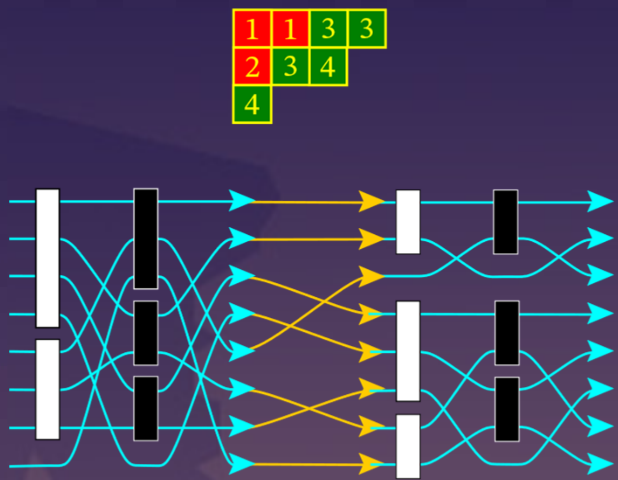
- We are left with the irreps that occur in the decomposition, with their multiplicities

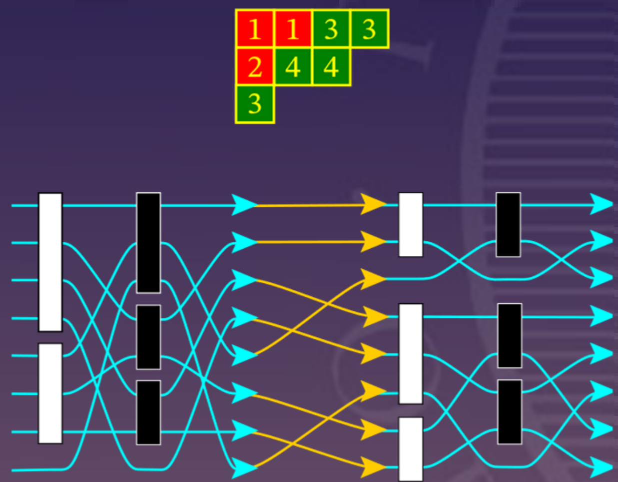


- This is the Littlewood—Richardson rule
  - The rule was first formulated in a 1934 paper by Littlewood and Richardson, but the first complete proofs were not published until the 1970s

Yamanouchi Vertices

- This suggests that Yamanouchi words should tell us the permutation that connects the rows





  - They are independent, but not orthogonal
    - What happens for hermitian Young projectors?
  - This seems to work empirically
  - But I do not have a proof!

Symmetric Group Representations

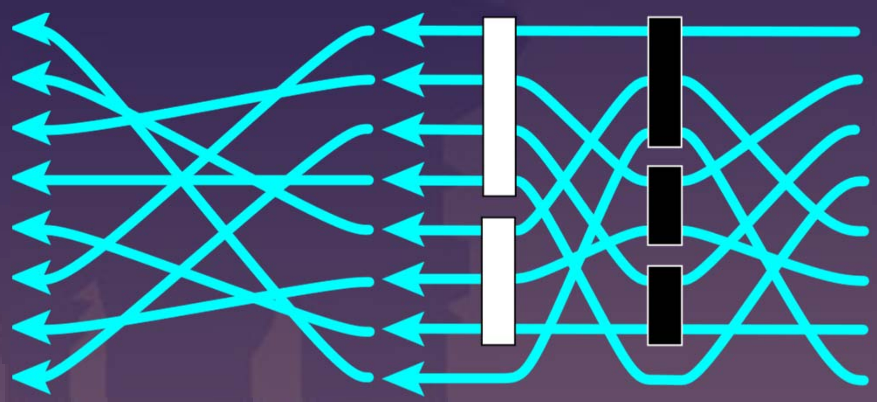
Elvang's Observation

- Any SU(n) $3n - j$ symbol can be expressed as a "sandwich" of Young projectors and vertex permutations between the largest projector

- It is therefore a multiple m of the dimension $d_{SU(n)}$ of the projector with the most legs
  - m may be computed by expanding the "filling" of the sandwich in permutations, each of which is ± 1 or 0 by von Neumann's argument
  - But this is exponentially painful in the number of legs!

Symmetric Group Irreps

- Consider a permutation acting on a Young projector



  - With some arbitrary conventions for left/right multiplication, arrows, etc.

- We may denote this by a Young Tableau, with the numbers corresponding to the permutation

1	2	3	4
5	6	7	
8			

$\mapsto$

6	3	8	4
2	7	5	
1			

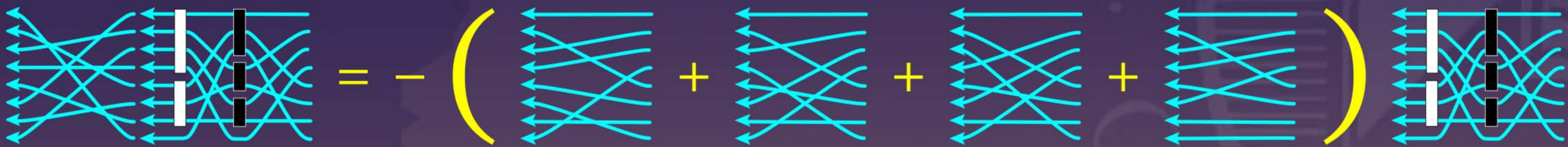
Standard Permutations

- The action of any permutation may be written as a linear combination of standard permutations
  - A standard permutation corresponds to a standard tableau, in which the numbers increase from left to right and top to bottom
  - The number of standard tableaux is the Kostka number $K = p!/h$
    - $|\mathcal{S}_p| = p!$ is the order of the symmetric group
    - h is the hook number
    - For our example with 8 legs/boxes the number of group elements is $8! = 40{,}320$ whereas the number of standard permutations is only 70

