High-precision calculations of the electron anomalous magnetic moment in quantum electrodynamics

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Electron anomalous magnetic moment: current status

Experiment:

a_e=0.00115965218073(28) [**2011**, D. Hanneke, S. Fogwell Hoogerheide, G. Gabrielse, Phys. Rev. A 83, 052122]

a_e=0.00115965218059(13) [**!!!NEW!!!**, **2022**, X. Fan, T. G. Myers, B. A. D. Sukra, G. Gabrielse, Phys. Rev. Lett. 130, 071801]

Theory:

$$a_e = a_e(QED) + a_e(hadronic) + a_e(electroweak),$$

$$a_e(QED) = \sum_{n\geqslant 1} \left(\frac{\alpha}{\pi}\right)^n a_e^{2n},$$

$$a_e^{2n} = A_1^{(2n)} + A_2^{(2n)}(m_e/m_\mu) + A_2^{(2n)}(m_e/m_\tau) + A_3^{(2n)}(m_e/m_\mu, m_e/m_\tau)$$

a_e=0.001159652181606(11)(12)(299)

2019, T. Aoyama, T. Kinoshita, M. Nio, Atoms, 7, 28

Uncertainties come from: $A_1^{(10)}$, hadronic+electroweak, α

α^{-1}=137.035999046(27) [**2018**, R. H. Parker et al., Science, V. 360, Is. 6385, pp. 191-195]

$A_1^{(10)}$[Aoyama,Hayakawa,Kinoshita,Nio(AHKN-2019)] = **6.737(159)**

$A_1^{(10)}$[S.Volkov+AHKN-2019] = **5.862(90)** [**2019**, S. Volkov, Phys. Rev. D 100, 096004]

(**4.8σ discrepancy**)

The history of the universal QED contributions $A_1^{(2n)}$ calculations

$$a_e(QED)=\sum_{n\geqslant 1}\left(\frac{\alpha}{\pi}\right)^n a_e^{2n},$$

$$a_e^{2n}=A_1^{(2n)}+A_2^{(2n)}(m_e/m_\mu)+A_2^{(2n)}(m_e/m_\tau)+A_3^{(2n)}(m_e/m_\mu,m_e/m_\tau)$$

- J.Schwinger [**1948**], analytically: $A_1^{(2)}$=**0.5**
- R. Karplus, N. Kroll [**1949**] – $A_1^{(4)}$ with a mistake
  A.Petermann [**1957**], C. Sommerfield [**1958**], analytically: $A_1^{(4)}$=**-0.328478966...**
- ~**1970**...~**1975**, $A_1^{(6)}$, numerically:
  1. M.Levine, J. Wright.
  2. R. Carroll, Y. Yao.
  3. T. Kinoshita, P. Cvitanović.
     T. Kinoshita, P. Cvitanović [**1974**]: $A_1^{(6)}$=**1.195±0.026**
- E. Remiddi, S. Laporta et al., ~**1965**..**1996**, analytically: $A_1^{(6)}$=**1.181241456**...
- T. Kinoshita, M. Nio et al., numerically, **2015**: $A_1^{(8)}$=**-1.91298(84)**
  (first estimations in 1980-x)
- S. Laporta, semianalytically, **2017**: $A_1^{(8)}$=**-1.9122457649...**
- T. Kinoshita, M. Nio et al., numerically, **2019**: $A_1^{(10)}$=**6.737(159)**
  $A_1^{(10)}$[no lepton loops]=**7.668(159),** $A_1^{(10)}$[with lepton loops]=**-0.931...** [**NOT DOUBLE-CHECKED**]
  S. Volkov, numerically, **2019**: $A_1^{(10)}$[no lepton loops]=**6.793(90)** [**DISCREPANCY 4.8σ!!!**]

The history of the mass-dependent QED contributions calculations

$$a_e(QED)=\sum_{n\geqslant 1}\left(\frac{\alpha}{\pi}\right)^n a_e^{2n},$$

$$a_e^{2n}=A_1^{(2n)}+A_2^{(2n)}(m_e/m_\mu)+A_2^{(2n)}(m_e/m_\tau)+A_3^{(2n)}(m_e/m_\mu,m_e/m_\tau)$$

- H.H.Elend [**1966**], analytically (**recently obtained masses are substituted into the analytical results**):
  for electron $A_2^{(4)}(m_e/m_\mu)=0.519738676(24)\cdot 10^{-6}$, $A_2^{(4)}(m_e/m_\tau)=0.183790(25)\cdot 10^{-8}$,
  for muon $A_2^{(4)}(m_\mu/m_e)=1.0942583093(76)$, $A_2^{(4)}(m_\mu/m_\tau)=0.000078076(11)$.
- ~**1970**...~**1990**, numerically, different research groups, increasing precision:
  J. Aldins, S. J. Brodsky, C. Chlouber, A. J. Dufner, T. Kinoshita, W. J. Marciano, B. Nizic, Y. Okamoto, M. A. Samuel...
  T. Kinoshita, W. J. Marciano [**1990**]: for muon $A_2^{(6)}(m_\mu/m_e)=\mathbf{22.8671(22)}$.
- M. A. Samuel, G. Li, S. Laporta, E. Remiddi [**1991-1993**], analytically:
  for electron $A_2^{(6)}(m_e/m_\mu)=\mathbf{-0.737394164(24)\cdot 10^{-5}}$, $A_2^{(6)}(m_e/m_\tau)=\mathbf{-0.658273(79)\cdot 10^{-7}}$,
  for muon $A_2^{(6)}(m_\mu/m_e)=\mathbf{22.86837998(20)}$, $A_2^{(6)}(m_\mu/m_\tau)=\mathbf{0.000360671(94)}$.
- S. Laporta [**1993**], semianalytically: for muon $A_3^{(6)}(m_\mu/m_e,m_\mu/m_\tau)=\mathbf{0.0005238(19)}$.
- A. Czarnecki, M. Skrzypek, B. Krause [**1999**], analytically: for muon $A_3^{(6)}(m_\mu/m_e,m_\mu/m_\tau)=\mathbf{0.000527738(75)}$.
- T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio [**2012**], numerically:
  for electron $A_2^{(8)}(m_e/m_\mu)=\mathbf{0.0009222(66)}$, $A_2^{(8)}(m_e/m_\tau)=\mathbf{8.24(12)\cdot 10^{-6}}$,
  for muon $A_2^{(8)}(m_\mu/m_e)=\mathbf{132.6852(60)}$, $A_2^{(8)}(m_\mu/m_\tau)=\mathbf{0.04234(12)}$, $A_3^{(8)}(m_\mu/m_e,m_\mu/m_\tau)=\mathbf{0.06272(4)}$,
  $A_2^{(10)}(m_\mu/m_e)=\mathbf{742.18(87)}$ [**NOT DOUBLE-CHECKED**], $A_2^{(10)}(m_\mu/m_\tau)=\mathbf{-0.068(5)}$, $A_3^{(10)}(m_\mu/m_e,m_\mu/m_\tau)=\mathbf{2.011(10)}$.
- A. Kurz, T. Liu, P. Marquard, M. Steinhauser [**2013**], semianalytically:
  for electron $A_2^{(8)}(m_e/m_\mu)=\mathbf{0.009161970703(372)}$, $A_2^{(8)}(m_e/m_\tau)=\mathbf{7.42924(118)\cdot 10^{-6}}$,
  for muon $A_2^{(8)}(m_\mu/m_\tau)=\mathbf{0.0424941(53)}$.
- A. Kurz, T. Liu, P. Marquard, V. A. Smirnov, A. V. Smirnov, M. Steinhauser [**2016**], semianalytically:
  for muon $A_2^{(8)}(m_\mu/m_e)=\mathbf{132.86(48)}$, $A_3^{(8)}(m_\mu/m_e,m_\mu/m_\tau)=\mathbf{0.0627220(100)}$.

