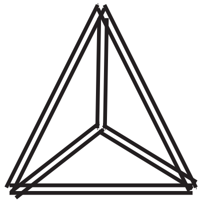
LUND UNIVERSITY

All you need is 

… and a little bit of chirality flow

Thanks to my collaborators:

Color: Judith Alcock-Zeilinger, Stefan Keppeler, Simon Plätzer, Johan Thorén

Chirality flow: Joakim Alnefjord, Andrew Lifson, Christian Reuschle, Simon Plätzer, Adam Warnerbring, Zenny Wettersten

Online
February 2024
Malin Sjödahl

This talk

- My motivation: With the LHC there is an increased interest in the treatment of color structure for processes with many colored partons
- This is applicable to fixed order calculations as well as parton showers and resummation
- Color structure of SU(N), in particular multiplet bases (transition operators of Heribert Weigert) – a pedagogical intro
- Calculating using basic group invariants, Wigner 6js (also known as 6j coefficients, 6j symbols, Racah coefficients, Racah W coefficients) and Wigner 3js
- I will talk about QCD (SU(N_c)), but similar methods can be applied more generally
- Chirality flow – flowing the Lorentz structure $\mathfrak{su}(2)_L$ $\mathfrak{su}(2)_R$

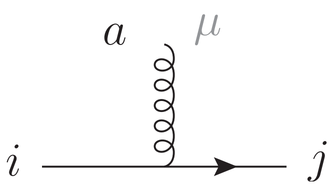
The QCD Lagrangian

The QCD Lagrangian

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + gA_\mu^a \overline{\psi}\gamma^\mu t^a \psi$$

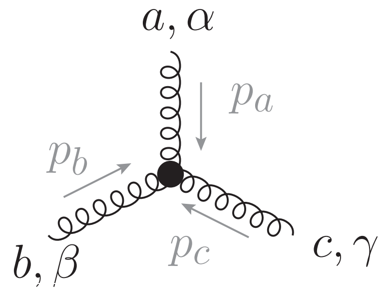
$$-gf^{abc}(\partial_\mu A_\nu^a)A^{\mu b}A^{\nu c} - \frac{1}{4}g^2(f^{eab}A_\mu^a A_\nu^b)(f^{ecd}A^{\mu c}A^{\nu d})$$

contains:

- quark-gluon vertex, 

$= (t^a)^i{}_j$
  Here $(t^a)^i{}_j$ are SU(3) generators and I take the graph to represent the color structure alone, no $ig\gamma^\mu$

- triple-gluon vertex,

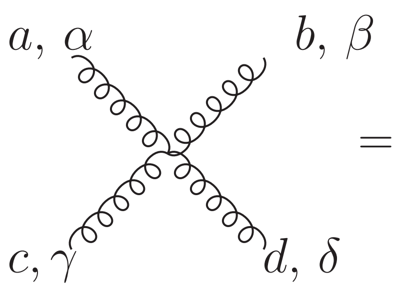


$$= i f^{abc}$$

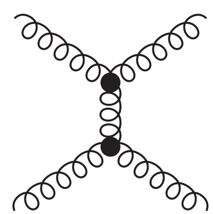
Here we use the convention of reading the indices counter clockwise in the SU(3) structure constants f^{abc}, and again I only mean the color structure, no $-ig(g^{\alpha\beta}(p_a - p_b)^\gamma +$ cyclic)

- four-gluon vertex, here color and kinematic factors are correlated

(so I cannot draw the color structure alone)

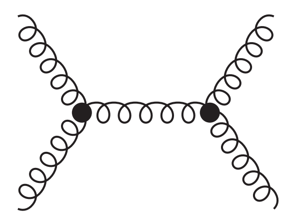


$=$



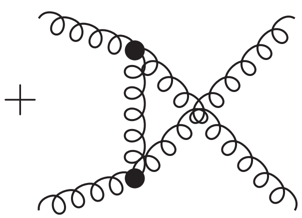
$\times i g_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta})$

$+$



$\times i g_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\beta} g^{\gamma\delta})$

$+$



$\times i g_s^2 (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta})$

$$= i f^{aeb}\, i f^{cde} \times i g_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta}) + i f^{ace}\, i f^{bed} \times i g_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\beta} g^{\gamma\delta}) + i f^{aed}\, i f^{cbe} \times i g_s^2 (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta})$$

but the color structure is just a linear combination of triple-gluon vertices

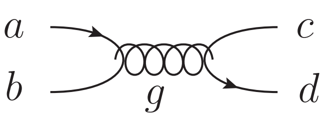
Dealing with color space

Due to confinement we never observe individual colors

- We average over incoming colors
- We sum over outgoing colors
- $\rightarrow$ we sum over the colors of all external partons
- As always in quantum mechanics we also sum over all degrees of freedom that can interfere with each other $\rightarrow$ we sum over the colors of all internal particles
- $\rightarrow$ We sum over **all** colors of all particles

So, if we for example consider

$$q\overline{q} \to q\overline{q}$$

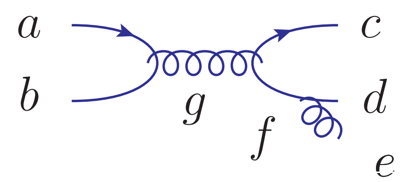


,

(let's pretend we have different flavors so we only have one Feynman diagram) we need the color sum

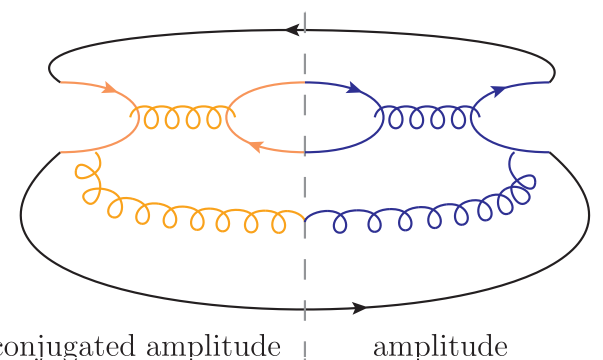
$$\frac{1}{3}\sum_{a=1}^{3}\frac{1}{3}\sum_{b=1}^{3}\sum_{c=1}^{3}\sum_{d=1}^{3}\left|\sum_{g=1}^{8}(t^g)^a{}_b(t^g)^c{}_d\right|^2$$

One way of dealing with this sum is to pick a particular representation of the generators, and sum over $3^4 * 8 = 648$ terms. Luckily there are better ways...

Example: If $A = (t^g)^a{}_b (t^g)^f{}_c (t^e)^d{}_f =$ , then

$$\langle A|A\rangle = \sum_{a,b,c,d,e,f,g,h,i} \left[(t^h)^a{}_b (t^h)^i{}_c (t^e)^d{}_i\right]^* (t^g)^a{}_b (t^g)^f{}_c (t^e)^d{}_f$$

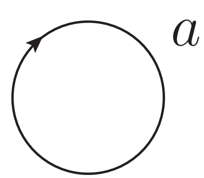
$$= \sum_{a,b,c,d,e,f,g,h,i} (t^h)^b{}_a (t^h)^c{}_i (t^e)^i{}_d (t^g)^a{}_b (t^g)^f{}_c (t^e)^d{}_f$$

$=$ 

The first equality holds since the generators are Hermitian, and the last holds since we always sum over the color of internal lines

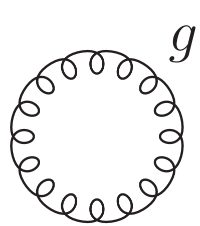
As seen above we can represent the squared amplitude with a picture. We can also calculate with graphs! To do so we need just a few rules

- There are N_c possible quark colors



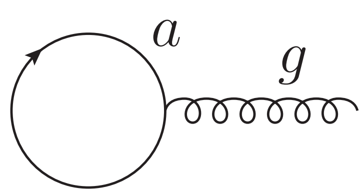
$$= N_c \qquad \sum_{a=1}^{N_c} \delta^a{}_a = N_c$$

- There are $N_g = N_c^2 - 1$ possible gluon colors



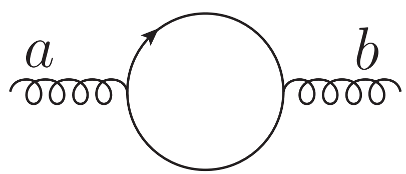
$$= N_c^2 - 1 \qquad \sum_{g=1}^{N_c^2-1} \delta^{gg} = N_c^2 - 1$$

- The generators are traceless

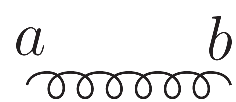


$$= 0 \qquad \sum_{a=1}^{N_c} (t^g)^a{}_a = 0$$

- Generator normalization

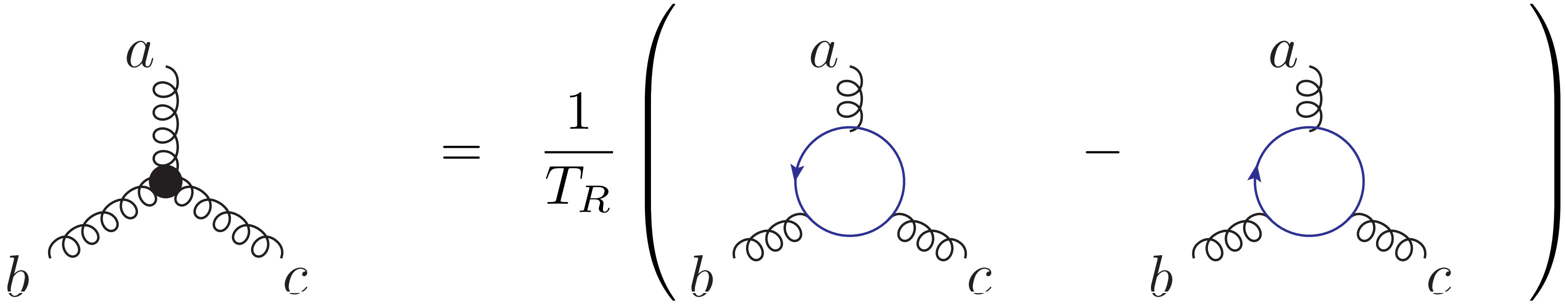


$$= T_R$$



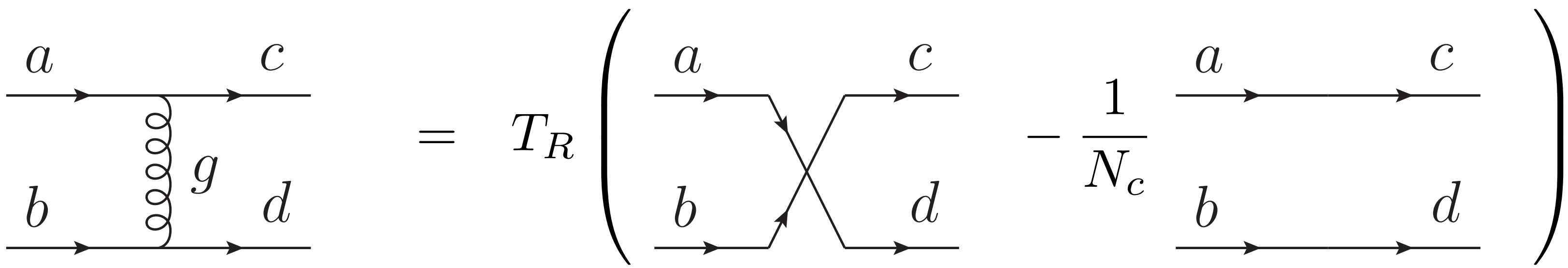
$$\mathrm{Tr}[t^a t^b] = T_R \delta^{ab}$$

- The algebra relation $[t^a, t^b] = if^{abc}t^c \Rightarrow$



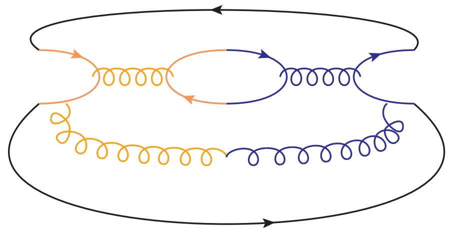
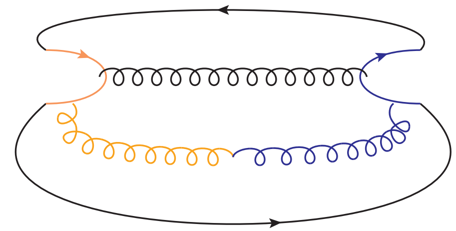
$$if^{abc} = \frac{1}{T_R}\left[\mathrm{Tr}[t^a t^b t^c] - \mathrm{Tr}[t^b t^a t^c]\right]$$

