Constructing generalized Hamiltonian lattice gauge theories using ~~a graphical calculus~~ Birdtracks!

T. V. Zache

University of Innsbruck & IQOQI Austrian Academy of Sciences

Online Birdtracks Meeting

28 February 2024

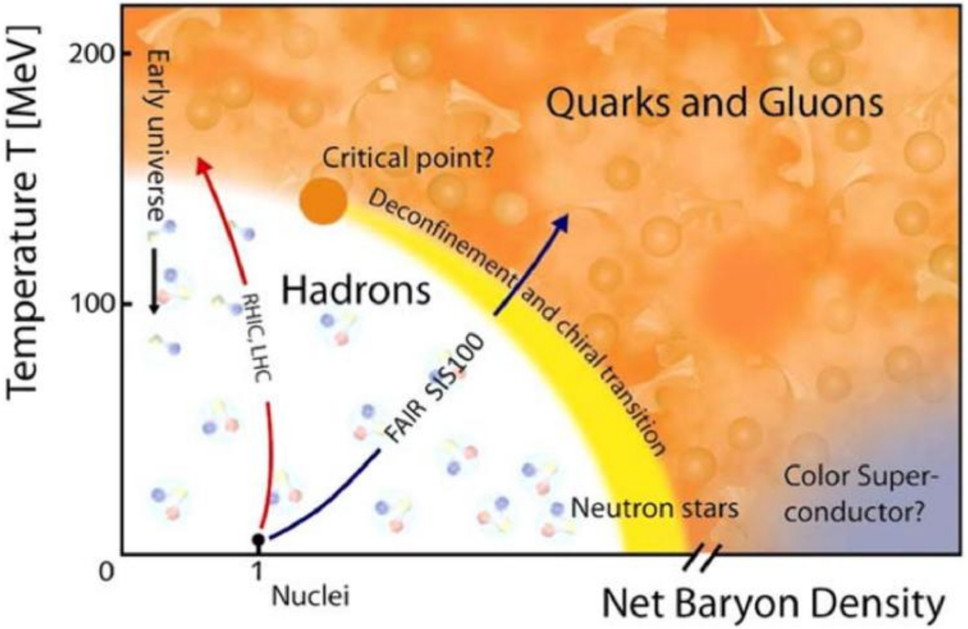
Phys. Rev. Lett. 131, 171902 (2023)

D. Gonzalez-Cuadra

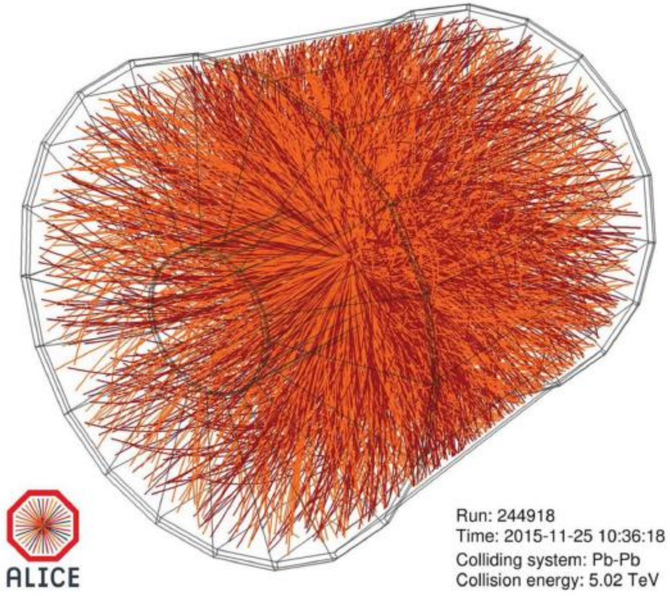
P. Zoller

Probing the Standard Model of particle physics

Strong force: quantum chromodynamics (QCD) → heavy-ion collisions



Durante et al., Physica Scripta, 94(3), 033001 (2019)



Computational challenges:
- Real-time dynamics
- Finite baryon density

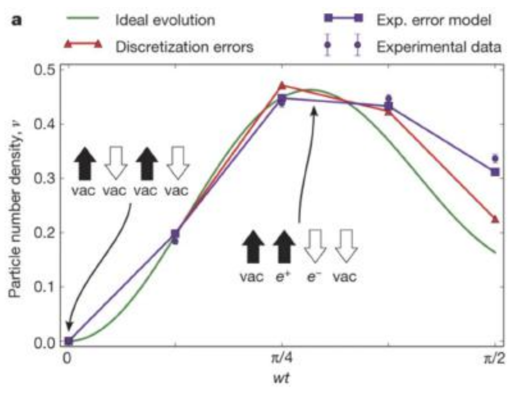
Troyer & Wiese, Physical review letters, 94(17), 170201 (2005)

→ **Quantum simulation!**

Büchler et al, Physical review letters, 95(4), 040402 (2005)
Byrnes & Yamamoto, Physical Review A, 73(2), 022328 (2006)
… Martinez et al., Nature, 534(7608), 516-519 (2016) …
Klco et al., Reports on Progress in Physics, 85(6), 064301 (2022)
Di Meglio, arXiv:2307.03236 (2023)
…

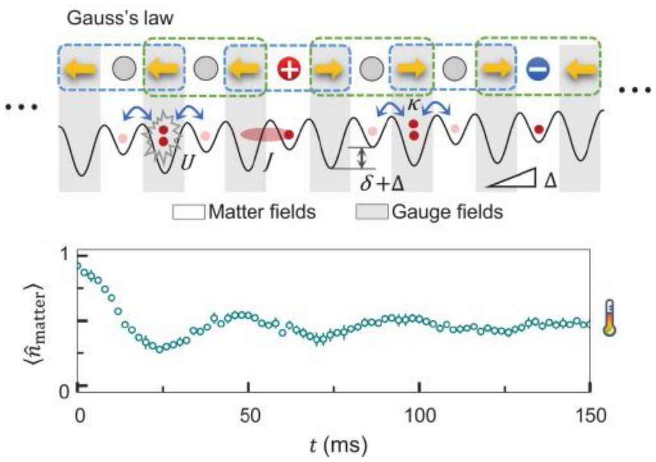
Digital and analog LGTs on quantum hardware

Trapped ions:

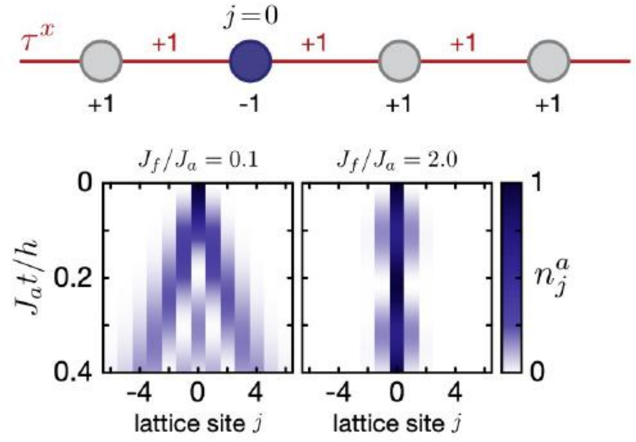


Martinez et al., Nature, 534, 516 (2016)

Atoms in optical lattices:

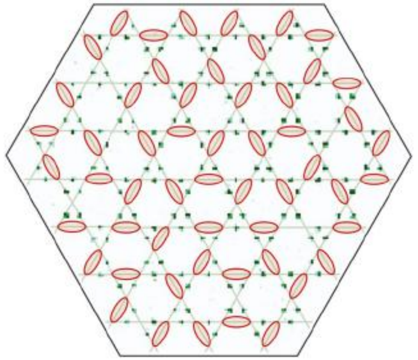


Zhou et al., Science, 377, 311 (2022)



