Matrix elements of the energy-momentum tensor in the hydrogen atom

based on work with Yizhuang Liu, Jagiellonian University, Poland

FFK 2023 Vienna

Andrzej Czarnecki  University of Alberta

May 22, 2023

Context

2208.05029

Gravitational Tensor-Monopole Moment of Hydrogen Atom To Order $\mathcal{O}(\alpha)$

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We calculate the gravitational tensor-monopole moment of the momentum-current density T^{ij} in the ground state of the hydrogen atom to order $\mathcal{O}(\alpha)$ in quantum electrodynamics (QED). The result is

$$\tau_H/\tau_0 - 1 = \frac{4\alpha}{3\pi}(\ln\alpha - 0.028)$$

where $\tau_0 = \hbar^2/4m_e$ is the leading-order moment. The physics of the next-to-leading-order correction is similar to that of the famous Lamb shift for energy levels.

- Intriguing: a new atomic observable
- Is the physics really similar to the Lamb shift?
- Practically important: related to hadronic structure
- Experiments ongoing at JLab and planned in Electron-Ion Collider

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Outline:

- Motivation: Ji & Liu's recent logarithmic correction to the graviton's interaction with the Hydrogen atom
- Experimental relevance: hadronic physics (a new observable)
- Gravitational form-factors: the D-term
- D-term's sign vs stability of the system
- Logarithmic corrections: D-term vs Lamb shift

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Misha Eides' recent work on energy-momentum tensor

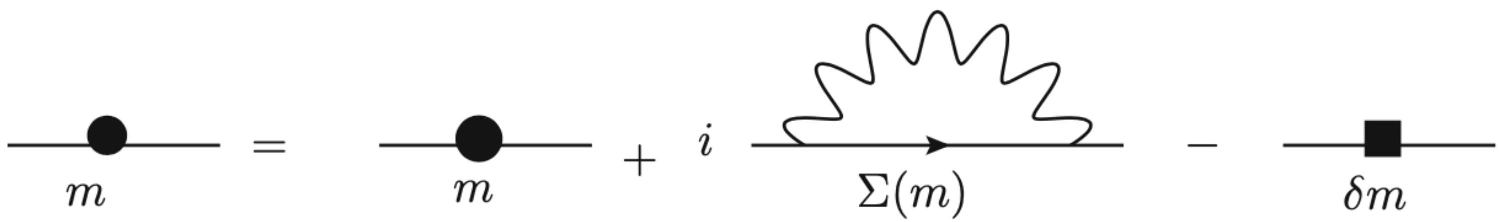
Motivation:

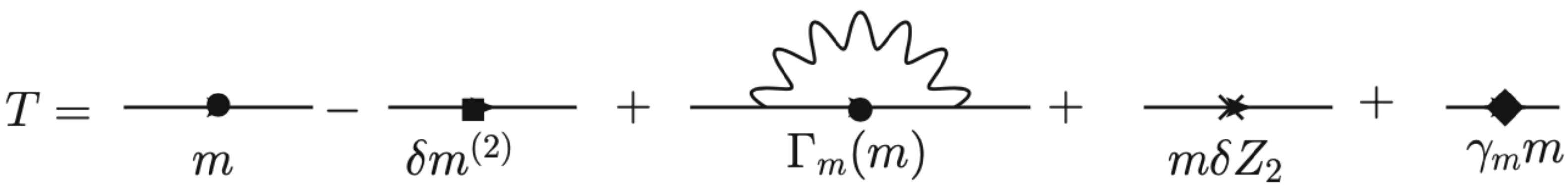
A new insight into the EMT properties could arise from consideration of EMT in theories which allow perturbative [treatment]

One-loop electron mass and QED trace anomaly

Michael I. Eides[a]