Measurements of the fine-structure constant α, their connection with $A_1^{(10)}$, current status

- $A_1^{(10)}$[no lepton loops, AHKN-2019] = 7.668(159)

  (AHKN = T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio)

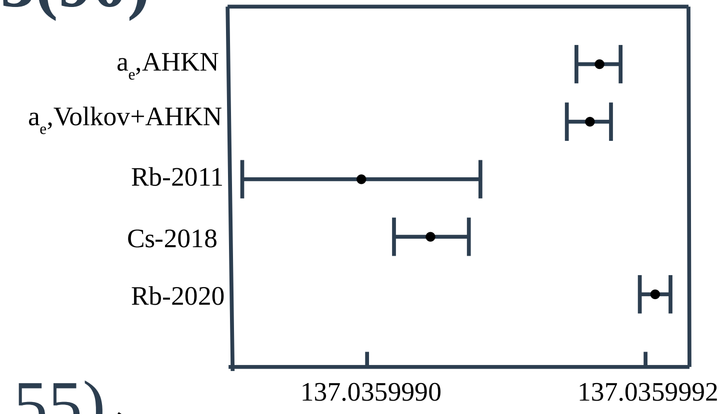
  $A_1^{(10)}$[no lepton loops, Volkov-2019] = 6.793(90)

  S. Volkov, Phys. Rev. D 100, 096004 (2019)

  $A_1^{(10)}$[AHKN] = 6.737(159)

  $A_1^{(10)}$[Volkov+AHKN] = 5.862(90)

  4.8σ



- $\alpha^{-1}[a_e$, AHKN] = 137.0359991663(155)

  $\alpha^{-1}[a_e$, Volkov+AHKN] = 137.0359991593(155)

- α^{-1}[Rb-2011] = 137.035998996(85)

  (PRL 106, 080801, 2011 + CODATA-2014)

  3.64σ

  3.86σ

  α^{-1}[Cs-2018] = 137.035999046(27)

  (Science 360, 191, 2018)

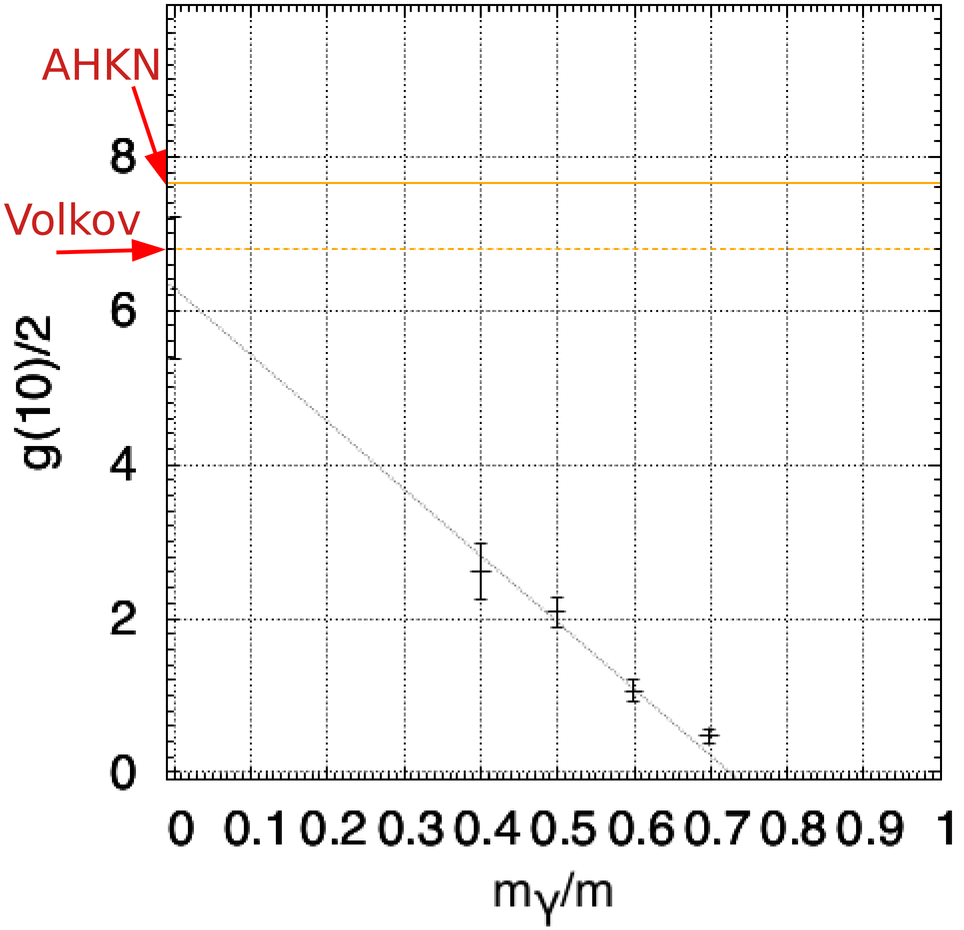
  5.4σ

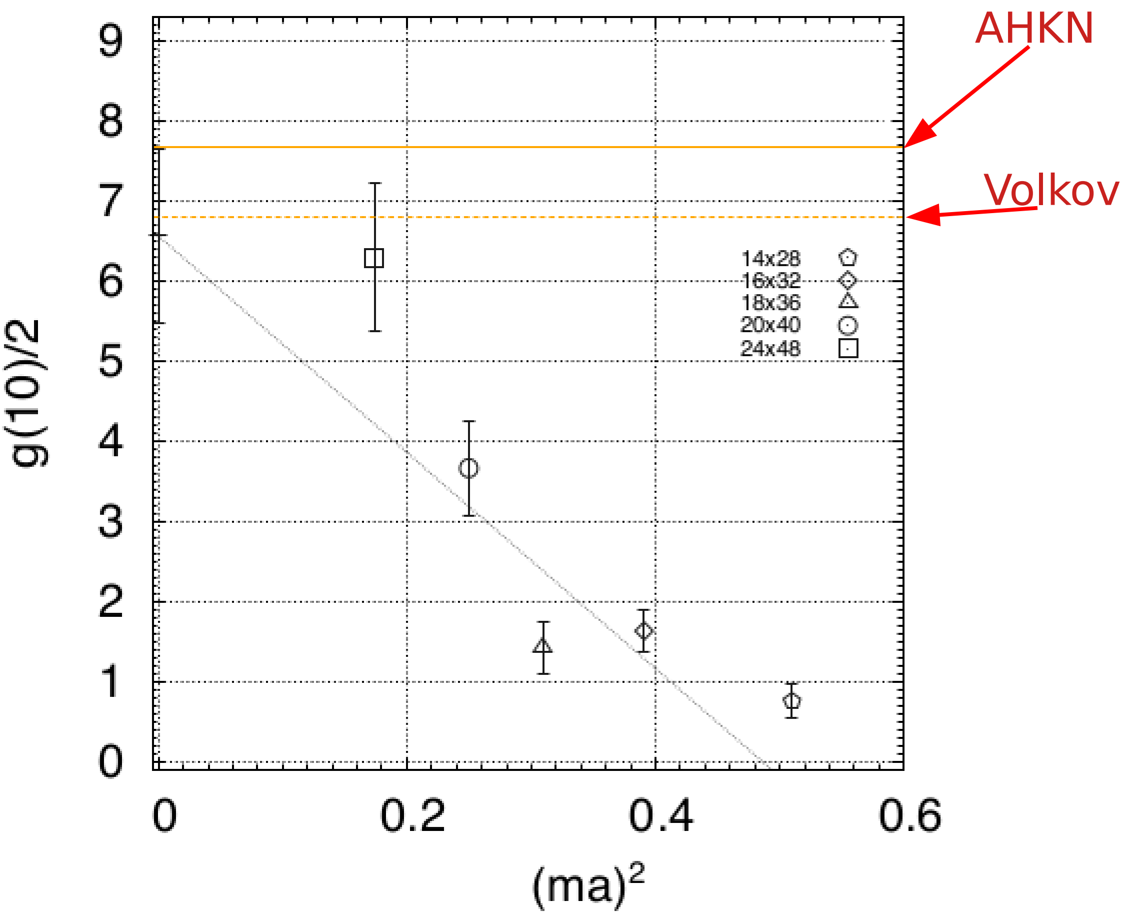
  α^{-1}[Rb-2020] = 137.035999206(11)

  (Nature 588, 61, 2020)

First independent check of $A_1^{(10)}$ [2022]

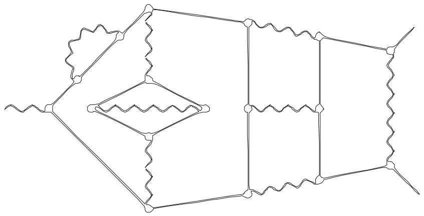
R. Kitano, H. Takaura, arXiv:2210.05569





Lepton anomalous magnetic moment, Feynman diagrams

- Lepton anomalous magnetic moments are extracted from Feynman diagrams with two external lepton lines and one external photon line.
- The corresponding intergrals have divergences (ultraviolet, infrared, mixed).
- The removal of the divergences is connected with the renormalization of physical parameters.
- However, the structure of divergences is complicated; their cancellation is inter-diagram.



an example of a Feynman diagram

Calculations with and without infinitesimal regularization parameters

Most of modern quantum field theory calculations use dimensional regularization. This has advantages and disadvantages:

- a possibility not to think about the structure of divergences;
- reduction to finite integrals requires an enormous amount of symbolic manipulations at higher orders.

