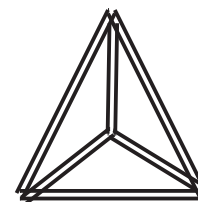




LUND
UNIVERSITY



All you need is



... and a little bit of chirality
flow

Thanks to my collaborators:

Color: Judith Alcock-Zeilinger, Stefan Keppeler,
Simon Plätzer, Johan Thorén

Chirality flow: Joakim Alnefjord, Andrew Lifson,
Christian Reuschle, Simon Plätzer, Adam
Warnerbring, Zenny Wettersten

Online
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Malin Sjö Dahl

This talk

- My motivation: With the LHC there is an increased interest in the treatment of color structure for processes with **many colored partons**
- This is applicable to **fixed order calculations** as well as **parton showers** and **resummation**
- Color structure of $SU(N)$, in particular **multiplet bases** (transition operators of Heribert Weigert) – a pedagogical intro
- Calculating using basic group invariants, **Wigner 6js** (also known as 6j coefficients, 6j symbols, Racah coefficients, Racah W coefficients) and **Wigner 3js**
- I will talk about QCD ($SU(N_c)$), but similar methods can be applied more generally
- **Chirality flow** – flowing the Lorentz structure $\mathfrak{su}(2)_L \mathfrak{su}(2)_R$

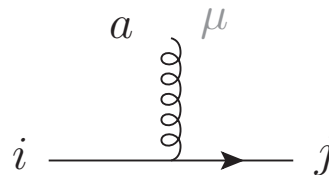


The QCD Lagrangian

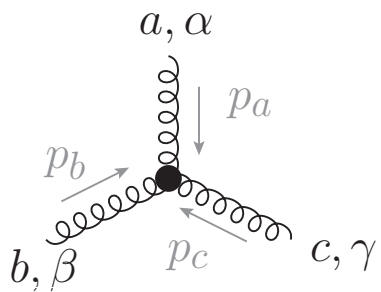
The QCD Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}(\partial_{\mu}A_{\nu}^a - \partial_{\nu}A_{\mu}^a)^2 + gA_{\mu}^a\bar{\psi}\gamma^{\mu}t^a\psi - g f^{abc}(\partial_{\mu}A_{\nu}^a)A^{\mu b}A^{\nu c} - \frac{1}{4}g^2(f^{eab}A_{\mu}^aA_{\nu}^b)(f^{ecd}A^{\mu c}A^{\nu d})$$

contains:

- quark-gluon vertex, $i \xrightarrow{\quad} j$  $= (t^a)^i_j$
Here $(t^a)^i_j$ are SU(3) generators and I take the graph to represent the color structure alone, no $ig\gamma^{\mu}$



- triple-gluon vertex,  $= i f^{abc}$

Here we use the convention of reading the indices counter clockwise in the SU(3) structure constants f^{abc} , and again I only mean the color structure, no $-ig(g^{\alpha\beta}(p_a - p_b)^\gamma + \text{cyclic})$

- four-gluon vertex, here color and kinematic factors are correlated (so I cannot draw the color structure alone)

$$\begin{aligned}
 & \begin{array}{c} a, \alpha \quad b, \beta \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ c, \gamma \quad d, \delta \end{array} = \begin{array}{c} \text{Diagram 1} \\ \times ig_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta}) \end{array} + \begin{array}{c} \text{Diagram 2} \\ \times ig_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\beta} g^{\gamma\delta}) \end{array} + \begin{array}{c} \text{Diagram 3} \\ \times ig_s^2 (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta}) \end{array} \\
 & = \begin{array}{c} i f^{aeb} \quad i f^{cde} \\ \times ig_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta}) \end{array} + \begin{array}{c} i f^{ace} \quad i f^{bed} \\ \times ig_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\beta} g^{\gamma\delta}) \end{array} + \begin{array}{c} i f^{aed} \quad i f^{cbe} \\ \times ig_s^2 (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta}) \end{array}
 \end{aligned}$$

but the color structure is just a linear combination of triple-gluon vertices

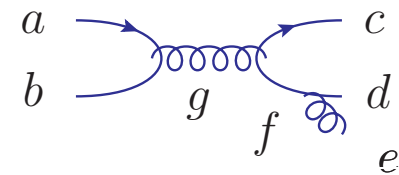


Dealing with color space

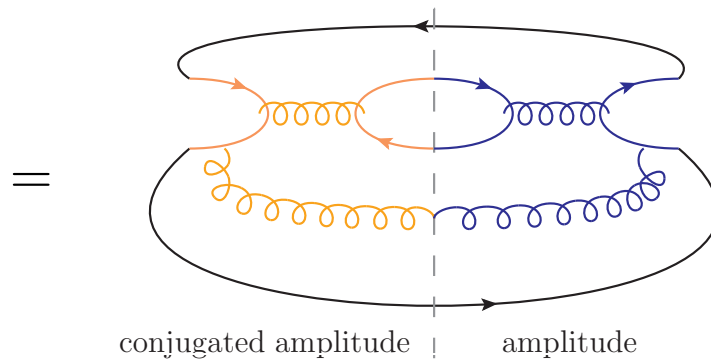
Due to confinement we never observe individual colors

- We average over incoming colors
- We sum over outgoing colors
- → we sum over the colors of all external partons
- As always in quantum mechanics we also sum over all degrees of freedom that can interfere with each other → we sum over the colors of all internal particles
- → We sum over **all** colors of all particles



Example: If $A = (t^g)^a_b (t^g)^f_c (t^e)^d_f =$  , then

$$\begin{aligned} \langle A|A \rangle &= \sum_{a,b,c,d,e,f,g,h,i} [(t^h)^a_b (t^h)^i_c (t^e)^d_i]^* (t^g)^a_b (t^g)^f_c (t^e)^d_f \\ &= \sum_{a,b,c,d,e,f,g,h,i} (t^h)^b_a (t^h)^c_i (t^e)^i_d (t^g)^a_b (t^g)^f_c (t^e)^d_f \end{aligned}$$



The first equality holds since the generators are Hermitian, and the last holds since we always sum over the color of internal lines



As seen above we can represent the squared amplitude with a picture. We can also calculate with graphs! To do so we need just a few rules

- There are N_c possible quark colors

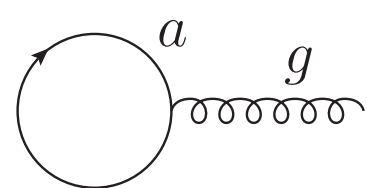
$$\begin{array}{c} \circlearrowleft \\ \text{a} \end{array} = N_c \quad \sum_{a=1}^{N_c} \delta^a_a = N_c$$

- There are $N_g = N_c^2 - 1$ possible gluon colors

$$\begin{array}{c} \circ \\ \text{g} \end{array} = N_c^2 - 1 \quad \sum_{g=1}^{N_c^2-1} \delta^{gg} = N_c^2 - 1$$



- The generators are traceless



$$= 0 \quad \sum_{a=1}^{N_c} (t^g)^a_a = 0$$

- Generator normalization



$$= T_R \text{a} \text{b}$$

$$\text{Tr}[t^a t^b] = T_R \delta^{ab}$$



- The algebra relation $[t^a, t^b] = if^{abc}t^c \Rightarrow$

$$\begin{aligned}
 \begin{array}{c} a \\ \uparrow \\ \bullet \\ \swarrow \quad \searrow \\ b \quad c \end{array} &= \frac{1}{T_R} \left(\begin{array}{c} a \\ \uparrow \\ \circlearrowleft \\ \swarrow \quad \searrow \\ b \quad c \end{array} - \begin{array}{c} a \\ \uparrow \\ \circlearrowright \\ \swarrow \quad \searrow \\ b \quad c \end{array} \right) \\
 if^{abc} &= \frac{1}{T_R} [\text{Tr}[t^a t^b t^c] - \text{Tr}[t^b t^a t^c]]
 \end{aligned}$$

- The Fierz identity (the completeness relation)

$$\begin{aligned}
 \begin{array}{c} a \quad c \\ \longrightarrow \quad \longrightarrow \\ \uparrow \quad \downarrow \\ b \quad d \\ \longrightarrow \quad \longrightarrow \end{array} &= T_R \left(\begin{array}{c} a \quad c \\ \longrightarrow \quad \longrightarrow \\ \searrow \quad \swarrow \\ b \quad d \\ \longrightarrow \quad \longrightarrow \end{array} - \frac{1}{N_c} \begin{array}{c} a \quad c \\ \longrightarrow \quad \longrightarrow \\ b \quad d \\ \longrightarrow \quad \longrightarrow \end{array} \right) \\
 (t^g)^a_c (t^g)^b_d &= T_R \left[\delta^a_d \delta^b_c - \frac{1}{N_c} \delta^a_c \delta^b_d \right]
 \end{aligned}$$



Let's apply the rules to our example

$$= T_R$$

To further simplify the color structure we note using Fierz

$$= T_R \left(\text{gluon loop} - \frac{1}{N_c} \text{ghost loop} \right) = T_R \left(N_c - \frac{1}{N_c} \right) \longrightarrow$$

$$= T_R \frac{N_c^2 - 1}{N_c} \longrightarrow \equiv C_F \longrightarrow$$

Giving, for the squared amplitude

$$= T_R C_F^2 \text{ (gluon loop)} = T_R C_F^2 N_c$$



- In this way we can square any color amplitude and calculate any interference term. In general we have interference terms between different Feynman diagrams/color structures, but these are treated in precisely the same way.
- I have written a Mathematica package, [ColorMath](#), (Eur. Phys. J. C 73:2310 (2013), 1211.2099)
- One way of dealing with color space is to just square the amplitudes one by one as one encounters them
- [Alternatively, we may use any basis](#) (spanning set)



Trace bases

- Every 4g vertex can be replaced by 3g vertices:

The diagram shows a four-gluon vertex with external lines labeled a, α , b, β , c, γ , and d, δ . This is equal to the sum of three three-gluon vertices:

- 1. A vertex with lines a, α and c, γ meeting at a central point, with a gluon line connecting to lines b, β and d, δ . Multiplied by $ig_s^2(g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\gamma}g^{\beta\delta})$.
- 2. A vertex with lines a, α and b, β meeting at a central point, with a gluon line connecting to lines c, γ and d, δ . Multiplied by $ig_s^2(g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\beta}g^{\gamma\delta})$.
- 3. A vertex with lines b, β and d, δ meeting at a central point, with a gluon line connecting to lines a, α and c, γ . Multiplied by $ig_s^2(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta})$.

