How I Learned to Stop Worrying and Love Birdtrack Projection Operators

(to construct orthogonal basis invariants)

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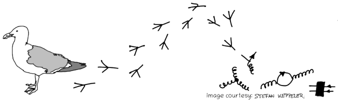
based on:

arXiv:1812.02614	JHEP 05 (2019) 208
arXiv:2002.12244	J.Phys.Conf.Ser. 1586 (2020) 012005
arXiv:2308.00019	JHEP 01 (2024) 024 with Miguel P. **Bento** and João P. **Silva**

Birdtracks 2024
Vienna / Virtual
27.2.24



MAX-PLANCK-GESELLSCHAFT

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OUTLINE

- NOTATION & CONVENTIONS
- WHAT AM I USING BIRDTRACKS FOR?
  → CONSTRUCTION OF BASIS INVARIANTS (BI'S)
- GENERAL ALGORITHM FOR CONSTRUCTION OF BI'S
- EXAMPLE 1: 2HDM
- EXAMPLE 2: STANDARD MODEL
- CONCLUSIONS & OPEN QUESTIONS

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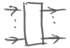
MY WORK WOULD NOT BE POSSIBLE W/O THESE:

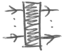
- CVITANOVIĆ PRD 14 (1976) 1536 & "THE BIBLE" (see talk by CVITANOVIĆ for more references)
- KEPPELER & SJÖDAHL J. MATH. PHYS 55 (2014) 1307.6147
  JHEP 09 (2012) 1207.0609
- KEPPELER 1707.07280 SCIPOST PHYS. LECT. NOTES 3 (2018)
- ALCOCK-ZEILINGER & WEIGERT 1610.08802, 1610.10088, 1610.08801
  J. MATH. PHYS. 58 (2017)

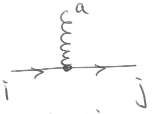
2A

NOTATION AND CONVENTIONS

FOLLOWING ESSENTIALLY KEPPELER 1707.07280

Sym. 

A-Sym. 

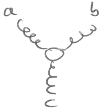


$[t^a]^i{}_j$

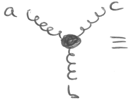
GENERATOR OF SU(N)

HERE ONLY: SU(2), SU(3)

[SU(3)]

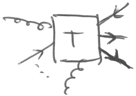


$\equiv d^{abc} := 2\,\mathrm{tr}\{\{t^a,t^b\}t^c\}$



$\equiv i f^{abc}$; $f^{abc} := 2\,\mathrm{tr}\{[t^a,t^b]t^c\}$

GENERAL TENSORS



NON-STANDARD NOTATION

$\varepsilon^{ab} =$ 

$\varepsilon_{ab} =$ 

PROJECTION OPERATORS:

$P^2 = P$; $\mathrm{tr}\, P = \dim \mathrm{Im}\, P$; $(P^\dagger = P)$

ORTHOGONALITY OF P's:

$P_\Phi \cdot P_\Theta = 0 \quad (\Phi \neq \Theta)$

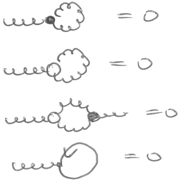
TRANSITION OPERATORS

$T_{\Phi\Theta} T_{\Theta\Phi} = P_\Phi$

$T_{\Phi\Theta} P_\Theta = T_{\Phi\Theta}$

$T_{\Phi\Theta}^\dagger = T_{\Theta\Phi}$

IDENTITIES



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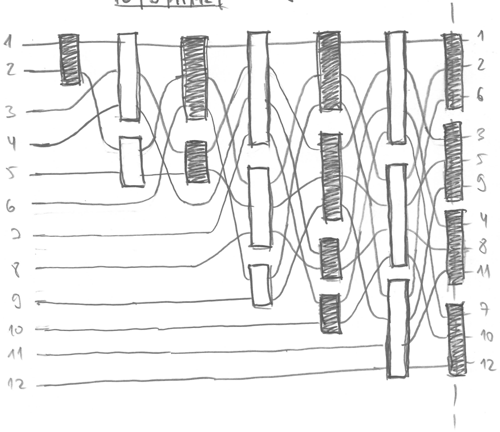
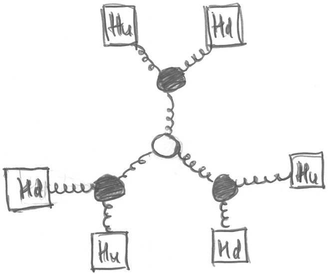
TEASER : WHAT YOU CAN EXPECT TO SEE

1	3	4	7
2	5	8	10
6	9	11	12

12-index Young Projection Operator :





$\propto$ CP-odd Jarlskog Invariant of the Standard Model

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WHAT AM I USING BIRDTRACKS FOR ?