Calculations without infinitesimal regularization parameters (point-by-point subtraction of divergences):

- finite integrals are required from the beginning;
- feasible at higher orders.

My calculation method: overview

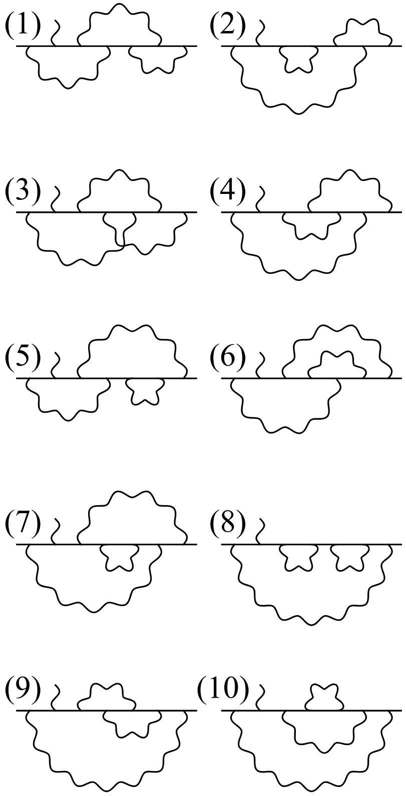
S. Volkov, J. Exp. Theor. Phys. 122, 1008 [2016]; S. Volkov, Phys. Part. Nuclei 53, 805 [2022]

- Fully automated at each order of the perturbation series.
- Very good agreement with known results up to 4 loops.
- Finite Feynman parametric integral for each individual Feynman diagram:
  $\int_{z\geq 0}\delta(z_1+\ldots+z_n-1)f(z_1,\ldots,z_n)dz_1\ldots dz_n$.
- The final values $A_1^{(2n)}$, $A_2^{(2n)}(m_e/m_\mu)$, $A_2^{(2n)}(m_e/m_\tau)$, $A_3^{(2n)}(m_e/m_\mu,m_e/m_\tau)$ are the sums of the corresponding diagram contributions. No residual renormalization is required. **My method is the first one satisfying this condition.**
- It preserves small gauge-invariant classes.
- **Currently it is the fastest method of high-order calculations in quantum field theory** (but it works currently only for lepton magnetic moments and QED).

Gauge-invariant classes

- Calculating the total 10-th order contribution requires too much computer resources.
- Splitting the total contribution into small parts is very useful for independent checking.
- Individual Feynman diagram contributions are strongly method-dependent and often don’t have sense.
- **Gauge-invariant classes help a lot!**
- Any class of one-particle-irreducible diagrams closed with respect to movement of the photon line ends along lepton paths and loops (without jumping over the external photon) is gauge invariant, if we consider a certain class of Lorentz-invariant gauges and the in-place on-shell renormalization.

  (P. Cvitanović, Nucl. Phys. B127 (1977), 176-188)
- Dimensional-regularization-based methods allow to obtain the contributions of individual gauge-invariant classes relatively easy.
- However, it is difficult in the frames of the regularization-free approach. **My method does it! It is the first (and the only) regularization-free method that allows us to obtain the individual contributions of all Predrag Cvitanović’s classes.**



an example of P. Cvitanović’s gauge-invariant class

My results: muon g-2, 6-th order, all QED contributions (A_1, A_2, A_3), comparison [2021]

$A_1^{(6)}=1.18108(18)$

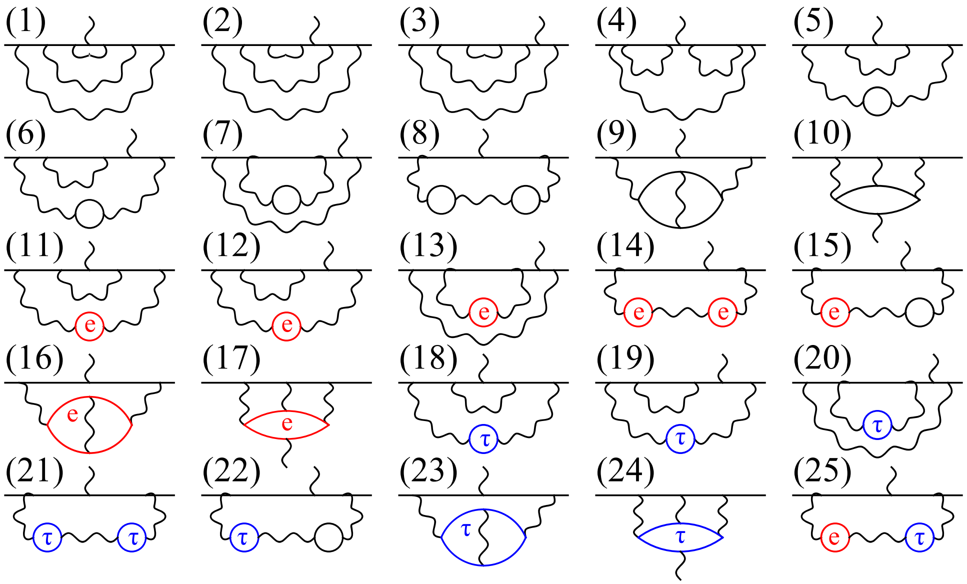
$A_2^{(6)}(m_\mu/m_e)=22.8691(18)$

$A_2^{(6)}(m_\mu/m_\tau)=0.0003614(13)$

$A_3^{(6)}(m_\mu/m_e,m_\mu/m_\tau)=0.0005245(70)$

We use the values: m_μ/m_e = 206.76828103, m_τ/m_μ = 16.81665 (without taking into account their uncertainty)

self-made Monte-Carlo,
~2 days on GPU NVidia Tesla V100 (“Govorun”, Dubna)



Analytical results:
1969-1999
R. Barbieri
D. Billi
M. Caffo
A. Czarnecki
L. L. DeRaad
B. Krause
S. Laporta
M. Levine
G. Li
J. Mignaco
K. A. Milton
R. Perisho
E. Remiddi
R. Roskies
M. A. Samuel
M. Skrzypek
W. Tsai
......

Class	My result	Analytical
1	0.448703(35)	0.44870
2	-0.498224(67)	-0.49825
3	0.533289(54)	0.53336
4	0.421080(43)	0.42117
5	0.050178(16)	0.0501487
6	-0.112324(21)	-0.112336
7	-0.087987(12)	-0.0879847
8	0.0025598(15)	0.0025585
9	0.052865(11)	0.05287
10	0.37094(15)	0.371005
11	1.61752(28)	?
12	-2.06183(39)	?
13	-1.94880(28)	?
11-13	-2.39311(56)	-2.39239181(7)
14	2.71885(62)	2.7186557(2)
15	0.10038(12)	0.100519296(3)
16	1.49545(92)	1.49367180(4)
17	20.9475(13)	20.9471(29)
18	0.00054410(20)	?
19	-0.00160008(30)	?
20	-0.00106137(17)	?
18-20	-0.00211735(40)	-0.00211713
21	0.0000002766(30)	0.000000277833
22	0.000038704(63)	0.0000386875
23	0.000295496(73)	0.000295557
24	0.0021443(12)	0.00214331
25	0.0005245(70)	0.000527761

My results: electron and muon g-2, 8-th order [2023]

Universal part:

$A_1^{(8)}$=-1.9118(41)

Electron, A_2:

$A_2^{(8)}(m_e/m_\mu)$=0.000924(11)

$A_2^{(8)}(m_e/m_\tau)$=0.00000710(60)

Electron, A_3:

$A_3^{(8)}(m_e/m_\mu,m_e/m_\tau)$=0.000000745(24)

Muon, A_2:

$A_2^{(8)}(m_\mu/m_e)$=132.673(84)

$A_2^{(8)}(m_\mu/m_\tau)$=0.04252(11)