- Every 3g vertex can be replaced using:

The diagram shows a three-gluon vertex with external lines a , b , and c . This is equal to $\frac{1}{T_R}$ times the difference of two loop diagrams:

- 1. A loop diagram with external lines a , b , and c , and a clockwise loop.
- 2. A loop diagram with external lines a , b , and c , and a counter-clockwise loop.

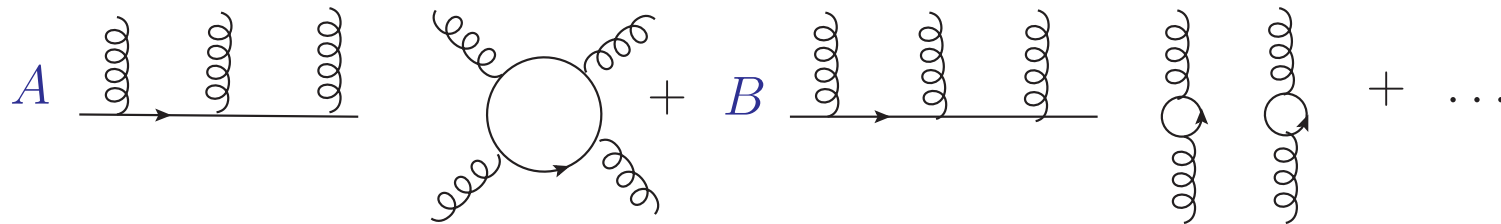
- After this every internal gluon can be removed using Fierz:

The diagram shows two quark lines, $a \rightarrow c$ and $b \rightarrow d$, with a gluon exchange between them. This is equal to T_R times the difference of two diagrams:

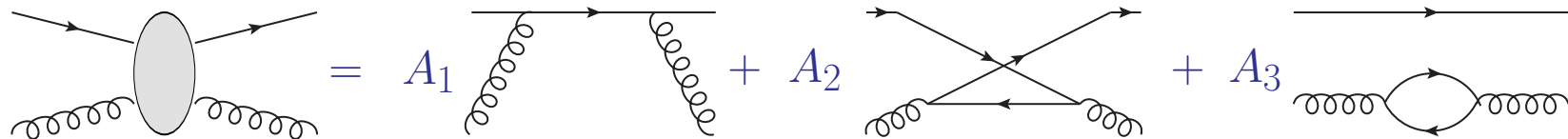
- 1. A diagram where the quark lines cross, representing a Fierz transformation.
- 2. A diagram where the quark lines do not cross, representing the identity $\frac{1}{N_c}$.



- This can be applied to any QCD amplitude, tree level or beyond
- In general an amplitude can be written as linear combination of different color structures, like



- For example for 2 (incoming + outgoing) gluons and one $q\bar{q}$ pair



(an incoming quark is the same as an outgoing anti-quark)

- The above type of color structure can be used as a spanning set, a trace basis



These bases have some nice properties

- Conceptual simplicity
- Taking the leading N_c limit is trivial \rightarrow a flow of colors and orthogonal basis vectors
- The effect of gluon emission and exchange is easily described

There are also drawbacks with trace bases

- Not orthogonal
 \rightarrow When squaring amplitudes almost all cross terms have to be taken into account $\rightarrow N_{\text{basis}}^2$ terms
- Overcomplete, for $N_g + N_{q\bar{q}} > N_c$ the bases are also overcomplete. The size of the vector space asymptotically grows as an exponential in the number of gluons/ $q\bar{q}$ -pairs for finite N_c



Example: Number of spanning vectors for N_g gluons (without imposing charge conjugation invariance). These numbers are representative also for N_g gluons plus $q\bar{q}$ -pairs.

N_g	Vectors $N_c = 3$	Vectors $N_c \rightarrow \infty$	LO Vectors $N_c \rightarrow \infty$
4	8	9	$3!=6$
5	32	44	$4!=24$
6	145	265	120
7	702	1 854	720
8	3 598	14 833	5 040
9	19 280	133 496	40 320
10	107 160	1 334 961	362 880
11	614 000	14 684 570	3 628 800
12	3 609 760	176 214 841	39 916 800

(Y. Du, M.S. & J. Thorén, JHEP 1505 (2015) 119, 1503.00530)



Color flow bases

- One way out is to give up exact treatment of color structure and run a [Monte Carlo over colors](#)
- This is particularly efficient in the [color flow basis](#)
- Here the adjoint representation indices are rewritten in terms of fundamental representation indices and new color flow Feynman rules are derived (Maltoni, Stelzer, Paul, Willenbrock, Phys.Rev. D67 (2003), hep-ph/0209271)
- Explicit colors (r, g, or b) are then assigned to the lines, and one may run a Monte Carlo sum over colors to sample color space
- This is not exact but the color structure treatment is much quicker (Comix, T. Gleisberg, S. Hoeche, JHEP 0812 (2008) 039, 0808.3674; S. Plätzer, Eur.Phys.J. C74 (2014) 6, 2907, 1312.2448; S. Prestel and J. Isaacson 1806.10102)



- quark-gluon vertex,

$$\begin{array}{c} a \\ \uparrow \\ \text{---} \mu \\ \downarrow \\ i \text{---} j \end{array} = ig_s \gamma^\mu (t^a)^i_j \rightarrow ig_s \gamma^\mu \delta^i_{a_2} \delta^{a_1}_j = \begin{array}{c} a_2 \quad a_1 \\ \uparrow \quad \downarrow \\ \mu \\ i \text{---} j \end{array}$$

- triple-gluon vertex,

$$\begin{array}{c} a, \alpha \\ \uparrow \\ p_a \\ \downarrow \\ b, \beta \quad c, \gamma \\ \swarrow \quad \searrow \\ p_b \quad p_c \end{array} = if^{abc} (-ig_s (g^{\alpha\beta} (p_a - p_b)^\gamma + \text{cyclic}))$$

$$\rightarrow \frac{1}{T_R} \left(\begin{array}{c} a_1 \quad a_2 \\ \uparrow \quad \uparrow \\ b_2 \quad b_1 \quad c_1 \quad c_2 \end{array} - \begin{array}{c} a_1 \quad a_2 \\ \uparrow \quad \downarrow \\ b_2 \quad b_1 \quad c_1 \quad c_2 \end{array} \right) (-ig_s (g^{\alpha\beta} (p_a - p_b)^\gamma + \text{cyclic}))$$

can easily be written in completely symmetric form...



- four-gluon vertex

$$\begin{aligned}
 & \begin{array}{c} a, \alpha \\ \text{wavy line} \\ b, \beta \\ \text{wavy line} \\ c, \gamma \\ \text{wavy line} \\ d, \delta \\ \text{wavy line} \end{array} = \begin{array}{c} \text{Diagram 1} \\ \times ig_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta}) \end{array} + \begin{array}{c} \text{Diagram 2} \\ \times ig_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\beta} g^{\gamma\delta}) \end{array} + \begin{array}{c} \text{Diagram 3} \\ \times ig_s^2 (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta}) \end{array} \\
 & = \begin{array}{c} i f^{aeb} i f^{cde} \\ \times ig_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta}) \end{array} + \begin{array}{c} i f^{ace} i f^{bed} \\ \times ig_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\beta} g^{\gamma\delta}) \end{array} + \begin{array}{c} i f^{aed} i f^{cbe} \\ \times ig_s^2 (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta}) \end{array}
 \end{aligned}$$

→

$$\begin{aligned}
 & ig_s^2 (2g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\beta} g^{\gamma\delta}) \frac{1}{T_R} \left(\begin{array}{c} a_2 \quad b_1 b_2 \\ \text{Diagram 4} \\ c_2 c_1 \quad d_1 \quad d_2 \end{array} + \begin{array}{c} a_2 \quad b_1 b_2 \\ \text{Diagram 5} \\ c_2 c_1 \quad d_1 \quad d_2 \end{array} \right) \\
 & + [c \leftrightarrow d] + [b \leftrightarrow d]
 \end{aligned}$$



- Color structure of propagator

$$\Delta^{ab} = a \text{---} b$$

$$\rightarrow \begin{array}{c} a_1 \\ a_2 \end{array} \text{---} \text{---} \begin{array}{c} b_2 \\ b_1 \end{array} = T_R \left(\begin{array}{c} a_1 \text{---} b_2 \\ a_2 \text{---} b_1 \end{array} - \frac{1}{N_c} \begin{array}{c} a_1 \text{---} \\ a_2 \text{---} \end{array} \begin{array}{c} \text{---} b_2 \\ \text{---} b_1 \end{array} \right)$$