Eur. Phys. J. C (2023) 83:356



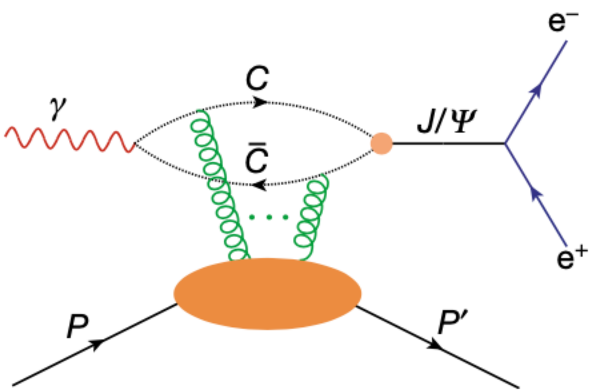


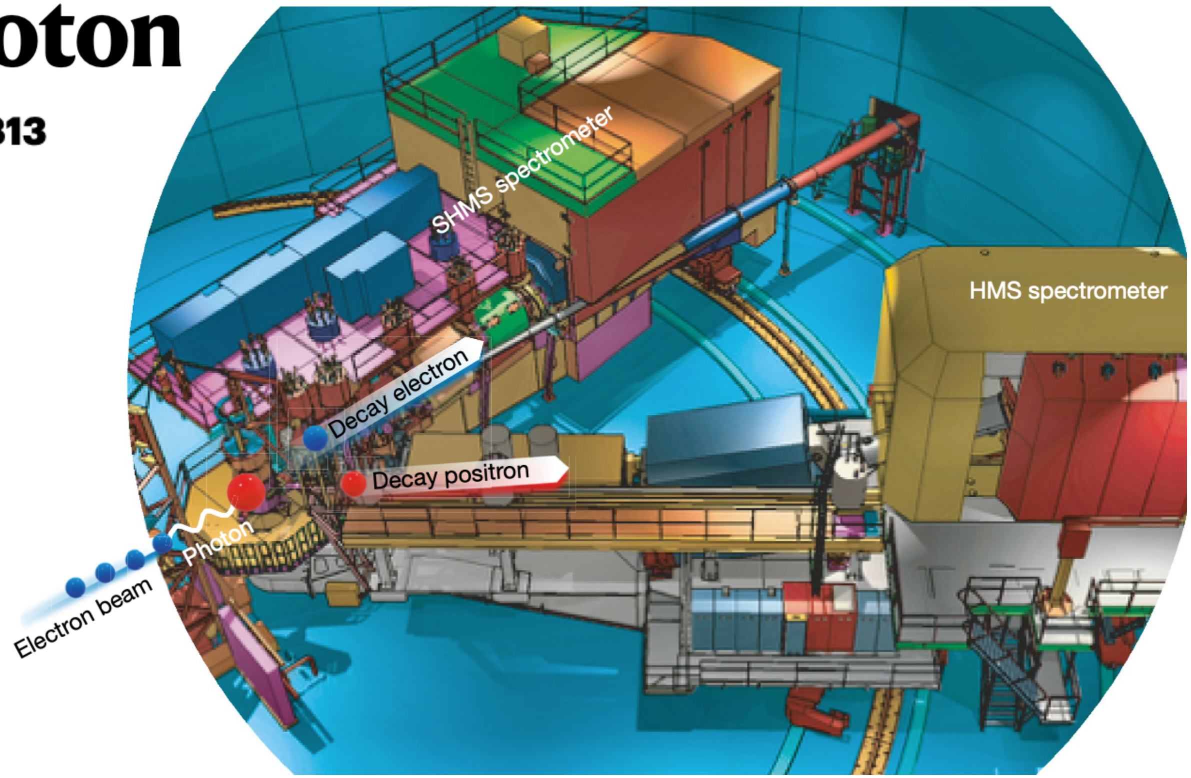
Graviton-atom interaction:
Of recent interest because can be probed in scattering experiments, via Generalized Parton Distributions [X. Ji; A. V. Radyushkin]

