Three-Loop Corrections to Lamb Shift in Muonium and Positronium

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M.E., Valery Shelyuto, PRA **105**, *012803 (2022),*
PLB **832**, *137247 (2022) and in preparation*

Outline

Muonium (MU= $\mu^+ e^-$) and Positronium (Ps= $e^+ e^-$)

- Muonium: for many years main emphasis of theoretical and experimental research was on hyperfine splitting, see, e.g., reviews: *Eides et al. (2007), Tiesinga et al. (2021)*
- New measurements of $1S - 2S$, Lamb shift and fine structure in muonium are going on and forthcoming: *MU-Mass at PSI; J-PARC MUSE , etc.*
- Positronium: for many years main emphasis of theoretical and experimental research was on hyperfine splitting, see, e.g., review *Adkins et al. (2022)*
- New measurements of $1S - 2S$, Lamb shift and fine structure are going on and forthcoming positronium experiments: *ETH Zurich, UC Riverside, University College London*

Hard three-loop corrections in muonium and positronium

Hard three-loop corrections arise as radiative insertions in two-photon exchange graphs

Muonium

- Some hard three-loop corrections were calculated long time ago (*reviews: M.E. et al. (2007); Tiesinga et al. (2021)*)
- Spin-dependent corrections
  1. Nonrecoil corrections of order $\alpha^2(Z\alpha)^5 m$
  2. Recoil corrections of order $\alpha^2(Z\alpha)^5(m/M)m$
- Spin-independent corrections
  1. Nonrecoil corrections of order $\alpha^2(Z\alpha)^5 m$
  2. Recoil corrections of order $\alpha^2(Z\alpha)^5(m/M)m$ - *this work*

Positronium

- Hard three-loop spin-dependent and spin-independent corrections, see review *Adkins et al. (2022)*
- We consider some hard spin-independent radiative-recoil corrections of order $\alpha^7 m$

Figure: Skeleton diagrams.

Skeleton integrals

- We consider corrections which arise as two-loop radiative insertions in skeleton diagrams

Skeleton integrals

- Recoil muonium skeleton integral

$$\Delta E^{(Mu)}_{skel-rec} = \frac{16(Z\alpha)^5 m}{\pi n^3(1-\mu^2)}\left(\frac{m_r}{m}\right)^3 \int_0^\infty \frac{kdk}{(k^2+\lambda^2)^2}$$
$$\times\left[\mu\sqrt{1+\frac{k^2}{4}}\left(\frac{1}{k}+\frac{k^3}{8}\right)-\sqrt{1+\frac{\mu^2k^2}{4}}\left(\frac{1}{k}+\frac{\mu^4k^3}{8}\right)\right.$$
$$\left.-\frac{\mu k^2}{8}\left(1+\frac{k^2}{2}\right)+\frac{\mu^3k^2}{8}\left(1+\frac{\mu^2k^2}{2}\right)+\frac{1}{k}\right]\delta_{\ell 0}$$

- m - electron mass, M - muon mass, $m_r = mM/(m+M)$ - reduced mass, $\mu = m/M$, λ - IR mass of the exchanged photon, n - principal quantum number, ℓ - orbital momentum, k - dimensionless momentum in units of m

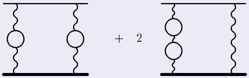
Skeleton integrals

- Positronium skeleton integral

$$\Delta E^{(Ps)}_{skel-rec} = \frac{2\alpha^5 m}{\pi n^3}\int_0^\infty dk\left[\frac{k^2}{8\sqrt{k^2+4}} + \frac{3}{8\sqrt{k^2+4}} - \frac{1}{\sqrt{k^2+4}k^4} - \frac{k}{8} - \frac{1}{8k}\right]\delta_{\ell 0}$$

- No separation into recoil and nonrecoil contributions

One-loop polarization insertions



- Substitution in the skeleton integral

$$\frac{1}{k^2} \to 3\left(\frac{\alpha}{\pi}\right)^2 k^2 I_1^2(k), \qquad I_1(k) = \int_0^1 dv \frac{v^2(1-v^2/3)}{4+(1-v^2)k^2}$$

Muonium

$$\Delta E_1^{(Mu)} = \frac{48(Z\alpha)^5 m}{\pi n^3(1-\mu^2)}\left(\frac{\alpha}{\pi}\right)^2\left(\frac{m_r}{m}\right)^3 \int_0^\infty kdk I_1^2(k)\Bigg\{\mu\sqrt{1+\frac{k^2}{4}}$$

$$\times\left(\frac{1}{k}+\frac{k^3}{8}\right) - \sqrt{1+\frac{\mu^2 k^2}{4}}\left(\frac{1}{k}+\frac{\mu^4 k^3}{8}\right) - \frac{\mu k^2}{8}\left(1+\frac{k^2}{2}\right)$$

$$+\frac{\mu^3 k^2}{8}\left(1+\frac{\mu^2 k^2}{2}\right)+\frac{1}{k}\Bigg\}\delta_{\ell 0},$$