QFT'S THAT HAVE SOME KIND OF "FLAVOR SPACE" (States w/ identical QN's)

EXAMPLES:

1) TWO-HIGGS-DOUBLET MODEL (2HDM): SU(2) REDUNDANCY $\Phi = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$

2) STANDARD MODEL OF PARTICLE PHYSICS: SU(3) REDUNDANCY IN REPETITION OF FERMION GENERATIONS

$$\underbrace{\begin{bmatrix} u & c & t \\ d & s & b \end{bmatrix}}\quad \underbrace{\begin{bmatrix} \nu_e & \nu_\mu & \nu_\tau \\ e & \mu & \tau \end{bmatrix}}$$

$\boxed{\text{I AM USING BIRDTRACKS TO CONSTRUCT OBJECTS WHICH ARE } \underline{\text{BASIS INVARIANTS}} \text{ OF THESE REDUNDANCIES.}}$

WHY IS THIS INTERESTING?

- PHYSICAL OBSERVABLES MUST BE BI.
- SYMMETRIES CORRESPOND TO RELATIONS OF BI'S.
  EG. CP VIOLATION (MATTER-ANTIMATTER SYMMETRY VIOLATION)
- ORTHOGONAL BI'S SIMPLIFY:
  (BI'S CONSTRUCTED VIA ORTHOGONAL PROJECTORS)
  - FORMULATION OF RGE'S
  - SYZYGIES (RELATION OF BI'S)
  - SYMMETRY DETECTION

HOPE: NEW PERSPECTIVE AND INSIGHTS INTO FLAVOR PUZZLE

(BECAUSE IT CONSISTS OF LARGE # OF BASIS DEPENDENT PARAMETERS AND IS DIRECT CONSEQUENCE OF MISALIGNMENT OF COVARIANTS)

(5) GENERAL ALGORITHM FOR CONSTRUCTION OF BI'S

STEPS 1) CONSTRUCTION OF ORTHOGONAL CO-VARIANTS (via B. Tracks)

2) FIND # OF ALGEBRAICALLY INDEPENDENT (A.I.) BI'S, AND THEIR COMPOSITION IN TERMS OF COVARIANTS (via HS & PL)
(+ RELATIONS OF BI'S)

3) EXPLICITLY CONSTRUCT (ORTHOGONAL) INVARIANTS FROM THE COVARIANTS (via B. Tracks)

REMARKS: 1) THIS IS CONSTRUCTION OF SU(N) INVARIANTS OUT OF GIVEN SET OF TENSORS (COUPLINGS OF THE QFT)

TECHNICALLY, THIS IS VERY SIMILAR TO CONSTRUCTION OF EFFECTIVE FIELD THEORY OPERATORS. (≡ SYMMETRY INVARIANTS)
see eg. [Henning, Lu, Melia, Murayama '16 & '17]

CONCEPTUALLY, HOWEVER, THESE ARE NOT THE SAME [DIFFERENCES: DERIVATIVES, IBP REDUNDANCY, ..]
HERE: NO KINEMATICS INVOLVED AT ALL.

2) I WILL FOCUS ON CONSTRUCTION AND RÔLE OF BIRDTRACKS THEREIN;
NOT FOCUS ON RESULTS (WHICH WOULD BE THE PHYSICALLY INTERESTING PART)

⑥ EXAMPLE 1 : 2HDM

[1812.02614]

HERMITICITY & GAUGE SYMMETRY

$y^a{}_b = (y^b{}_a)^*$

$z^{ab}_{cd} = (z^{cd}_{ab})^*$

$z^{ab}_{cd} = z^{ba}_{dc}$

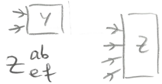
$\Phi = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \updownarrow SU(2) \quad V(\Phi) = \Phi^\dagger_a \, y^a{}_b \, \Phi^b + \Phi^\dagger_a \Phi^\dagger_b \, z^{ab}{}_{cd} \, \Phi^c \Phi^d$

$(a,b,c,d,\ldots = 1,2)$

STEP 1) WANT TO WORK W/ YOUNG PROJECTORS
[need all upper or lower indices]
[cf. talk by KENNEDY]