Experimental consequences

Determining the gluonic gravitational form factors of the proton

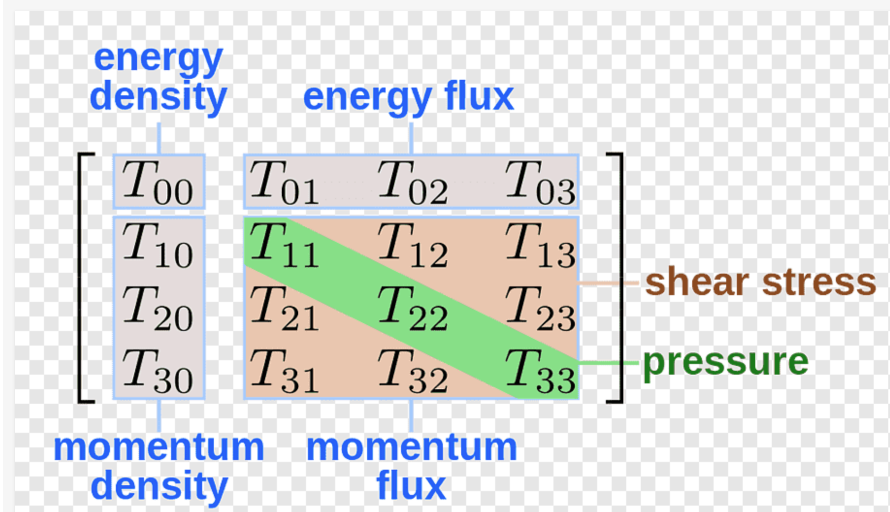
Nature | Vol 615 | 30 March 2023 | 813





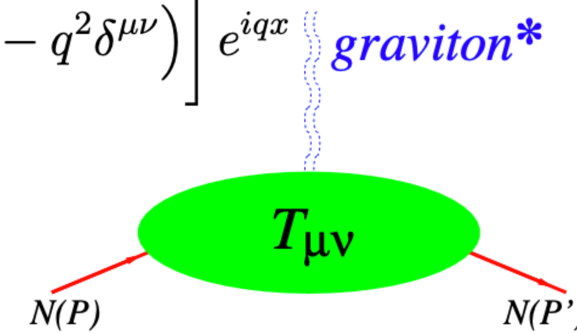
- Beyond earlier studies of the charge and spin distributions in the proton;
- New parameter: proton mass radius 0.52(3) fm.

Graviton's interaction and the EMT



$$\left\langle P+\frac{q}{2}\right|T^{\mu\nu}(x)\left|P-\frac{q}{2}\right\rangle=\left[\frac{A(t)}{m}P^{\mu}P^{\nu}+\frac{D(t)}{4m}\left(q^{\mu}q^{\nu}-q^{2}\delta^{\mu\nu}\right)\right]e^{iqx}$$

$$t=q^2$$



The sign of the D-term and system's stability

On LHCb pentaquarks as a baryon-$\psi(2S)$ bound state – prediction of isospin $\frac{3}{2}$ pentaquarks with hidden charm *

Irina A. Perevalova,[1] Maxim V. Polyakov,[2,3] and Peter Schweitzer[4,5]

arXiv:1607.07008

A less trivial local criterion can be obtained by considering that at any chosen distance r the force exhibited by the system on an infinitesimal piece of area $\mathrm{d}A\, e_r^i$ must be directed outwards. If this was not the case, the system would collapse. Since this force is $F^i(\boldsymbol{r}) = T^{ij}(\boldsymbol{r})\mathrm{d}A\, e_r^j = [\frac{2}{3}\, s(r) + p(r)]\mathrm{d}A\, e_r^i$ we obtain the criterion

$$\frac{2}{3}\, s(r) + p(r) > 0\,. \tag{18}$$

We checked that the condition (18) is satisfied in all systems we are aware of where EMT densities were studied [9, 10, 25–31]. As this includes unstable systems, apparently also (18) is a necessary but not sufficient condition for stability. Due to its local character, it provides a stronger criterion than the von Laue condition (14) and will play an important role below. Interestingly, the criterion (18) allows one to draw a conclusion on the sign of the D-term. We see that

$$0 < 4\pi \int_0^\infty \mathrm{d}r\, r^4 \left(\frac{2}{3}\, s(r) + p(r)\right) = -\frac{2d_1}{M_N} + \frac{4d_1}{5M_N} = -\,\frac{6d_1}{5M_N}\,. \tag{19}$$

Thus, if a system satisfies the local stability criterion (18), then it must necessarily have a negative D-term (but a negative D-term does not imply that $s(r)$ and $p(r)$ satisfy (18), so the opposite is in general not true). Indeed, in all systems studied so far the D-terms were found to be negative [9, 10, 25–31].

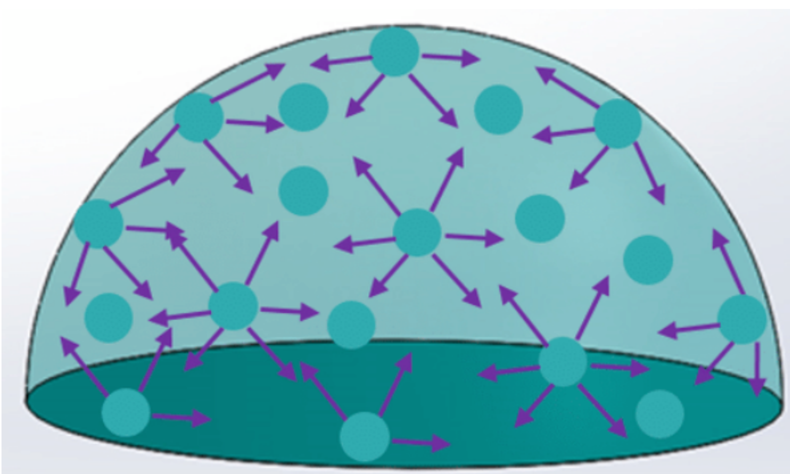
Two integrals:

$$0 = \int p(r)\mathrm{d}^3 r$$ Max von Laue's stability condition

$$D \sim \int p(r) r^2 \mathrm{d}^3 r$$ assuming spherical symmetry

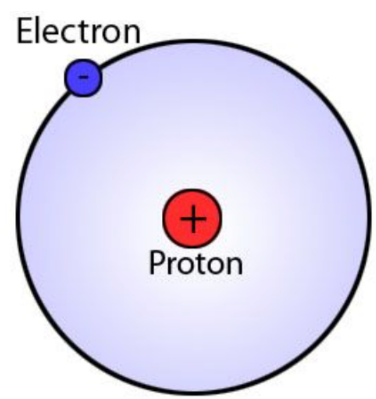
Two types of stable systems:

"Liquid droplet"



tension at large r: D < 0

"Atom"



tension at small r: D > 0

Max von Laue's stability condition

Energy-momentum conservation → in terms of the EMT $\partial^{\mu} T_{\mu\nu} = 0$

In a stationary state: no time dependence, $\nabla^{i} T_{i\nu} = 0$

Integral form (n: normal to an enclosing surface): $\int_{\sigma} T^{ij} n_j \mathrm{d}\sigma = 0$

Choose the surface as a cross-section of the system in the x-plane, closed at infinity:

$$\int T^{xi} \mathrm{d}y \mathrm{d}z = 0$$

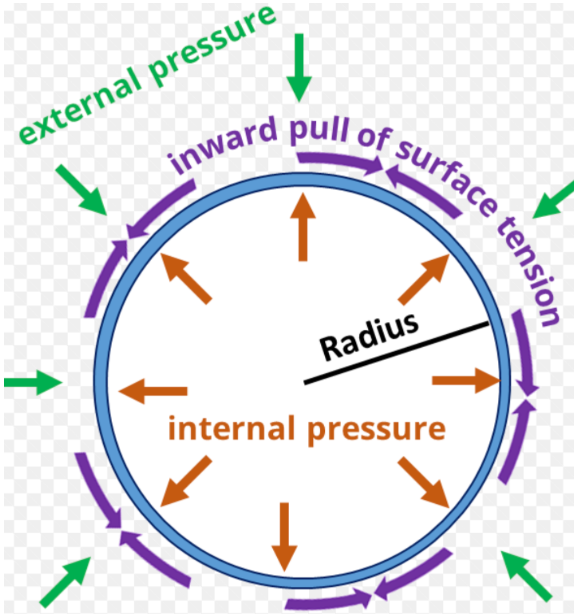
Finally, integrate over x:

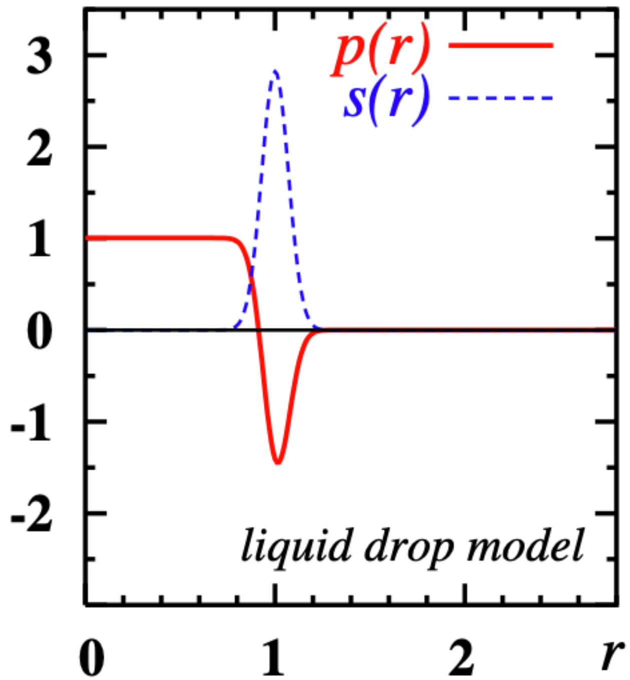
$$\int T^{xi} \mathrm{d}^3 r = 0$$

Diagonal element like i=x: pressure $0 = \int p(r) \mathrm{d}^3 r$

Zur Dynamik der Relativitätstheorie
von M. Laue. 1911

von Laue's stability example 1: liquid droplet





$$p(r) = p_0\theta(r-R) - \frac{p_0 R}{3}\delta(r-R)$$

$$\int p(r)\mathrm{d}^3r = \frac{4\pi R^3}{3}p_0 - \frac{p_0 R}{3}4\pi R^2 = 0$$

Forces inside hadrons: pressure, surface tension, mechanical radius, and all that