- The Fierz identity (the completeness relation)

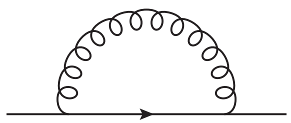
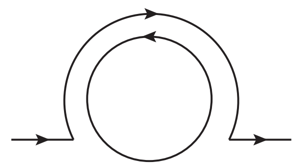


$$(t^g)^a{}_c (t^g)^b{}_d = T_R\left[\delta^a{}_d \delta^b{}_c - \frac{1}{N_c}\delta^a{}_c \delta^b{}_d\right]$$

Let's apply the rules to our example

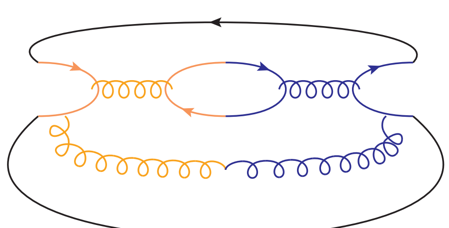
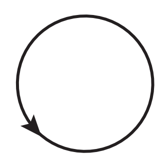
 $= T_R$ 

To further simplify the color structure we note using Fierz

 $= T_R \Big($  $- \frac{1}{N_c}$  $\Big) = T_R \left(N_c - \frac{1}{N_c} \right)$ 

$= T_R \frac{N_c^2 - 1}{N_c}$  $\equiv C_F$ 

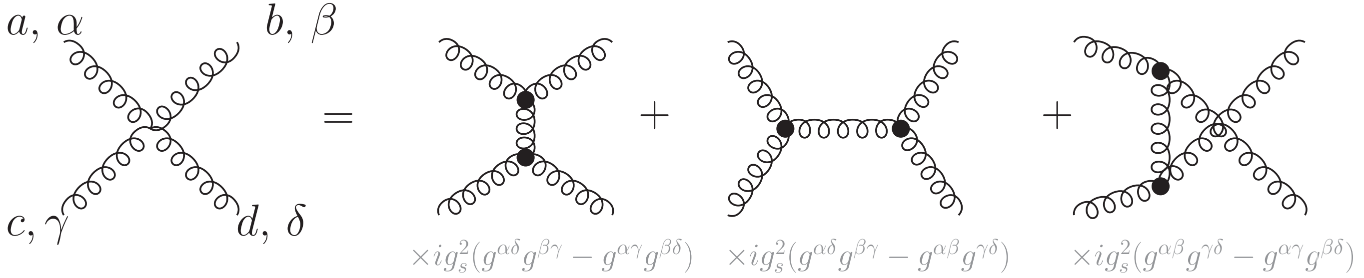
Giving, for the squared amplitude

 $= T_R C_F^2$  $= T_R C_F^2 N_c$

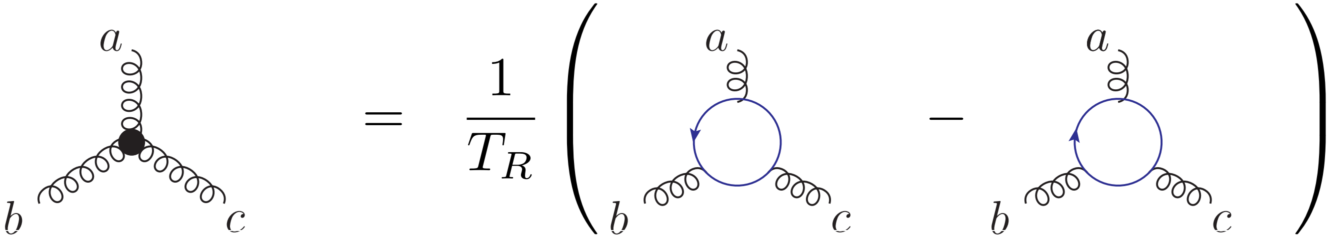
- In this way we can square any color amplitude and calculate any interference term. In general we have interference terms between different Feynman diagrams/color structures, but these are treated in precisely the same way.
- I have written a Mathematica package, ColorMath, (Eur. Phys. J. C 73:2310 (2013), 1211.2099)
- One way of dealing with color space is to just square the amplitudes one by one as one encounters them
- Alternatively, we may use any basis (spanning set)

Trace bases

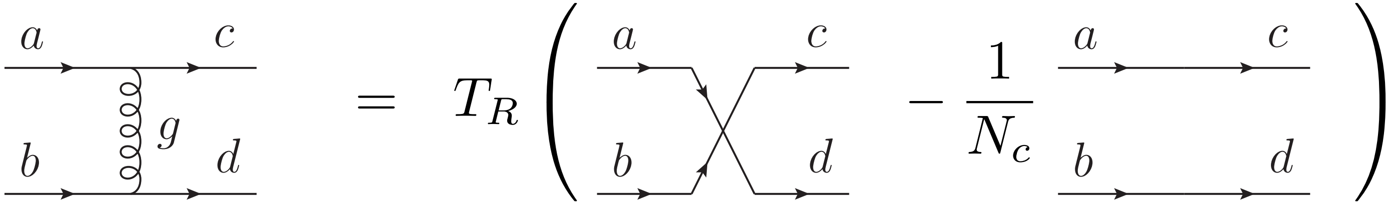
- Every 4g vertex can be replaced by 3g vertices:



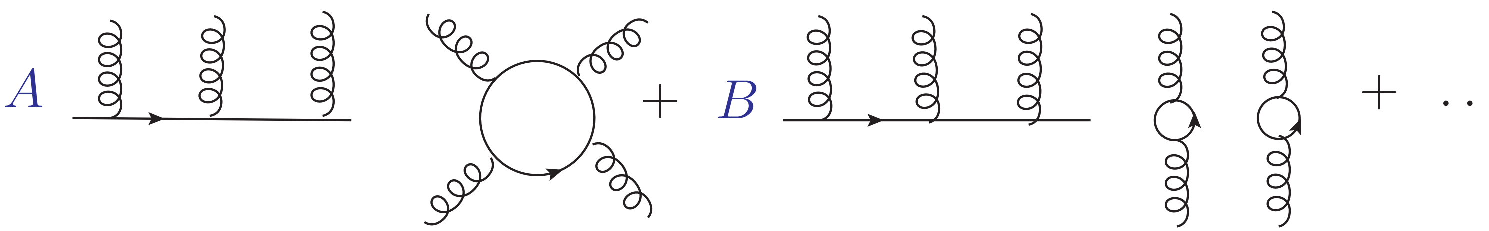
- Every 3g vertex can be replaced using:



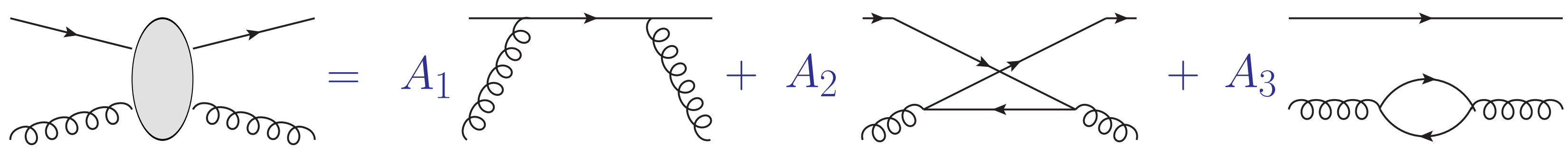
- After this every internal gluon can be removed using Fierz:



- This can be applied to any QCD amplitude, tree level or beyond
- In general an amplitude can be written as linear combination of different color structures, like



- For example for 2 (incoming + outgoing) gluons and one $q\bar{q}$ pair



(an incoming quark is the same as an outgoing anti-quark)

- The above type of color structure can be used as a spanning set, a trace basis

These bases have some nice properties

- Conceptual simplicity
- Taking the leading N_c limit is trivial $\rightarrow$ a flow of colors and orthogonal basis vectors
- The effect of gluon emission and exchange is easily described

There are also drawbacks with trace bases

- Not orthogonal
  $\rightarrow$ When squaring amplitudes almost all cross terms have to be taken into account $\rightarrow N_{\text{basis}}^2$ terms
- Overcomplete, for $N_g + N_{q\overline{q}} > N_c$ the bases are also overcomplete. The size of the vector space asymptotically grows as an exponential in the number of gluons/$q\overline{q}$-pairs for finite N_c

Example: Number of spanning vectors for N_g gluons (without imposing charge conjugation invariance). These numbers are representative also for N_g gluons plus $q\bar{q}$-pairs.

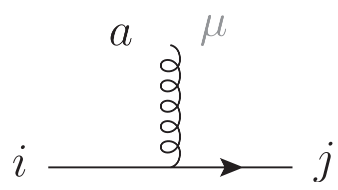
N_g	Vectors $N_c = 3$	Vectors $N_c \to \infty$	LO Vectors $N_c \to \infty$
4	8	9	3!=6
5	32	44	4!=24
6	145	265	120
7	702	1 854	720
8	3 598	14 833	5 040
9	19 280	133 496	40 320
10	107 160	1 334 961	362 880
11	614 000	14 684 570	3 628 800
12	3 609 760	176 214 841	39 916 800

(Y. Du, M.S. & J. Thorén, JHEP 1505 (2015) 119, 1503.00530)

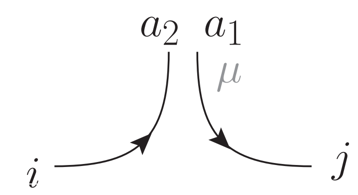
Color flow bases

- One way out is to give up exact treatment of color structure and run a Monte Carlo over colors
- This is particularly efficient in the color flow basis
- Here the adjoint representation indices are rewritten in terms of fundamental representation indices and new color flow Feynman rules are derived (Maltoni, Stelzer, Paul, Willenbrock, Phys.Rev. D67 (2003), hep-ph/0209271)
- Explicit colors (r, g, or b) are then assigned to the lines, and one may run a Monte Carlo sum over colors to sample color space
- This is not exact but the color structure treatment is much quicker (Comix, T. Gleisberg, S. Hoeche, JHEP 0812 (2008) 039, 0808.3674; S. Plätzer, Eur.Phys.J. C74 (2014) 6, 2907, 1312.2448; S. Prestel and J. Isaacson 1806.10102)

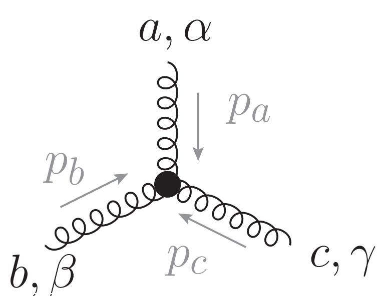
- quark-gluon vertex,



$$= ig_s\gamma^\mu (t^a)^i{}_j \to \quad ig_s\gamma^\mu \delta^i{}_{a_2}\delta^{a_1}{}_j =$$

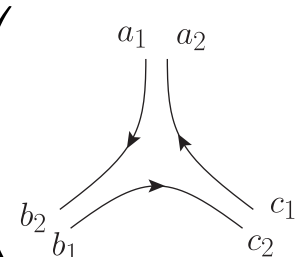


- triple-gluon vertex,

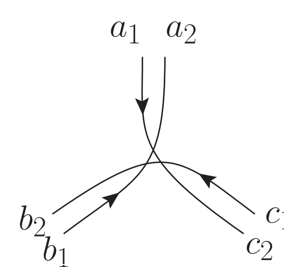


$$= if^{abc}(-ig_s(g^{\alpha\beta}(p_a - p_b)^\gamma + \text{cyclic}))$$

$$\to \frac{1}{T_R}\Bigg($$



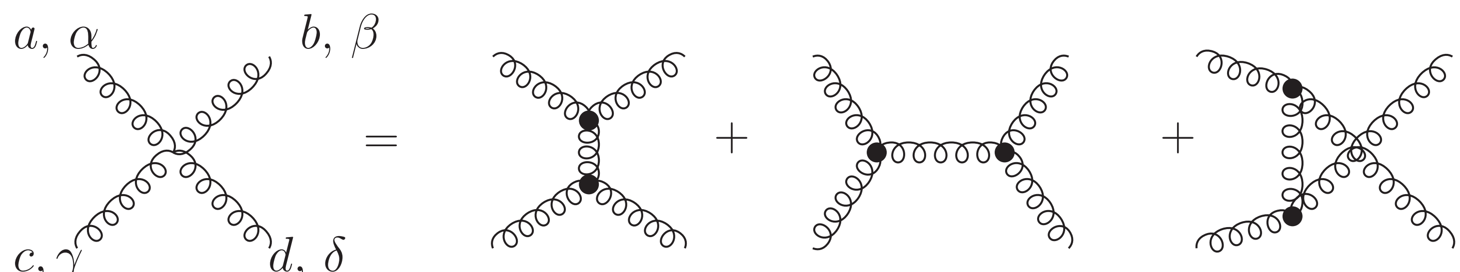
$$-$$



$$\Bigg)(-ig_s(g^{\alpha\beta}(p_a - p_b)^\gamma + \text{cyclic}))$$

can easily be written in completely symmetric form...