$y^{ab} := \varepsilon^{bc} \, y^a{}_c$

$z^{abcd} := \varepsilon^{ce} \varepsilon^{df} \, z^{ab}{}_{ef}$

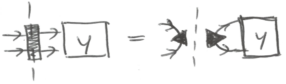


$Y: \quad 2 \otimes 2 = 1 \oplus 3$

$\square \otimes \square = \boxminus \oplus \square\!\square$

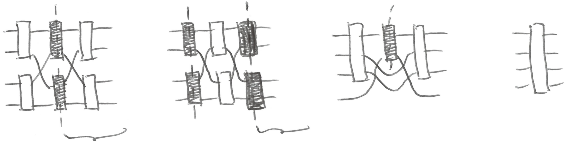
$Z: \quad 2 \otimes 2 \otimes 2 \otimes 2 = 1_1 \oplus 1_2 \oplus 3 \oplus 5$

$\square \otimes \square \otimes \square \otimes \square = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline a & c \\ \hline b & d \\ \hline\end{array} \oplus \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline\end{array}$



FACTORIZATION!

$\hat{=}\, Y_3 \,\hat{=}\, $ "Y"

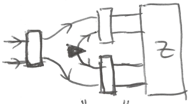


Z_{11} $\quad$ Z_{12}

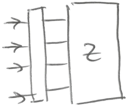
MENTION:

- TRIVIAL SINGLETS
- FACTORIZATION OF PO's
- ORTHOGONALITY OF PO's

↷ ORTHOGONAL COVARIANTS
"BUILDING BLOCKS"



$Z_3 \,\hat{=}\,$ "T"



$Z_5 \,\hat{=}\,$ "Q"

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EXAMPLE 1 : 2HDM

STEP 2) BI'S IN PRINCIPLE: "WIRE UP" INDICES IN ALL POSSIBLE WAYS

BUT: – HOW MANY ALGEBRAICALLY INDEPENDENT (A.I.) POSSIBILITIES?

– WHEN TO STOP ?

– WHAT IS THE COVARIANT CONTENT ?

GENERAL ANSWER: HILBERT SERIES (HS) [Noether 1916] & PLETHYSTIC LOGARITHM (PL) [Getzler & Kapranov 1994]

(A WHOLE TALK BY THEMSELVES)

$\mathcal{H}S(3_Y, 3_T, 5_Q)$ of $SU(2)$ (+ PL) $\Rightarrow$ A.I. INVARIANTS & COVARIANT CONTENT

$\Rightarrow$ SYZYGIES

PRIMARY INVARIANTS: (NOT UNIQUE)

$$\boxed{\begin{array}{l} Y^2,\ T^2,\ YT,\ Q^2 \\ Q^3,\ QY^2,\ QT^2 \\ Q^2YT \end{array}}$$

SECONDARY INVARIANTS: $QYT;\ Q^2Y^2,\ Q^2T^2$
$QY^2T,\ QYT^2$
$Q^2Y^2T,\ Q^2YT^2$
$Q^3Y^3,\ Q^3T^3$
$Q^3Y^2T,\ Q^3YT^2$

THIS DOES NOT TELL US HOW TO "WIRE THINGS UP" $\longrightarrow$ BIRDTRACK PROJECTION OP'S.

ALGEBRAIC (IN)DEPENDENCE: $\exists$ POL$(I_1, I_2, \dots) = 0$?
PRIMARY INVARIANTS: MAXIMAL SET OF A.I. INVARIANTS (# PHYSICAL PARAMETERS = # PRIMARY INVARIANTS)
SECONDARY INVARIANTS: ADDITIONAL INVARIANTS NEEDED TO SOLVE FOR ANY INVARIANT I, I = POL(PRIMARIES, SECONDARIES)

EXAMPLE 1: 2HDM

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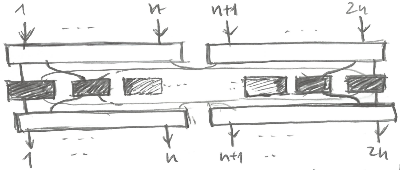
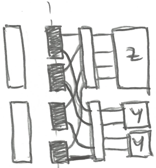
STEP 3) TURNS OUT IN THIS CASE ALL NEEDED PROJECTION OPERATORS ARE VERY SIMPLE AND GIVEN BY:

"COMPLETE CHOCOLATE BAR SHAPE"



E.G. $QY^2 \hat{=} (Q_5 \otimes Y_3 \otimes Y_3)_{1_0}$



ALL A.I. (PRIMARY) INVARIANTS ARE:

$Q^2 =$  $Y^2 =$  $T^2 =$  $YT =$ 

$Q^3 =$  $QY^2 =$  $QT^2 =$ 

$Q^2YT =$ 

INVARIANTS CONSTRUCTED LIKE THIS GIVE RISE TO:

- SHORTEST POSSIBLE SYZYGIES
- NECESSARY & SUFFICIENT CONDITIONS FOR CPV [AT '18]
- NECESSARY & SUFFICIENT CONDITIONS FOR ALL OTHER GLOBAL SYMMETRIES OF $V(\Phi)$ [BENTO, BOTO, SILVA, AT '20]
- ...