Maxim V. Polyakov[1,2] and Peter Schweitzer[3] 1805.06596

von Laue's stability example 2: hydrogen atom

$$T^{ij}(\vec{r}) = m v^i v^j \delta^3\left(\vec{r} - \vec{x}(t)\right) - E^i E^j + \frac{\delta^{ij}}{2}\vec{E}^2$$

Electron's motion

Electron's and proton's electric field

We want to show $\int d^3\vec{r}\, T^{ii}(\vec{r}) = 2T + V = 0$ as in virial theorem

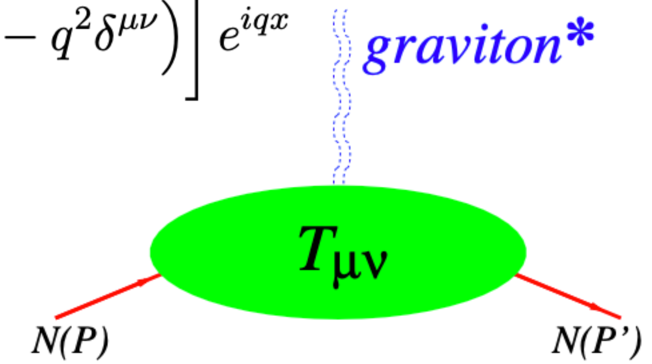
$$\vec{E} \approx \frac{e}{4\pi}\frac{\vec{r}}{r^3} - \frac{e}{4\pi}\frac{\vec{r} - \vec{x}(t)}{|\vec{r} - \vec{x}(t)|^3} \equiv \vec{E}_p + \vec{E}_e$$

$$\int d^3\vec{r}\, \vec{E}_e \cdot \vec{E}_p = -\frac{\alpha}{|\vec{x}(t)|} = -\frac{\alpha}{R} = V$$

D-term vs EMT elements

$$\left\langle P+\frac{q}{2}\right| T^{\mu\nu}(x) \left| P-\frac{q}{2}\right\rangle = \left[\frac{A(t)}{m}P^\mu P^\nu + \frac{D(t)}{4m}\left(q^\mu q^\nu - q^2\delta^{\mu\nu}\right)\right] e^{iqx}$$

$$t = q^2$$



Determine the EMT as the Fourier transform
of this matrix element
with respect to the momentum transfer
in the Breit frame (no energy transfer).

On the other hand, decompose in terms of pressure *p* and shear *s*,

$$T^{ij}(\boldsymbol{r}) = \left(\frac{r^i r^j}{r^2} - \frac{\delta^{ij}}{3}\right) s(\boldsymbol{r}) + \delta^{ij} p(\boldsymbol{r})$$

$$D \equiv D(t=0) = m\int \mathrm{d}^3r\; r^2 p(r)$$

Example: D-term of a liquid drop

$$p(r) = p_0\theta(r-R) - \frac{p_0 R}{3}\delta(r-R)$$

$$\begin{aligned} D &= m4\pi^2 \left(\int_0^R \mathrm{d}r\ r^4 p_0 - \frac{p_0 R}{3} R^4 \right) \\ &= -\frac{8\pi^2}{15} m p_0 \end{aligned}$$

It is negative because the negative pressure region is at the outer boundary.

Example: D-term of the hydrogen atom

Consider $\int d^3\vec{r}\; r^2 T^{ii}$

In dimensional regularization, terms homogeneous in r vanish.
Potential energy contributions give two integrals,

$$I = I_1 + I_2 \equiv \int d^3\vec{r}\left(-\frac{2}{|\vec{r}||\vec{r}-\vec{R}|} + \frac{2\vec{R}\cdot(\vec{r}-\vec{R})}{r|\vec{r}-\vec{R}|^3} \right)$$

$$I_1(D) = -\frac{2}{\pi}\int \frac{d\alpha_1 d\alpha_2}{\sqrt{\alpha_1\alpha_2}} \int d^D\vec{r} e^{-(\alpha_1+\alpha_2)r^2 - \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}R^2}$$
$$\to 4\pi R|_{D=3}\;,$$
$$I_2(D) = -2R^2 \frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{\pi}}\int \frac{d\alpha_1 d\alpha_2 \sqrt{\alpha_1\alpha_2}}{\alpha_1+\alpha_2} \int d^D\vec{r} e^{-(\alpha_1+\alpha_2)r^2 - \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}R^2}$$
$$\to -4\pi R\;.$$

Together with the kinetic energy contribution, we get

$$\int d^3\vec{r}\; r^2\, T^{ii}(\vec{r}) = mv^2R^2 + \frac{e^2}{32\pi^2}\Big(I_1 + I_2\Big) = \alpha R$$

This is positive, reflecting electron-proton attraction (rather than surface tension).