$\mathcal{S}_8$ Representation

- For our example the desired decomposition is

$$\begin{array}{|c|c|c|c|}\hline 6&3&8&4\\\hline 2&7&5\\\cline{1-3} 1\\\cline{1-1}\end{array} = -\begin{array}{|c|c|c|c|}\hline 1&3&4&8\\\hline 2&5&7\\\cline{1-3} 6\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|c|}\hline 1&3&6&8\\\hline 2&5&7\\\cline{1-3} 4\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|c|}\hline 1&4&6&8\\\hline 2&5&7\\\cline{1-3} 3\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|c|}\hline 1&3&4&6\\\hline 2&5&7\\\cline{1-3} 8\\\cline{1-1}\end{array}$$



- This immediately gives a column of the 70×70 matrix representing the permutation in the irrep specified by the tableau's shape
  - The other columns are obtained by inserting standard permutations just before the projector

Garnir Algorithm

- The proof that this can be done is constructive
  - Garnir's algorithm recursively computes the decomposition
  - Clearly, we can always sort the rows as they directly correspond to symmetrizers

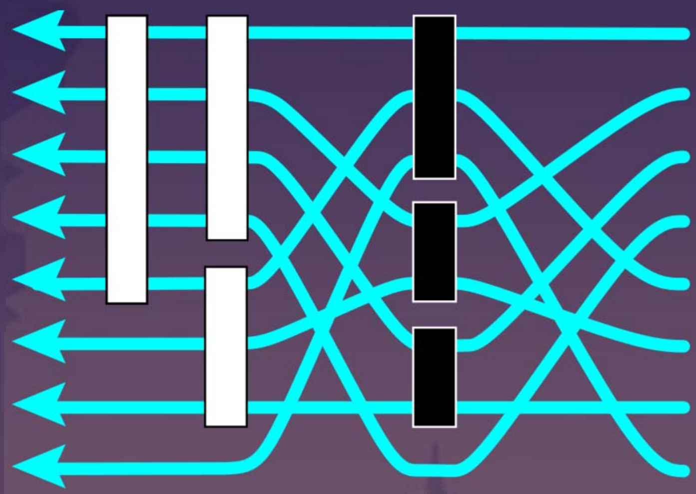
$$\begin{array}{|c|c|c|c|}\hline 6&3&8&4\\\hline 2&7&5\\\cline{1-3} 1\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 3&4&6&8\\\hline 2&5&7\\\cline{1-3} 1\\\cline{1-1}\end{array}$$

  - We find the first "strip" (red) which is not in standard order
  - Use the fact that the sum over all permutations of the numbers in the strip vanishes to move the disorder to the right or down

$$\begin{array}{|c|c|c|c|}\hline \color{red}3&\color{red}4&\color{red}6&\color{red}8\\\hline \color{red}2&5&7\\\cline{1-3} 1\\\cline{1-1}\end{array} = -\begin{array}{|c|c|c|c|}\hline 2&4&6&8\\\hline \color{red}3&\color{red}5&\color{red}7\\\cline{1-3} \color{red}1\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|c|}\hline 2&3&6&8\\\hline \color{red}4&\color{red}5&\color{red}7\\\cline{1-3} \color{red}1\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|c|}\hline 2&3&4&8\\\hline \color{red}6&\color{red}5&\color{red}7\\\cline{1-3} \color{red}1\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|c|}\hline 2&3&4&6\\\hline \color{red}8&\color{red}5&\color{red}7\\\cline{1-3} \color{red}1\\\cline{1-1}\end{array}$$

Garnir Algorithm

- The symmetrization of the strip vanishes becomes obvious by applying von Neumann's argument to the following birdtrack



$= 0$

  – The strip is always one box longer than von Neumann allows

$3n - j$ Computation

- We need not compute all the representation matrices
  - It suffices to compute a generating set, such as neighbour transpositions
- The representation extends naturally to a representation of the group algebra
  - We can represent symmetrizers, antisymmetrizers, and Young projectors recursively
  - We can also compute the matrices for vertex permutations
- It is easy to compute the coefficient m of the SU(n) dimension from the matrix representing the "filling of the sandwich"

Conclusions

1. The multiplicity of irreps in the Clebsch—Gordan series (the Littlewood—Richardson numbers) can be greater than one, and we need to compute the permutations corresponding to all such vertices
2. We can compute Wigner $3n-j$ coefficients efficiently using irreps of the symmetric group

- Future work
  - This all should be easy to extend to Hermitian Young projectors
  - Prove the relation between Yamanouchi words and vertex permutations
    - Jeu de Taquin, Knuth equivalence, Plactic monoids,…