⑨

EXAMPLE 2: THE STANDARD MODEL (QUARK SECTOR) [2308.00019]

Flavor degeneracy

$$\left[\, SU(3)_{Q_L} \otimes SU(3)_{u_R} \otimes SU(3)_{d_R} \,\right]$$

$$\boxed{-\mathcal{L}_{Yuk} = \bar{Q}_L \tilde{H} Y_u U_R + \bar{Q}_L H Y_d D_R + h.c.}$$

$SU(3)_{Q_L}$: $\quad H_u := Y_u Y_u^\dagger \qquad H_d := Y_d Y_d^\dagger$

$$H^{abc}_{u,d} := \epsilon^{bcd} [H_{u,d}]^a{}_d$$

$SU(3)_{Q_L}$ $\quad \bar{3} \otimes 3 = 1 \oplus 8_u \qquad \bar{3} \otimes 3 = 1 + 8_d$

(All upper/lower indices see talk by Kennedy and yesterday's discussion)

I COULD WORK IN ADJOINT OR FUNDAMENTAL SPACE TO CONSTRUCT FURTHER P.O.'S.

FUNDAMENTAL SPACE: PROLIFERATION OF INDICES (gets worse for SU(N>3))

$8^4 \longrightarrow 1$  $\rightsquigarrow$ twelve index P.O.

$8^6 \longrightarrow 1$ $\qquad \rightsquigarrow$ 18 index P.O. ...

PROBLEM IS NOT THE CONSTRUCTION OF THE OPERATORS ON PAPER, PROBLEM IS THE IMPLEMENTATION ON COMPUTER! MEMORY REQUIREMENT GROWS $\gtrsim n!$

THOSE OPERATORS ARE HARD TO COMPUTE BUT CHEAP TO STORE, CHECK AND (RE)USE.

COMPLEXITY CLASS?

FOR NOW: RESIDE TO ADJOINT SPACE OF $SU(3)_{Q_L}$.

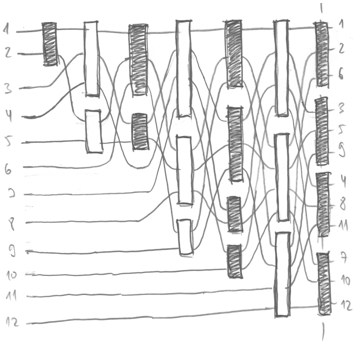
(9A)

EXAMPLE 2: THE STANDARD MODEL

EXAMPLE OF CONSTRUCTION OF LARGE-ISH (FUNDAMENTAL SPACE) YOUNG P.O.

$8^4 \to 1$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 4 & 6 \\ \hline 5 \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 7 & 9 \\ \hline 8 \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 10 & 12 \\ \hline 11 \\ \cline{1-1}\end{array} \supset \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 7 \\ \hline 2 & 5 & 8 & 10 \\ \hline 6 & 9 & 11 & 12 \\ \hline\end{array} \;\hat{=}\; \mathbb{1} \text{ in } SU(3))$$



PROBLEM IS NOT THE CONSTRUCTION OF SUCH OPERATORS ON PAPER; PROBLEM IS THE IMPLEMENTATION ON COMPUTER! MEMORY REQUIREMENTS GROW $\propto n!$ (# of indices but also for N)

↪ THOSE OPERATORS ARE HARD TO EXPLICITLY COMPUTE. BUT THEY ARE CHEAP TO STORE, CHECK AND (RE-) USE. ⇝ COMPLEXITY CLASS ?

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$SU(3)_{RL}$ IN ADJOINT SPACE

EXAMPLE 2: THE STANDARD MODEL

STEP 1)

"δ_u" "δ_d"

$u^a := \mathrm{Tr}\{t^a H_u\} =$  ; $u^d := \mathrm{tr}\{t^a H_d\} =$ 

NEED: $8_u^{\otimes n} \otimes 8_d^{\otimes m} \supset \mathbb{1}_1 \oplus \mathbb{1}_2 \oplus \cdots$ SINGLET PROJECTION OPERATORS

STEP 2) $\mathcal{H}S(8_u, 8_d) \Longrightarrow$

PRIMARY INVARIANTS

$$\boxed{\begin{array}{l} u^2,\ d^2,\ ud \\ u^3,\ d^3,\ u^2d,\ ud^2 \\ u^2d^2 \end{array}}$$