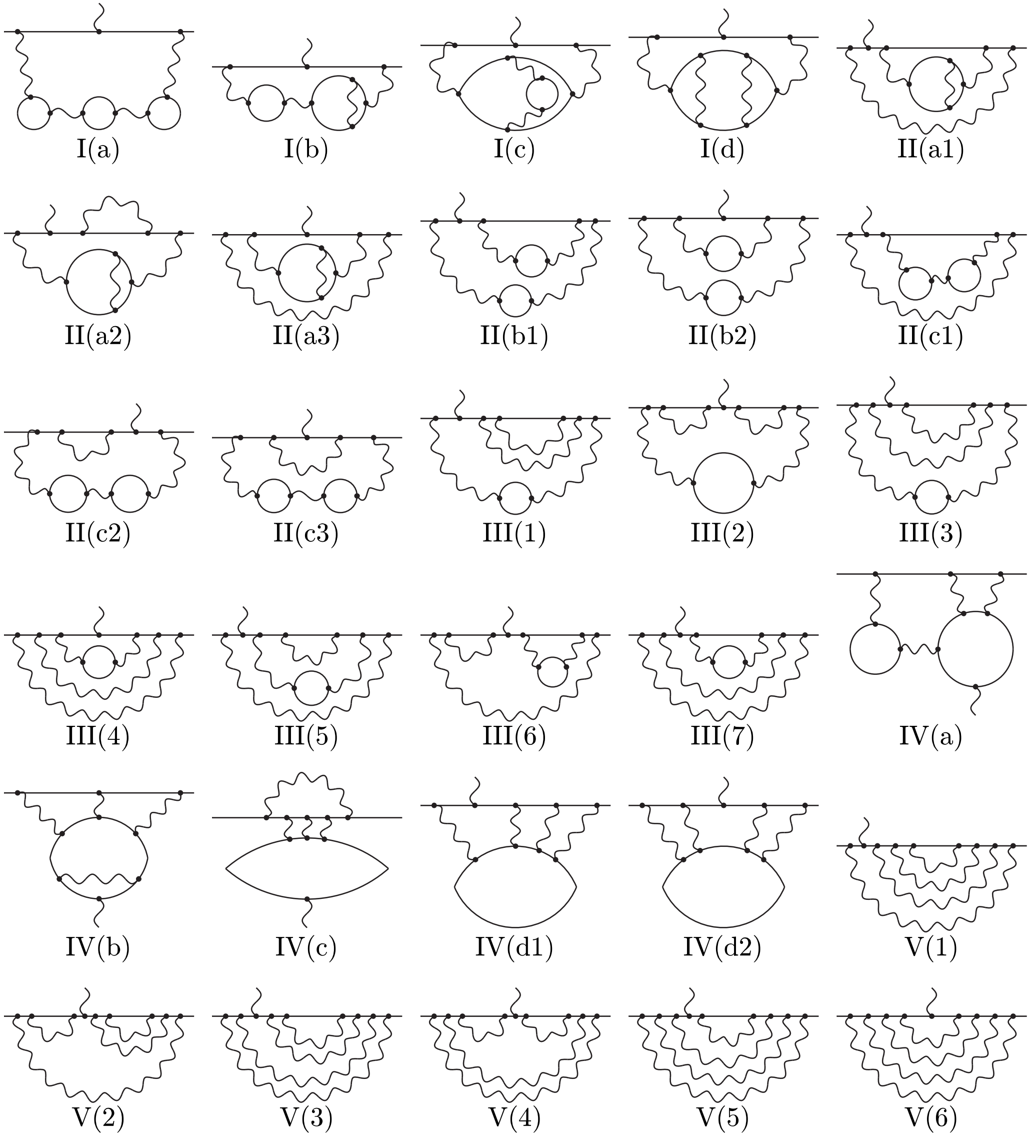
Muon, A_3:

$A_3^{(8)}(m_\mu/m_e,m_\mu/m_\tau)$=0.0622(23)

We use the values: m_μ/m_e = 206.76828103, m_τ/m_μ = 16.81665
(without taking into account their uncertainty)

self-made Monte-Carlo, ~3 weeks on GPU NVidia A100 (ITP/TTP KIT computing cluster)

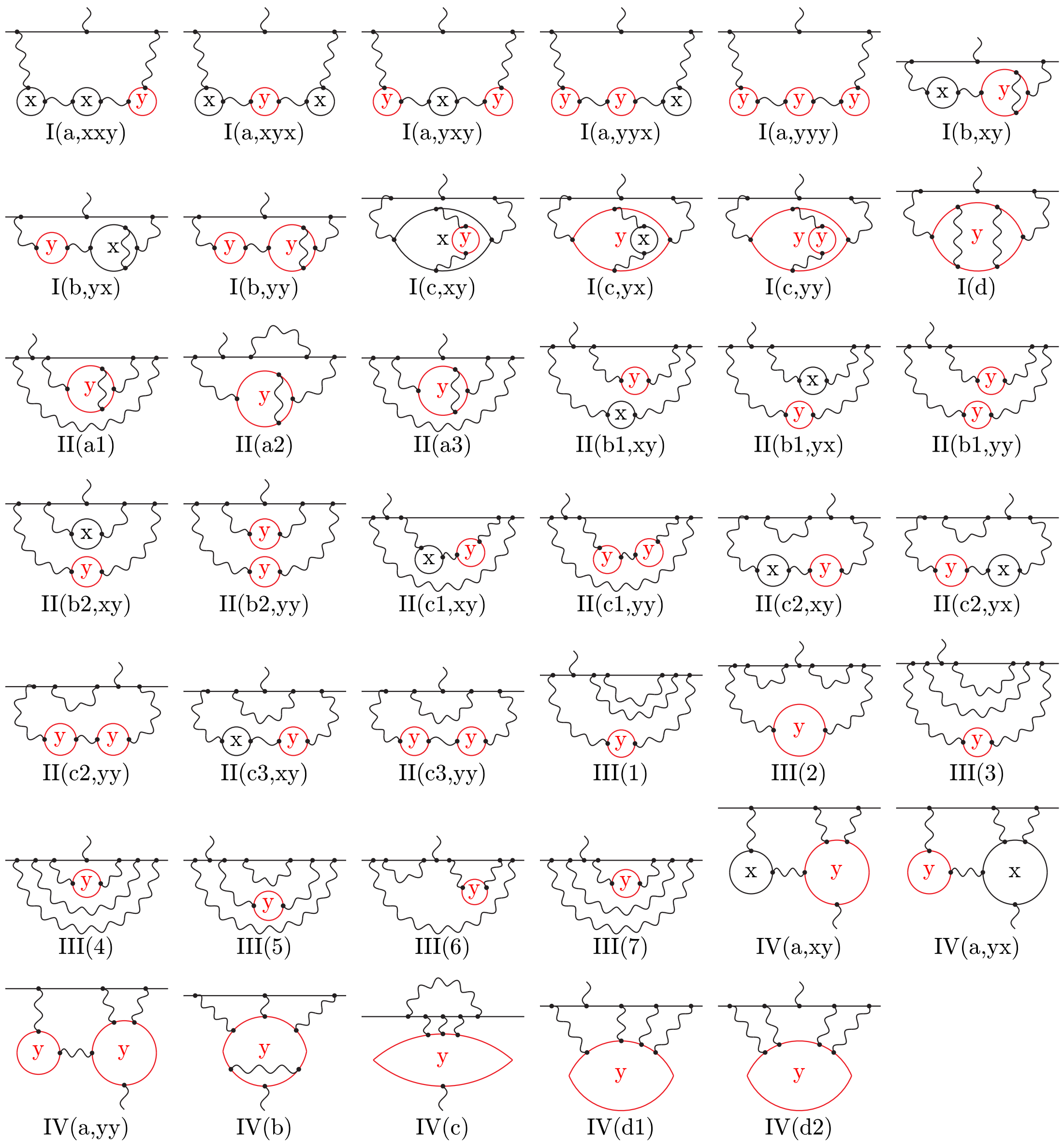
My results: electron and muon g-2, 8-th order, universal (mass-independent) contributions ($A_1^{(8)}$) [2023], comparison with Laporta's values



Class	My value	Laporta's value
I(a)	0.00087614(64)	0.0008768659
I(b)	0.0153300(39)	0.0153252829
I(c)	0.0111324(13)	0.0111309140
I(d)	0.049544(60)	0.0495132026
II(a1)	-0.255082(24)	-
II(a2)	-0.317402(29)	-
II(a3)	0.151986(16)	0.1519895997
II(b1)	-0.0341771(30)	-0.034179376
II(b2)	0.0065025(10)	0.0065041484
II(c1)	-0.0376103(54)	-
II(c2)	-0.0536815(84)	-
II(c3)	0.0178533(40)	0.0178536865
III(1)	0.128258(83)	-
III(2)	0.242948(42)	-
III(3)	-0.04388(10)	-
III(4)	0.374317(49)	0.3743579348
III(5)	0.562521(92)	-
III(6)	0.166229(64)	-
III(7)	-0.012485(63)	-
IV(a)	0.598864(69)	0.598842072
IV(b)	0.82236(20)	0.8222844858
IV(c)	-1.13789(52)	-1.1388228765
IV(d1)	-0.87251(21)	-0.872657392
IV(d2)	-0.117978(94)	-0.1179498688
V(1)	-1.9723(12)	-1.9710756168
V(2)	-0.6227(20)	-0.6219210635
V(3)	-0.1412(21)	-0.1424873798
V(4)	1.0865(15)	1.0866983948
V(5)	-1.0397(19)	-1.04054241
V(6)	0.51153(71)	0.512462048
II(a1)+II(a2)	-0.572485(38)	-0.5724718621
II(c1)+II(c2)	-0.091292(10)	-0.09130584
III(1)+III(5)	0.69078(12)	0.6904483476
III(2)+III(6)	0.409178(76)	0.4092170285
III(3)+III(7)	-0.05637(12)	-0.0563360902