- Similarly the $q\bar{q}$ -pairs corresponding to external gluons have to be forced to be in octets when squaring amplitudes (Conventions differ from those in hep-ph/0209271)
- ... but these bases are not orthogonal



Wanted: Orthogonal bases

How can we construct an orthogonal basis? Symmetrize!

$$\begin{aligned}
 v_1 &= \frac{1}{2} \left[\begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \begin{array}{c} \nearrow \\ \searrow \end{array} \right] = \begin{array}{c} \rightarrow \\ \rightarrow \end{array} \text{ (with a white vertical bar) } = \begin{array}{c} \diagdown \\ \diagup \end{array} \text{ (with a white horizontal bar and '6' below) } \\
 v_2 &= \frac{1}{2} \left[\begin{array}{c} \rightarrow \\ \rightarrow \end{array} - \begin{array}{c} \nearrow \\ \searrow \end{array} \right] = \begin{array}{c} \rightarrow \\ \rightarrow \end{array} \text{ (with a black vertical bar) } = \begin{array}{c} \diagdown \\ \diagup \end{array} \text{ (with a black horizontal bar and '3' above) } \\
 \square \otimes \square &= \begin{array}{c} \square \square \\ \square \square \end{array} \oplus \begin{array}{c} \square \\ \square \end{array}
 \end{aligned}$$

Here the [birdtrack notation](#) is used. These color tensors are orthogonal both when seen as qq -projectors, and when seen as basis vectors on the 4-parton space



Orthogonal multiplet bases

In collaboration with Stefan Keppeler and Johan Thorén

- The color space may be decomposed into irreducible representations, enumerated using Young tableaux multiplication
- For quarks we can construct orthogonal projectors and basis vectors using Young tableaux ...at least from the Hermitian quark projectors (S.K. and MS, 1307.6147, J.Math.Phys.)
- In fact the $qq \rightarrow qq$ color space is the same as for $q\bar{q} \rightarrow q\bar{q}$,

$$\square \otimes \bar{\square} = \bullet \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$$

and we could as well have used the basis:

$$\mathbf{V}^1 = \delta^a_b \delta^c_d = \begin{array}{c} a \\ \curvearrowright \\ b \end{array} \begin{array}{c} \curvearrowleft \\ c \\ d \end{array}, \quad \mathbf{V}^8 = (t^g)^a_b (t^g)^c_d = \begin{array}{c} a \\ \curvearrowright \\ b \end{array} \begin{array}{c} \text{gluon} \\ c \\ d \end{array}$$

- In general we may “comb” the involved particles as incoming and outgoing as we wish



The simplest gluon example, $gg \rightarrow gg$

- In QCD we have quarks, anti-quarks and gluons
 → No obvious way to construct projectors
- Basis vectors can be enumerated using Young tableaux multiplication

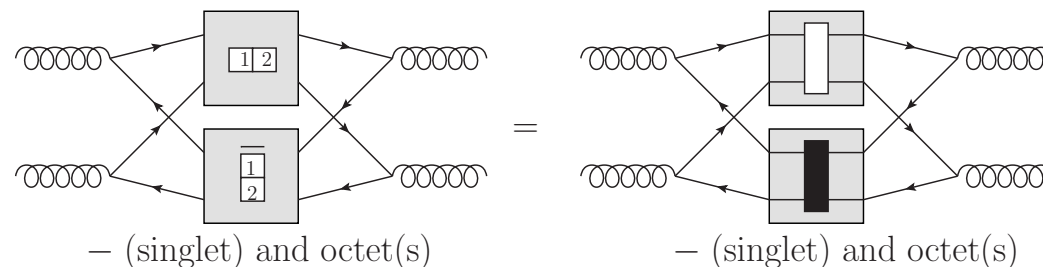
$$\begin{array}{cccccccccccc}
 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} = & \bullet & \oplus & \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} & \oplus & \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} & \oplus & \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} & \oplus & \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} & \oplus & \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} & \oplus & \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} & \oplus & 0 & \\
 & 1 & & 8 & & 8 & & 10 & & \bar{10} & & 27 & & 0 & \\
 & & & & & & & & & & & & & & 0 &
 \end{array}$$

- As color is conserved an incoming multiplet of a certain kind can only go to an outgoing multiplet of the same kind,
 $1 \rightarrow 1, 8 \rightarrow 8 \dots$

Charge conjugation implies that some vectors only occur together... (MacFarlane, Sudbery, and Weisz 1968, Butera, Cicuta and Enriotti 1979, Cvitanović 1984, Dokshitzer and Marchesini 2006)



- For two gluons, there are two octet projectors, one singlet projector, and 4 new projectors, $10, \overline{10}, 27$, and for general N_c , “0”
- It turns out that the new projectors can be seen as corresponding to different symmetries w.r.t. quark and anti-quark units, for example the decuplet can be seen as corresponding to



Similarly the anti-decuplet corresponds to $\frac{1}{2} \otimes \overline{12}$, the 27-plet corresponds to $\overline{12} \otimes \overline{12}$ and the 0-plet to $\frac{1}{2} \otimes \overline{12}$. (MacFarlane, Sudbery, and Weisz 1968, Butera, Cicuta and Enriotti 1979, Cvitanović 1984, Dokshitzer and Marchesini 2006)



$$\mathbf{P}^1 = \frac{1}{N_c^2 - 1} \text{ (two gluon loops) }, \quad \mathbf{P}^{8s} = \frac{N_c}{2T_R(N_c^2 - 4)} \text{ (gluon exchange) }, \quad \mathbf{P}^{8a} = \frac{1}{2N_c T_R} \text{ (gluon exchange with dots) },$$

$$\mathbf{P}^{10} = \frac{1}{2} \text{ (gluon exchange with black bar) } + \frac{1}{2T_R^2} \text{ (gluon exchange with black bar and loop) } - \frac{1}{2} \mathbf{P}^{8a}$$

$$\mathbf{P}^{\bar{10}} = \frac{1}{2} \text{ (gluon exchange with black bar) } - \frac{1}{2T_R^2} \text{ (gluon exchange with black bar and loop) } - \frac{1}{2} \mathbf{P}^{8a}$$

$$\mathbf{P}^{27} = \frac{1}{2} \text{ (gluon exchange with white bar) } + \frac{1}{2T_R^2} \text{ (gluon exchange with white bar and loop) } - \frac{N_c - 2}{2N_c} \mathbf{P}^{8s} - \frac{N_c - 1}{2N_c} \mathbf{P}^1$$

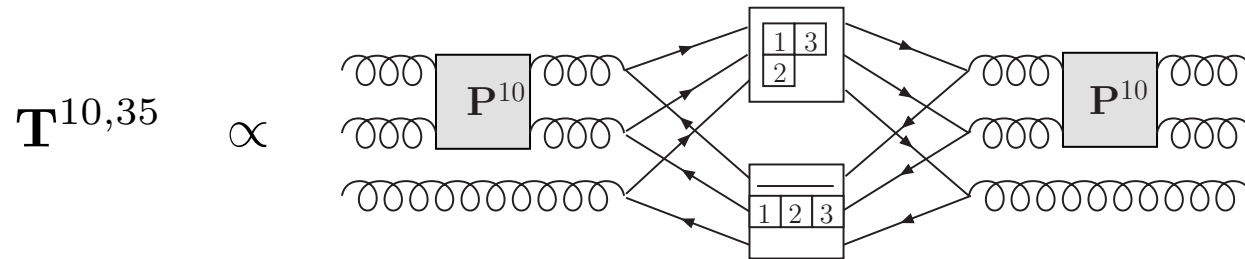
$$\mathbf{P}^0 = \frac{1}{2} \text{ (gluon exchange with white bar) } - \frac{1}{2T_R^2} \text{ (gluon exchange with white bar and loop) } - \frac{N_c + 2}{2N_c} \mathbf{P}^{8s} - \frac{N_c + 1}{2N_c} \mathbf{P}^1$$



Idea: Could this work in general?

$$g_1 \otimes g_2 \otimes \dots \otimes g_n \subseteq (q_1 \otimes \bar{q}_1) \otimes (q_2 \otimes \bar{q}_2) \otimes \dots \otimes (q_n \otimes \bar{q}_n)$$

- Construct the tensors which will give rise to “new” projectors



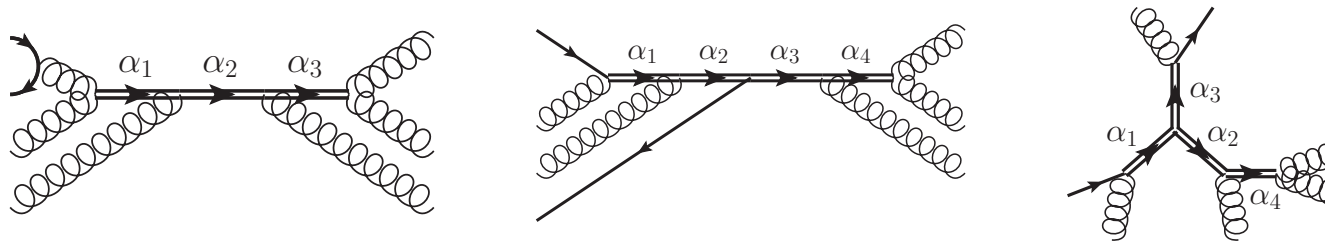
$$\mathbf{P}^{10,35} \propto \mathbf{T}^{10,35} - \sum_{m \subseteq 10 \otimes 8} \mathbf{P}^m \mathbf{T}^{10,35}$$

- From projectors construct basis vectors (S. Keppeler and M.S. 1207.0609 (JHEP))
- Care to find all multiplets, care with going from general N_c to $N_c = 3$, issues with multiple occurrence



Multiplet bases

- QCD is based on $SU(3)$ \rightarrow the color space may be decomposed into irreducible representations
- Orthogonal basis vectors corresponding to irreducible representations may be constructed, in many different ways...