SECONDARY INVARIANTS

u^3d^3

STEP 3) PROJECTORS OF $8_u^n \otimes 8_d^m \leftrightarrow 8_u^n \otimes 8_d^m$

SINGLET P.O.'S CHARACTERIZED BY FACTORIZATION [see talk by WEIGERT] ; ALL P.O.'S ARE ORTHOGONAL TO EACH OTER

NEEDED P.O.'S:

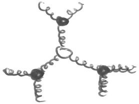
$8^2 \to 1$: 

$8^3 \to 1$: 

$8^4 \to 1$:

"transition" operators

$8^6 \to 1$: ... MANY ... ONLY RELEVANT:



from: $8 \times 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27$ → transition operators

see e.g. [KEPPELER, SJÖDAHL '13]

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EXAMPLE 2: THE STANDARD MODEL

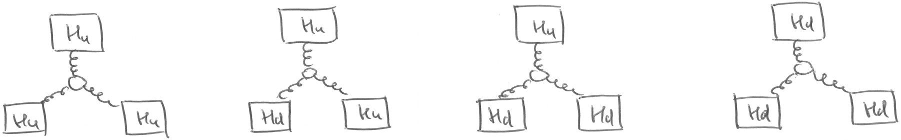
10 ALGEBRAICALLY INDEPENDENT (PRIMARY) INVARIANTS

"trivial"

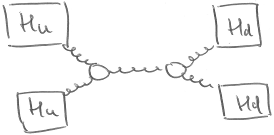
;

"quadratic"

"cubic"

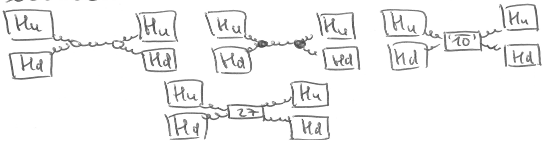


"quartic"

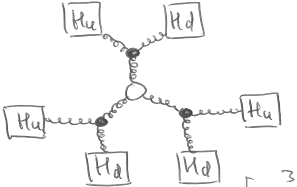


QUARTIC IS NOT UNIQUE!?

ALTERNATIVES:



UNIQUE SECONDARY INVARIANT



[3 of tensors ⇒ CP-odd]

EXAMPLE 2: THE STANDARD MODEL

BENEFITS OF THE (ORTHOGONAL) INVARIANTS OF SM FLAVOR SECTOR

> ALLOW FOR QUANTITATIVE ANALYSIS OF FLAVOR PUZZLE IN INVARIANTS
>
> [REVEALS STRONGLY CORRELATED INVARIANTS AT THE PHYSICAL POINT (NATURE)]
>
> (SEE BACKUP SLIDES)

> MOST TRANSPARENT ACCESS & ANALYSIS OF (APPROXIMATE) FLAVOR SYMMETRIES

> UNAMBIGUOUS ANALYSIS OF CP VIOLATION ($u^3d^3 \triangleq$ JARLSKOG INVARIANT)

> SHORTEST POSSIBLE SYZYGY RELATING $(u^3d^3)^2 = f$ (PRIMARIES)

MANY OPEN QUESTIONS:

> RELATION TO OBSERVABLES?

> RGE EVOLUTION OF INVARIANTS DIRECTLY IN TERMS OF INVARIANTS?

> AMBIGUITY IN I_{22}?

> EXPLANATION OF FLAVOR STRUCTURE?

The Syzygy

With our orthogonal invariants, the syzygy is given by

$$
\begin{aligned}
(J_{33})^2 = &-\frac{4}{27}I_{22}^3 + \frac{1}{9}I_{22}^2 I_{11}^2 + \frac{1}{9}I_{22}^2 I_{02} I_{20} + \frac{2}{3}I_{22}I_{30}I_{03}I_{11} - \frac{2}{3}I_{22}I_{21}I_{12}I_{11} - \frac{1}{9}I_{22}I_{11}^2 I_{20}I_{02} \\
&+ \frac{2}{3}I_{22}I_{21}^2 I_{02} + \frac{2}{3}I_{22}I_{12}^2 I_{20} - \frac{2}{3}I_{22}I_{30}I_{12}I_{02} - \frac{2}{3}I_{22}I_{03}I_{21}I_{20} \\
&- \frac{1}{3}I_{30}^2 I_{03}^2 + I_{21}^2 I_{12}^2 + 2I_{30}I_{03}I_{21}I_{12} - \frac{4}{9}I_{30}I_{03}I_{11}^3 \\
&+ \frac{1}{18}I_{30}^2 I_{02}^3 + \frac{1}{18}I_{03}^2 I_{20}^3 - \frac{4}{3}I_{30}I_{12}^2 - \frac{4}{3}I_{03}I_{21}^2 \\
&- \frac{1}{3}I_{30}I_{21}I_{11}I_{02}^2 - \frac{1}{3}I_{03}I_{12}I_{11}I_{20}^2 + \frac{2}{3}I_{30}I_{12}I_{11}^2 I_{02} + \frac{2}{3}I_{03}I_{21}I_{11}^2 I_{20} \\
&- \frac{2}{3}I_{21}I_{12}I_{20}I_{02}I_{11} - \frac{1}{108}I_{20}^3 I_{02}^3 + \frac{1}{36}I_{20}^2 I_{02}^2 I_{11}^2 + \frac{1}{6}I_{21}^2 I_{20} I_{02}^2 + \frac{1}{6}I_{12}^2 I_{02} I_{20}^2 \ .
\end{aligned}
$$