One-loop polarization insertions

Muonium

- Numerically with account of all orders in $\mu = m/M$

$$\Delta E_1^{(Mu)} = 0.959540854(3)\ldots \frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3}\frac{m}{M}\delta_{\ell 0}$$

- Analytic result, first order in μ

$$\Delta E_1^{(Mu)} \approx \frac{48(Z\alpha)^5 m}{\pi n^3}\left(\frac{\alpha}{\pi}\right)^2\left(\frac{m_r}{m}\right)^3 \mu \int_0^\infty k dk I_1^2(k)\left[\sqrt{1+\frac{k^2}{4}}\right.$$

$$\left.\times\left(\frac{1}{k}+\frac{k^3}{8}\right)-\frac{k^2}{8}\left(1+\frac{k^2}{2}\right)\right]\delta_{l0}$$

$$=\left(\frac{1541}{486}-\frac{172}{2835}\pi^2-\frac{4}{3}\zeta(3)\right)\frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m\delta_{l0}$$

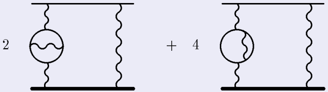
$$=0.9692\ldots\frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m\delta_{\ell 0}.$$

One-loop polarization insertions

Positronium

$$\Delta E_1^{(Ps)} = \left(-\frac{\zeta(3)}{6} + \frac{1709}{3888} - \frac{11\pi^2}{405}\right)\frac{\alpha^7}{\pi^3 n^3} m\delta_{\ell 0} = -0.028848\ldots \frac{\alpha^7}{\pi^3 n^3} m\delta_{\ell 0}$$

Two-Loop Polarization Insertions



- Substitution in the skeleton integral

$$\frac{1}{k^2} \to 2\left(\frac{\alpha}{\pi}\right)^2 I_2(k), \qquad I_2(k) = \int_0^1 dv \frac{\frac{3}{4}v^2\left(1-\frac{v^2}{3}\right) + R(v)}{4 + (1-v^2)k^2}$$

Two-loop polarization insertions

Muonium

$$\Delta E_2^{(Mu)} = \frac{32(Z\alpha)^5 m}{\pi n^3(1-\mu^2)}\left(\frac{\alpha}{\pi}\right)^2\left(\frac{m_r}{m}\right)^3\int_0^\infty \frac{dk}{k}\left\{I_2(k)\left[\mu\sqrt{1+\frac{k^2}{4}}\right.\right.$$
$$\times\left(\frac{1}{k}+\frac{k^3}{8}\right)-\sqrt{1+\frac{\mu^2k^2}{4}}\left(\frac{1}{k}+\frac{\mu^4k^3}{8}\right)-\frac{\mu k^2}{8}\left(1+\frac{k^2}{2}\right)$$
$$\left.\left.+\frac{\mu^3k^2}{8}\left(1+\frac{\mu^2k^2}{2}\right)+\frac{1}{k}\right]-\frac{41}{162}\frac{\mu}{k}\right\}\delta_{\ell 0}.$$

- Numerically with account of all orders in $\mu = m/M$

$$\Delta E_2^{(Mu)} = -3.133412(3)\ldots\frac{\alpha^2(Z\alpha)^5}{\pi^3n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m\delta_{\ell 0}$$

Two-loop polarization insertions

Muonium

- Analytic result, first order in μ

$$\Delta E_2^{(Mu)} \approx \frac{32(Z\alpha)^5 m}{\pi n^3}\left(\frac{\alpha}{\pi}\right)^2\left(\frac{m_r}{m}\right)^3 \mu\int_0^\infty \frac{dk}{k}\left\{I_2(k)\left[\sqrt{1+\frac{k^2}{4}}\right.\right.$$

$$\left.\left.\times\left(\frac{1}{k}+\frac{k^3}{8}\right)-\frac{k^2}{8}\left(1+\frac{k^2}{2}\right)\right]-\frac{41}{162k}\right\}\delta_{\ell 0}$$

$$=\left(\frac{6589}{7560}+\frac{145756\pi^2}{99225}+\frac{7\pi^4}{270}-\frac{296\pi^2}{315}\ln 2+\frac{4\pi^2}{9}\ln^2 2\right.$$

$$\left.-\frac{4}{9}\ln^4 2-\frac{32}{3}\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{11597}{1260}\zeta(3)\right)\frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m\delta_{\ell 0}$$

$$=-3.1121\ldots\frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m\delta_{\ell 0}.$$