- The construction of the corresponding basis vectors is non-trivial, and a general strategy was presented relatively recently, (S. Keppeler and M.S. JHEP09(2012)124, 1207.0609 generalized by MS and J.Thorén in 1809.05002)
- These vectors are **orthogonal** by construction \rightarrow can potentially speed up squaring of color structure very significantly



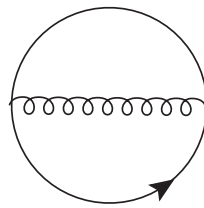
Decomposing color structure in multiplet bases

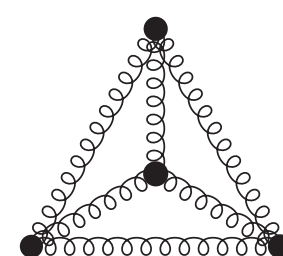
- But before squaring, amplitudes must be decomposed in multiplet bases
- One way of decomposing color structure into multiplet bases would be to simply **evaluate the scalar product** between each possible Feynman diagram and each possible vector as we have seen in the first half of this talk.
- The problem is that this **scales badly**, a factorial from the number of diagrams, an exponential from the number of basis vectors and another (growing) factor from each single scalar product evaluation
- → **We need a better strategy**



Group invariants!

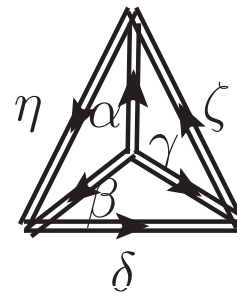
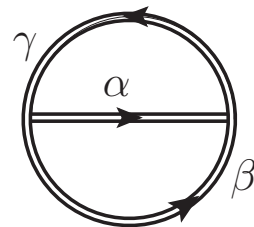
- Fortunately there is one: Any group invariant quantity can be evaluated using Wigner 3j and 6j coefficients



$$= T_R(N_c^2 - 1)$$


$$= 2 T_R^2 N_c^2 (N_c^2 - 1)$$

(using standard normalization of vertices)



- Using the multiplet basis we can calculate the needed 3j and 6j coefficients for higher representations



- Furthermore, only a small number of such coefficients are needed, up to NLO

N_g	4	6	8	10	12
$N_c = 3$	29	120	272	476	733
$N_c \geq N_g$	44	389	2 023	8 077	27 631

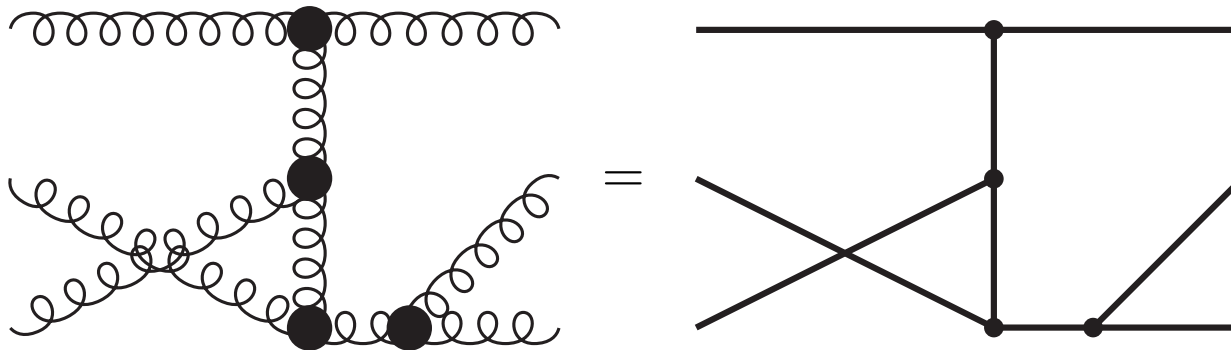
and they can be evaluated once and for all

(Numbers could be slightly reduced by additional symmetries, and smart choices of vertices)



Decomposing color with 6_j and 3_j coefficients

As an example consider the color structure of the Feynman diagram:



The scalar product between the color structure and a basis vector is given by:

$$A(\alpha_1, \alpha_2, \alpha_3) = \text{[Diagram of a loop with internal lines and arrows labeled } \alpha_1, \alpha_2, \alpha_3 \text{]} = \text{[Diagram of a planar graph with vertices and edges labeled } \alpha_1, \alpha_2, \alpha_3 \text{]} =$$



To fully contract any color structure we need four simple rules:

- Dimension relation

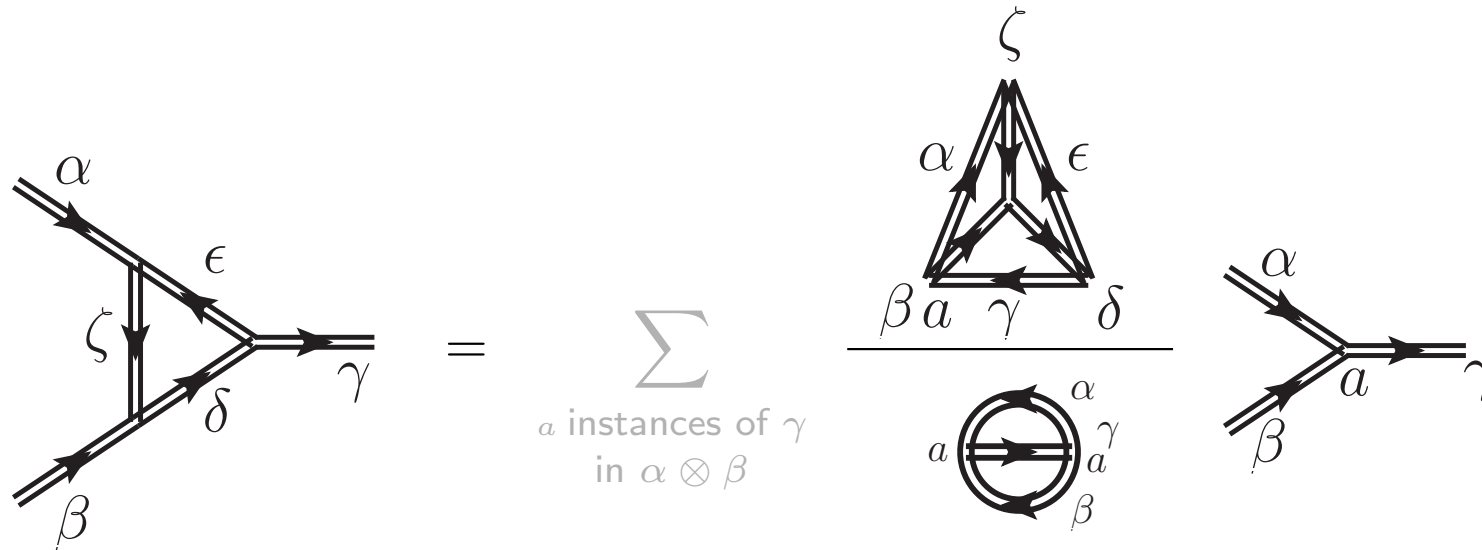
$$\text{loop}_\alpha = d_\alpha$$

- Two vertex loops give just a constant

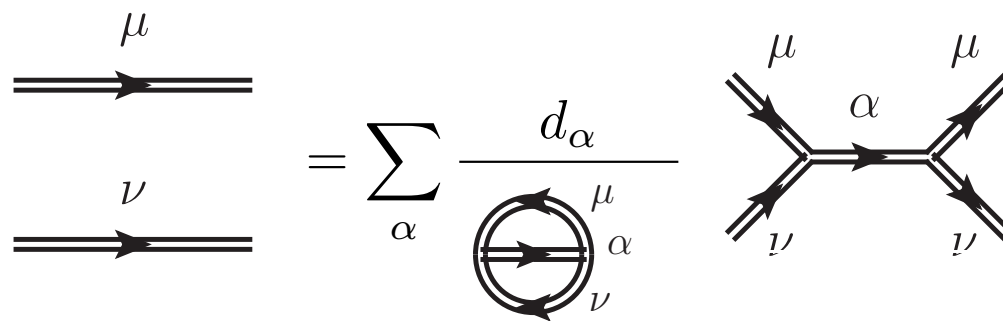
$$\text{loop}_\beta \text{ loop}_\gamma \text{ with external lines } \alpha, \delta = \frac{d_\beta d_\gamma}{d_\alpha} \text{ loop}_\alpha \text{ with external lines } \alpha, \delta$$



- The vertex correction relation



- The completeness relation



In our color structure we note that we have a vertex correction:

$$A(\alpha_1, \alpha_2, \alpha_3) = \text{Diagram}$$

In our case the vertex correction is:

$$\text{Diagram} = \sum_a \frac{\text{Diagram}_1}{\text{Diagram}_2} = \text{Diagram}_3$$

Where the sum runs over vertices a connecting the three representations α_2 , α_3 and 8 . For $\alpha_2 \neq \alpha_3$ there is only one such vertex, and for $\alpha_2 = \alpha_3$, there can be up to $N_c - 1 = 2$.