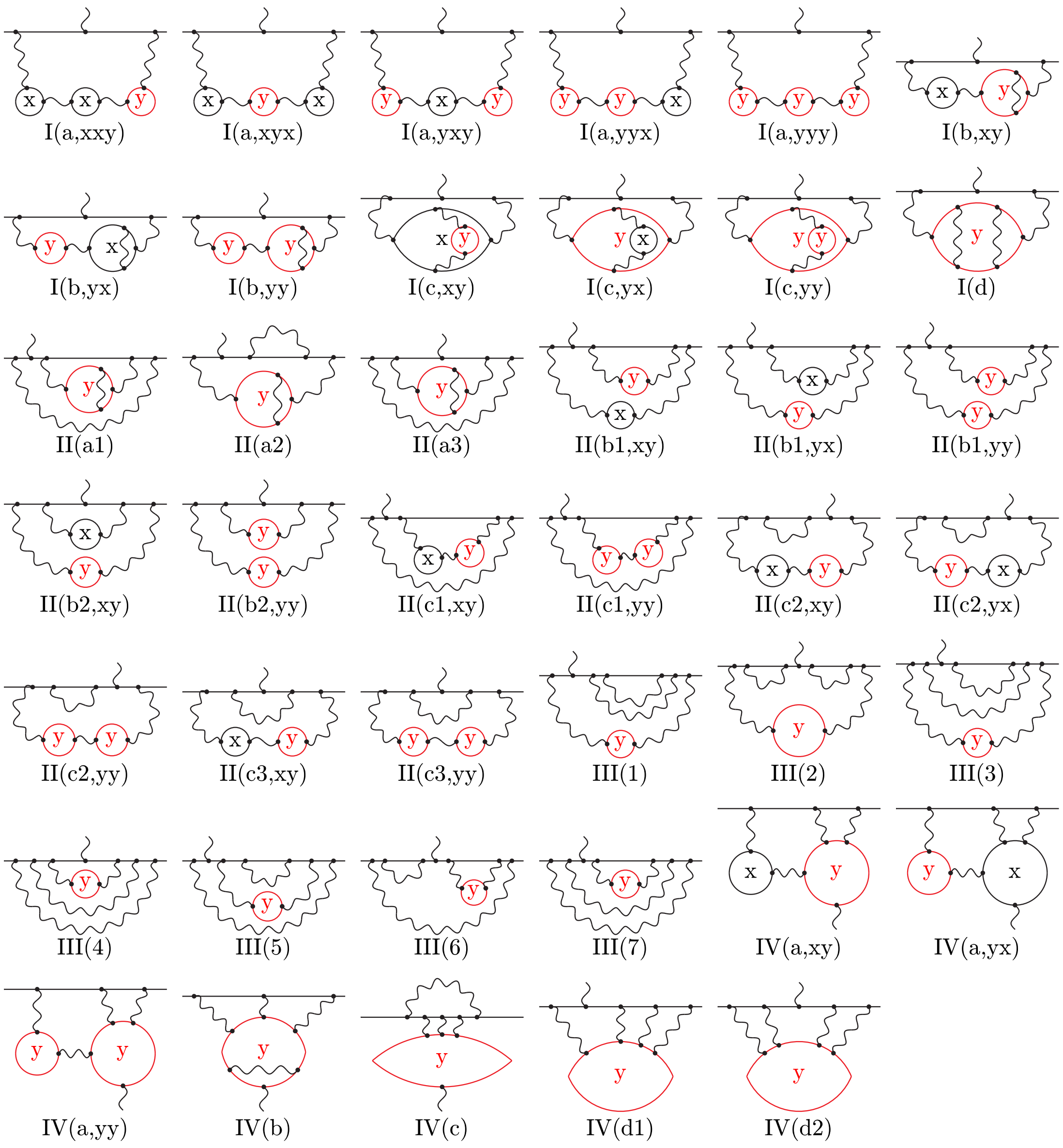
S. Laporta, J. Phys: Conf. Ser. 1085 052008 (2018)

My results: electron and muon g-2, 8-th order, dependent on one mass relation contributions ($A_2^{(8)}$) [2023]



Class	$A_2^{(8)}(m_e/m_\mu)\times 10^3$ x=e, y=μ	$A_2^{(8)}(m_e/m_\tau)\times 10^5$ x=e, y=τ	$A_2^{(8)}(m_\mu/m_e)$ x=μ, y=e	$A_2^{(8)}(m_\mu/m_\tau)$ x=μ, y=τ
I(a,xxy)	0.000168(35)	0.00003(37)	0.01854(19)	0.00002116(31)
I(a,xyx)	0.000077(18)	0.00008(22)	0.009378(80)	0.00001057(16)
I(a,yxy)	0.0000009(22)	-0.000004(13)	0.16473(32)	0.000000274(38)
I(a,yyx)	0.0000034(45)	0.000012(27)	0.32986(60)	0.000000612(77)
I(a,yyy)	0.000000006(22)	0.0000000035(78)	7.2235(18)	0.0000000180(35)
I(b,xy)	0.000963(51)	0.00050(78)	0.1195(10)	0.00014580(54)
I(b,yx)	0.000811(79)	0.0010(11)	0.33397(53)	0.00010541(69)
I(b,yy)	0.00000038(20)	0.000000053(79)	7.1267(62)	0.000001950(32)
I(c,xy)	0.004754(57)	0.00346(41)	0.16205(25)	0.00027729(30)
I(c,yx)	0.005936(32)	0.00361(37)	0.02156(40)	0.00040798(35)
I(c,yy)	0.0003543(26)	0.000127(10)	1.4372(25)	0.000053149(67)
I(d)	0.002457(32)	0.000915(62)	-0.234(15)	0.0003681(11)
II(a1)	-0.041120(60)	-0.02104(22)	-2.3873(57)	-0.0037194(13)
II(a2)	-0.07041(11)	-0.04147(57)	-2.4684(73)	-0.0054854(21)
II(a3)	0.025403(57)	0.01748(23)	2.0651(80)	0.0018698(11)
II(b1,xy)	-0.017667(63)	-0.01333(56)	-0.37697(57)	-0.00098634(55)
II(b1,yx)	-0.027301(55)	-0.02399(68)	-0.31493(55)	-0.00126397(53)
II(b1,yy)	-0.00090489(12)	-0.000319911(44)	-6.2229(30)	-0.000136912(20)
II(b2,xy)	0.006692(24)	0.00684(22)	0.16083(31)	0.00033086(24)
II(b2,yy)	0.000133460(43)	0.000047178(15)	2.1911(19)	0.0000204864(67)
II(c1,xy)	-0.04082(29)	-0.0306(33)	-0.6564(10)	-0.0022605(23)
II(c1,yy)	-0.0009045(88)	-0.000336(36)	-6.0791(24)	-0.00014068(19)
II(c2,xy)	-0.04159(30)	-0.0373(30)	-0.4278(10)	-0.0019306(26)
II(c2,yx)	-0.04159(30)	-0.0373(30)	-0.4278(10)	-0.0019306(26)
II(c2,yy)	-0.002274(22)	-0.000921(84)	-6.7191(37)	-0.00027518(40)
II(c3,xy)	0.03193(20)	0.0337(19)	0.36033(65)	0.0011914(16)
II(c3,yy)	0.0010686(85)	0.000539(36)	4.6173(27)	0.00009622(17)
III(1)	-0.0384(11)	-0.0679(70)	1.7884(74)	0.000526(12)
III(2)	0.23468(44)	0.2280(34)	1.7049(78)	0.0098734(51)
III(3)	0.0860(23)	0.073(10)	-2.703(14)	0.003569(14)
III(4)	0.2010(17)	0.1436(61)	4.375(14)	0.0111379(75)
III(5)	0.2768(11)	0.2027(75)	4.400(10)	0.016024(11)
III(6)	0.05621(29)	0.0395(15)	2.565(10)	0.0035162(48)
III(7)	0.00390(33)	-0.0013(16)	-1.3382(82)	0.0005628(56)
IV(a,xy)	0.1540(17)	0.054(16)	2.6961(53)	0.012611(17)
IV(a,yx)	0.4639(13)	0.364(13)	4.3409(74)	0.023262(12)
IV(a,yy)	0.01755(10)	0.00658(34)	116.802(27)	0.0026298(27)
IV(b)	0.04174(90)	0.0118(30)	-0.423(52)	0.006132(20)
IV(c)	-0.1954(50)	-0.114(25)	2.860(47)	-0.018337(83)
IV(d1)	-0.1693(70)	-0.092(38)	-3.5037(79)	-0.015203(55)
IV(d2)	-0.0051(45)	-0.000(28)	-0.9187(49)	-0.000555(22)