- four-gluon vertex



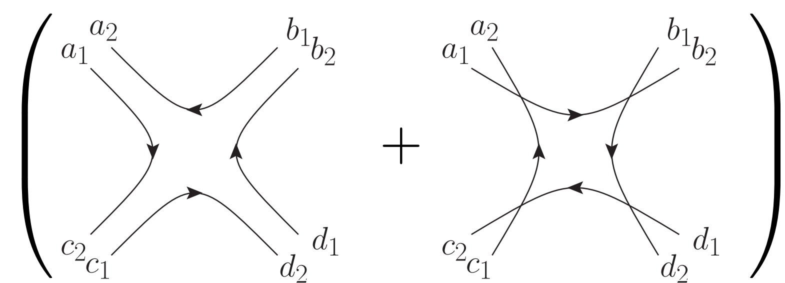
$$\times i g_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta}) \quad \times i g_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\beta} g^{\gamma\delta}) \quad \times i g_s^2 (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta})$$

$$= i f^{aeb}\, i f^{cde} + i f^{ace}\, i f^{bed} + i f^{aed}\, i f^{cbe}$$

$$\times i g_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta}) \quad \times i g_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\beta} g^{\gamma\delta}) \quad \times i g_s^2 (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta})$$

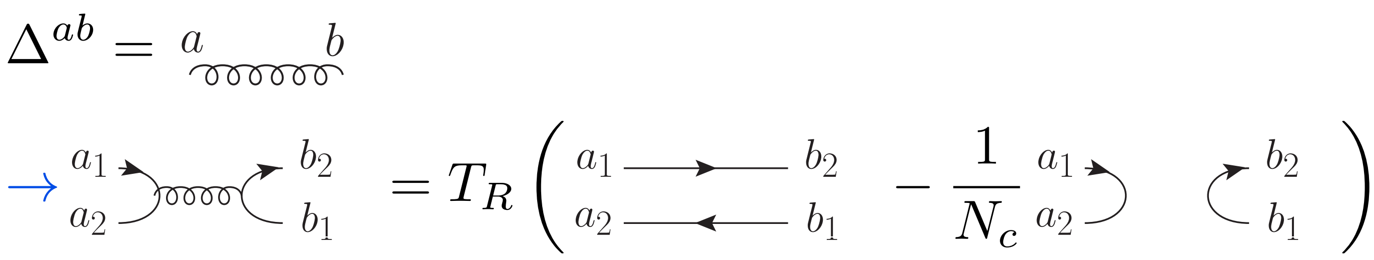
$\rightarrow$

$$i g_s^2 \left(2 g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\beta} g^{\gamma\delta}\right) \frac{1}{T_R}$$



$$+[c \leftrightarrow d] \quad + [b \leftrightarrow d]$$

- Color structure of propagator

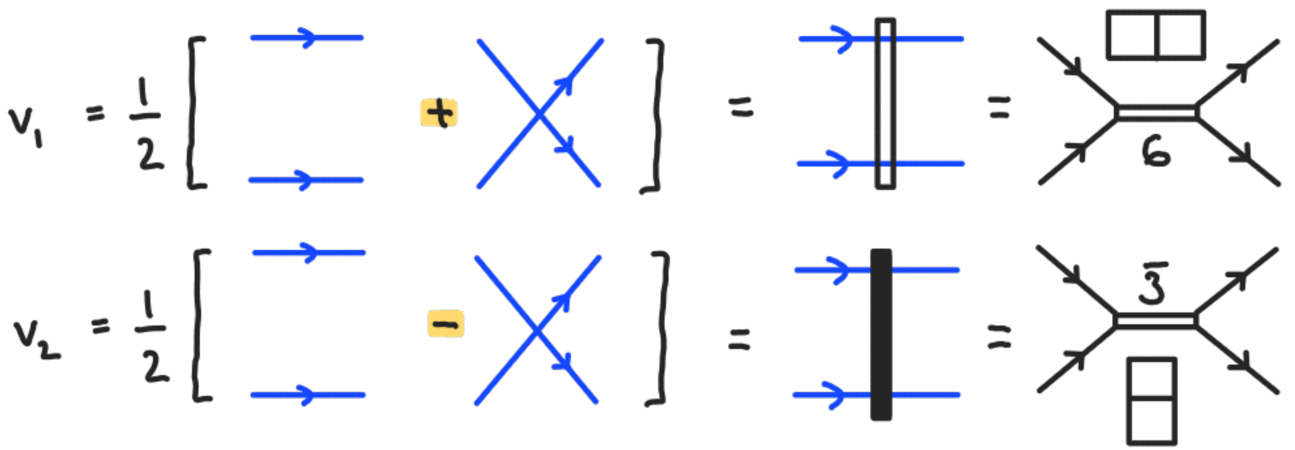


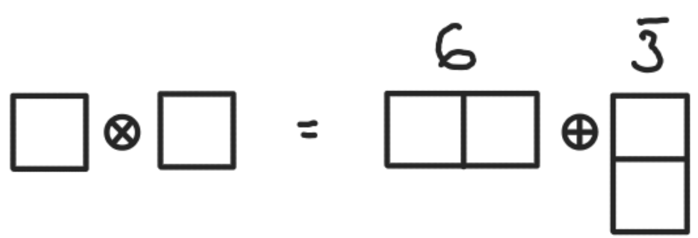
- Similarly the $q\bar{q}$-pairs corresponding to external gluons have to be forced to be in octets when squaring amplitudes (Conventions differ from those in hep-ph/0209271)

- ... but these bases are not orthogonal

Wanted: Orthogonal bases

How can we construct an orthogonal basis? Symmetrize!



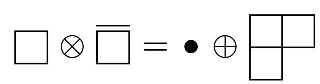


Here the birdtrack notation is used. These color tensors are orthogonal both when seen as qq-projectors, and when seen as basis vectors on the 4-parton space

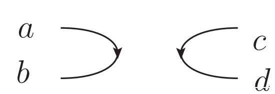
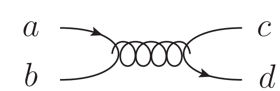
Orthogonal multiplet bases

In collaboration with Stefan Keppeler and Johan Thorén

- The color space may be decomposed into irreducible representations, enumerated using Young tableaux multiplication
- For quarks we can construct orthogonal projectors and basis vectors using Young tableaux ...at least from the Hermitian quark projectors (S.K. and MS, 1307.6147, J.Math.Phys.)
- In fact the $qq \to qq$ color space is the same as for $q\bar{q} \to q\bar{q}$,



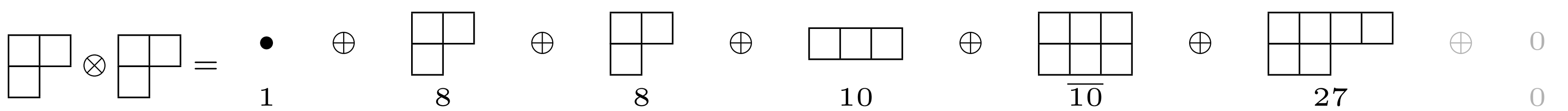
and we could as well have used the basis:

$\mathbf{V}^1 = \delta^a{}_b \delta^c{}_d =$  , $\mathbf{V}^8 = (t^g)^a{}_b (t^g)^c{}_d =$ 

- In general we may "comb" the involved particles as incoming and outgoing as we wish

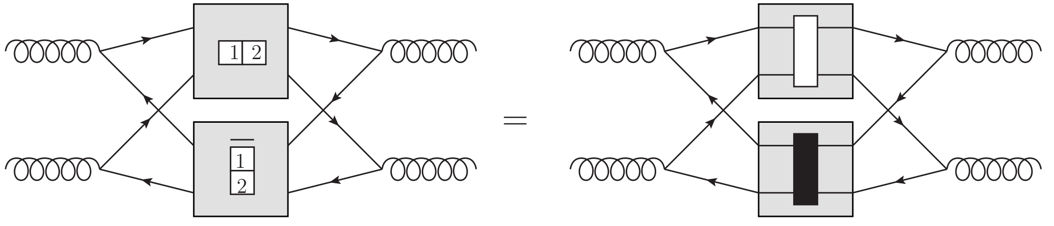
The simplest gluon example, $gg \to gg$

- In QCD we have quarks, anti-quarks and gluons
  $\to$ No obvious way to construct projectors
- Basis vectors can be enumerated using Young tableaux multiplication



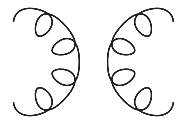
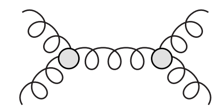
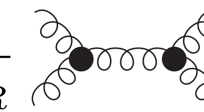
- As color is conserved an incoming multiplet of a certain kind can only go to an outgoing multiplet of the same kind,
  $1 \to 1, 8 \to 8...$
  Charge conjugation implies that some vectors only occur together... (MacFarlane, Sudbery, and Weisz 1968, Butera, Cicuta and Enriotti 1979, Cvitanović 1984, Dokshitzer and Marchesini 2006)

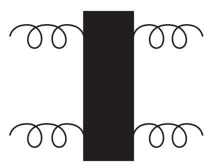
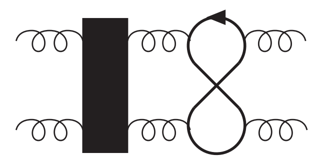
- For two gluons, there are two octet projectors, one singlet projector, and 4 new projectors, 10, $\overline{10}, 27$, and for general N_c, "0"
- It turns out that the new projectors can be seen as corresponding to different symmetries w.r.t. quark and anti-quark units, for example the decuplet can be seen as corresponding to

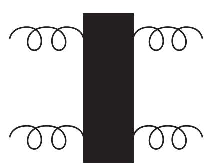
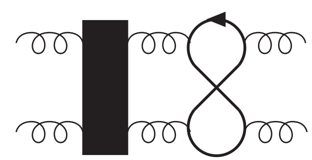


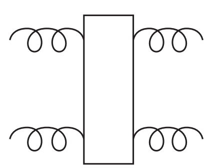
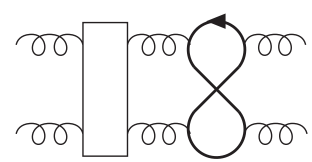
− (singlet) and octet(s) − (singlet) and octet(s)

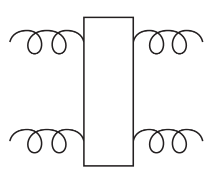
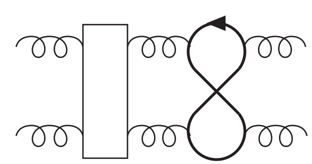
Similarly the anti-decuplet corresponds to $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \overline{\begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}}$, the 27-plet corresponds to $\begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array} \otimes \overline{\begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}}$ and the 0-plet to $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \overline{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}}$. (MacFarlane, Sudbery, and Weisz 1968, Butera, Cicuta and Enriotti 1979, Cvitanović 1984, Dokshitzer and Marchesini 2006)

$\mathbf{P}^1 = \frac{1}{N_c^2-1}$  , $\mathbf{P}^{8s} = \frac{N_c}{2T_R(N_c^2-4)}$  , $\mathbf{P}^{8a} = \frac{1}{2N_cT_R}$  ,