Using the vertex correction results in:

$$\begin{aligned}
 A(\alpha_1, \alpha_2, \alpha_3) &= \text{Diagram 1} \\
 &= \sum_a \frac{\text{Diagram 2}}{\text{Diagram 3}} \text{Diagram 4}
 \end{aligned}$$

Diagram 1: A complex diagram with vertices and lines. A red triangle is highlighted on the left, with vertices labeled α_3 and α_2 . A double line at the bottom is labeled α_2 . A double line on the right is labeled α_1 .

Diagram 2: A triangle with a red line from the top vertex to the bottom edge. The bottom edge is labeled α_2 and α_3 . The left side is labeled a .

Diagram 3: A circle with a horizontal line through the center. The line is labeled a on both sides and α_2 in the middle.

Diagram 4: A diagram similar to Diagram 1, but with a single line at the bottom labeled α_2 and a double line on the right labeled α_1 . The left side is labeled a .



Now there is no trivial color structure, but we can pick any loop...

$$A(\alpha_1, \alpha_2, \alpha_3) = \sum_a \frac{\begin{array}{c} a \\ \alpha_3 \\ \alpha_2 \\ a \end{array}}{\begin{array}{c} a \\ \alpha_2 \\ a \end{array}} \begin{array}{c} a \\ \alpha_1 \\ a \end{array}$$

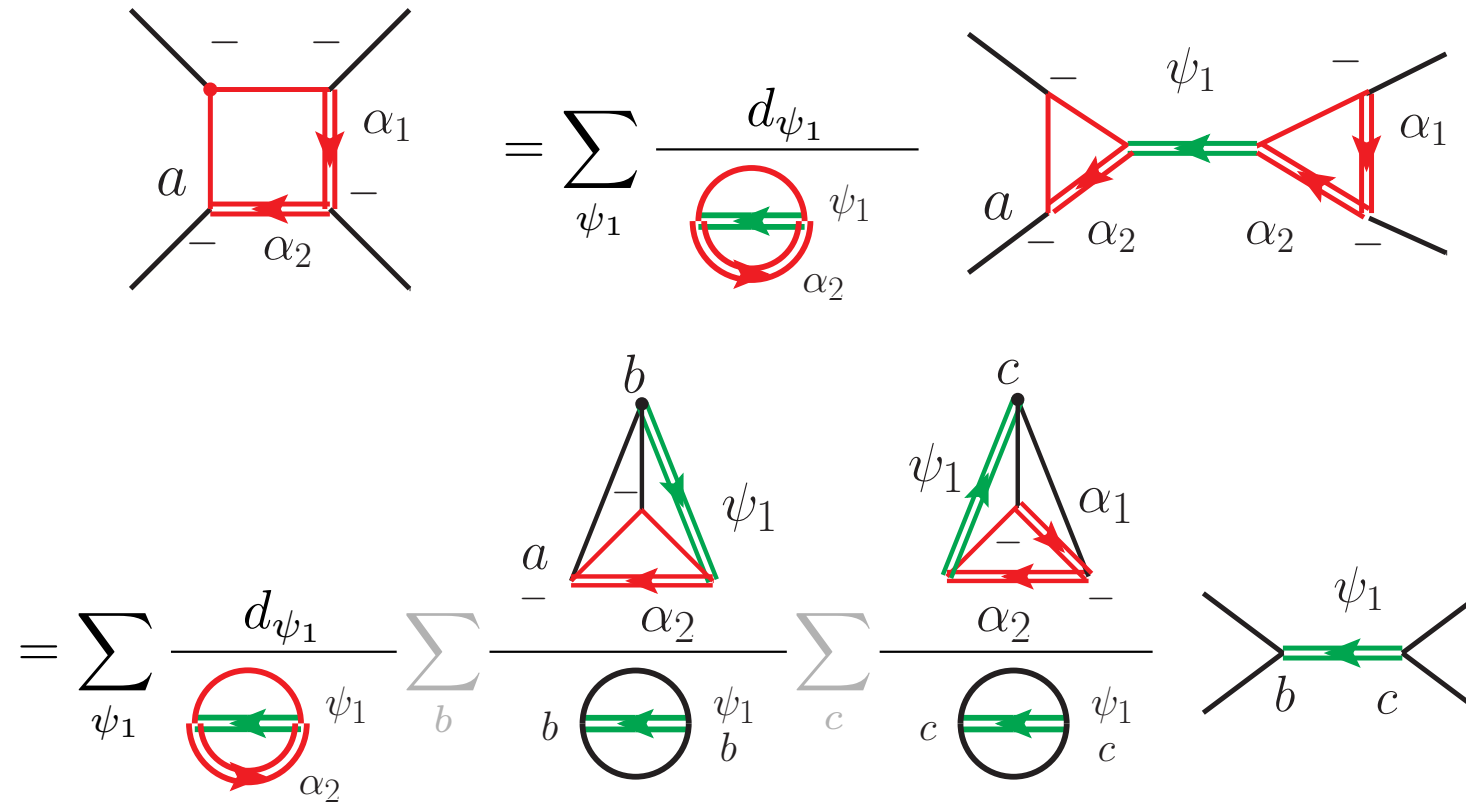
and use the completeness relation

$$\begin{array}{c} \mu \\ \nu \end{array} = \sum_{\alpha} \frac{d_{\alpha}}{\begin{array}{c} \mu \\ \alpha \\ \nu \end{array}} \begin{array}{c} \mu \\ \alpha \\ \nu \end{array}$$

to remove it



Applying the completeness relation and removing vertex corrections:



Removing the 4-vertex loop we get:

$$\begin{aligned}
 A(\alpha_1, \alpha_2, \alpha_3) &= \sum_a \frac{\begin{array}{c} \text{triangle with } \alpha_3 \text{ and } \alpha_2 \end{array}}{a \begin{array}{c} \text{circle with } \alpha_2 \end{array}} a \begin{array}{c} \text{pentagon with } \alpha_1 \text{ and } \alpha_2 \end{array} \\
 &= \sum_a \frac{\begin{array}{c} \text{triangle with } \alpha_3 \text{ and } \alpha_2 \end{array}}{a \begin{array}{c} \text{circle with } \alpha_2 \end{array}} \sum_{\psi_1, b, c} \frac{d\psi_1}{\begin{array}{c} \text{circle with } \psi_1 \end{array}} \frac{\begin{array}{c} \text{triangle with } \psi_1 \text{ and } \alpha_2 \end{array}}{b \begin{array}{c} \text{circle with } \psi_1 \end{array}} \frac{\begin{array}{c} \text{triangle with } \psi_1 \text{ and } \alpha_1 \end{array}}{c \begin{array}{c} \text{circle with } \psi_1 \end{array}} \begin{array}{c} \text{triangle with } \psi_1 \end{array}
 \end{aligned}$$



The final expression is:

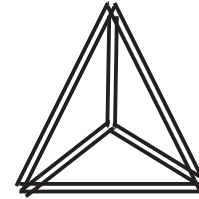
$$A(\alpha_1, \alpha_2, \alpha_3) = \sum_{\psi_1, a, b, c} d_{\psi_1} \frac{\text{[Triangle Diagrams]}}{\text{[Circle Diagrams]}}$$


The diagram shows four terms in the numerator (triangles) and four terms in the denominator (circles).
 Triangle 1: Red lines, vertices labeled a, b, c. Edges labeled α_2 (bottom), α_3 (right), and α_1 (left).
 Triangle 2: Red lines, vertices labeled a, b, c. Edges labeled α_2 (bottom), ψ_1 (right), and α_1 (left).
 Triangle 3: Green lines, vertices labeled a, b, c. Edges labeled α_2 (bottom), ψ_1 (right), and α_1 (left).
 Triangle 4: Green lines, vertices labeled a, b, c. Edges labeled ψ_1 (bottom), ψ_1 (right), and ψ_1 (left).
 Circle 1: Red line, vertex labeled a. Edges labeled α_2 (top) and α_2 (bottom).
 Circle 2: Red line, vertex labeled b. Edges labeled ψ_1 (top) and ψ_1 (bottom).
 Circle 3: Red line, vertex labeled c. Edges labeled ψ_1 (top) and ψ_1 (bottom).
 Circle 4: Red line, vertex labeled c. Edges labeled ψ_1 (top) and α_2 (bottom).