My results: electron g-2, 8-th order, $A_2^{(8)}$, comparison with known values



$A_2^{(8)}(m_e/m_\mu)$, $x=e$, $y=\mu$

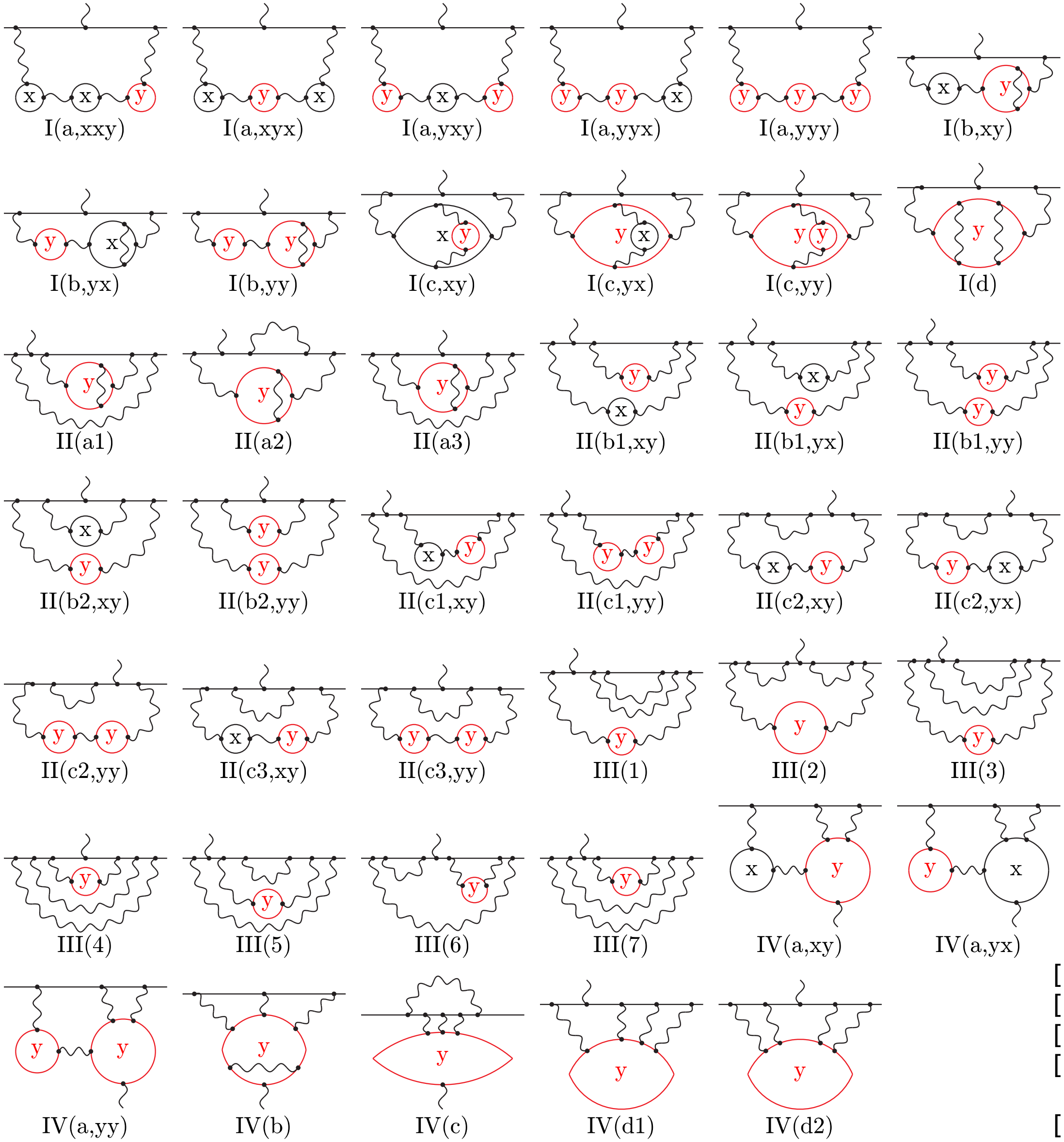
Class	My value ×1000	Previously known value	Ref.
I(a)	0.000250(40)	0.0002264474	[2]
I(b)	0.001774(94)	0.001704139(76)	[1]
I(c)	0.011044(66)	0.0110072(15)	[1]
I(d)	0.002457(32)	0.0024726876	[2]
II(a)	-0.08613(14)	-0.0864460(90)	[1]
II(b)	-0.039048(87)	-0.0390003(27)	[1]
II(c)	-0.09418(55)	-0.095097(24)	[1]
III	0.8203(33)	0.8171525156	[2]
IV(a)	0.6354(21)	0.6357881037	[2]
IV(b)	0.04174(90)	0.0415736717	[2]
IV(c)	-0.1954(50)	-0.19548262120(20)	[2]
IV(d)	-0.1744(83)	-0.1778(12)	[2]
I(a,yyy)	0.000000006(22)	0.000000001033	[3]
I(a,yxy)+I(a,yyx)	0.0000043(51)	0.0000001346	[3]
I(a,xxy)+I(a,xyx)	0.000246(40)	0.0002263118	[3]

$A_2^{(8)}(m_e/m_\tau)$, $x=e$, $y=\tau$

Class	My value ×100000	Previously known value	Ref.
I(a)	0.00012(43)	0.0000802467	[2]
I(b)	0.0015(14)	0.000602805(26)	[1]
I(c)	0.00720(55)	0.0069819(12)	[1]
I(d)	0.000915(62)	0.0008745506	[2]
II(a)	-0.04503(65)	-0.0456480(70)	[1]
II(b)	-0.03075(90)	-0.0303937(42)	[1]
II(c)	-0.0721(57)	-0.071697(25)	[1]
III	0.617(16)	0.6059301962	[2]
IV(a)	0.425(21)	0.4510496216	[2]
IV(b)	0.0118(30)	0.0147081582	[2]
IV(c)	-0.114(25)	-0.0978886097	[2]
IV(d)	-0.092(47)	-0.0927(13)	[2]
I(a,yyy)	0.0000000035(78)	0.00000000000000001293	[3]
I(a,yxy)+I(a,yyx)	0.000008(30)	0.00000000000000395366	[3]
I(a,xxy)+I(a,xyx)	0.00011(43)	0.0000802462	[3]

[1] T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, Phys. Rev. Lett. 109, 111807 (2012).
[2] A. Kurz, T. Liu, P. Marquard, M.Steinhauser, Nucl. Phys. B 879 (2014) 1-18.
[3] O. Solovtsova, V. Lashkevich, A. Sidorov, EPJ Web of Conferences 222, 03007 (2019).