Two-loop polarization insertions

Positronium

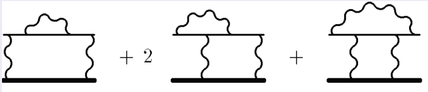
$$\Delta E_2^{(Ps)} = \frac{4\alpha^7 m}{\pi^3 n^3}\int_0^\infty dk I_2(k)\left[\frac{k^4}{8\sqrt{k^2+4}} + \frac{3k^2}{8\sqrt{k^2+4}} - \frac{k^3+k}{8} - \frac{1}{\sqrt{k^2+4}k^2} + \frac{41}{324k^2}\right]\delta_{\ell 0}$$

$$= \left(-\frac{4\mathrm{Li}_4\left(\frac{1}{2}\right)}{3} - \frac{17921\zeta(3)}{10080} + \frac{26347}{60480} + \frac{311233\pi^2}{793800} + \frac{7\pi^4}{2160} + \frac{1}{18}\pi^2\ln^2 2 - \frac{\ln^4 2}{18} - \frac{76}{315}\pi^2\ln 2\right)\frac{\alpha^7 m}{\pi^3 n^3}\delta_{\ell 0}$$

$$= 0.393966\ldots\frac{\alpha^7}{\pi^3 n^3} m\delta_{\ell 0}.$$

One-loop electron factor and polarization

Previous order



$$\Delta E = -\frac{(Z\alpha)^5}{\pi n^3} m_r^3 \int \frac{d^4k}{i\pi^2 k^4} \frac{1}{4} Tr\Big[(1+\gamma_0)L_{\mu\nu}\Big] \frac{1}{4} Tr\Big[(1+\gamma_0)H_{\mu\nu}\Big] \delta_{\ell 0}$$

- $L_{\mu\nu} = L^{\Sigma}_{\mu\nu} + 2L^{\Lambda}_{\mu\nu} + L^{\Xi}_{\mu\nu}$

$$\frac{1}{4} Tr\Big[(1+\gamma_0)L_{\mu\nu}\Big] \equiv \frac{\alpha}{\pi m} \mathcal{L}_{\mu\nu}\left(\frac{k}{m}\right)$$
$$= \frac{\alpha}{\pi m}\left[\mathcal{L}^{\Sigma}_{\mu\nu}\left(\frac{k}{m}\right) + 2\mathcal{L}^{\Lambda}_{\mu\nu}\left(\frac{k}{m}\right) + \mathcal{L}^{\Xi}_{\mu\nu}\left(\frac{k}{m}\right)\right]$$

One-loop electron factor and polarization

Previous order

- $H_{\mu\nu} = \gamma_\mu \frac{\not P + \not k + M}{k^2+2Mk_0+i0}\gamma_\nu + \gamma_\nu \frac{\not P - \not k + M}{k^2-2Mk_0+i0}\gamma_\mu$
- $\frac{1}{4}\, Tr\left[(1+\gamma_0)H_{\mu\nu}\right] =$
  $-\frac{1}{M}\left[k^2 g_{\mu 0} g_{\nu 0} - k_0\left(g_{\mu 0}k_\nu + g_{\nu 0}k_\mu\right) + k_0^2 g_{\mu\nu}\right]\frac{1}{k_0^2 - \frac{k^4}{4M^2}}$

Muonium

- Principal value definition $k^2 \wp\left(\frac{1}{k_0^2}\right) = k^2 \lim_{\frac{k}{M}\to 0} \frac{k_0^2 + \frac{k^4}{4M^2}}{\left(k_0^2 - \frac{k^4}{4M^2}\right)^2}$
- Linear in $\mu = m/M$ approximation

$$\frac{1}{4}\, Tr\left[(1+\gamma_0)H_{\mu\nu}\right] \to -\frac{1}{M}\left[k^2 g_{\mu 0} g_{\nu 0}\wp\left(\frac{1}{k_0^2}\right) - \left(g_{\mu 0}k_\nu + g_{\nu 0}k_\mu\right)\frac{1}{k_0} + g_{\mu\nu}\right]$$

$$\equiv -\frac{1}{M}\mathcal{H}_{\mu\nu}(k)$$

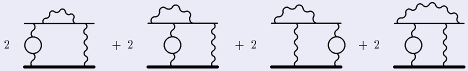
One-loop electron factor and polarization

Muonium

- Linear in mass ratio radiative-recoil contribution of previous order

$$\Delta E_{rec} = \frac{\alpha(Z\alpha)^5}{\pi^2 n^3}\frac{m_r^3}{Mm}\int \frac{d^4k}{i\pi^2 k^4}\mathcal{L}_{\mu\nu}\left(\frac{k}{m}\right)\mathcal{H}_{\mu\nu}(k)$$

