Ultracold Neutrons beyond Newton: Remarks on next-to-leading order effects[1]

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[1] Koch, B., Muñoz, E., Santoni, A. (2023). Ultracold Neutrons in the Low Curvature Limit: Remarks on the post-Newtonian effects. arXiv preprint arXiv:2401.00277.

What makes Ultracold Neutrons (UCN) special?

UCN are neutrons with energies around $10^{-7}eV$

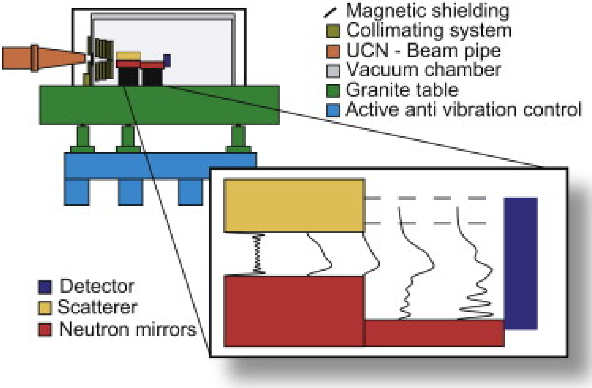
- Temperature $\sim 1\,mK$
- Velocity $\sim 10\,m/s$

$$\lambda = \frac{h}{p} \sim 10^{-8}m >> \text{nuclear spacing} \sim 10^{-10}m \tag{1}$$

UCN cannot penetrate solid surfaces since they act as mirrors!

This makes UCNs **much easier to store** and study.

In the “quantum bouncing ball” experiment, **qBounce**



UCNs are confined by a bottom mirror and Earth’s gravity.

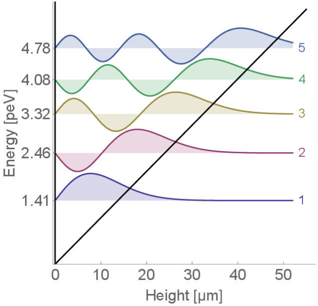
Schrodinger equation with a **Newtonian linear potential**

$$\overbrace{\left(-\frac{\hbar^2}{2m}\vec{\partial}_Z^2 + maZ\right)}^{H_N}\psi_n = E_n^{(0)}\psi_n \quad \text{with} \quad a = \text{local acceleration}\,,$$

Leading to Airy functions

$$\psi_n \propto Ai\left(\frac{Z - Z_n}{Z_0}\right)$$

where $Z_n = \frac{E_n^{(0)}}{ma}\,,\; Z_0 = \left(\frac{\hbar^2}{2m^2 a}\right)^{\frac{1}{3}}$



Experiments have already reached sensitivity around $10^{-6} \times E_n^{(0)}$

Is the Newtonian prescription sufficiently precise?.

Figure:
www.deviantart.com/astrocony/art/Einstein-vs-Newton-701825098

Starting point: Dirac equation in curved space

$$(i\hbar \underline{\gamma}^{\mu} D_{\mu} - mc)\psi = 0\,, \tag{2}$$

Multiplying by $(g^{tt})^{-1}\underline{\gamma}^{t}$, we obtain a Schrödinger-like equation

$$\mathcal{H}_D \psi = i\hbar \frac{\partial \psi}{\partial t}\,, \tag{3}$$

where we used $x^t = ct$ and $\mathcal{H}_D$ is the Dirac Hamiltonian

$$\mathcal{H}_D = mc^2 (g^{tt})^{-1}\underline{\gamma}^{t} - i\hbar c\,\Gamma_t - i\hbar c\,(g^{tt})^{-1}\underline{\gamma}^{t}\underline{\gamma}^{i} D_i\,, \tag{4}$$

for a generic spacetime.