This is the **shortest relation ever** expressed for the SM quark flavor ring and has 27 terms.

(this should be compared to result of [Jenkins&Manohar'09] with 241 terms using non-orthogonal invariants).

CP transformation of covariants and invariants

CP is trafo under $\mathrm{Out}\,(\mathrm{SU}(N)) = \mathbb{Z}_2$.
Covariants:

$$\begin{aligned} \boldsymbol{u}^a &\mapsto -R^{ab}\,\boldsymbol{u}^b\,, \\ \boldsymbol{d}^a &\mapsto -R^{ab}\,\boldsymbol{d}^b\,, \end{aligned}$$

e.g. in Gell-Mann basis for the generators:
$R = \mathrm{diag}(-1, +1, -1, -1, +1, -1, +1, -1)$.

$\mathrm{SU}(3)$ tensors (projection ops.):

$$\begin{aligned} f^{abc} &\mapsto R^{aa'}\,R^{bb'}\,R^{cc'}\,f^{a'b'c'} = f^{abc}\,, \\ d^{abc} &\mapsto R^{aa'}\,R^{bb'}\,R^{cc'}\,d^{a'b'c'} = -\,d^{abc}\,. \end{aligned}$$

CP trafo of invariants is easily read-off from birdtrack projection operator:

Invariants are **CP even** / **CP odd** iff their projection operator contains and **even** / **odd # of** f **tensors**.

CP transformation of covariants and invariants

CP is trafo under $\mathrm{Out}\,(\mathrm{SU}(N)) = \mathbb{Z}_2$.
Covariants:

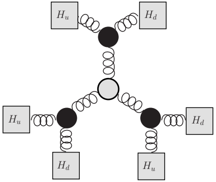
$$\boldsymbol{u}^a \;\mapsto\; -\,R^{ab}\,\boldsymbol{u}^b\,,$$
$$\boldsymbol{d}^a \;\mapsto\; -\,R^{ab}\,\boldsymbol{d}^b\,,$$

e.g. in Gell-Mann basis for the generators:
$R = \mathrm{diag}(-1,+1,-1,-1,+1,-1,+1,-1)$.

$\mathrm{SU}(3)$ tensors (projection ops.):

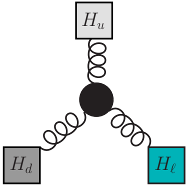
$$f^{abc} \;\mapsto\; R^{aa'}\,R^{bb'}\,R^{cc'}\,f^{a'b'c'} \;=\; f^{abc}\,,$$
$$d^{abc} \;\mapsto\; R^{aa'}\,R^{bb'}\,R^{cc'}\,d^{a'b'c'} \;=\; -\,d^{abc}\,.$$

$\Rightarrow$ Only CP-odd in SM: $J_{33} \propto$



BSM: CPV at order 3 ?

$\mathrm{i}f^{abc}\;\mathrm{Tr}[t^a\,H_u]\,\mathrm{Tr}[t^b\,H_d]\,\mathrm{Tr}[t^c\,\boldsymbol{H_\ell}]$



CP trafo of invariants is easily read-off from birdtrack projection operator:

Invariants are **CP even** / **CP odd** iff their projection operator contains and **even** / **odd # of f tensors**.

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CONCLUSIONS

* BIRDTRACKS ARE A WONDERFUL TECHNIQUE THAT ALLOWED ME TO SYSTEMATICALLY CONSTRUCT BASIS INVARIANTS FOR 2HDM, SM, ...