My results: muon g-2, 8-th order, electron contributions, $A_2^{(8)}(m_\mu/m_e)$, comparison with known values

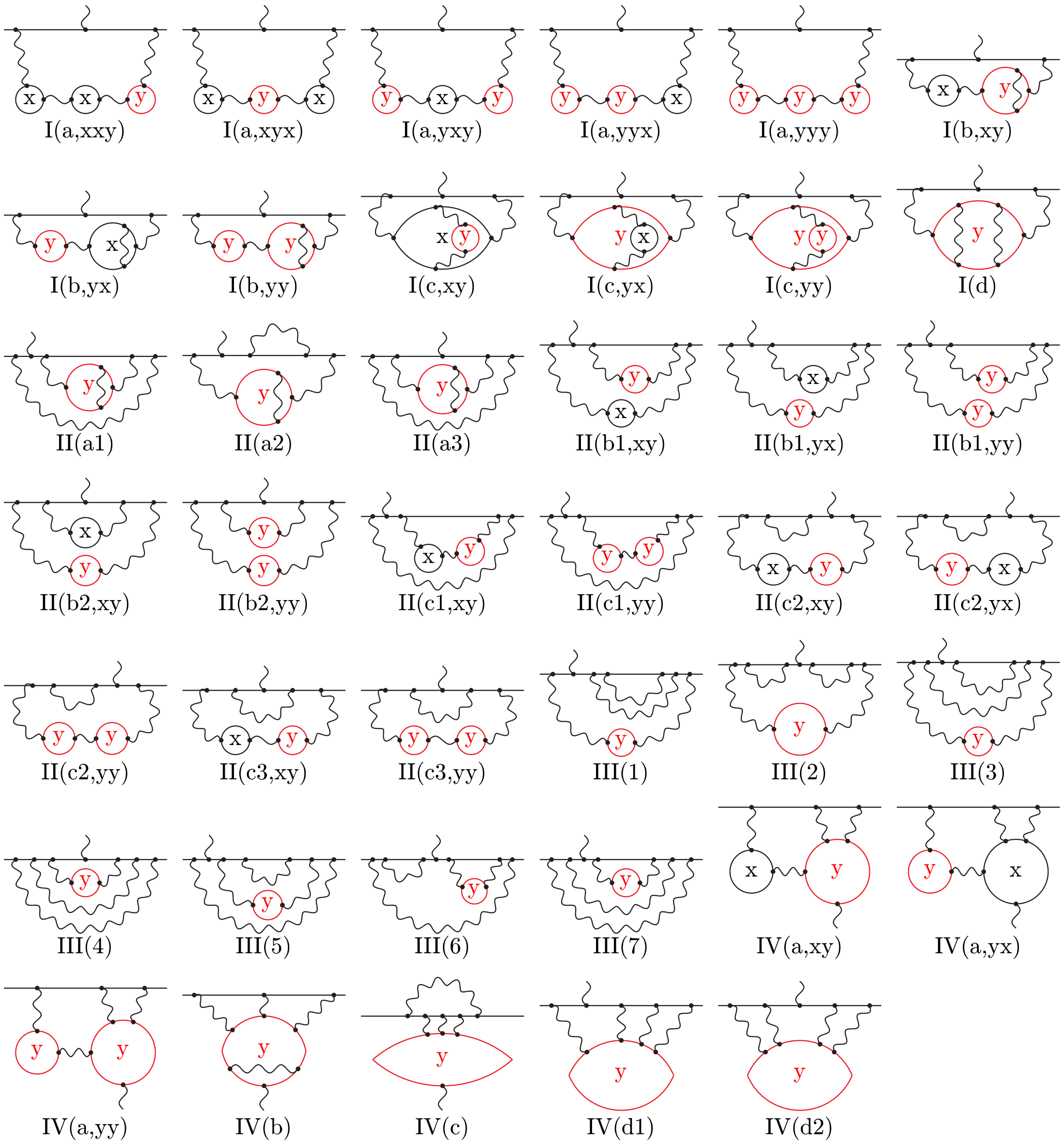


x=μ, y=e

Class	My value	Previously known value	Ref.
I(a)	7.7460(19)	7.7451360000	[5]
I(b)	7.5802(63)	7.58201(71)	[4]
I(c)	1.6208(25)	1.624307(40)	[4]
I(d)	-0.234(15)	-0.2303620(50)	[2]
II(a)	-2.791(12)	-2.77885	[5]
II(b)	-4.5628(37)	-4.55277(30)	[4]
II(c)	-9.3325(55)	-9.34180(83)	[4]
III	10.792(28)	10.7934(27)	[4]
IV(a)	123.839(29)	123.78551(44)	[4]
IV(b)	-0.423(52)	-0.4170(37)	[4]
IV(c)	2.860(47)	2.9072(44)	[4]
IV(d)	-4.4225(93)	-4.43243(58)	[4]
I(a,yyy)	7.2235(18)	7.22307640(80)	[1]
I(b,xy)	0.1195(10)	0.1196024600(20)	[1]
I(b,yx)	0.33397(53)	0.333664680(10)	[1]
I(b,yy)	7.1267(62)	7.12800840(20)	[1]
I(c,xy)	0.16205(25)	0.161982(11)	[3]
I(c,yx)	0.02156(40)	0.0215830(20)	[3]
I(c,yy)	1.4372(25)	1.440744(16)	[3]
IV(a,xy)	2.6961(53)	2.69(14)	[5]
IV(a,yx)	4.3409(74)	4.33(17)	[5]
IV(a,yy)	116.802(27)	116.760(20)	[5]
I(a,yxy)+I(a,yyx)	0.49459(68)	0.494072030(30)	[1]
I(a,xxy)+I(a,xyx)	0.02792(20)	0.0279883220(70)	[1]
II(b1,yy)+II(b2,yy) +II(c1,yy)+II(c2,yy) +II(c3,yy)	-12.2126(63)	-12.2126310000	[5]
II(b1,xy)+II(b1,yx) +II(b2,xy)+II(c1,xy) +II(c2,xy)+II(c2,yx) +II(c3,xy)	-1.6828(20)	-1.683165(13)	[5]

[1] S. Laporta, Phys. Lett. B 312 (1993), 495-500.
[2] P. A. Baikov, D. J. Broadhurst (1995), arXiv:hep-ph/9504398.
[3] T. Kinoshita, M. Nio, Phys. Rev. D 70, 113001 (2004).
[4] T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, Phys. Rev. Lett. 109, 111808 (2012).
[5] A. Kurz, T. Liu, P. Marquard, A. V. Smirnov, V. A. Smirnov, M. Steinhauser, Phys. Rev. D 93, 053017 (2016) .

My results: muon g-2, 8-th order, tauon contributions, $A_2^{(8)}(m_\mu/m_\tau)$, comparison with known values

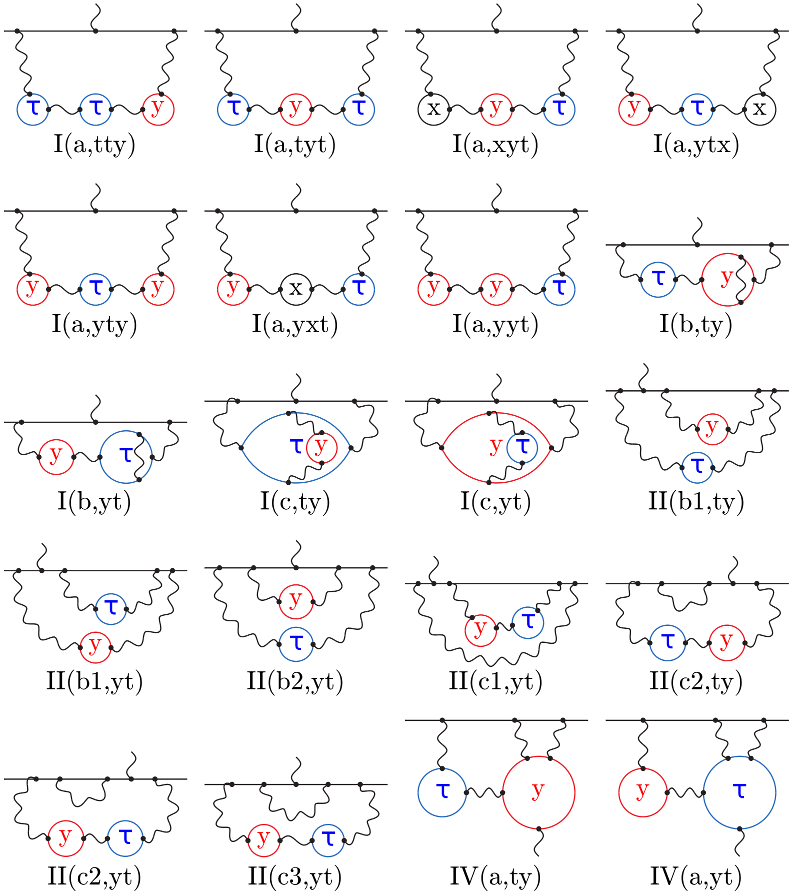


$x=\mu$, $y=\tau$

Class	My value	Previously known value	Ref.
I(a)	0.00003263(36)	0.00003242810(20)	[2]
I(b)	0.00025316(87)	0.000252	[1]
I(c)	0.00073843(47)	0.000737	[1]
I(d)	0.0003681(11)	0.0003677960(40)	[2]
II(a)	-0.0073350(26)	-0.0073290(10)	[1]
II(b)	-0.00203588(80)	-0.002036	[1]
II(c)	-0.0052501(46)	-0.0052460(10)	[1]
III	0.045209(24)	0.0452089860(60)	[2]
IV(a)	0.038504(20)	0.038519670(30)	[2]
IV(b)	0.006132(20)	0.006126610(50)	[2]
IV(c)	-0.018337(83)	-0.01830100(10)	[2]
IV(d)	-0.015757(59)	-0.015868(37)	[2]
I(a,yyy)	0.000000037(14)	0.0000000232	[3]
I(a,yxy)+I(a,yyx)	0.000000885(86)	0.0000008757	[3]
I(a,xxy)+I(a,xyx)	0.00003173(35)	0.0000315291	[3]

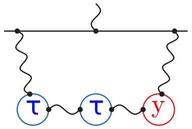
[1] T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, Phys. Rev. Lett. 109, 111807 (2012).
[2] A. Kurz, T. Liu, P. Marquard, M.Steinhauser, Nucl. Phys. B 879 (2014) 1-18.
[3] O. Solovtsova, V. Lashkevich, A. Sidorov, EPJ Web of Conferences 222, 03007 (2019).