$\mathbf{P}^{10} = \frac{1}{2}$  $+ \frac{1}{2T_R^2}$  $- \frac{1}{2}\mathbf{P}^{8a}$

$\mathbf{P}^{\overline{10}} = \frac{1}{2}$  $- \frac{1}{2T_R^2}$  $- \frac{1}{2}\mathbf{P}^{8a}$

$\mathbf{P}^{27} = \frac{1}{2}$  $+ \frac{1}{2T_R^2}$  $- \frac{N_c-2}{2N_c}\mathbf{P}^{8s} - \frac{N_c-1}{2N_c}\mathbf{P}^1$

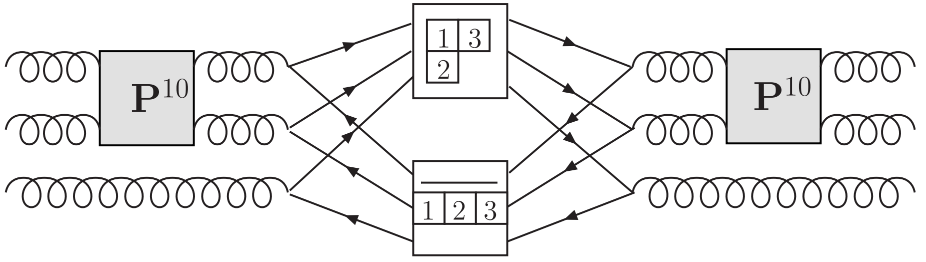
$\mathbf{P}^{0} = \frac{1}{2}$  $- \frac{1}{2T_R^2}$  $- \frac{N_c+2}{2N_c}\mathbf{P}^{8s} - \frac{N_c+1}{2N_c}\mathbf{P}^1$

Idea: Could this work in general?

$$g_1 \otimes g_2 \otimes .... \otimes g_n \subseteq (q_1 \otimes \bar{q}_1) \otimes (q_2 \otimes \bar{q}_2) \otimes ... \otimes (q_n \otimes \bar{q}_n)$$

- Construct the tensors which will give rise to "new" projectors

$$\mathbf{T}^{10,35} \propto$$

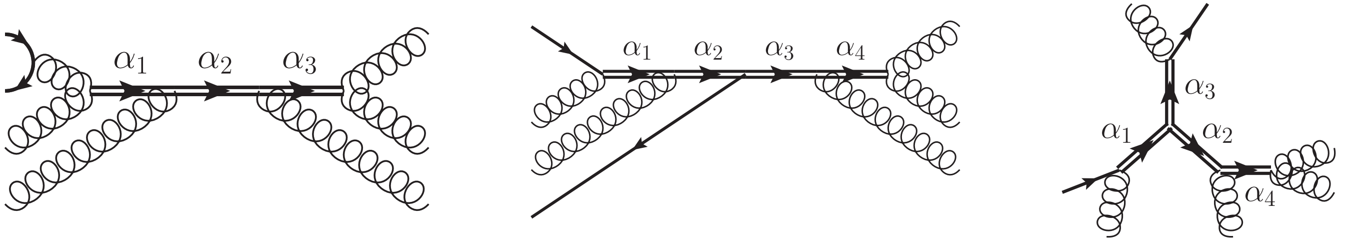


$$\mathbf{P}^{10,35} \propto \mathbf{T}^{10,35} - \sum_{m \subseteq 10 \otimes 8} \mathbf{P}^m \, \mathbf{T}^{10,35}$$

- From projectors construct basis vectors (S. Keppeler and M.S. 1207.0609 (JHEP))
- Care to find all multiplets, care with going from general N_c to $N_c = 3$, issues with multiple occurrence

Multiplet bases

- QCD is based on SU(3) $\rightarrow$ the color space may be decomposed into irreducible representations
- Orthogonal basis vectors corresponding to irreducible representations may be constructed, in may different ways...



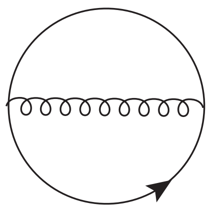
- The construction of the corresponding basis vectors is non-trivial, and a general strategy was presented relatively recently, (S. Keppeler and M.S. JHEP09(2012)124, 1207.0609 generalized by MS and J.Thorén in 1809.05002)
- These vectors are orthogonal by construction $\rightarrow$ can potentially speed up squaring of color structure very significantly

Decomposing color structure in multiplet bases

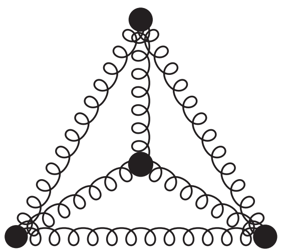
- But before squaring, amplitudes must be decomposed in multiplet bases
- One way of decomposing color structure into multiplet bases would be to simply evaluate the scalar product between each possible Feynman diagram and each possible vector as we have seen in the first half of this talk.
- The problem is that this scales badly, a factorial from the number of diagrams, an exponential from the number of basis vectors and another (growing) factor from each single scalar product evaluation
- → We need a better strategy

Group invariants!

- Fortunately there is one: Any group invariant quantity can be evaluated using Wigner 3j and 6j coefficients

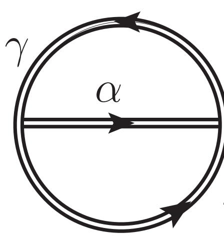


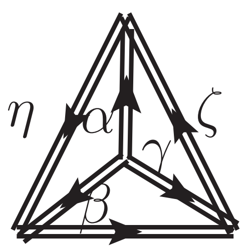
$=T_R(N_c^2-1)$



$=2\,T_R^2 N_c^2(N_c^2-1)$

(using standard normalization of vertices)





- Using the multiplet basis we can calculate the needed 3j and 6j coefficients for higher representations

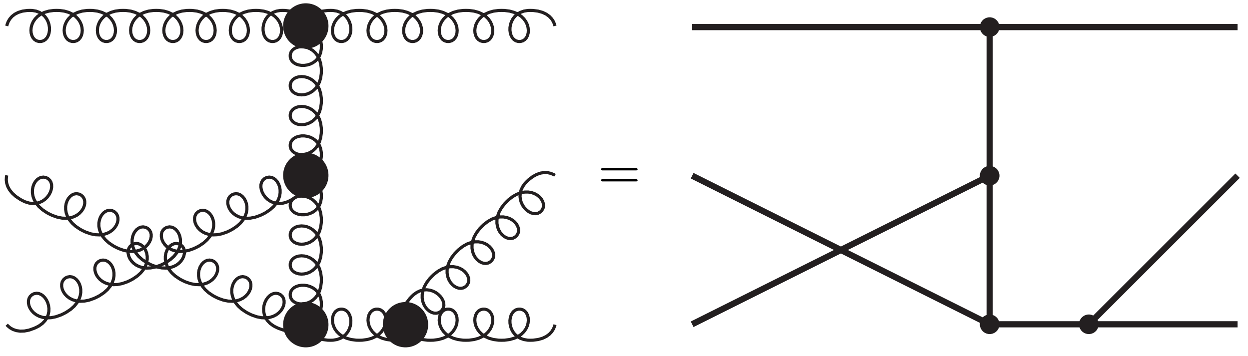
- Furthermore, only a small number of such coefficients are needed, up to NLO

N_g	4	6	8	10	12
$N_c = 3$	29	120	272	476	733
$N_c \geq N_g$	44	389	2 023	8 077	27 631

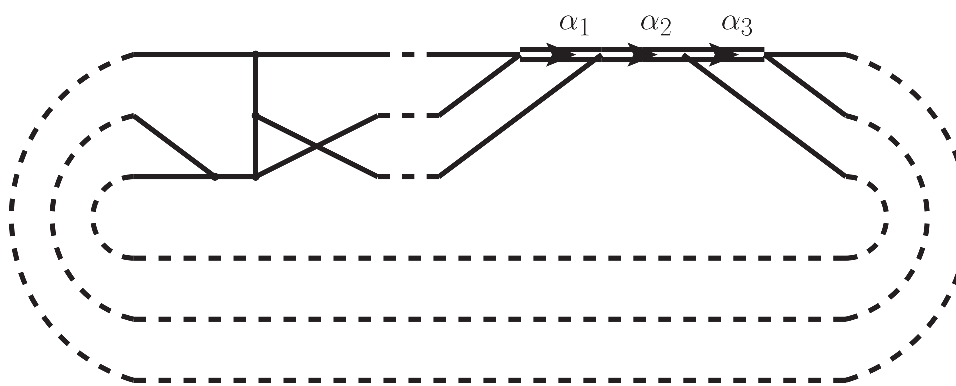
and they can be evaluated once and for all
(Numbers could be slightly reduced by additional symmetries, and smart choices of vertices)

Decomposing color with 6j and 3j coefficients

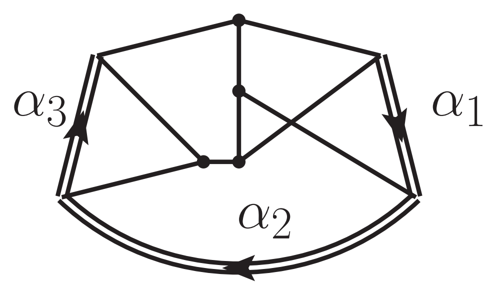
As an example consider the color structure of the Feynman diagram:



The scalar product between the color structure and a basis vector is given by:

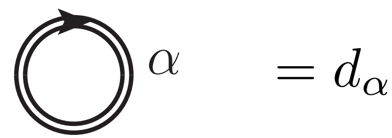
$A(\alpha_1, \alpha_2, \alpha_3) =$ 

$=$

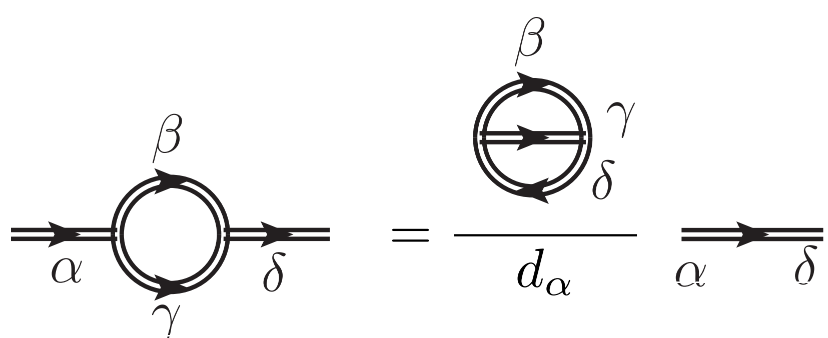
$=$ 

To fully contract any color structure we need four simple rules:

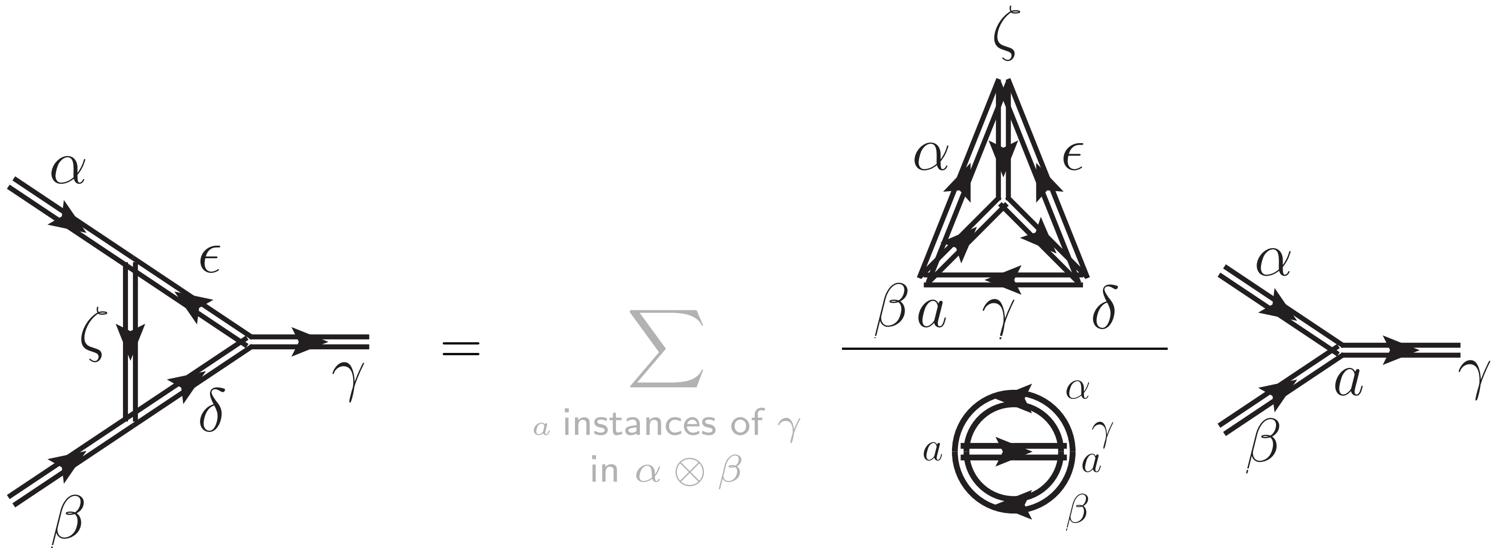
- Dimension relation



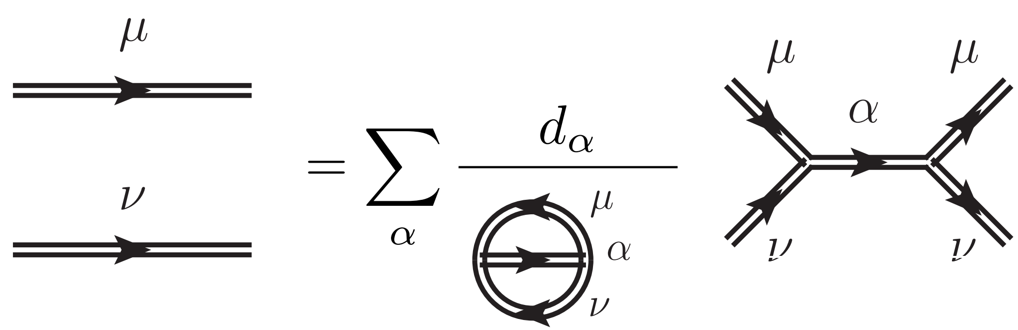
- Two vertex loops give just a constant



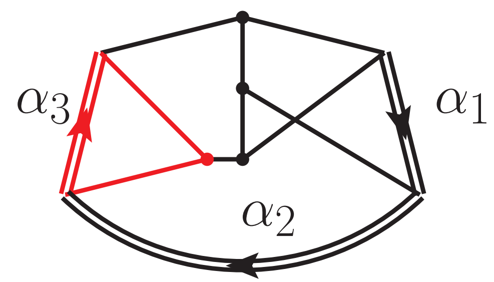
- The vertex correction relation



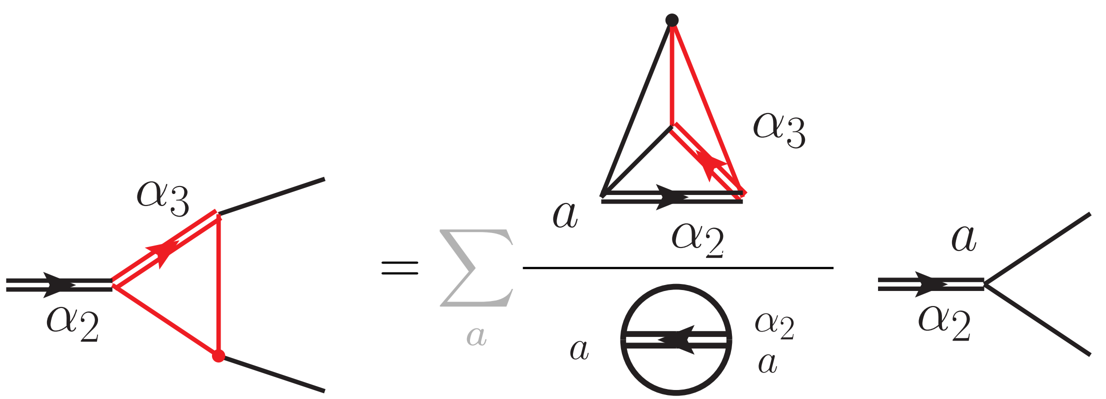
- The completeness relation



In our color structure we note that we have a vertex correction:

$A(\alpha_1, \alpha_2, \alpha_3) \quad =$ 

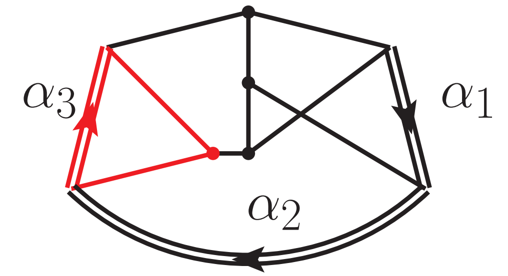
In our case the vertex correction is:



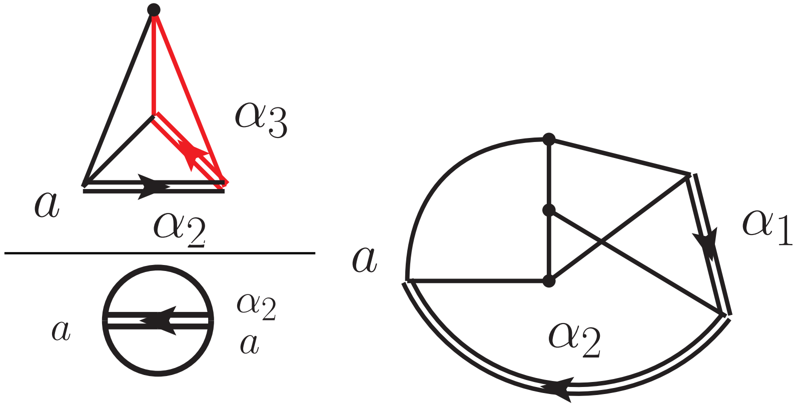
Where the sum runs over vertices a connecting the three representations α_2, α_3 and 8. For $\alpha_2 \neq \alpha_3$ there is only one such vertex, and for $\alpha_2 = \alpha_3$, there can be up to $N_c - 1 = 2$.

Using the vertex correction results in:

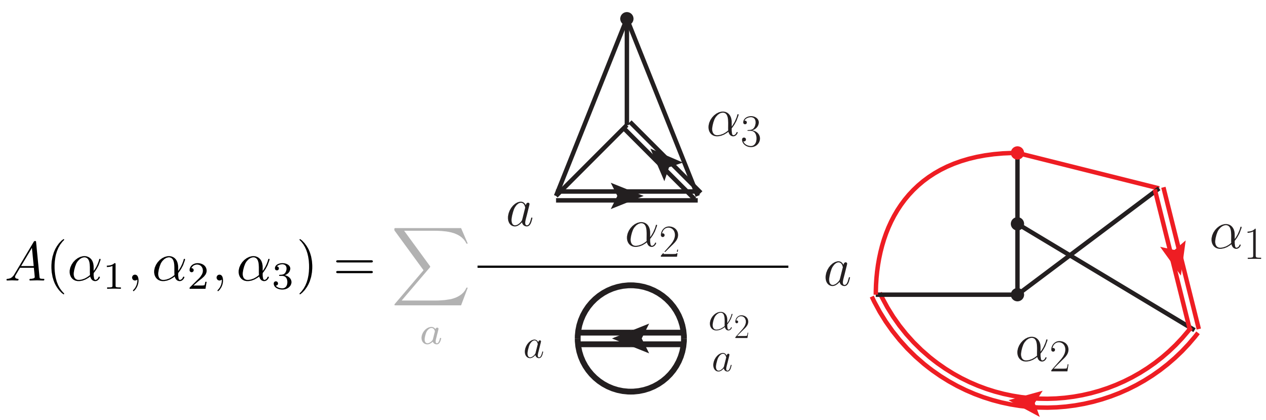
$A(\alpha_1, \alpha_2, \alpha_3) \quad =$



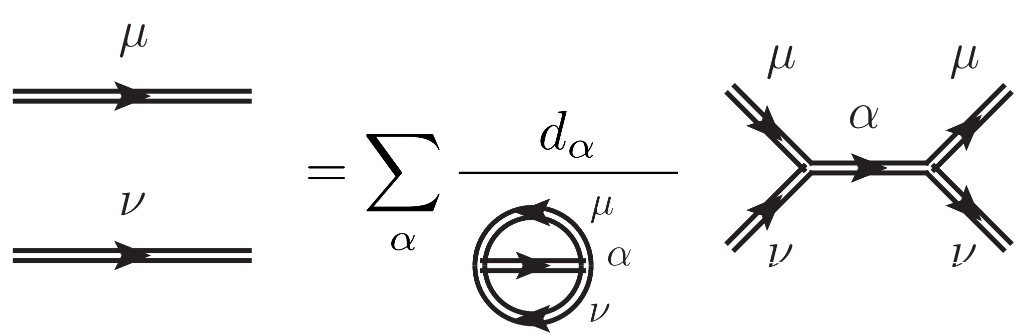
$= \quad \sum_a$



Now there is no trivial color structure, but we can pick any loop...