There are many technicalities involved:

1. Static spacetime $ds^2 = V^2(c\,dt)^2 + g_{ij}dx^i dx^j$

2. **Weak-gravity** $\rightarrow \frac{1}{c}-$expansion

$$\begin{aligned} h_{\mu\nu} &\equiv g_{\mu\nu} - \eta_{\mu\nu} \sim O(c^{-2})\,, \\ \epsilon_a{}^{\mu} &\equiv e_a{}^{\mu} - \delta_a{}^{\mu} \sim O(c^{-2})\,. \end{aligned}$$

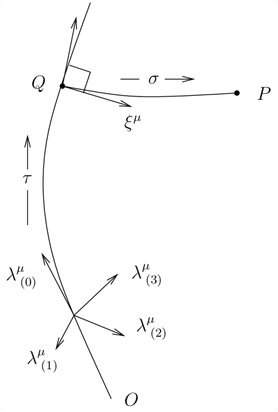
3. **Non-relativistic (NR) Limit** $\rightarrow$ Foldy-Wouthuisen transformation

$$\mathcal{H}_D = \beta mc^2 + \mathcal{E} + \Theta\,, \tag{5}$$

$$[\mathcal{E}\,,\beta] = 0\,,\;\; \{\Theta, \beta\} = 0\,,$$

$$U = e^{iS} \;\text{ with }\; S = -i\frac{\beta}{mc^2}\Theta\,.$$

Fermi Coordinates



Approximate a small region of spacetime around the worldline of an observer by constructing an euclidean grid $\{X^i\}$ comoving with the lab.

Fermi metric:

$$\begin{aligned} g^F_{tt} &\simeq (1 + \frac{\vec{a}\cdot\vec{X}}{c^2})^2 - \frac{1}{2}\bar{R}^F_{tltm}X^l X^m \,, \\ g^F_{ij} &\simeq -\delta_{ij} - \frac{1}{3}\bar{R}^F_{ijlm}X^l X^m \,. \end{aligned}$$

Fermi Hamiltonian:

$$H_{NR} \underset{Fermi}{\longrightarrow} H_F = \underbrace{maZ - \frac{\hbar^2}{2m}\partial_i^2}_{H_N} + H_{NLO} \,.$$

Decoupling $xy-$dynamics and Effective $Z-$Hamiltonian

The qBounce settings + leading-order free $XY-$motion imply

- Wavefunction factorization $\Psi(\vec{x}) = \phi(x, y)\varphi(z)$
- $XY-$dynamics is well described by semi-classical laws

We model UCN as **Gaussian wave packet** in XY

$$\phi(\vec{x}_\perp) = \frac{1}{\sqrt{\pi}\sigma} e^{\frac{i}{\hbar}\vec{k}_\perp \cdot \vec{x}_\perp - \frac{\vec{x}_\perp^2}{2\sigma^2}}\,,\ \text{with}\ \vec{X}_\perp = (X, Y)\,,\ \vec{k}_\perp = (k^x, k^y) \tag{6}$$

Effective one-dimensional Hamiltonian guiding the $Z-$evolution

$$H^{(Z)} = \int d^2x_\perp \phi^*(\vec{x}_\perp)\, H\, \phi(\vec{x}_\perp) = H_N^{(Z)} + H_{NLO}^{(Z)}\,. \tag{7}$$

NLO introduce spin mixing → degenerate perturbation theory

For each n → two-dimensional eigenspace,
spanned by the unperturbed degenerate eigenvectors

$$\begin{aligned} \langle Z | \varphi_n, \uparrow \rangle &= C_n \, Ai\left(\frac{Z - Z_n}{Z_0}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \\ \langle Z | \varphi_n, \downarrow \rangle &= C_n \, Ai\left(\frac{Z - Z_n}{Z_0}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \end{aligned} \tag{8}$$

1. Perturbation **matrix elements** $W^n_{\alpha\beta}$ within the eigenspaces

$$W^n_{\alpha\beta} \equiv \langle \varphi_n, \, \alpha | \, H^{(Z)}_{pN} \, | \varphi_n, \, \beta \rangle \;\; \text{with} \;\; \alpha, \beta = \uparrow, \downarrow \, , \tag{9}$$

2. Diagonalize the W^n-matrices to find their eigenvalues

Secular equation for the corrections

$$(W^n_{\uparrow\uparrow} -{}^{(1)}_{n})^2 - |W^n_{\uparrow\downarrow}|^2 = 0\,, \tag{10}$$