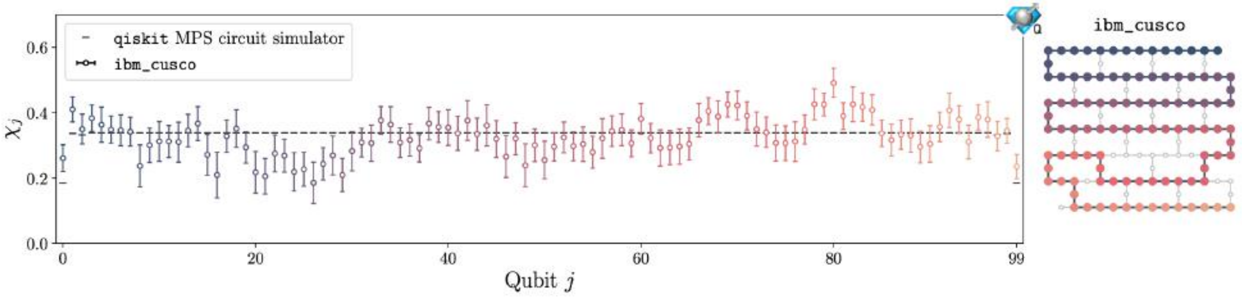
Schweizer et al., Nature Physics, 15, 1168 (2019)

Rydberg tweezer arrays:



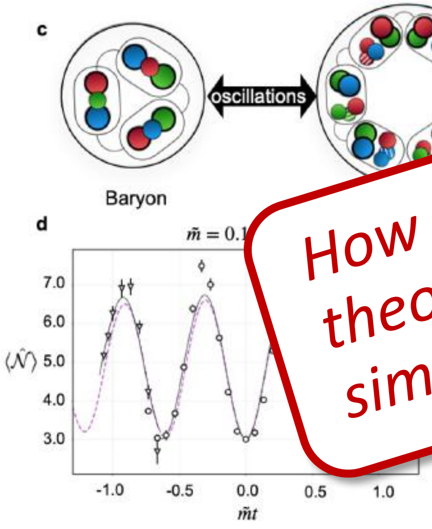
Semeghini et al., Science, 374, 1242 (2021)

Superconducting qubits:

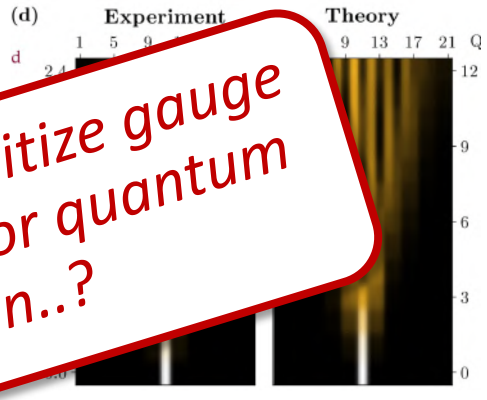


Farrell et al., arXiv:2308.04481 (2023)

and many more proposals …



Atas et al., arXiv:2207.03473 (2022)



Mildenberger et al., arXiv:2203.08905 (2022)

How to digitize gauge theories for quantum simulation..?

Outline

Motivation

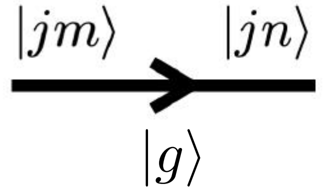
Standard Kogut-Susskind LGTs & graphical calculus for Lie groups

Generalized LGTs & graphical calculus for tensor categories

Quantum algorithm for real-time evolution

Conclusion & outlook

A single link of SU(2) LGT



Hilbert space: $\mathcal{H} = L^2(S^3) \simeq L^2(\mathrm{SU}(2)) \simeq \bigoplus_{j\ \mathrm{irrep}} V_j^* \otimes V_j$

Group element basis (``*position space*''): $|g\rangle\,,\ g \in \mathrm{SU}(2)$

Link operators: $\hat{U}^{(J)}_{MN}|g\rangle = D^{(J)}_{MN}(g)|g\rangle\,,\ J \in \{0, \frac{1}{2}, 1, \dots\}\,,\ M, N \in \{-J, \dots, J\}$

Irrep basis (``*momentum space*''): $|jmn\rangle = (-1)^{j-m}\sqrt{d_j}\int_{\mathrm{SU}(2)} dg\, D^{(j)}_{-m,n}(g)|g\rangle\,,\ d_j = 2j+1$

Electric field operators: $\langle j'm'n'|\hat{L}^\alpha|jmn\rangle = \delta_{j'j}\delta_{n'n}t^{(j)\alpha}_{m'm}\,,\ \langle j'm'n'|\hat{R}^\alpha|jmn\rangle = \delta_{j'j}\delta_{m'm}t^{(j)\alpha}_{n'n}$

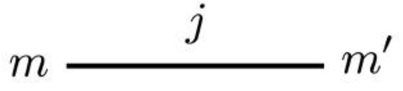
$$\hat{E}_{MN} = \frac{1}{2}\sum_\alpha \hat{E}^\alpha \sigma^\alpha_{MN}\ (E = L/R)\,,\ \hat{L} = -\hat{U}^\dagger \hat{R}\hat{U}\ (\hat{U} = \hat{U}^{(1/2)})\,,\ \hat{E}^2 = \sum_\alpha \hat{L}^\alpha \hat{L}^\alpha = \sum_\alpha \hat{R}^\alpha \hat{R}^\alpha$$

Canonical commutation relations:

$$\left[\hat{E}^\alpha, \hat{E}^\beta\right] = i\epsilon^{\alpha\beta\gamma}\hat{E}^\gamma\,,\ \left[\hat{L}^\alpha, \hat{U}^{(J)}_{MN}\right] = -\sum_{M'} t^{(J)}_{MM'}\hat{U}^{(J)}_{M'N}\,,\ \left[\hat{R}^\alpha, \hat{U}^{(J)}_{MN}\right] = +\sum_{N'} \hat{U}^{(J)}_{MN'}t^{(J)}_{N'N}$$

Kogut & Susskind, Physical Review D, 11(2), 395 (1975)

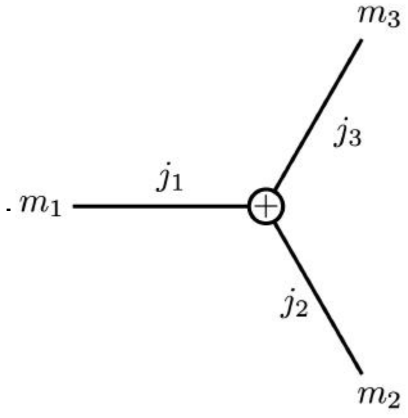
Graphical calculus for the group SU(2)

Lines: $\delta^{(j)}_{mm'} =$ 

$\epsilon^{(j)}_{mm'} = (-1)^{j-m}\delta^{(j)}_{m,-m'} =$ 

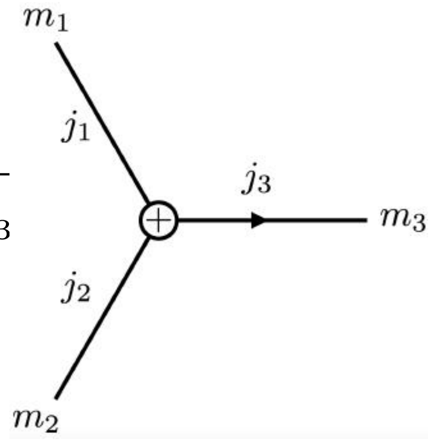
Vertices (3j):