- This only has to be done once for each Feynman diagram, and the scalar product with most basis vectors vanishes
- We only need to care about non-zero scalar products, we could list the non-zero 6j-coefficients
- Each sum over representations contains at most 8 terms for SU(3), at most $N_c^2 - 1$ for SU(N_c)
- **Knowing the 3j and 6j Wigner coefficients we can immediately write down any scalar product!**



All you need is



- In the above example we saw that we could decompose the color structure fully using only d_α , \ominus , 

- We can normalize $\ominus=1$, so we really only need 

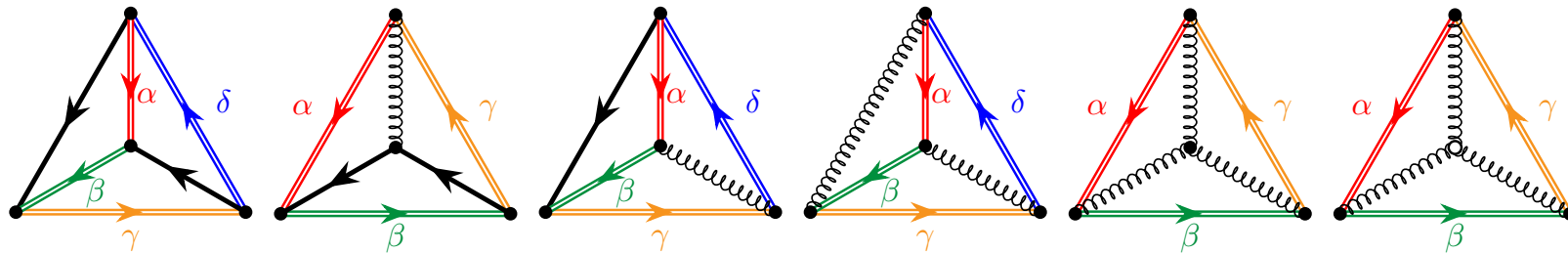
- **Question:** If we can get all the color structure as a function of δ_j s can we then also get the δ_j s as a function of δ_j s?

- Can we calculate δ_j s (recursively)?

$$\text{tetrahedron} = \text{tetrahedron} \left(\text{other tetrahedron}, d_\alpha \right)?$$

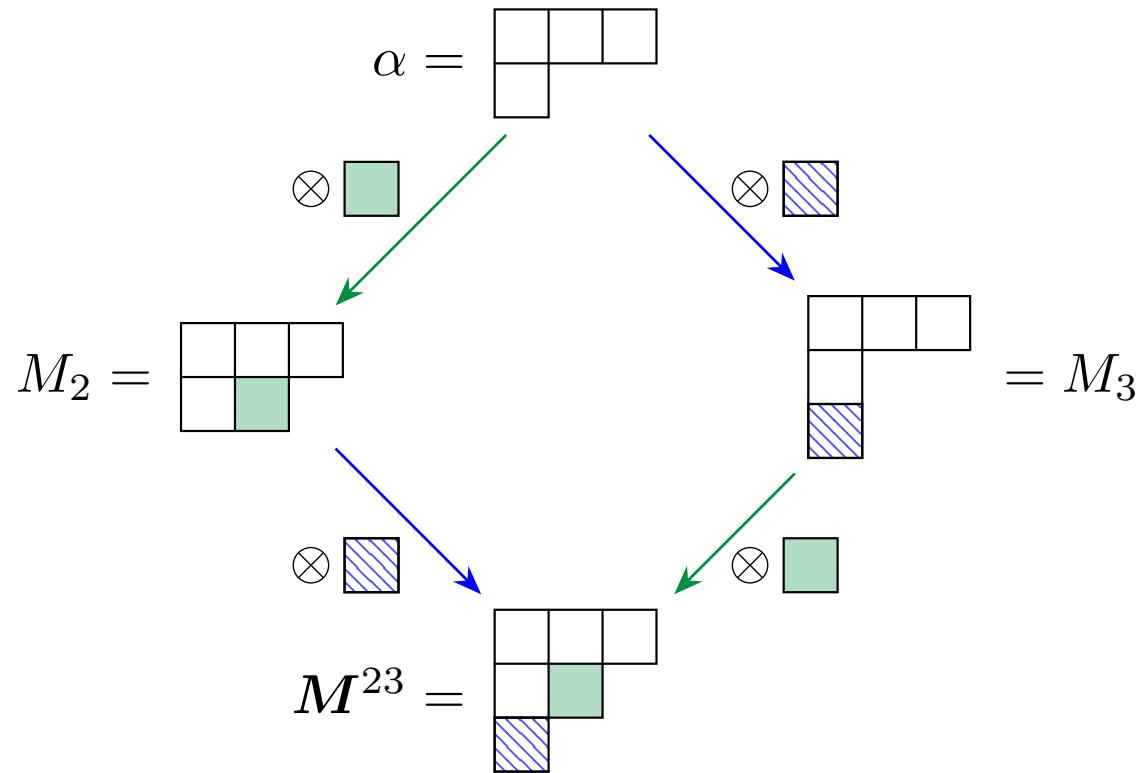
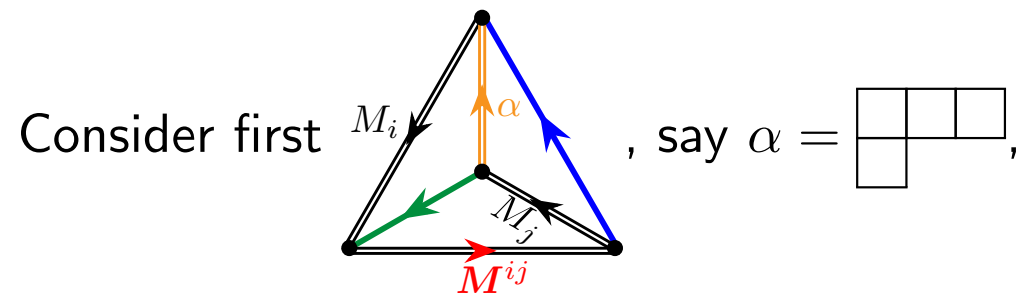


- For QCD, where every representation is 8 , 3 or $\bar{3}$, it turns out that we only need $6j$ s of form

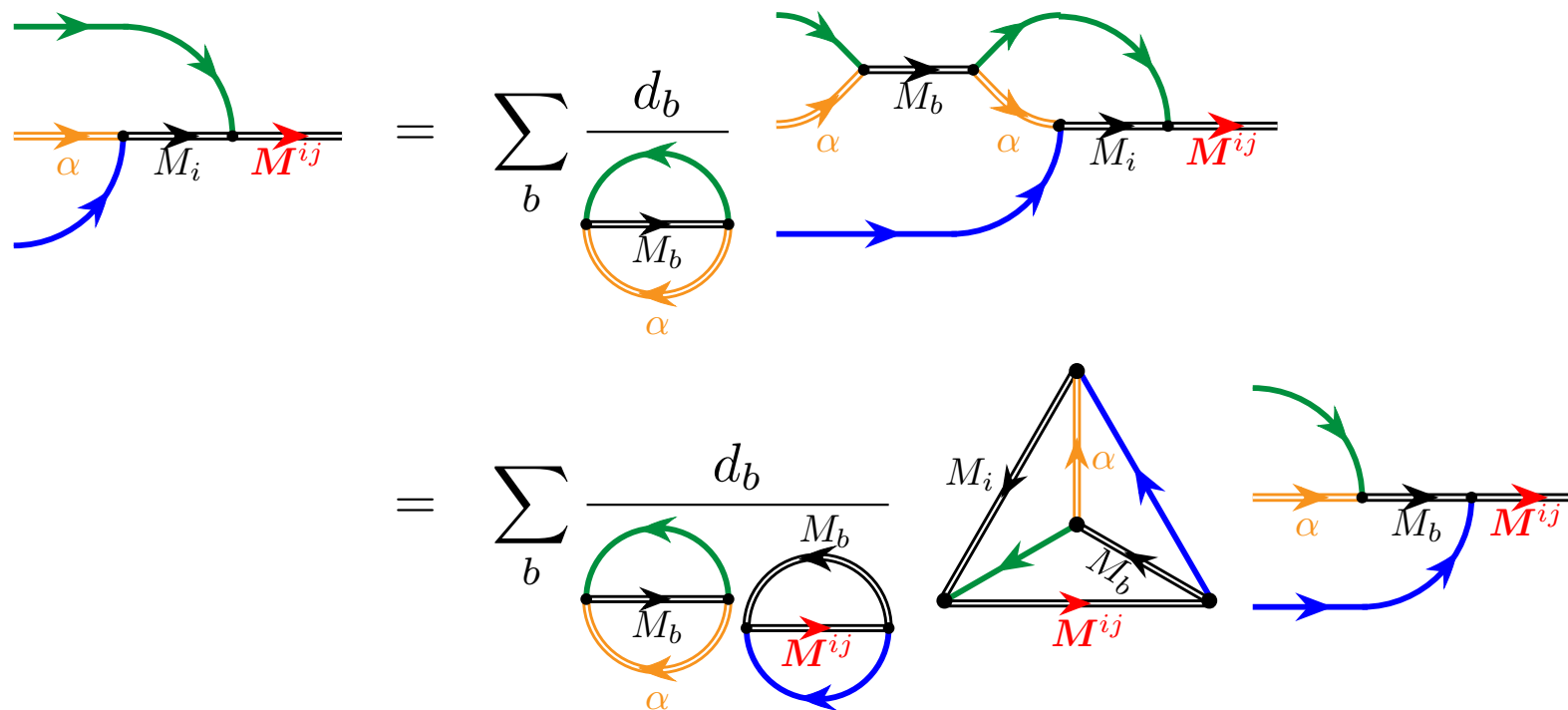


- Wigner $6j$ and $3j$ coefficients and their values can be calculated once and for all (M.S. & J. Thorén, 1507.03814 (JHEP), 1809.05002 (JHEP))
 ... but this still builds on constructing bases which builds on symmetrizers and anti-symmetrizers