The **next-to-leading order corrections** to the spectrum will be

$$\delta E^{(1)}_{n,\pm\, n,\pm} = W^n_{\uparrow\uparrow} \pm W^n_{\uparrow\downarrow}\,. \tag{11}$$

Neglecting constant shift terms, we obtain

$$\begin{aligned}
\delta E^{(1)}_{n,\pm} \;=\; & \left[\frac{1}{6c^2}\left(v_\perp^2 + \frac{\hbar^2}{m^2\sigma^2} + (3-2d)\frac{GM\sigma^2}{3R^3}\right)\right] E_n \\
& -\frac{1}{5m}\left[\frac{8\,GM}{3\,a^2R^3} - \frac{1}{6c^2} - (5-6b-5d)\frac{4\,G^2M^2}{3a^2c^2R^4}\right. \\
& \left. +(d+3)\frac{4\,GM}{9\,c^2a^2R^3}\left(v_\perp^2 + \frac{\hbar^2}{m^2\sigma^2}\right)\right] E_n^2 \qquad (12) \\
& +\frac{8\,GM\,E_n^3}{105\,ma^2c^2R^3} \pm \frac{\hbar\, v_\perp}{4c^2}\left((d+2)\frac{2\,GM\,E_n}{3\,maR^3} - a\right).
\end{aligned}$$

Let's finally give some estimations in the **qBounce context**.
for **post-Newtnonian (PN)** corrections[2]

qBounce Parameters	Values
Longitudinal velocity $\upsilon_\perp$	$\sim 4 - 10\ \frac{m}{s}$
Local acceleration a	$9.8049\ \frac{m}{s^2}$
Spatial spread σ	$\sim 10^{-8} m$

Table: Typical parameters for the qBounce experiment.

Largest spectrum's perturbations are of order $10^{-12} \times E_n^{(0)}$
→ still too small to detect in qBounce setup!

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A few takeaways...

- Ultracold Neutrons are a great experimental tool to study gravity and its possible extensions
- post-Newtonian corrections does not seem to play a role in the near future of qBounce experiment
- Systematic shifts in the qBounce value of $a \to$ new Physics?

Thanks for your attention!

Restricting our analysis to **generic static spacetimes**

$$ds^2 = (1+v)^2(c\,dt)^2 + (-\delta_{ij} + h_{ij})dx^i dx^j\,, \tag{13}$$

Our first result is the compact expression for the NR Hamiltonian

$$\begin{aligned} H_{NR} &= mc^2(1+v) - \frac{\hbar^4\partial_i^4}{8m^3c^2} \\ &\quad -\frac{\hbar^2}{2m}\left(\partial_i^2 + \tilde{h}_{ij}\partial_i\partial_j + \partial_i\tilde{h}_{ij}\partial_j + \frac{1}{4}\partial_i\partial_j\tilde{h}_{ij}\right. \\ &\qquad \left. + \frac{i}{2}\epsilon^{ijk}\sigma^k\partial_i\tilde{h}_{jl}\partial_l + \frac{i}{4}\epsilon^{ijk}\sigma^k\partial_i\partial_l\tilde{h}_{jl}\right) + O(c^{-3})\,, \end{aligned} \tag{14}$$

where we defined the shifted metric perturbation $\tilde{h}_{ij} \equiv h_{ij} + v\delta_{ij}$.

Backup: Eddington-Robertson metric

Eddington–Robertson (ER) parametrised post-Newtonian metric

$$ds^2_{ER} = (1 + \frac{2\Phi}{c^2} + \frac{2\beta\,\Phi^2}{c^4})(c\,dt)^2 - (1 - \frac{2\gamma\Phi}{c^2})\,\vec{dx}^{\,2}\,, \qquad (15)$$

where the parameters β and γ account for possible deviations from GR, in which $\beta = \gamma = 1$.

Outside a spherical symmetric configuration

$$ds^2_{ERS} = (1 - \frac{2GM}{c^2 r} + \frac{2b\,G^2 M^2}{c^4 r^2})(c\,dt)^2 - (1 + \frac{2d\,GM}{c^2 r})\,\vec{dx}^{\,2}, \quad (16)$$

with $r = \sqrt{x^2 + y^2 + z^2}$ and being M the mass of the Earth.