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} =$$



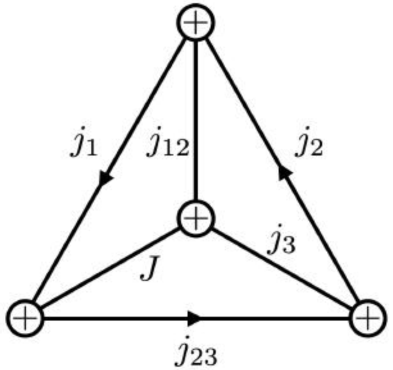
Clebsch-Gordan coeffs.

$$C^{j_3 m_3}_{j_1 m_1 j_2 m_2} = (-1)^{j_1-j_2-j_3}\sqrt{d_{j_3}}$$



Recoupling (6j):

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{Bmatrix} =$$



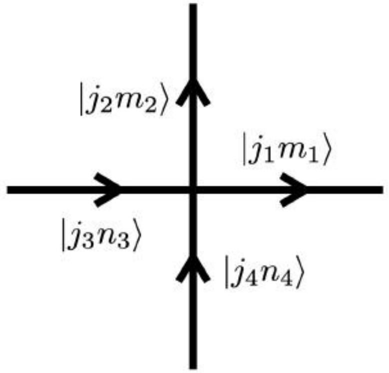
F-symbols

$$F^{j_1 j_2 j_{12}}_{j_3 J j_{23}} = \sqrt{d_{j_{12}} d_{j_{23}}}(-1)^{j_1+j_2+j_3+J} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{Bmatrix}$$

$$\langle JM, j_1 j_{23}(j_2 j_3) | J'M', j_{12}(j_1 j_2) j_3 \rangle = \delta_{JJ'}\delta_{MM'} F^{j_1 j_2 j_{12}}_{j_3 J j_{23}}$$

Messiah, QM II (1962)
Alesci, Mäkinen, & Yang, arXiv:2304.00268 (2023)

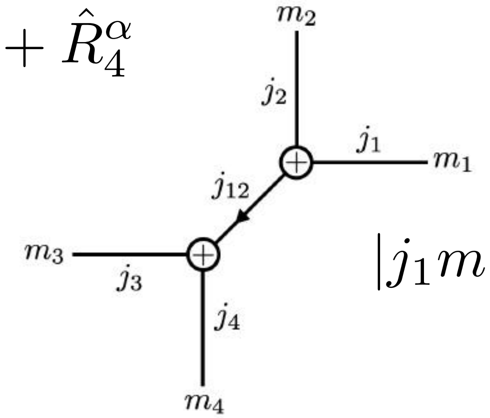
Gauge-invariant spin network states in 2D



Gauss' law: $\hat{G}_x^\alpha |\psi\rangle = 0\,,\ \hat{G}_x^\alpha = \hat{L}_1^\alpha + \hat{L}_2^\alpha + \hat{R}_3^\alpha + \hat{R}_4^\alpha$

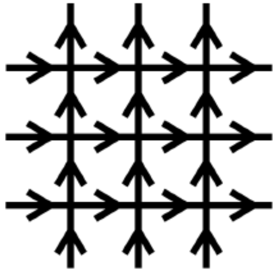
Local solution:

$$|j_{12} j_1 j_2 j_3 j_4\rangle = \sum_{m_1 m_2 n_3 n_4} (-1)^{j_1 - j_2 - j_3 + j_4} \sqrt{d_{j_{12}}}$$

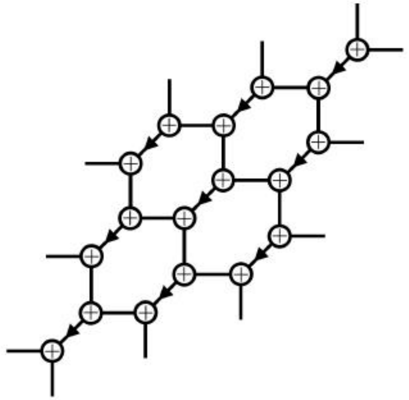


$$|j_1 m_1\rangle |j_2 m_2\rangle |j_3 n_3\rangle |j_4 n_4\rangle$$

Global solution:



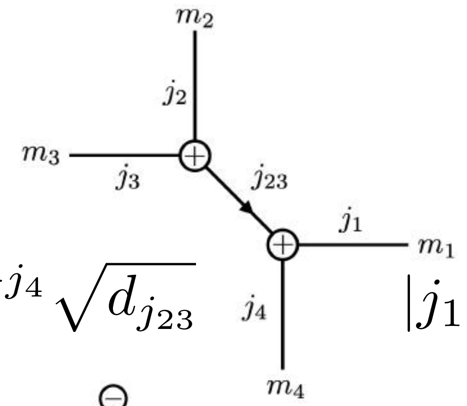
$\rightarrow$



Spin-network basis (subject to triangle constraints)

$$|\boldsymbol{j}\rangle = \bigotimes_\ell{}' \, |j_\ell\rangle$$

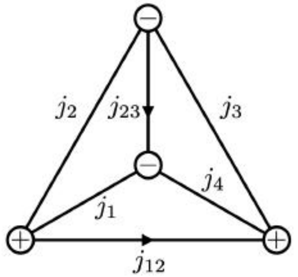
Another option: $|j_{23} j_1 j_2 j_3 j_4\rangle = \sum_{m_1 m_2 n_3 n_4} (-1)^{j_1 - j_2 + j_3 + j_4} \sqrt{d_{j_{23}}}$



$|j_1 m_1\rangle |j_2 m_2\rangle |j_3 n_3\rangle |j_4 n_4\rangle$

Local basis change:

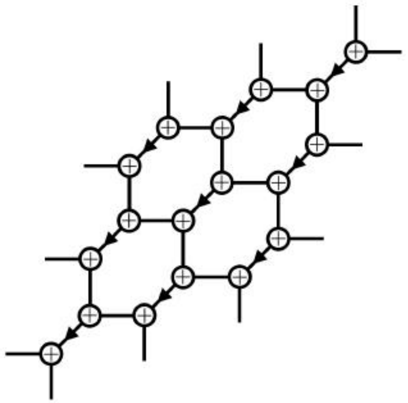
$$\langle j_{23} j_1 j_2 j_3 j_4 | j_{12} j_1 j_2 j_3 j_4 \rangle = \sqrt{d_{j_{12}} d_{j_{23}}} (-1)^{2j_1 + 2j_2 + 2j_4}$$



$$= F^{j_2 j_3 j_{23}}_{j_4 j_1 j_{12}}$$

Robson & Webber, Zeitschrift für Physik C Particles and Fields, 15, 199-226 (1982)

Gauge-invariant operators & Hamiltonian

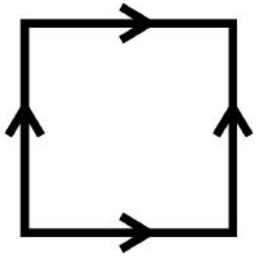


Gauge transformations: $\hat{V}(\boldsymbol{\theta}) = \prod_x e^{i\sum_\alpha \theta_x^\alpha \hat{G}_x^\alpha}$ $\quad \hat{V}(\boldsymbol{\theta})|\boldsymbol{j}\rangle = |\boldsymbol{j}\rangle \ \forall\theta$