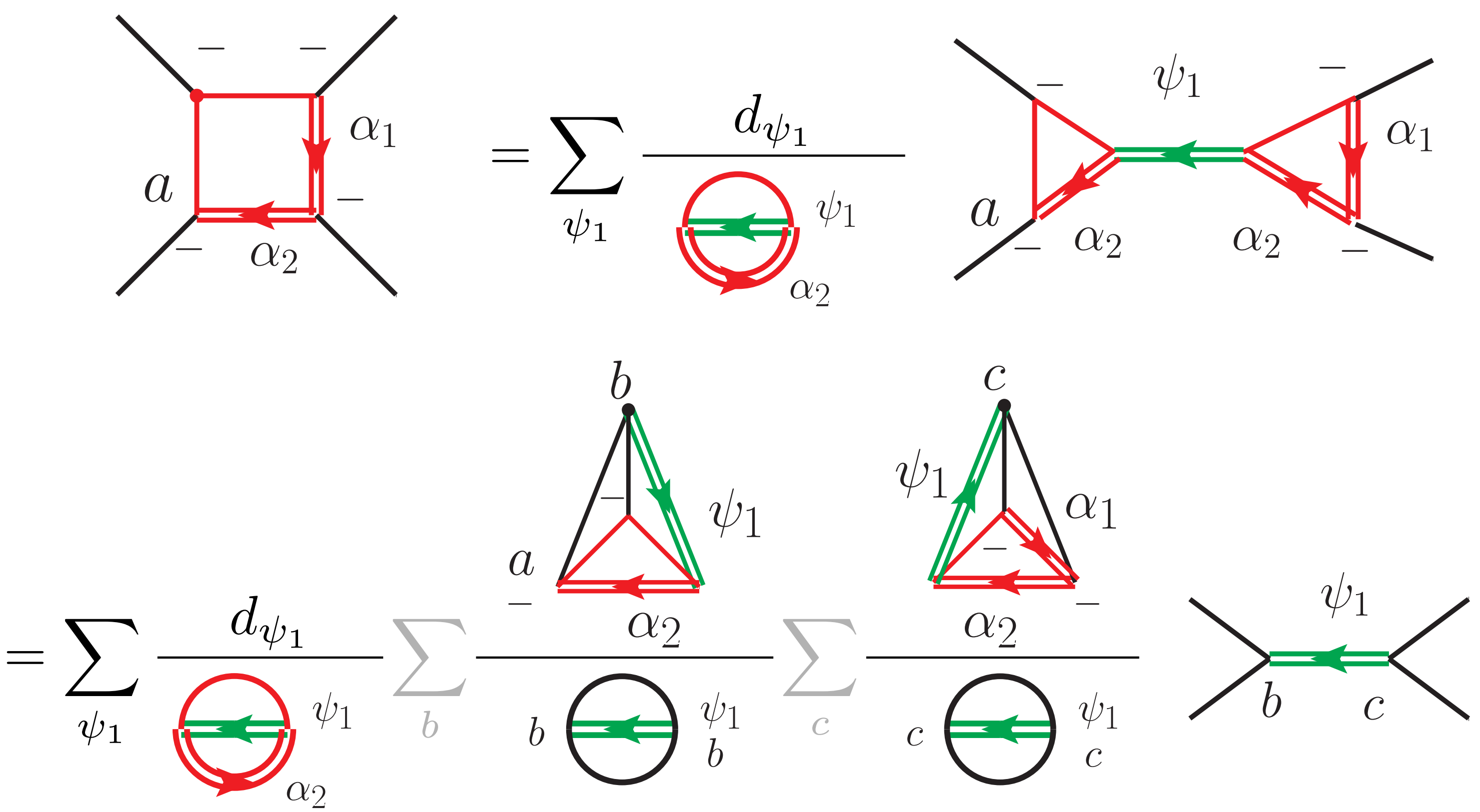


and use the completeness relation

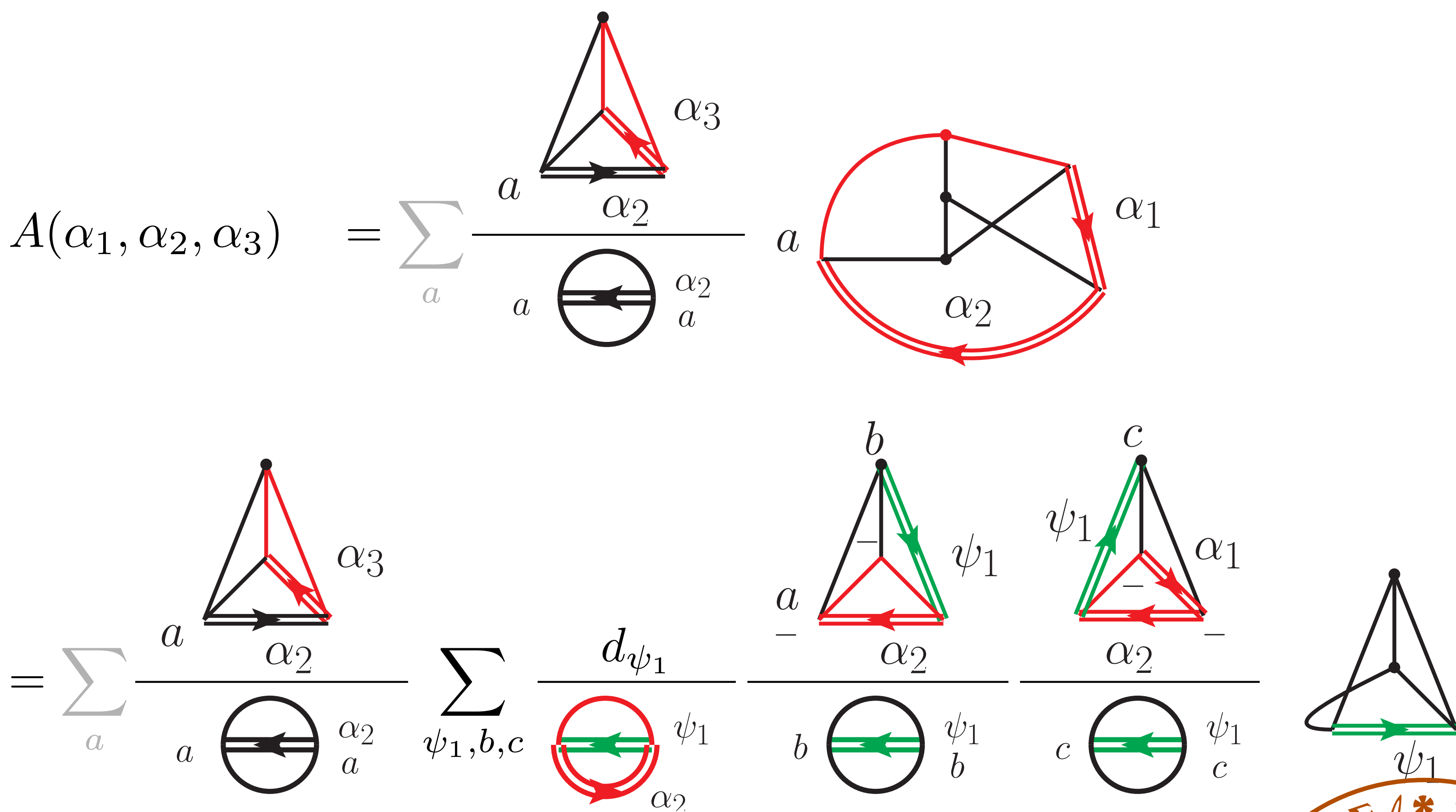


to remove it

Applying the completeness relation and removing vertex corrections:

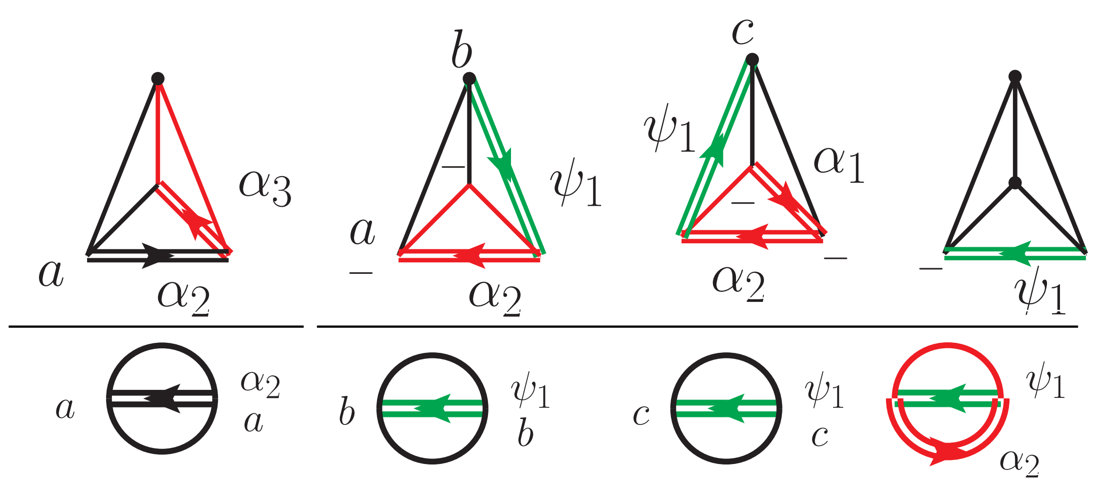


Removing the 4-vertex loop we get:

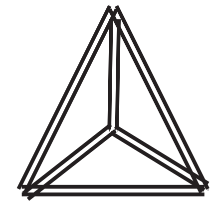


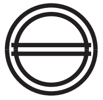
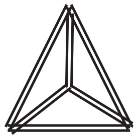
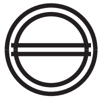
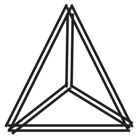
The final expression is:

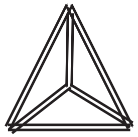
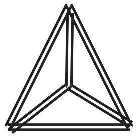
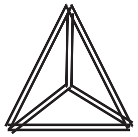
$$A(\alpha_1, \alpha_2, \alpha_3) = \sum_{\psi_1, a, b, c} d_{\psi_1}$$



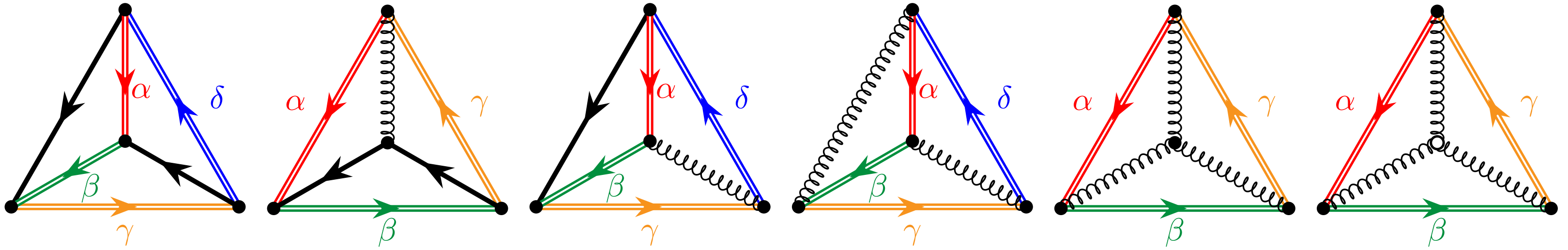
- This only has to be done once for each Feynman diagram, and the scalar product with most basis vectors vanishes
- We only need to care about non-zero scalar products, we could list the non-zero 6j-coefficients
- Each sum over representations contains at most 8 terms for SU(3), at most $N_c^2 - 1$ for SU(N_c)
- **Knowing the 3j and 6j Wigner coefficients we can immediately write down any scalar product!**

All you need is 

- In the above example we saw that we could decompose the color structure fully using only d_α,  , 
- We can normalize  $=1$, so we really only need 
- **Question:** If we can get all the color structure as a function of 6js can we then also get the 6js as a function of 6js?
- Can we calculate 6js (recursively)?

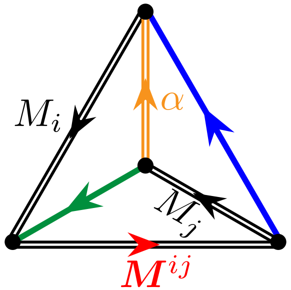
 $=$  $\Big($other  $, d_\alpha\Big)$?

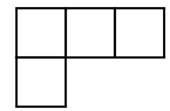
- For QCD, where every representation is 8, 3 or $\overline{3}$, it turns out that we only need 6js of form

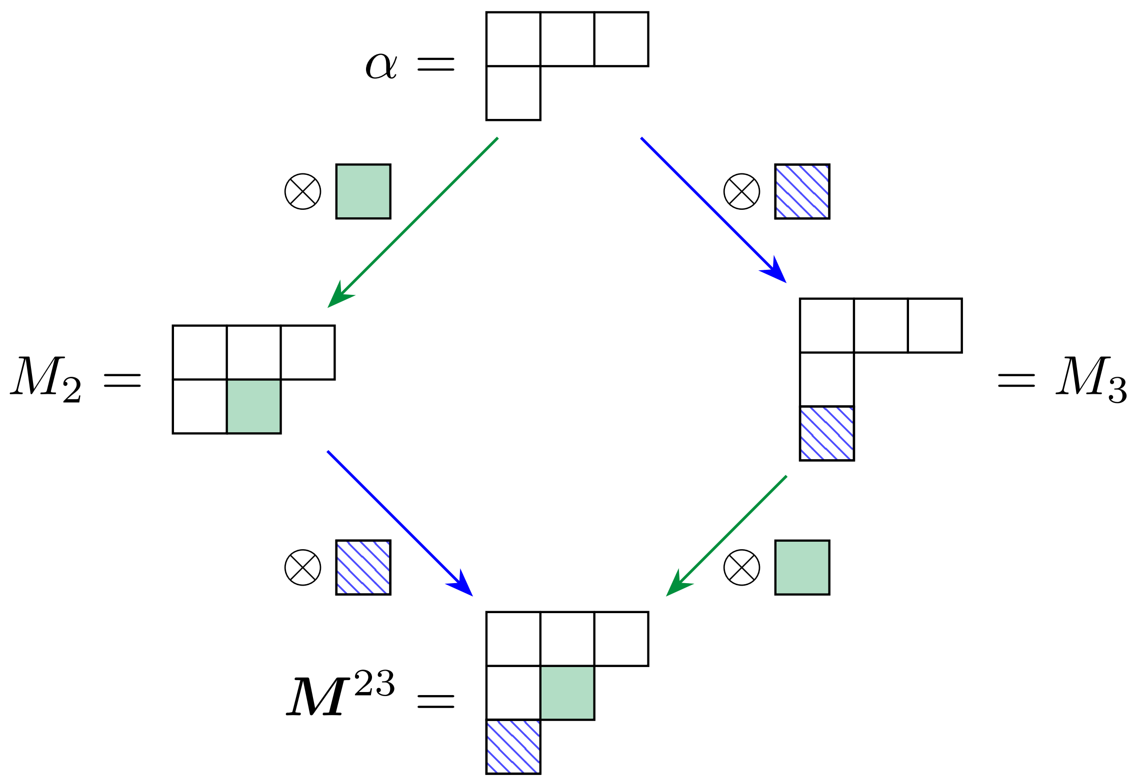


- Wigner 6j and 3j coefficients and their values can be calculated once and for all (M.S. & J. Thorén, 1507.03814 (JHEP), 1809.05002 (JHEP))