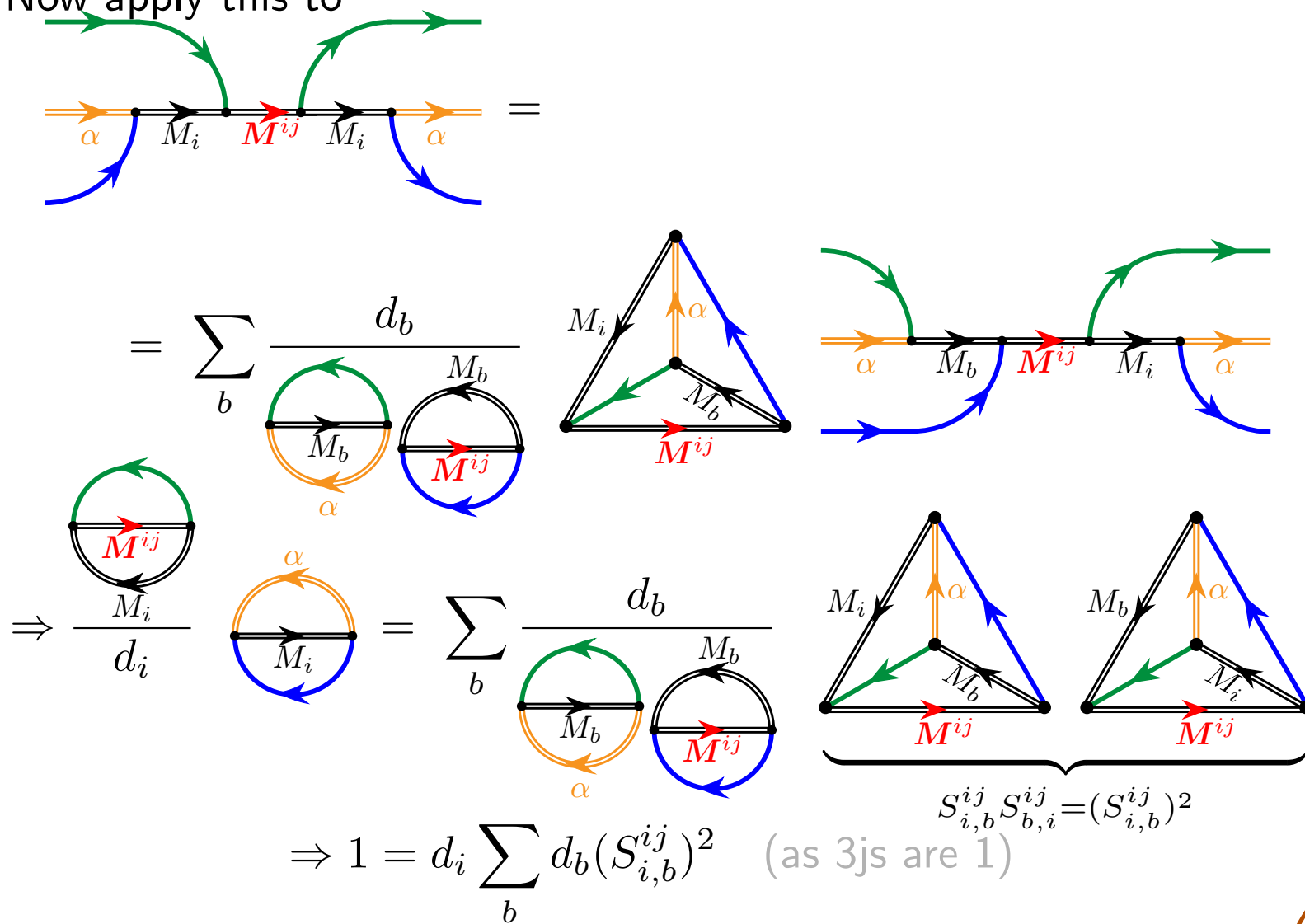




By repeated use of the completeness relation and the vertex correction relation (giving δ_{js}), we can constrain the δ_{js} . Consider for example



Now apply this to



By similar methods we find a set of equations, for $N_c = 3$

1. For a given representation M^{ij} , we obtain

$$1 = (d_i)^2 (S_{i,i}^{ij})^2 + d_i d_j (S_{i,j}^{ij})^2 \quad 0 = d_i S_{i,i}^{ij} S_{i,j}^{ij} + d_j S_{i,j}^{ij} S_{j,j}^{ij}$$

2. For two given representations M_i and M_j , we obtain

$$\frac{1}{d_\alpha} = \sum_{M^{ab}} d_{ab} (S_{i,j}^{ab})^2 ,$$

where d_{ab} is the dimension of the representation M^{ab} .

3. For a given representation M_i , we have

$$1 = \sum_b d_{ib} S_{i,i}^{ib} .$$

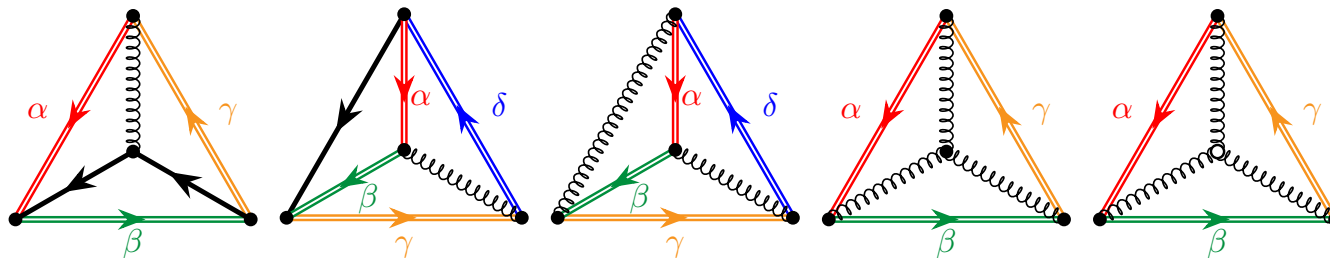


- This equation system can be solved giving

$$\begin{aligned}
 \begin{array}{c} \text{Triangle with } M_i \text{ on } \alpha, M_j \text{ on } \beta, M^{ij} \text{ on } \gamma \end{array} &= \frac{1}{d_i}, & \begin{array}{c} \text{Triangle with } M_i \text{ on } \alpha, M_j \text{ on } \beta, M^{ij} \text{ on } \gamma \end{array} &= \pm \frac{1}{\sqrt{d_\alpha d_{ij}}} \\
 d_i \begin{array}{c} \text{Triangle with } M_i \text{ on } \alpha, M_j \text{ on } \beta, M^{ij} \text{ on } \gamma \end{array} &= \pm \sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}} = d_j \begin{array}{c} \text{Triangle with } M_j \text{ on } \alpha, M_i \text{ on } \beta, M^{ij} \text{ on } \gamma \end{array}
 \end{aligned}$$

(Judith Alcock-Zeilinger, Stefan Keppeler, Simon Plätzer and MS, 2209.15013 (J. Math. Phy.))

- Also need the 6js with gluons



- Idea: split gluon into $q\bar{q}$ -pair, for example we have

The diagram shows the expansion of a triangle diagram with a gluon loop into a sum of diagrams with a quark-antiquark pair loop. The original diagram has external lines labeled α (red), β (green), and δ (blue), and vertices a and b . The gluon loop is represented by a wavy line. The expansion is given by:

$$\begin{aligned}
 &= \sum_{j=1}^a \sum_{k=1}^b C_{aj}^{\beta\alpha} C_{bk}^{\delta\gamma} \text{ (Diagram with } q\bar{q} \text{ loop)} \\
 &= \sum_{j=1}^a \sum_{k=1}^b \frac{C_{aj}^{\beta\alpha} C_{bk}^{\delta\gamma}}{N^2 - 1} \left(\text{Diagram 1} - \frac{1}{N} \text{Diagram 2} \right) \\
 &= \sum_{j=1}^a \sum_{k=1}^b \frac{C_{aj}^{\beta\alpha} C_{bk}^{\delta\gamma}}{N^2 - 1} \left(\text{Diagram 3} - \frac{\delta_{\alpha\beta} \delta_{\gamma\delta}}{N d_\alpha d_\gamma} \text{Diagram 4} \right)
 \end{aligned}$$

The diagrams in the expansion are:

- Diagram 1: Triangle with a quark-antiquark pair loop (solid lines) and a gluon loop (wavy line).
- Diagram 2: Triangle with a quark-antiquark pair loop and a gluon loop, with a ghost loop (dashed line) attached to the gluon loop.
- Diagram 3: Triangle with a quark-antiquark pair loop and a gluon loop, with a ghost loop attached to the quark lines.
- Diagram 4: Triangle with a quark-antiquark pair loop and a gluon loop, with a ghost loop attached to the gluon line.