Electric flux: $\hat{V}(\boldsymbol{\theta})\hat{E}_\ell^2\hat{V}^\dagger(\boldsymbol{\theta}) = \hat{E}_\ell^2$ $\quad \hat{H}_E = {\sum_\ell}' \hat{E}_\ell^2$ $\quad \hat{E}_\ell^2|\boldsymbol{j}\rangle = j_\ell(j_\ell+1)|\boldsymbol{j}\rangle$

Plaquette interactions: $\hat{V}(\boldsymbol{\theta})\hat{U}_\ell^{(J)}\hat{V}^\dagger(\boldsymbol{\theta}) = e^{-i\sum_\alpha \theta_x^\alpha t^{(J)\alpha}}\, \hat{U}_\ell^{(J)}\, e^{+i\sum_\beta \theta_y^\beta t^{(J)\beta}}\,, \ \ell = (x,y)$

$$\hat{\mathcal{U}}_\square^{(J)} = \mathrm{tr}\left[\hat{U}_1^{(J)}\hat{U}_2^{(J)}\hat{U}_3^{(J)\dagger}\hat{U}_4^{(J)\dagger}\right]\,,\ \hat{V}(\boldsymbol{\theta})\hat{\mathcal{U}}_\square^{(J)}\hat{V}^\dagger(\boldsymbol{\theta}) = \hat{\mathcal{U}}_\square^{(J)}$$



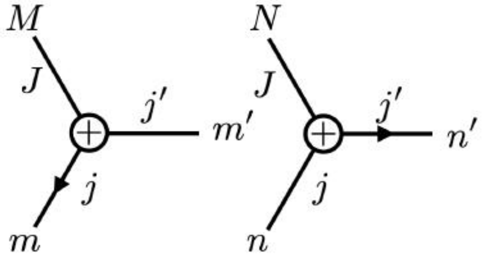
$$\hat{H}_B = \sum_\square \left(\hat{\mathcal{U}}_\square + \hat{\mathcal{U}}_\square^\dagger\right) = 2\sum_\square \hat{\mathcal{U}}_\square \qquad \left(\hat{\mathcal{U}}_\square = \hat{\mathcal{U}}_\square^{(1/2)}\right)$$

Kogut-Susskind Hamiltonian: $\hat{H}_{KS} = \frac{g^2}{2a}\hat{H}_E - \frac{1}{2ag^2}\hat{H}_B$

Kogut & Susskind, Physical Review D, 11(2), 395 (1975)
Zohar & Burrello, Physical Review D, 91(5), 054506 (2015)

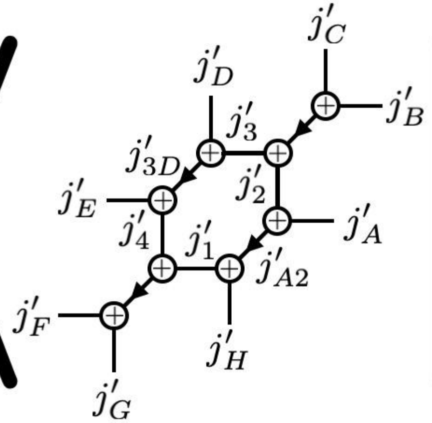
Calculating matrix elements in the SN basis

Link operators: $\langle j'm'n'|\mathcal{U}^{(J)}_{MN}|jmn\rangle = \sqrt{d_j d_{j'}}(-1)^{2j}$

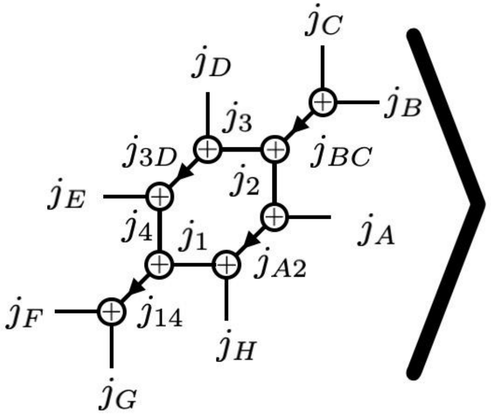


Plaquette operators:

$\langle \boldsymbol{j}'|\mathcal{U}^{(J)}_{\square}|\boldsymbol{j}\rangle =$

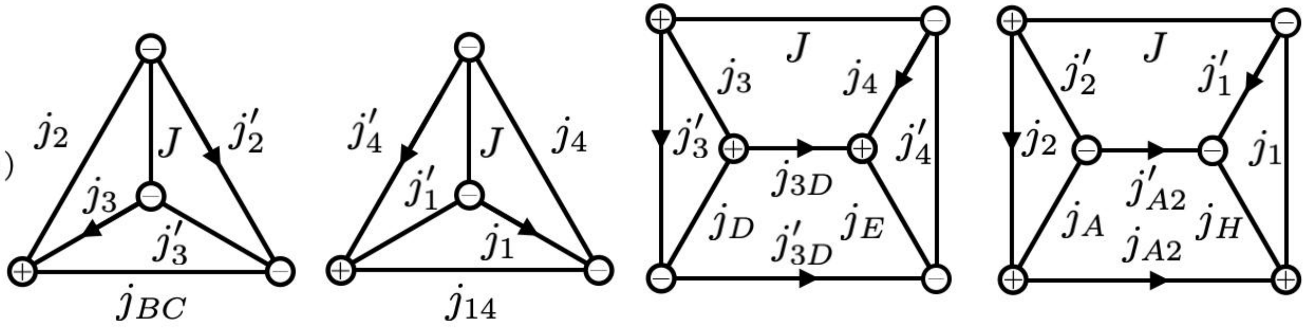


$\mathrm{tr}\left[U_1^{(J)}U_2^{(J)}U_3^{(J)\dagger}U_4^{(J)\dagger}\right]$



$$= \prod_{\ell \notin \square}\left[\delta_{j'_\ell j_\ell}\right] F^{j_H j_1 j_{A2}}_{J j'_{A2} j'_1} F^{j_A j_{A2} j_2}_{J j'_2 j_{A2}} \cdots F^{j_{14} j_4 j_1}_{J j'_1 j'_4}$$

$$= \prod_{\ell \notin \square}\left[\delta_{j'_\ell j_\ell}\right] \sqrt{d_{j_1} d_{j'_1} \cdots d_{j_4} d_{j'_4}}(-1)^{2(j_H+j_A+j_D+j_E)}$$



Robson & Webber, Zeitschrift für Physik C Particles and Fields, 15, 199-226 (1982)

Outline

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Standard Kogut-Susskind LGTs & graphical calculus for Lie groups

Generalized LGTs & graphical calculus for tensor categories

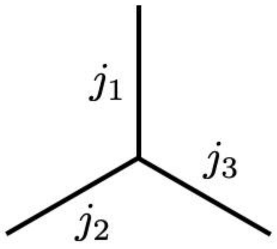
Quantum algorithm for real-time evolution

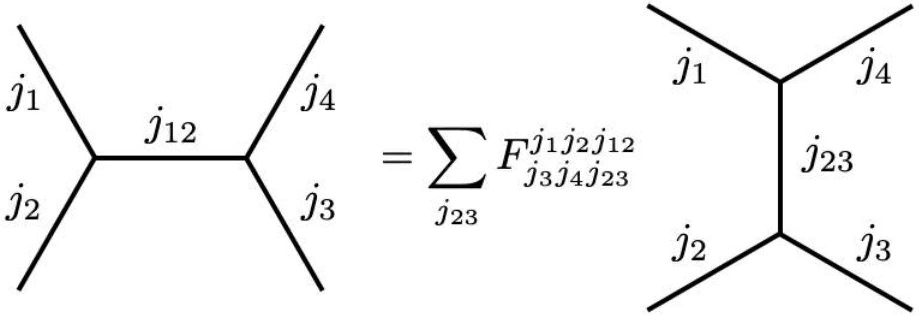
Conclusion & outlook

Graphical calculus for the tensor category of SU(2)