Muonium: linear in mass ratio contribution, order $\alpha^2(Z\alpha)^5\mu m$



- Insert $(\alpha/\pi)k^2 I_1(k)$ in ΔE_{rec}

$$\Delta E_3^{(Mu)} = \frac{\alpha^2(Z\alpha)^5}{\pi^2 n^3}\frac{m_r^3}{Mm}\int \frac{d^4k}{i\pi^2 k^2} 2I_1(k)\mathcal{L}_{\mu\nu}\left(\frac{k}{m}\right)\mathcal{H}_{\mu\nu}(k)$$

$$= (J_{\Sigma P} + 2J_{\Lambda P} + J_{\Xi P})\frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m\delta_{\ell 0}$$

$$= -9.2569(2)\frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m\delta_{\ell 0}$$

One-loop electron factor and polarization

Positronium

- Unexpected connection between the total (recoil + nonrecoil) & linear in μ heavy particle factor at $m = M$

$$\mathcal{H}_{\mu\nu}(k)\frac{k_0^2}{k_0^2-\frac{k^4}{4m^2}} = \frac{k^2 g_{\mu 0}g_{\nu 0}-(g_{\mu 0}k_\nu+g_{\nu 0}k_\mu)k_0+g_{\mu\nu}k_0^2}{k_0^2-\frac{k^4}{4m^2}}$$

- Dimensionless interpolating factor

$$\widetilde{G}(k,M) = \frac{k_0^2\left(\frac{k^4}{4M^2}+k_0^2\right)}{\left(k_0^2-\frac{k^4}{4M^2}\right)^2} - \frac{k_0^2k^4m^2}{2M^4\left(k_0^2-\frac{k^4}{4M^2}\right)^2}$$

$$\widetilde{G}(k,m) = k_0^2/(k_0^2-k^4/4m^2), \qquad \widetilde{G}(k,M)_{|k/M\to 0} \to 1$$

- Substitution $\mathcal{H}_{\mu\nu}(k) \to \mathcal{H}_{\mu\nu}(k)\widetilde{G}(k,M)$
- $\mathcal{H}_{\mu\nu}(k)\widetilde{G}(k,M)_{|k/M\to 0}$ – integrand for linear in μ corrections for muonium
- $\mathcal{H}_{\mu\nu}(k)\widetilde{G}(k,m)$ – integrand for total (recoil + nonrecoil) corrections for positronium

One-loop electron factor and polarization

Positronium

- Universal expression

$$\Delta E_{pol} = \frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3}\frac{m_r^3}{mM}\int\frac{d^4k}{i\pi^2k^4}\mathcal{L}_{\mu\nu}(k)\,\mathcal{H}_{\mu\nu}(k)2k^2 I_1(k)\widetilde{G}(km,M)\delta_{\ell 0}$$
$$\equiv \frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3}\frac{m_r^3}{mM}\Delta\mathcal{E}_{pol}(M),$$

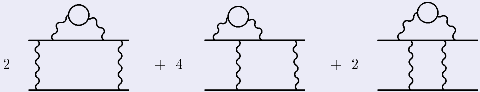
- Once again linear in μ contribution for muonium

$$\Delta E_3^{(Mu)} = \frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3}\frac{m_r^3}{mM}\Delta\mathcal{E}_{pol}(M\to\infty)$$

- Total (recoil and nonrecoil) spin-independent contribution in positronium

$$\Delta E_3^{(Ps)} = \frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3}\frac{m}{4}\Delta\mathcal{E}_{pol}(M=m) \equiv (J_{\Sigma P} + 2J_{\Lambda P} + J_{\Xi P})\frac{\alpha^7 m}{\pi^3 n^3}\delta_{\ell 0}$$
$$= 0.5701(2)\frac{\alpha^7}{\pi^3 n^3}m\delta_{\ell 0}$$

Radiatively corrected electron factor



- Linear in mass ratio contribution in muonium

$$\Delta E_4^{(Mu)} = -0.0799(2)\frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m\delta_{\ell 0}$$

- Positronium

$$\Delta E_4^{(Ps)} = -0.4147(2)\frac{\alpha^7 m}{\pi^3 n^3}\delta_{\ell 0}$$

A bit on phenomenology

- Numerically the contributions to the Lamb shift in muonium calculated above are at the level of a few tenths of kHz
- Numerically the contributions to the Lamb shift in positronium calculated above are at the level of a few kHz
- These contributions are too small to be relevant for the results of the ongoing experiments, but will hopefully become relevant in the future

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Thank you!