  THESE CAN BE USED TO OBTAIN PHYSICAL INSIGHTS (THIS IS ONLY THE BEGINNING)
  - SYMMETRIES & VIOLATION
  - PARAMETER SPACE / PARAMETER CORRELATIONS

* THIS WOULD NOT BE POSSIBLE W/O (ORTHOGONAL) BIRDTRACK PROJECTORS.

OPEN QUESTIONS (UP TO DISCUSSION)

* COMPLEXITY CLASS OF COMPUTATION OF MANY-INDEX PROJECTION OPERATORS?
* IS THIS A USEFUL BENCHMARK PROBLEM FOR (QUANTUM) COMPUTERS?
* FUNDAMENTAL VS. ADJOINT (VS. ...) ORTHOGONAL BASES?
* HOW TO (BEST) TAKE INTO ACCOUNT MULTIPLE (IDENTICAL) TENSORS IN B.I. CONSTRUCTION?
* IS THERE ANY CONSTRUCTION (OF P.O.'S) THAT TAKES INTO ACCOUNT UPPER/LOWER INDICES? (IN THE STYLE OF YOUNG DIAGRAMS, TO REDUCE THE NUMBER OF BOXES) [see also discussion yesterday]

Thank You!

Backup slides

CP transformation of the building blocks in 2HDM

$$Y_{\mathbf{1}} \mapsto Y_{\mathbf{1}} \,, \qquad Z_{\mathbf{1}_{(1)}} \mapsto Z_{\mathbf{1}_{(1)}} \,, \qquad Z_{\mathbf{1}_{(2)}} \mapsto Z_{\mathbf{1}_{(2)}} \,,$$

$$Y_{\mathbf{3}}^{ab} \mapsto -(Y_{\mathbf{3}})_{ab} \,, \qquad Z_{\mathbf{3}}^{ab} \mapsto -(Z_{\mathbf{3}})_{ab} \,,$$

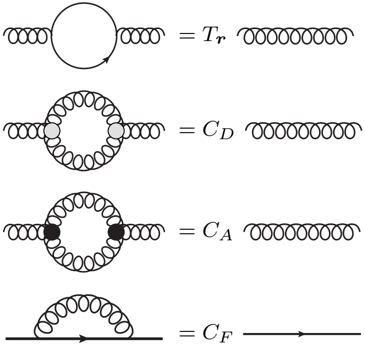
$$Z_{\mathbf{5}}^{abcd} \mapsto (Z_{\mathbf{5}})_{abcd} \,.$$

For basis invariants ($\rightarrow$ all indices contracted):

> A basis invariant is CP $\left\{\begin{array}{c}\text{even}\\ \text{odd}\end{array}\right\}$ *iff* it contains an $\left\{\begin{array}{c}\text{even}\\ \text{odd}\end{array}\right\}$ number of triplet building blocks ($Y_{\mathbf{3}}$, $Z_{\mathbf{3}}$).

Birdtrack Identities

We mostly use the conventions of [Keppeler '17] with the following identities

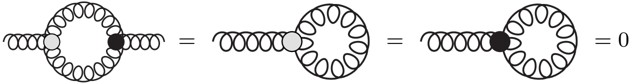


with $T_{\boldsymbol{r}}\delta^{ab} = \mathrm{Tr}[t^a t^b]$,

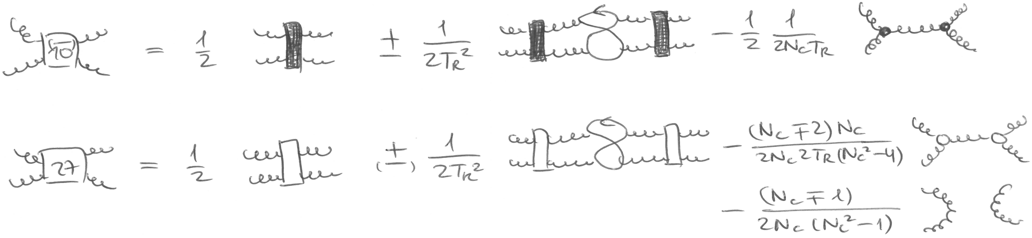
with $C_D = \dfrac{N^2-4}{N}$,

with $C_A = 2T_{\boldsymbol{r}}N$.

with $C_F = T_{\boldsymbol{r}}\dfrac{N^2-1}{N}$,



(B) BACKUP SLIDES



Hilbert Series and Plethystic Logarithm

Covariant building blocks as **input** for the ring:

$$\boxed{\mathbf{8}_u \ \widehat{=}\ u\,, \quad \mathbf{8}_d \ \widehat{=}\ d.}$$