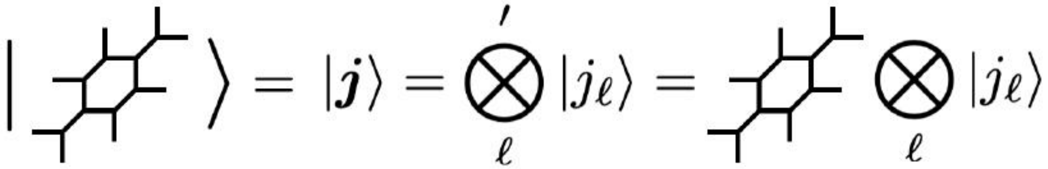
Irrep labels: $j = 0, \frac{1}{2}, 1, \ldots$

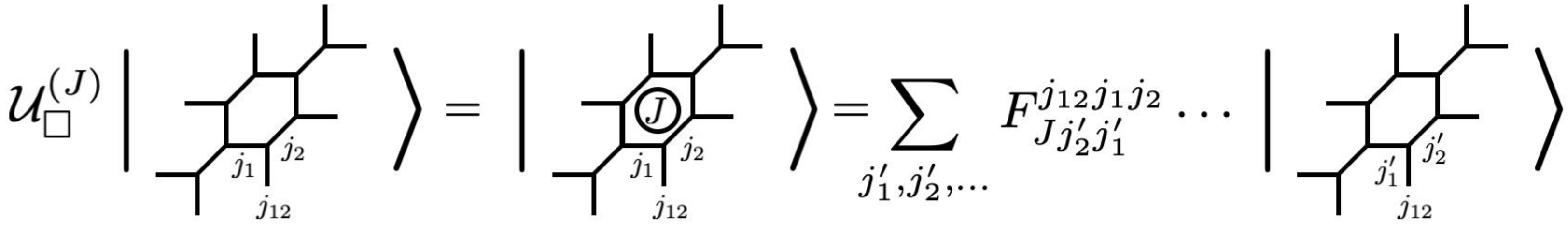
Fusion rules: $j_1 \times j_2 = \sum_k \delta_{j_1 j_2 j_3} j_3$

$\delta_{j_1 j_2 j_3} =$ 

Recoupling: 

(also braiding & more ..)

Gauge-invariant states: 

Plaquette operator: 

Levin & Wen, Physical Review B, 71(4), 045110 (2005)

Q-deformation as a truncation of Lie groups

$q = e^{2\pi i/(k+2)}$

q-numbers: $[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = \frac{\sin\left(\frac{\pi}{k+2} n\right)}{\sin\left(\frac{\pi}{k+2}\right)}$, $[n]! = [n][n-1]\cdots[1][0]$, $[0]! = [0] = 1$

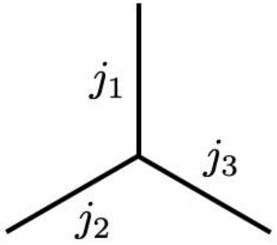
Quantum group $\mathrm{SU}(2)_k$: $\left[J^+, J^-\right] = [2J^z] = \frac{q^{J^z} - q^{-J^z}}{q^{1/2} - q^{-1/2}}$, $\left[J^z, J^\pm\right] = \pm J$ $(k = 1, 2, 3, \dots)$

Tensor category: $j = 0, \frac{1}{2}, 1, \dots, \frac{k}{2}$

quantum dimension

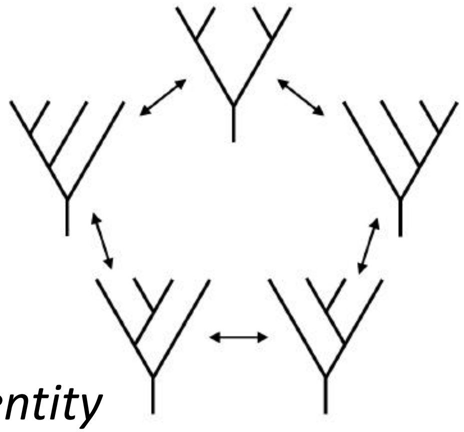
$d_j = [2j+1]$

$\mathrm{SU}(2)_k \overset{k\to\infty}{\longrightarrow} \mathrm{SU}(2)$



$$= \begin{cases} 1 & j_1 + j_2 + j_3 \in \mathbb{N}\,,\ j_1 + j_2 + j_3 \leq k\,,\ j_a + j_b \leq j_c \\ 0 & \text{else} \end{cases}$$

$$F^{j_1 j_2 j_{12}}_{j_3 j_4 j_{23}} = (-1)^{j_1+j_2+j_3+j_4} \sqrt{d_{j_{12}} d_{j_{23}}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix}_q$$



Biedenharn-Elliot identity

(& more properties/axioms..)

Kirillow & Reshetikhin, Infinite dimensional Lie algebras and groups, 7, 285 (1989)
Keller, letters in mathematical physics, 21, 273-286 (1991)
Dittrich & Geiller, New Journal of Physics, 19(1), 013003 (2017)

Q-deformed Kogut-Susskind lattice gauge theories

replace everything by q-deformed analogs (here for KS Hamiltonian LGT)!

TVZ, D. Gonzalez & P. Zoller, Phys. Rev. Lett. 131, 171902 (2023)
Hayata & Hidaka, arXiv:2305.05950 (2023)

Not a completely new idea:

- Quantum gravity (spin-foam state sums)
- Condensed matter (string-net models)
- Quantum computing (topological codes)
- …

Turaev & Viro, Topology, 31(4), 865-902 (1992)
Dittrich & Geiller, New Journal of Physics, 19(1), 013003 (2017)
…
Levin & Wen, Physical Review B, 71(4), 045110 (2005)
…
Kitaev, Annals of physics, 303(1), 2-30 (2003)
Koenig, Kuperberg, & Reichardt, Annals of Physics, 325(12), 2707-2749 (2010)
Schotte, Zhu, Burgelman & Verstraete, Physical Review X, 12(2), 021012 (2022)
…