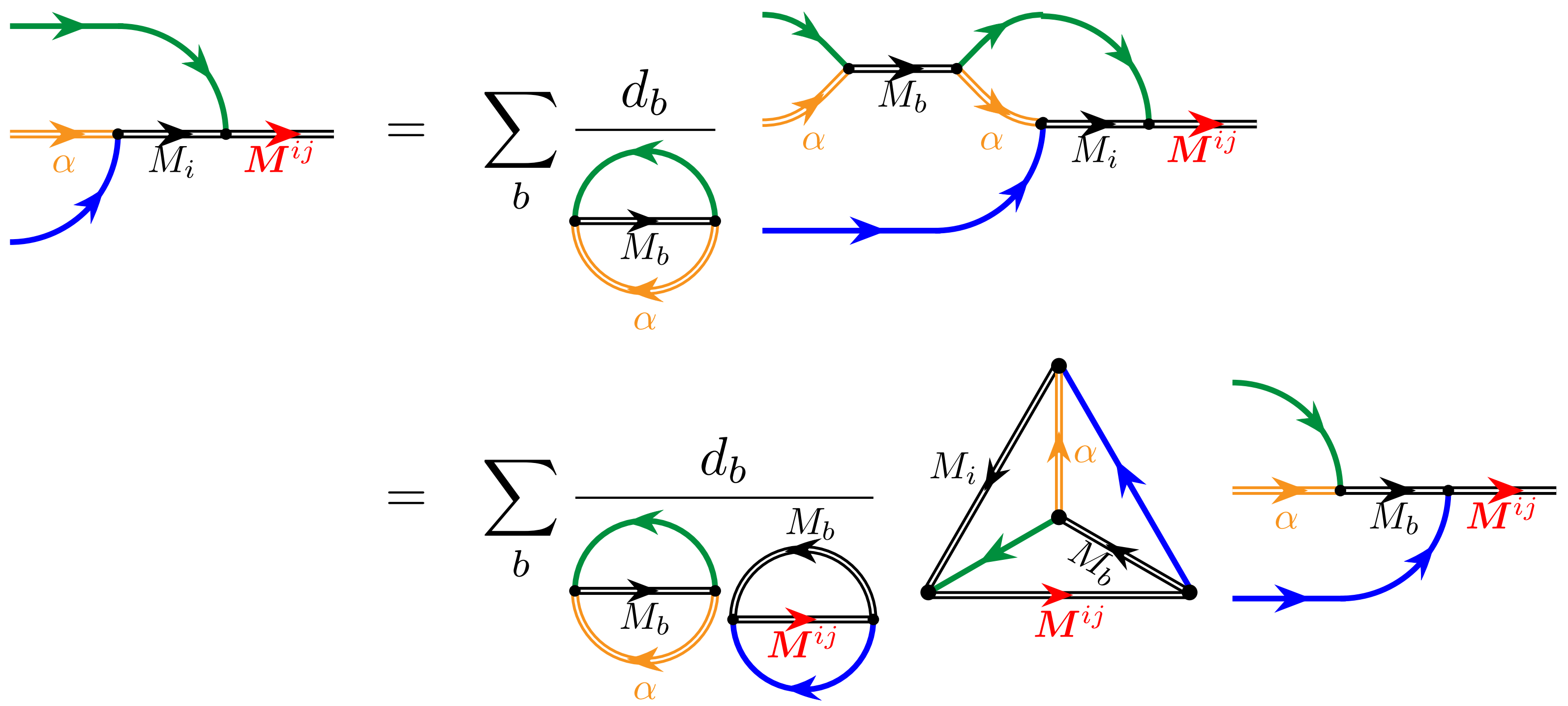
  ... but this still builds on constructing bases which builds on symmetrizers and anti-symmetrizers

Consider first 

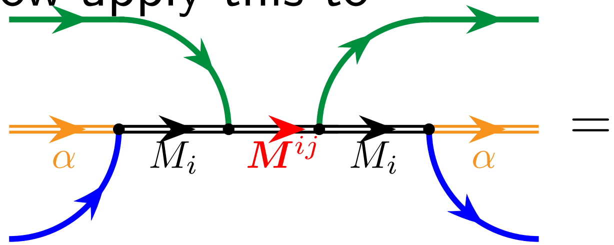
, say $\alpha =$ ,

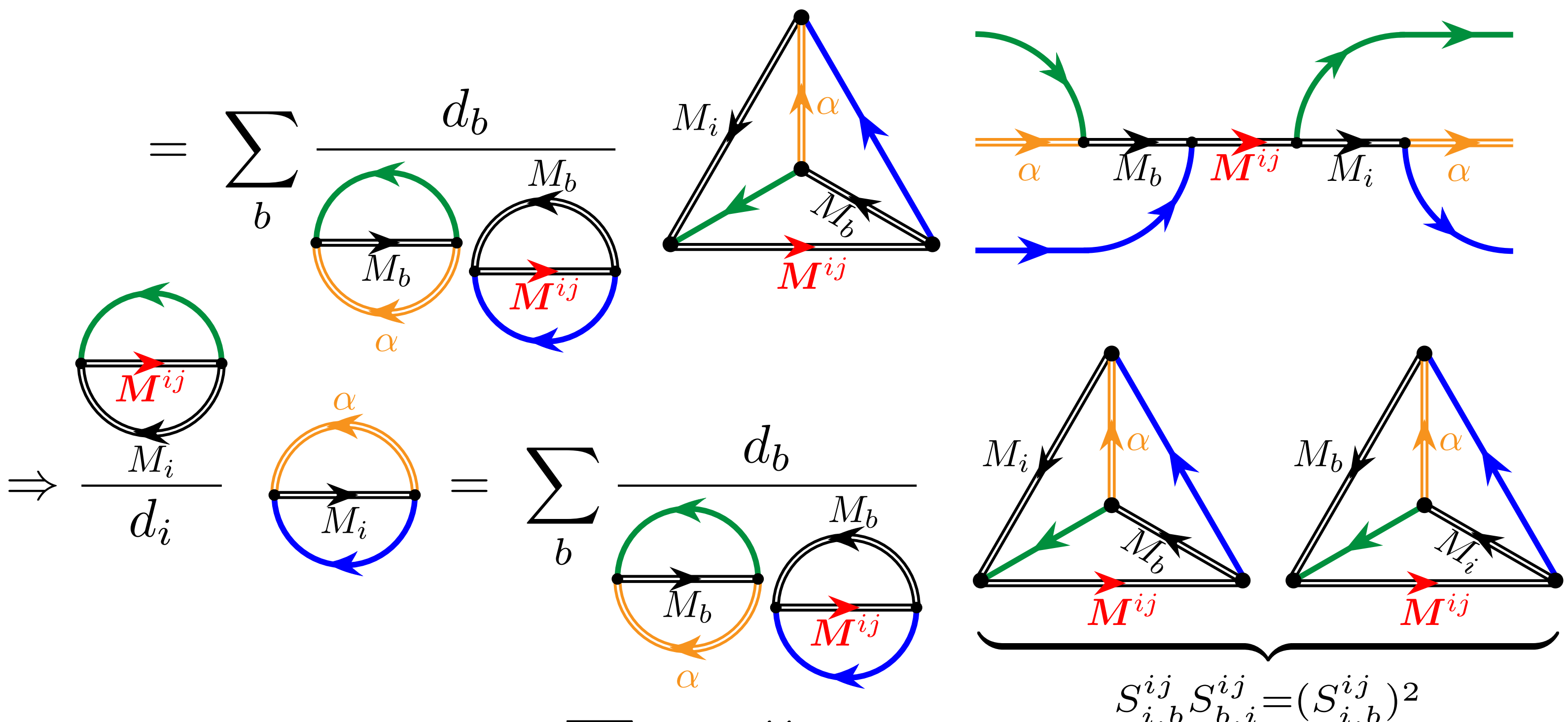


By repeated use of the completeness relation and the vertex correction relation (giving 6js), we can constrain the 6js. Consider for example



Now apply this to





$$\Rightarrow 1 = d_i \sum_b d_b (S^{ij}_{i,b})^2$$ (as 3js are 1)

By similar methods we find a set of equations, for $N_c = 3$

1. For a given representation M^{ij}, we obtain

$$1 = (d_i)^2 (S^{ij}_{i,i})^2 + d_i d_j (S^{ij}_{i,j})^2 \quad 0 = d_i S^{ij}_{i,i} S^{ij}_{i,j} + d_j S^{ij}_{i,j} S^{ij}_{j,j}$$

2. For two given representations M_i and M_j, we obtain

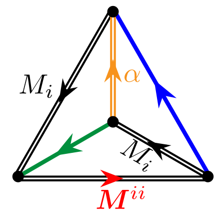
$$\frac{1}{d_\alpha} = \sum_{M^{ab}} d_{ab} (S^{ab}_{i,j})^2 \ ,$$

   where d_{ab} is the dimension of the representation M^{ab}.

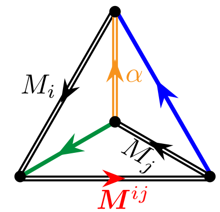
3. For a given representation M_i, we have

$$1 = \sum_b d_{ib} S^{ib}_{i,i} \ .$$

- This equation system can be solved giving

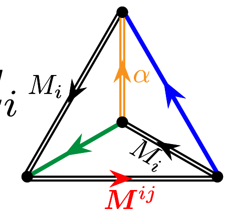


$$= \frac{1}{d_i} \,,$$

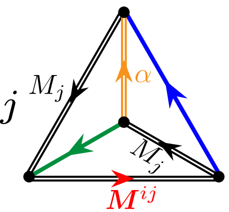


$$= \pm \frac{1}{\sqrt{d_\alpha d_{ij}}}$$

$$d_i$$

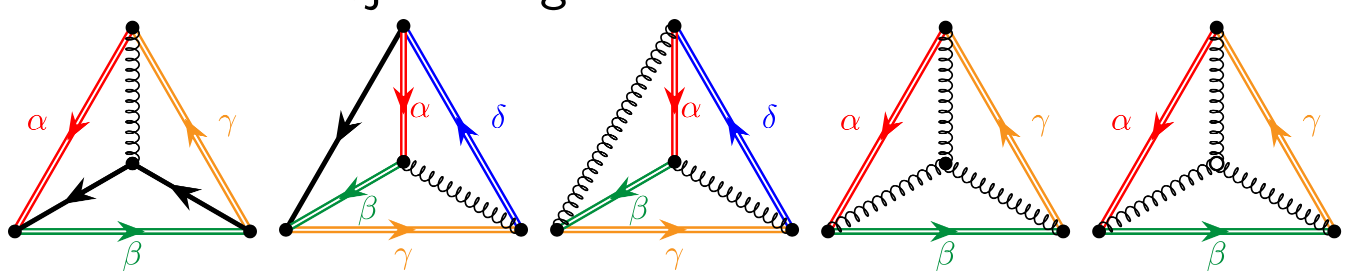


$$= \pm \sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}} = d_j$$

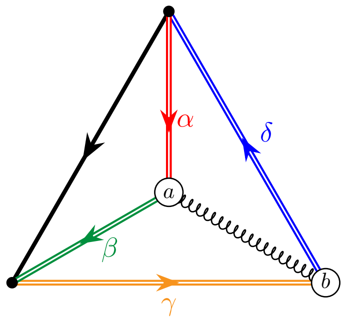


(Judith Alcock-Zeilinger, Stefan Keppeler, Simon Plätzer and MS, 2209.15013 (J. Math. Phy.))