From input, compute Hilbert series (HS) and Plethystic logarithm (PL):

introduced in math: [Getzler, Kapranov '94], physics [Benvenuti, Feng, Hanany, He '06]

$$\mathfrak{H}(u,d) = \int_{\mathrm{SU}(3)} d\mu_{\mathrm{SU}(3)}\, \mathrm{PE}\left[z_1, z_2; u; \mathbf{8}\right] \mathrm{PE}\left[z_1, z_2; d; \mathbf{8}\right]\ ,$$

$$\mathrm{PL}\left[\mathfrak{H}\left(u,d\right)\right] := \sum_{k=1}^{\infty} \frac{\mu(k)\ \ln \mathfrak{H}\left(u^k, d^k\right)}{k}\ .$$

$$\mathfrak{H}(u,d) = \frac{1 + u^3 d^3}{(1-u^2)(1-d^2)(1-ud)(1-u^3)(1-d^3)(1-ud^2)(1-u^2 d)(1-u^2 d^2)}\ .$$

$$\mathrm{PL}\left[\mathfrak{H}(u,d)\right] = u^2 + ud + d^2 + u^3 + d^3 + u^2 d + ud^2 + u^2 d^2 + u^3 d^3 - u^6 d^6\ .$$

Möbius function $\mu(n) = \begin{cases} \substack{+ \\ (-)}1, & \text{if } n \text{ is square free with even(odd) \# number of prime factors,} \\ 0, & \text{else.} \end{cases}$

Orthogonal Invariants

The 10 algebraically independent and orthogonal invariants are given by:

$$I_{10} := \mathrm{Tr}\,\widetilde{H}_u \qquad \text{and} \qquad I_{01} := \mathrm{Tr}\,\widetilde{H}_d\ .$$

$$I_{20} := \mathrm{Tr}(H_u^2)\,, \quad I_{02} := \mathrm{Tr}(H_d^2)\,, \quad I_{11} := \mathrm{Tr}(H_u H_d)\,,$$

$$I_{30} := \mathrm{Tr}(H_u^3)\,, \quad I_{03} := \mathrm{Tr}(H_d^3)\,, \quad I_{21} := \mathrm{Tr}(H_u^2 H_d)\,, \quad I_{12} := \mathrm{Tr}(H_u H_d^2)\,,$$

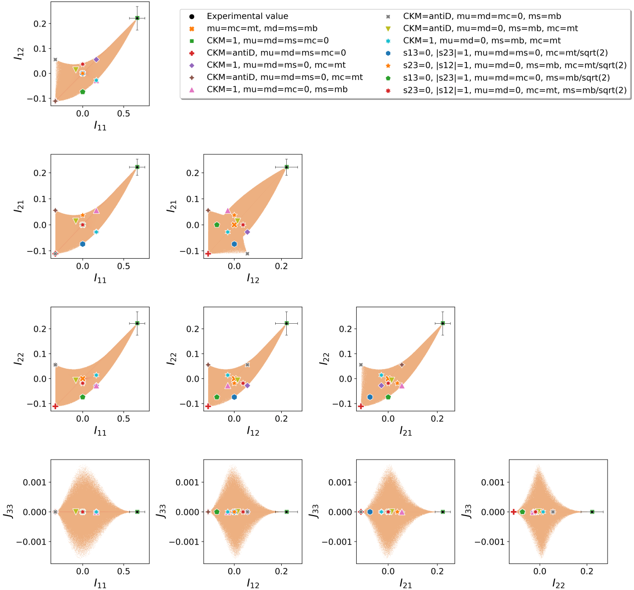
$$I_{22} := 3\,\mathrm{Tr}(H_u^2 H_d^2) - \mathrm{Tr}(H_u^2)\,\mathrm{Tr}(H_d^2)\,.$$

Secondary invariant: exactly the Jarlskog invariant,

$$J_{33} := \mathrm{Tr}(H_u^2 H_d^2 H_u H_d) - \mathrm{Tr}(H_d^2 H_u^2 H_d H_u) \equiv \frac{1}{3}\,\mathrm{Tr}\,[H_u, H_d]^3\ .$$

Note: Here $\widetilde{H}_u \equiv Y_u Y_u^\dagger$, $\widetilde{H}_d \equiv Y_d Y_d^\dagger$, and $H_{u,d} \equiv \widetilde{H}_{u,d} - \mathbb{1}\,\mathrm{Tr}\frac{\widetilde{H}_{u,d}}{3}$. **“Traces of traceless matrices”**

Parameter space and experimental values



Choice of normalization:

$$\hat{I}^{\mathrm{alt}}_{ij} := \frac{I_{ij}}{I^{i}_{10}\, I^{j}_{01}} \,.$$