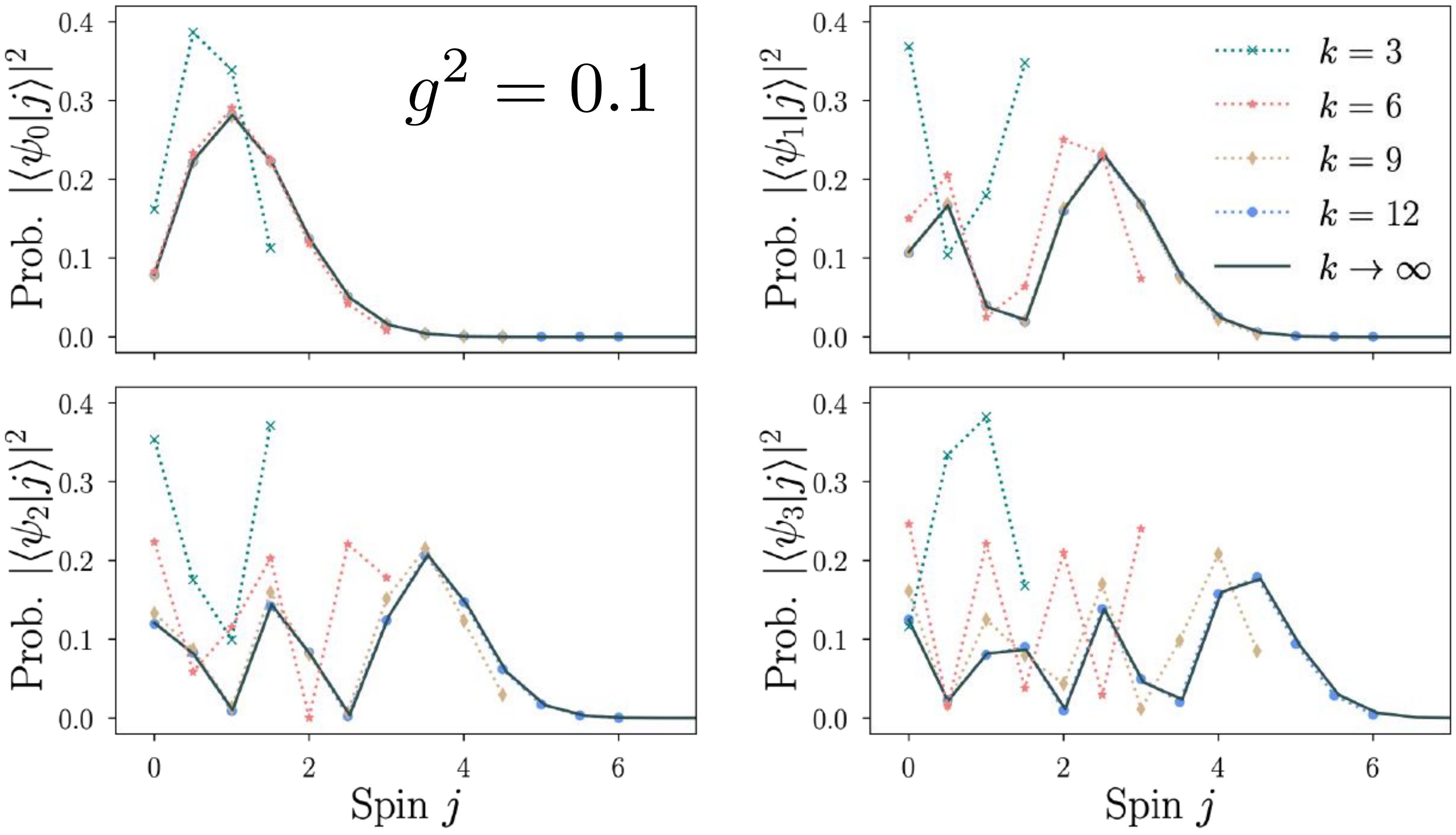
Extensions:

- more general Lie groups, in particular SU(3)
- higher dimensions, in particular 3D
- matter fields, in particular fermions
- …

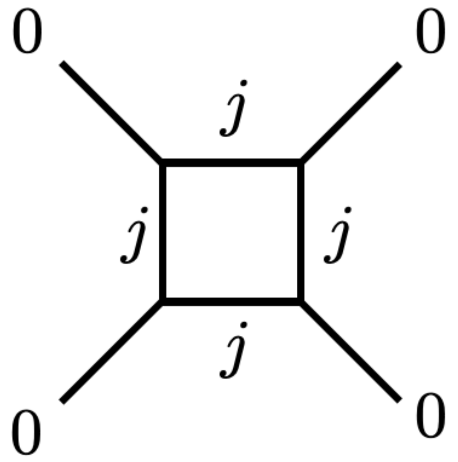
Liegener & Thiemann, Physical Review D, 94(2), 024042 (2016)
Hayata & Hidaka, arXiv:2306.12324 (2023)
…
Walker & Wang, Frontiers of Physics, 7, 150-159 (2012)
…
..work in progress..

A simple test of convergence

Low-lying eigenfunctions on a single plaquette



TVZ, D. Gonzalez & P. Zoller, Phys. Rev. Lett. 131, 171902 (2023)



0, 1, ..., k/2

$$H = \frac{g^2}{2a}\begin{pmatrix} 0 & -\frac{2}{g^4} & 0 & 0 & \cdots \\ -\frac{2}{g^4} & \frac{3}{4} & -\frac{2}{g^4} & 0 & \cdots \\ 0 & -\frac{2}{g^4} & \frac{8}{4} & -\frac{2}{g^4} & \cdots \\ 0 & 0 & -\frac{2}{g^4} & \frac{15}{4} & \cdots \\ \cdots & & & & \end{pmatrix}$$

exact solution at $k \to \infty$:

Mathieu functions

Robson & Webber, Zeitschrift für Physik C, 7, 53 (1980)

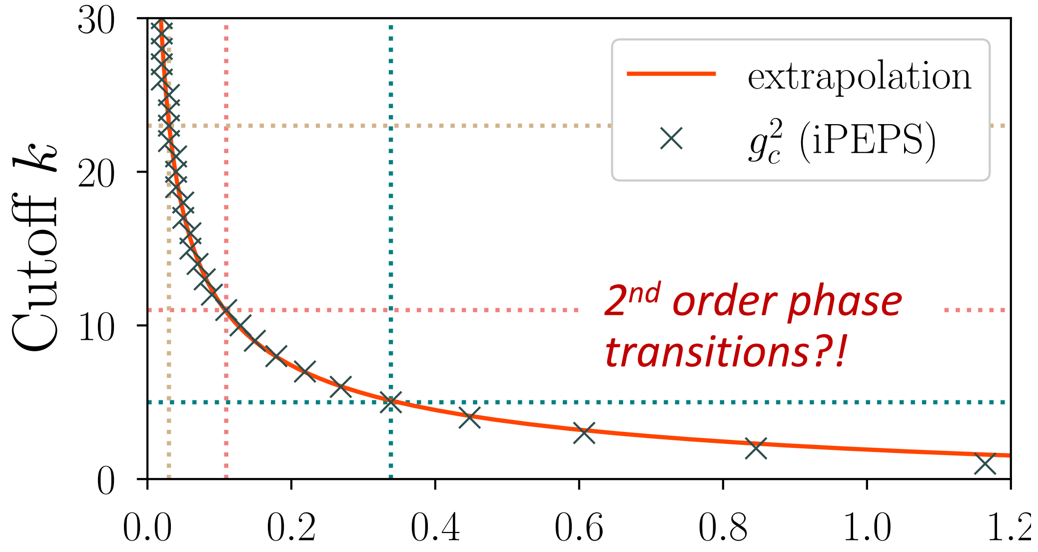
Phase diagram from tensor network states