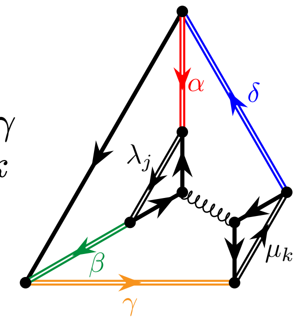
- Also need the 6js with gluons



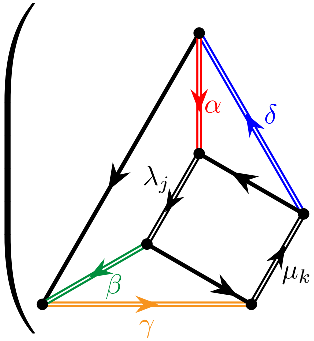
- Idea: split gluon into $q\overline{q}$-pair, for example we have



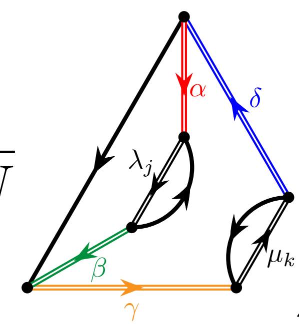
$$= \sum_{j=1}^{a} \sum_{k=1}^{b} C_{aj}^{\beta\alpha} C_{bk}^{\delta\gamma}$$



$$= \sum_{j=1}^{a} \sum_{k=1}^{b} \frac{C_{aj}^{\beta\alpha} C_{bk}^{\delta\gamma}}{N^2 - 1} \Bigg($$

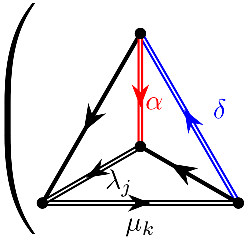


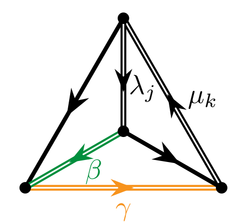
$$- \frac{1}{N}$$



$$\Bigg)$$

$$= \sum_{j=1}^{a} \sum_{k=1}^{b} \frac{C_{aj}^{\beta\alpha} C_{bk}^{\delta\gamma}}{N^2 - 1} \Bigg($$





$$- \frac{\delta_{\alpha\beta}\,\delta_{\gamma\delta}}{N d_\alpha\, d_\gamma} \Bigg)$$

- By similar methods the other 6js with gluons are derived
  2312.16688 (submitted to JHEP), Stefan Keppeler, Simon Plätzer and MS
- Not more complicated to calculate 6js for high representations
- Multiple occurrence is an issue... but can be addressed
- $\rightarrow$ We in principle have all the ingredients for using representation based orthogonal bases for QCD also for very high multiplicities

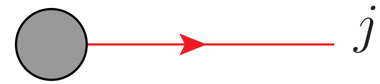
Lorentz structure using chirality flow

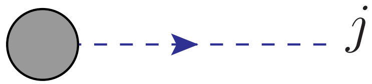
- At the (complexified) algebra level, the Lorentz group consists of two copies of $\mathfrak{su}(2)$, $\mathfrak{so}(3,1) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ $\rightarrow$ **Can we treat the Lorentz structure similarly?**
- The Dirac spinor structure transforms under the direct sum representation $\underbrace{(\frac{1}{2},0)}_{\text{left}} \oplus \underbrace{(0,\frac{1}{2})}_{\text{right}}$ in the chiral/Weyl basis

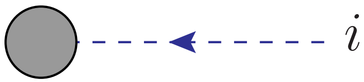
$$\begin{pmatrix} u_L \\ u_R \end{pmatrix} \rightarrow \begin{pmatrix} e^{-i\bar{\theta}\cdot\frac{\bar{\sigma}}{2}+\bar{\eta}\cdot\frac{\bar{\sigma}}{2}} & 0 \\ 0 & e^{-i\bar{\theta}\cdot\frac{\bar{\sigma}}{2}-\bar{\eta}\cdot\frac{\bar{\sigma}}{2}} \end{pmatrix} \begin{pmatrix} u_L \\ u_R \end{pmatrix}$$

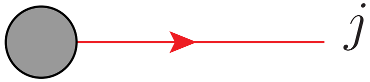
i.e. actually two copies of SL(2,$\mathbb{C}$), generated by the complexified $\mathfrak{su}(2)$ algebra

- Consider massless particles: (mass just gives a linear combination)

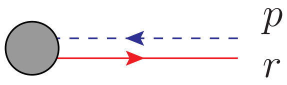
$$u^{R}(p_j) = \begin{pmatrix} 0 \\ |p_j\rangle \end{pmatrix} =$$ 

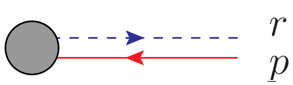
$$u^{L}(p_j) = \begin{pmatrix} |p_j] \\ 0 \end{pmatrix} =$$ 

$$\bar{u}^{L}(p_i) = \begin{pmatrix} [p_i| & 0 \end{pmatrix} =$$ 

$$\bar{u}^{R}(p_j) = \begin{pmatrix} 0 & \langle p_j| \end{pmatrix} =$$ 

- Polarization vectors

$$\epsilon_L{}^{\mu}(p,r) \to \frac{|r\rangle[p|}{\langle rp\rangle} = \frac{1}{\langle rp\rangle}$$ 

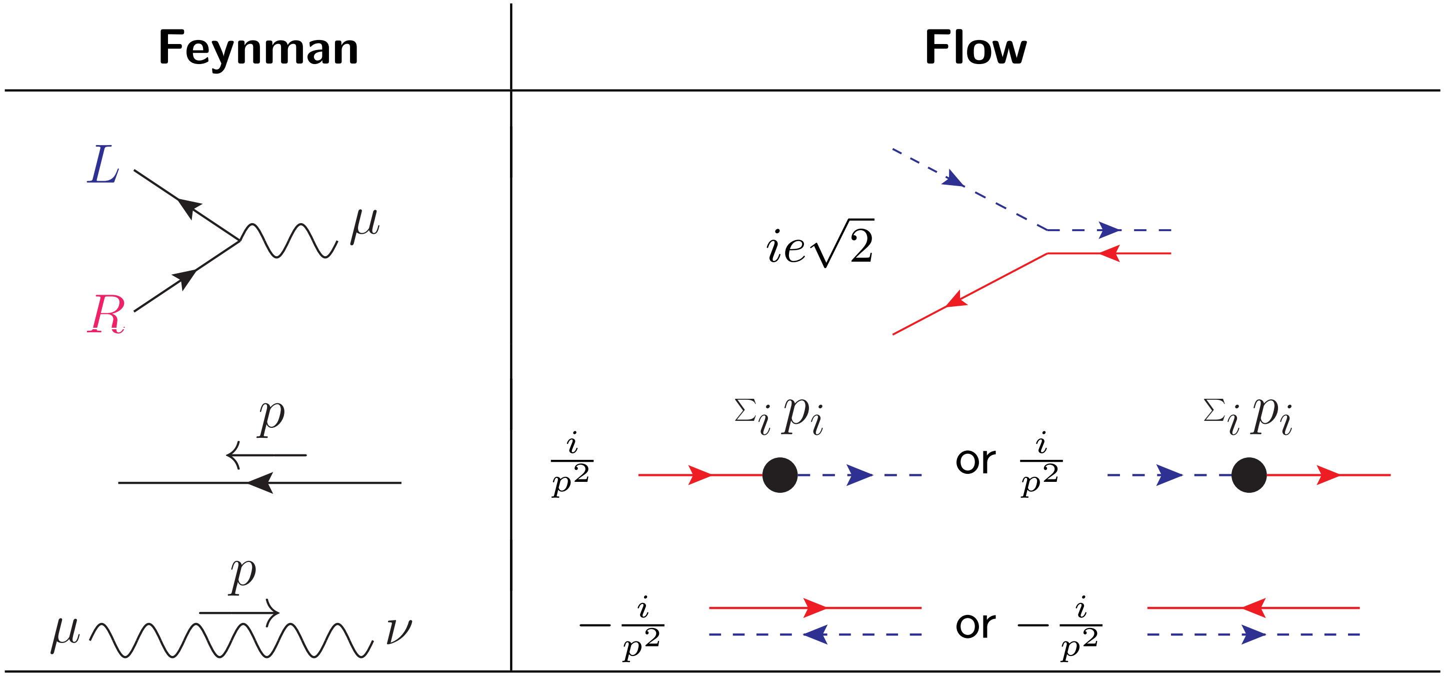
$$\epsilon_R{}^{\mu}(p,r) \to \frac{|r]\langle p|}{[pr]} = \frac{1}{[pr]}$$ 

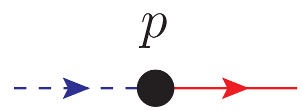
where ϵ_L is for incoming negative helicity or outgoing positive helicity and ϵ_R is for incoming positive helicity or outgoing negative helicity

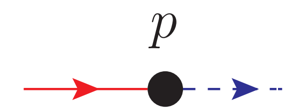
- Amplitudes built up from Lorentz invariant inner products
- Lorentz inner products formed using **the only SL(2,$\mathbb{C}$) invariant object** $\epsilon^{\alpha\beta}$, $\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12}$

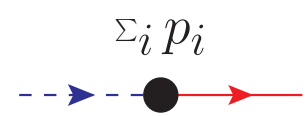
$$\underbrace{\epsilon^{\alpha\beta}|i\rangle_\beta}_{\equiv\langle i|^\alpha}|j\rangle_\alpha = \langle i|^\alpha|j\rangle_\alpha = \langle ij\rangle, \quad \underbrace{\epsilon_{\dot\alpha\dot\beta}|i]^{\dot\beta}}_{\equiv[i|_{\dot\alpha}}|j]^{\dot\alpha} = [i|_{\dot\alpha}|j]^{\dot\alpha} = [ij] \ ,$$

- $\rightarrow$ Amplitudes are built up of contractions of form $\langle ij\rangle, [ij] \sim \sqrt{s_{ij}}$
- In the flow picture, the "flow" must contract dotted and undotted indices separately



Here $\not{p} \equiv p^{\mu}\sigma_{\mu}^{\dot{\alpha}\beta} =$ 

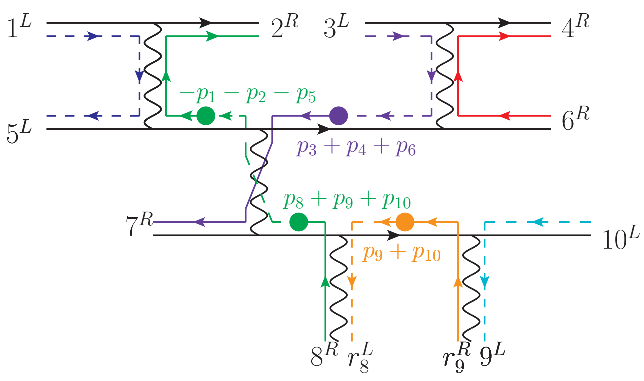
, $\bar{\not{p}} \equiv p_{\mu}\bar{\sigma}^{\mu}_{\alpha\dot{\beta}} =$ 

with $p = \sum p_i$, $p_i^2 = 0$, $\not{p} =$ 

$= \sum_i |i]\langle i|$ etc.

(A. Lifson, C. Reuschle and MS 2003.05877 (EPJC), full SM: J. Alnefjord, A. Lifson, C. Reuschle and MS 2011.10075 (EPJC))

Example: Lorentz structure using chirality flow



- Stitch together such that arrow direction match.
- Here all particles crossed to outgoing, and consistent arrow directions are picked

$$= \underbrace{(\sqrt{2}ei)^8}_{\text{vertices}} \underbrace{\frac{(-i)^3}{s_{1\,2}\ s_{3\,4}\ s_{8\,9\,10}}}_{\text{photon propagators}} \underbrace{\frac{(i)^4}{s_{1\,2\,5}\ s_{3\,4\,6}\ s_{8\,9\,10}\ s_{9\,10}}}_{\text{fermion propagators}} \underbrace{\frac{1}{[8r_8]\langle r_9 9\rangle}}_{\text{polarization vectors}} \quad [15]\langle 64\rangle[10\ 9]$$

$$\times \quad (\langle r_9 9\rangle[9 r_8] + \langle r_9 10\rangle[10 r_8])(\underbrace{[33]}_{0}\langle 37\rangle + [34]\langle 47\rangle + [36]\langle 67\rangle)$$

$$\times \quad (-\langle 89\rangle[91]\langle 12\rangle - \langle 89\rangle[95]\langle 52\rangle - \langle 8\,10\rangle[10\ 1]\langle 12\rangle - \langle 8\,10\rangle[10\ 5]\langle 52\rangle)$$

Conclusion and outlook

Color:

- In this presentation, I have shown how the color structure can be dealt with using 6j coefficients
- We only need a limited number of 6js which we know how to calculate efficiently
- Looking forward to applying them for heavy QCD calculations
- Still... one might benefit from more general 6js, with arbitrary representations everywhere. Can one find them similarly?

Chirality flow:

- We can flow the Lorentz structure as well ... which often makes it trivial to write down the value of Feynman diagrams!