Logarithmic corrections: Lamb vs D-term

Vacuum fluctuations smear electron's position, $\overline{\delta^2} = \frac{4\alpha}{\pi m^2} \ln \frac{1}{\alpha}$

Lamb

$$E = E^{(0)} \left(1 - \frac{16\alpha^3}{3\pi} \ln \frac{1}{\alpha}\right)$$

$$\frac{\Delta E}{E^{(0)}} \sim \frac{\overline{\delta^2}}{a_B^2}$$

Only S-states are affected

D-term

$$D = D^{(0)} \left(1 - \frac{4\alpha}{3\pi} \ln \frac{1}{\alpha}\right)$$

$$\frac{\Delta D}{D^{(0)}} \sim \frac{\overline{\delta^2}}{\lambda_C^2}$$

Universal log-correction: all states

Welton's interpretation of Lamb shift

Electron's position in the H-atom modified by vacuum fluctuations.
This changes the potential experienced by the electron,

$$\langle U_c(\boldsymbol{r}+\boldsymbol{q})\rangle = U_c(r) + \underbrace{\langle \boldsymbol{q}\rangle}_{0} \cdot \boldsymbol{\nabla} U_c + \frac{1}{2}\underbrace{\left\langle q^i q^j\right\rangle}_{\frac{\delta_{ij}}{3}\langle q^2\rangle} \nabla^i \nabla^j U_c + \ldots,$$

$$\delta U = \langle U_c(\boldsymbol{r}+\boldsymbol{q})\rangle - U_c(r) \simeq \frac{1}{6}\left\langle q^2\right\rangle \nabla^2 U_c = \frac{1}{6}\left\langle q^2\right\rangle \alpha 4\pi\delta^3(\boldsymbol{r}).$$

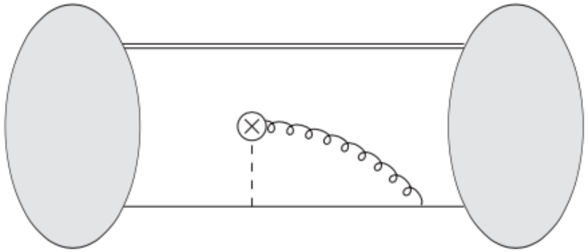
Mean-squared displacement $<q^2>$:

$$\begin{aligned}\left\langle q^2\right\rangle &= 2\int\left(\frac{e}{m\omega^2}\right)^2 \frac{V\mathrm{d}^3k}{(2\pi)^3} E_{\boldsymbol{k}}^2 \\ &= \frac{2\alpha}{\pi m^2}\int\frac{\mathrm{d}k}{k}.\end{aligned}$$

$$\left\langle q^2\right\rangle = \frac{2\alpha}{\pi m^2}\ln\frac{1}{\alpha^2} + \text{non-logarithmic terms.}$$

$$\langle\delta U\rangle_{2S} = \frac{m}{3\pi}\alpha^5 \ln\frac{1}{\alpha}$$

Log correction to the D-term



Ji & Liu 2022

$$D_{\rm NLO} = \frac{\alpha}{6\pi}\sum_M \frac{2\vec{v}_{0M}\cdot\vec{v}_{M0}}{D(E_M-E_0)}\left(\ln\frac{4(E_M-E_0)^2}{m_e^2}-\frac{1}{4}\right)$$

dimension $D = 3$

Coefficient of the log: $$\sum_M \frac{2\vec{v}_{0M}\cdot\vec{v}_{M0}}{D(E_M-E_0)} \equiv \frac{1}{m_e}$$

Summary

- There are interesting observables in atoms, related to the energy-momentum tensor, in addition to the usually studied electromagnetic current.
- Atomic examples help understand properties of the EMT
- Sign of the D-term can be positive for a stable system
- Logarithmic corrections to the D-term are universal, affecting not only S-states.