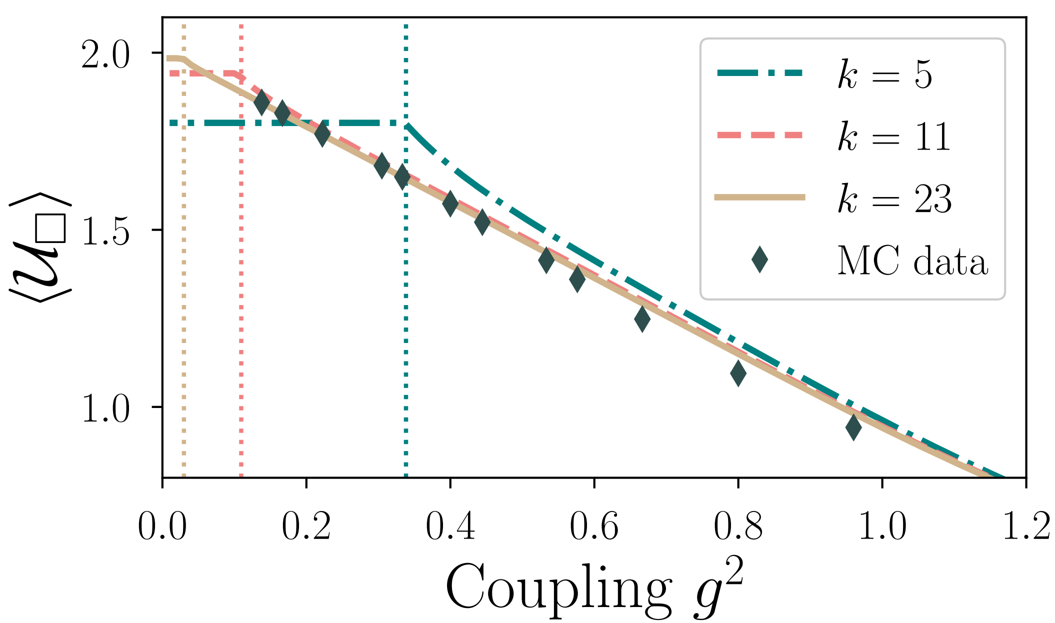
Ansatz (iPEPS / MERA):

$$|\psi\rangle = \prod_{\square} \frac{\sum_{j=0}^{k/2} \psi_j \mathcal{U}_{\square}^{(j)}}{\sqrt{\sum_{j=0}^{k/2} |\psi_j|^2}} |\mathbf{0}\rangle$$

Contains exact limiting ground states!

generalization of: Dusuel & Vidal,
Physical Review B, 92(12), 125150 (2015)





Spurious phase (topological order!) vanishes as $k \to \infty$

TVZ, D. Gonzalez & P. Zoller, Phys. Rev. Lett. 131, 171902 (2023)

See also: Hayata & Hidaka, arXiv:2306.12324

MC data from: Teper Phys. Rev. D 59, 014512 (1998)

Outline

Motivation

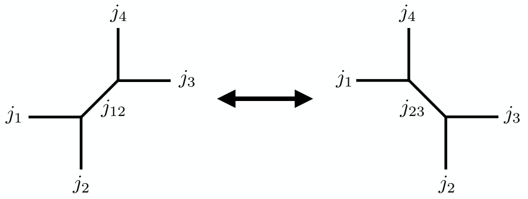
Standard Kogut-Susskind LGTs & graphical calculus for Lie groups

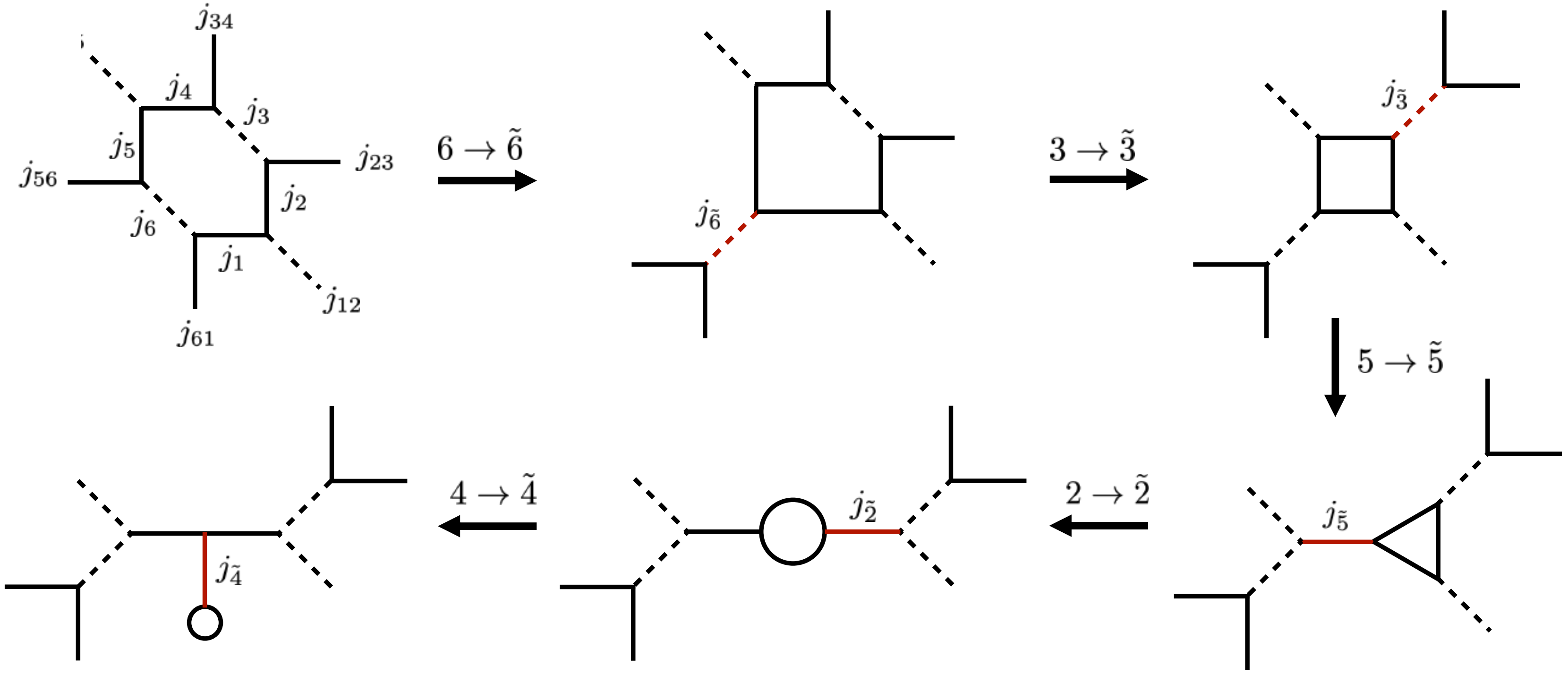
Generalized LGTs & graphical calculus for tensor categories

Quantum algorithm for real-time evolution

Conclusion & outlook

Diagonalization of the plaquette operator





Exact, local decomposition in the SN basis!

$\mathrm{SU}(2)_k$ retains unitarity of F-moves!

$$F^{j_{12}j_1j_2}_{\frac{1}{2}j_2'j_1'}\,F^{j_{23}j_2j_3}_{\frac{1}{2}j_3'j_2'}\,F^{j_{34}j_3j_4}_{\frac{1}{2}j_4'j_3'}\,F^{j_{45}j_4j_5}_{\frac{1}{2}j_5'j_4'}\,F^{j_{56}j_5j_6}_{\frac{1}{2}j_6'j_5'}\,F^{j_{61}j_6j_1}_{\frac{1}{2}j_1'j_6'} \to F^{j_4j_1j_1}_{\frac{1}{2}j_1'j_1'}$$