- By similar methods the other 6js with gluons are derived
2312.16688 (submitted to JHEP), Stefan Keppeler, Simon Plätzer and MS
- Not more complicated to calculate 6js for high representations
- Multiple occurrence is an issue... but can be addressed
- → We in principle have all the ingredients for using representation based orthogonal bases for QCD also for very high multiplicities



Lorentz structure using chirality flow

- At the (complexified) algebra level, the Lorentz group consists of two copies of $\mathfrak{su}(2)$, $\mathfrak{so}(3, 1) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \rightarrow$ **Can we treat the Lorentz structure similarly?**
- The Dirac spinor structure transforms under the direct sum representation $\underbrace{\left(\frac{1}{2}, 0\right)}_{\text{left}} \oplus \underbrace{\left(0, \frac{1}{2}\right)}_{\text{right}}$ in the chiral/Weyl basis

$$\begin{pmatrix} u_L \\ u_R \end{pmatrix} \rightarrow \begin{pmatrix} e^{-i\bar{\theta} \cdot \frac{\bar{\sigma}}{2} + \bar{\eta} \cdot \frac{\bar{\sigma}}{2}} & 0 \\ 0 & e^{-i\bar{\theta} \cdot \frac{\bar{\sigma}}{2} - \bar{\eta} \cdot \frac{\bar{\sigma}}{2}} \end{pmatrix} \begin{pmatrix} u_L \\ u_R \end{pmatrix}$$

i.e. actually two copies of $SL(2, \mathbb{C})$, generated by the complexified $\mathfrak{su}(2)$ algebra



- Consider massless particles: (mass just gives a linear combination)

$$\begin{aligned}
 u^R(p_j) &= \begin{pmatrix} 0 \\ |p_j\rangle \end{pmatrix} = \text{grey circle} \xrightarrow{\text{red arrow}} j &
 u^L(p_j) &= \begin{pmatrix} |p_j] \\ 0 \end{pmatrix} = \text{grey circle} \xrightarrow{\text{blue dashed arrow}} j \\
 \bar{u}^L(p_i) &= \begin{pmatrix} [p_i| \\ 0 \end{pmatrix} = \text{grey circle} \xleftarrow{\text{blue dashed arrow}} i &
 \bar{u}^R(p_j) &= \begin{pmatrix} 0 & \langle p_j| \end{pmatrix} = \text{grey circle} \xrightarrow{\text{red arrow}} j
 \end{aligned}$$

- Polarization vectors

$$\epsilon_L^\mu(p, r) \rightarrow \frac{|r\rangle [p|}{\langle rp\rangle} = \frac{1}{\langle rp\rangle} \text{grey circle} \begin{matrix} \xleftarrow{\text{blue dashed arrow}} p \\ \xrightarrow{\text{red arrow}} r \end{matrix} \quad
 \epsilon_R^\mu(p, r) \rightarrow \frac{|r\rangle \langle p|}{[pr]} = \frac{1}{[pr]} \text{grey circle} \begin{matrix} \xrightarrow{\text{blue dashed arrow}} r \\ \xleftarrow{\text{red arrow}} p \end{matrix}$$

where ϵ_L is for incoming negative helicity or outgoing positive helicity and ϵ_R is for incoming positive helicity or outgoing negative helicity

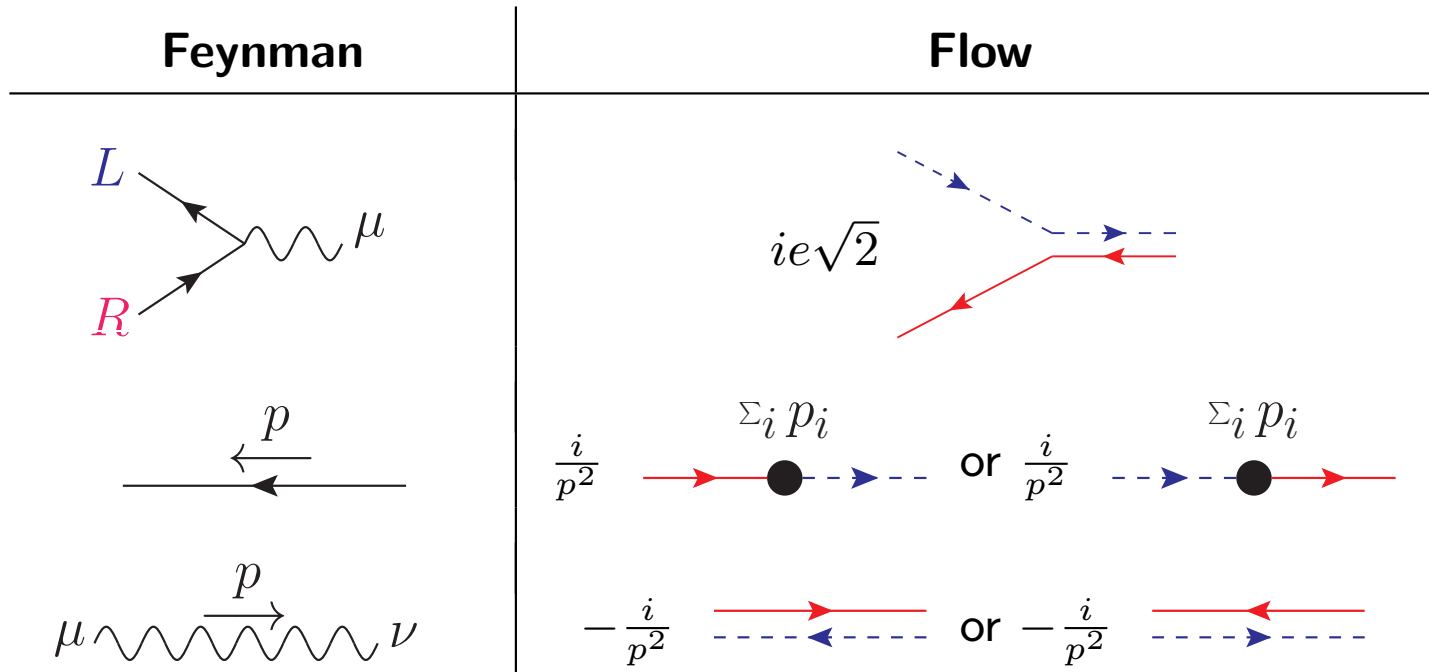


- Amplitudes built up from Lorentz invariant inner products
- Lorentz inner products formed using **the only $SL(2, \mathbb{C})$ invariant object** $\epsilon^{\alpha\beta}$, $\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12}$

$$\underbrace{\epsilon^{\alpha\beta} |i\rangle_\beta |j\rangle_\alpha}_{\equiv \langle i |^\alpha} = \langle i |^\alpha |j\rangle_\alpha = \langle ij \rangle, \quad \underbrace{\epsilon_{\dot{\alpha}\dot{\beta}} [i]^{\dot{\beta}} [j]^{\dot{\alpha}}}_{\equiv [i]_{\dot{\alpha}}} = [i]_{\dot{\alpha}} [j]^{\dot{\alpha}} = [ij],$$

- \rightarrow Amplitudes are built up of contractions of form $\langle ij \rangle, [ij] \sim \sqrt{s_{ij}}$
- In the flow picture, the “flow” must contract **dotted** and **undotted** indices separately





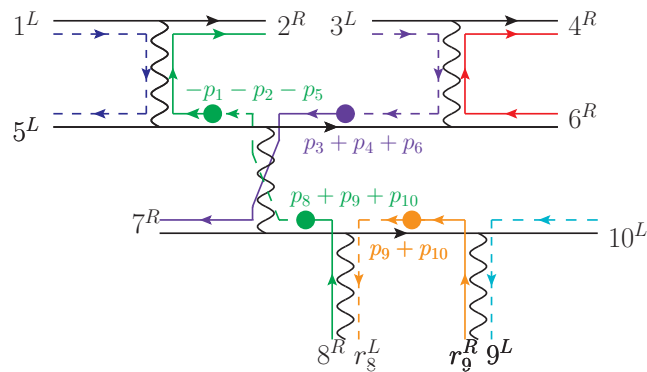
Here $\not{p} \equiv p^\mu \sigma_\mu^{\dot{\alpha}\beta} = \text{---} \bullet \text{---}$, $\bar{\not{p}} \equiv p_\mu \bar{\sigma}^{\mu}_{\alpha\beta} = \text{---} \bullet \text{---}$

with $p = \sum_i p_i$, $p_i^2 = 0$, $\not{p} = \text{---} \bullet \text{---} = \sum_i |i\rangle \langle i|$ etc.

(A. Lifson, C. Reuschle and MS 2003.05877 (EPJC), full SM: J. Alnefjord,
 A. Lifson, C. Reuschle and MS 2011.10075 (EPJC))



Example: Lorentz structure using chirality flow



- Stitch together such that arrow direction match.
- Here all particles crossed to outgoing, and consistent arrow directions are picked

$$\begin{aligned}
 &= \underbrace{(\sqrt{2}ei)^8}_{\text{vertices}} \underbrace{\frac{(-i)^3}{s_{12} s_{34} s_{8910}}}_{\text{photon propagators}} \underbrace{\frac{(i)^4}{s_{125} s_{346} s_{8910} s_{910}}}_{\text{fermion propagators}} \underbrace{\frac{1}{[8r_8]\langle r_99\rangle}}_{\text{polarization vectors}} \quad [15]\langle 64\rangle[10\ 9] \\
 &\times (\langle r_99\rangle[9r_8] + \langle r_910\rangle[10r_8]) \underbrace{([33]\langle 37\rangle + [34]\langle 47\rangle + [36]\langle 67\rangle)}_0 \\
 &\times (-\langle 89\rangle[91]\langle 12\rangle - \langle 89\rangle[95]\langle 52\rangle - \langle 810\rangle[10\ 1]\langle 12\rangle - \langle 810\rangle[10\ 5]\langle 52\rangle)
 \end{aligned}$$



Conclusion and outlook

Color:

- In this presentation, I have shown how the color structure can be dealt with using $6j$ coefficients
- We only need a limited number of $6j$ s which we know how to calculate efficiently
- Looking forward to applying them for heavy QCD calculations
- Still... one might benefit from more general $6j$ s, with arbitrary representations everywhere. Can one find them similarly?

Chirality flow:

- We can flow the Lorentz structure as well ... which often makes it trivial to write down the value of Feynman diagrams!