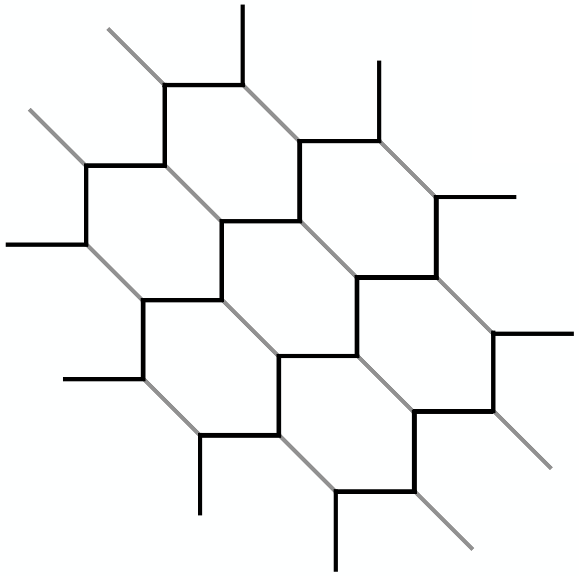
(from Biedenharn-Elliot)

TVZ, D. Gonzalez & P. Zoller, Phys. Rev. Lett. 131, 171902 (2023)

Inspired by: Robson & Webber, Zeitschrift für Physik C, 15(3), 199-226 (1982).   see also: N. Bonesteel and D. DiVincenzo, Physi. Rev. B 86, 165113 (2012)

Digital (Trotter) simulation

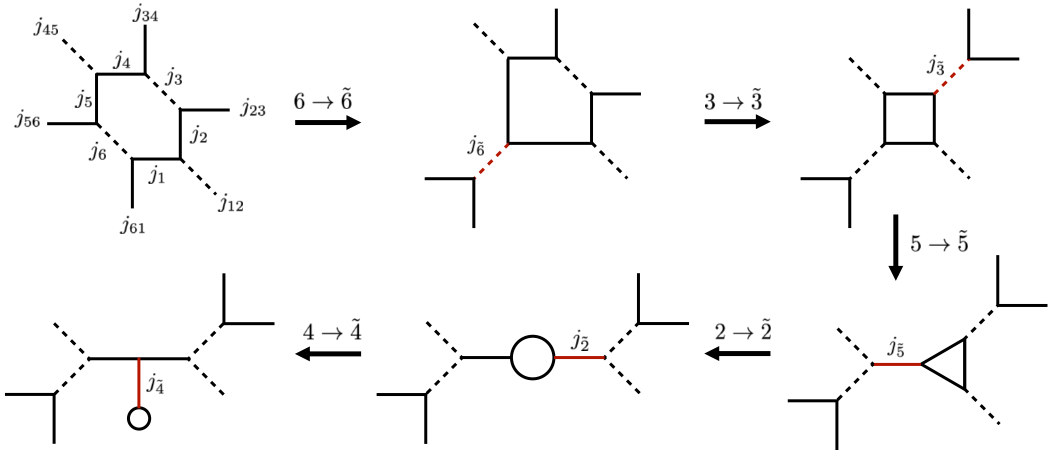
truncated hexagonal SN basis with $j \leq j_{\max}$

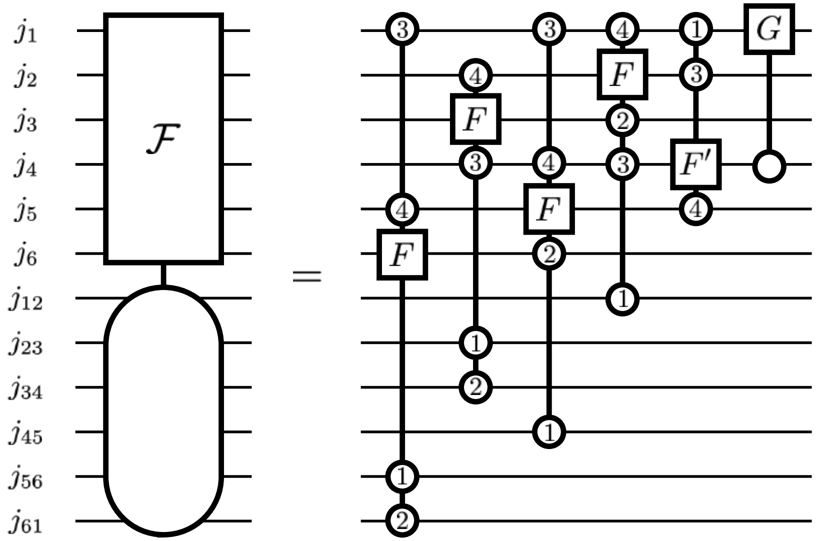


Exact Trotter decomposition:

$$e^{-i\tau H_{\mathrm{KS}}} \approx \prod_{\ell} e^{-i\tau g^2/(2a)\mathbf{E}_\ell^2} \prod_{\square} e^{+i\tau/(ag^2)\mathcal{U}_\square}$$

exact, gauge-invariant, preserves locality (parallelizable)
& natural implementation with qudits





TVZ, D. Gonzalez & P. Zoller, Phys. Rev. Lett. 131, 171902 (2023)

Conclusion & outlook

○ q-deformed Kogut-Susskind LGTs as an algebraic truncation using quantum groups

→ for quantum simulation & tensor networks!

next:

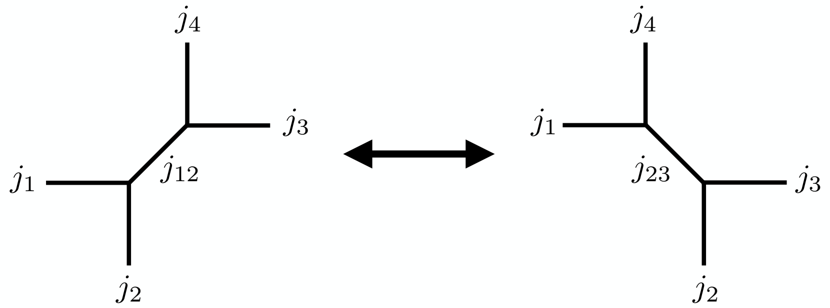
○ efficient simulation with iPEPS?

○ natural implementation with qudits

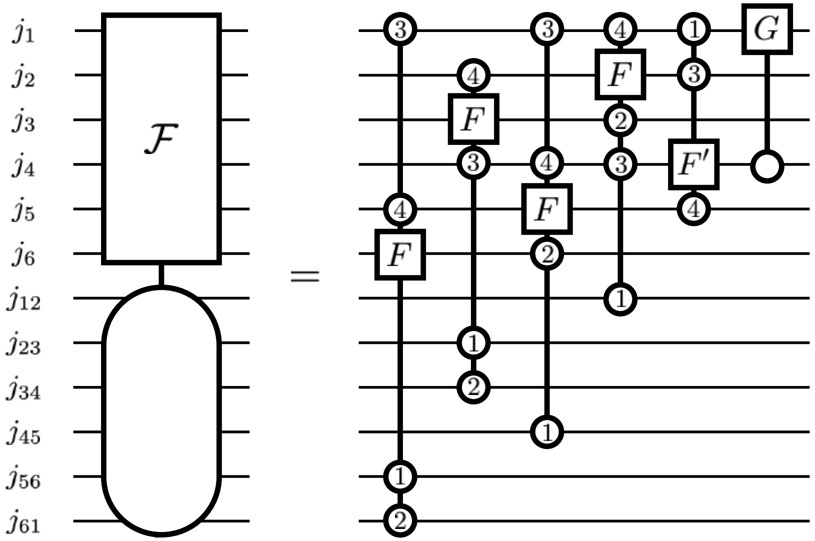
○ inclusion of fermionic matter

○ extension to general SU(N)

○ …



TVZ, D. Gonzalez & P. Zoller,
Phys. Rev. Lett. 131, 171902 (2023)



Thanks for listening!