My results: electron and muon g-2, 8-th order, dependent on two mass relations contributions ($A_3^{(8)}$) [2023]

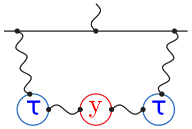


Class	$A_3^{(8)}(m_e/m_\mu, m_e/m_\tau)\times 10^7$ x=e, y=μ, t=τ	$A_3^{(8)}(m_\mu/m_e, m_\mu/m_\tau)$ x=μ, y=e, t=τ
I(a,tty)	-0.000003(17)	-0.0000012(85)
I(a,tyt)	0.0000063(85)	0.0000038(35)
I(a,xyt)	-0.0113(95)	0.000155(34)
I(a,ytx)	-0.0033(93)	0.000133(30)
I(a,yty)	0.000037(36)	0.000884(51)
I(a,yxt)	0.008(11)	0.000147(30)
I(a,yyt)	-0.000022(71)	0.00184(11)
I(b,ty)	0.00025(20)	0.00069(13)
I(b,yt)	0.00001(14)	0.001944(81)
I(c,ty)	0.0965(52)	0.001389(36)
I(c,yt)	0.0797(60)	0.000383(24)
II(b1,ty)	-0.30252(13)	-0.006434(89)
II(b1,yt)	-0.22691(14)	-0.00457(10)
II(b2,yt)	0.070470(61)	0.002062(38)
II(c1,yt)	-0.533(33)	-0.00972(23)
II(c2,yt)	-0.718(63)	-0.00753(20)
II(c2,ty)	-0.718(63)	-0.00753(20)
II(c3,yt)	0.771(29)	0.00498(12)
IV(a,ty)	5.94(16)	0.0482(12)
IV(a,yt)	2.99(15)	0.0351(20)

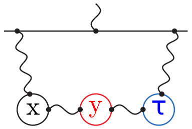
Electron, muon g-2, 8-th order, $A_3^{(8)}$, comparison with known values



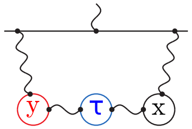
I(a,tty)



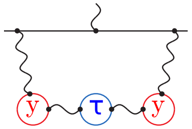
I(a,tyt)



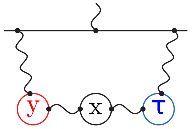
I(a,xyt)



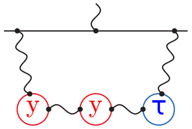
I(a,ytx)



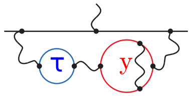
I(a,yty)



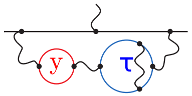
I(a,yxt)



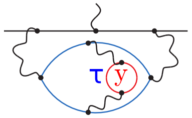
I(a,yyt)



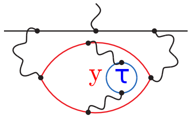
I(b,ty)



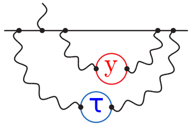
I(b,yt)



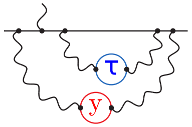
I(c,ty)



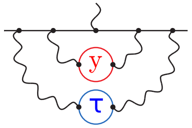
I(c,yt)



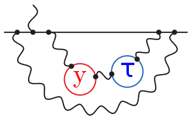
II(b1,ty)



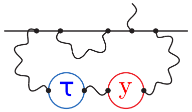
II(b1,yt)



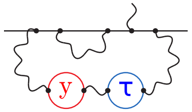
II(b2,yt)



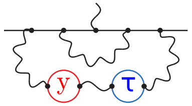
II(c1,yt)



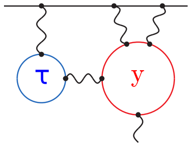
II(c2,ty)



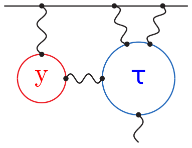
II(c2,yt)



II(c3,yt)



IV(a,ty)



IV(a,yt)

$A_3^{(8)}(m_e/m_\mu, m_e/m_\tau)$, $x=e$, $y=\mu$, $t=\tau$

Class	My value ×1000000	Previously known value	Ref.
I(a)	-0.007(17)	0.0000119956	[4]
I(b)	0.00025(24)	0.0000140970(10)	[2]
I(c)	0.1763(79)	0.172860(21)	[2]
II(b)	-0.45897(20)	-0.458968(17)	[2]
II(c)	-1.20(10)	-1.18969(67)	[2]
IV(a)	8.93(22)	8.9432(25)	[4]

$A_3^{(8)}(m_\mu/m_e, m_\mu/m_\tau)$, $x=\mu$, $y=e$, $t=\tau$

Class	My value	Previously known value	Ref.
I(a)	0.00316(14)	0.003209050(10)	[5]
I(b)	0.00264(16)	0.0026110000	[3]
I(c)	0.001772(43)	0.001807	[3]
II(b)	-0.00895(14)	-0.0090080(10)	[3]
II(c)	-0.01980(38)	-0.0196420(20)	[3]
IV(a)	0.0834(23)	0.083747570(90)	[4]
I(a,yty)+I(a,yyt)	0.00273(12)	0.00274860(90)	[1]

[1] J-P Aguilar, E. de Rafael, D. Greynat, Phys. Rev. D 77, 093010 (2008).
[2] T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, Phys. Rev. Lett. 109, 111807 (2012).
[3] T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, Phys. Rev. Lett. 109, 111808 (2012).
[4] A. Kurz, T. Liu, P. Marquard, M.Steinhauser, Nucl. Phys. B 879 (2014) 1-18.
[5] A. Kurz, T. Liu, P. Marquard, A. V. Smirnov, V. A. Smirnov, M. Steinhauser, Phys. Rev. D 93, 053017 (2016) .

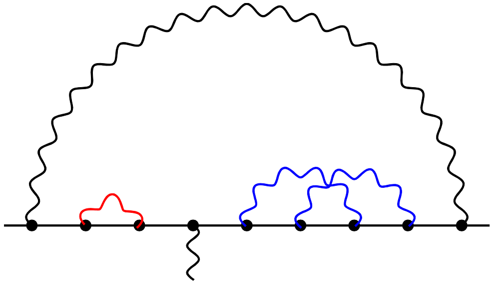
My results: 10-th order, universal (mass-independent) contributions, diagrams without lepton loops [2019]

- **$A_1^{(10)}$[no lepton loops]=6.793(90)**
- Supercomputer “Govorun” (JINR, Dubna), GPU NVidia Tesla V100
- 4.5 GPU-years
- 3213 undirected Feynman diagrams
- 13-dimensional integrals
- 1.9×10^{14} Monte Carlo samples
- 500 GB of the integrands code (compiled)
- **The contributions of the 9 gauge-invariant classes separately had been obtained for the first time!**

Class	Value
(1,4,0)	6.172(42)
(2,3,0)	-0.724(54)
(1,3,1)	0.895(43)
(3,2,0)	-0.396(43)
(2,2,1)	-2.160(46)
(4,1,0)	-1.017(26)
(1,2,2)	0.301(25)
(3,1,1)	2.624(30)
(5,0,0)	1.0889(80)

S. Volkov, Phys. Rev. D 100, 096004

Example of a diagram from (1,2,1):



(k,m,n):
m and n photon lines to the right and to the left from the external photon (or vice versa),
k photon lines with ends on different sides
(P. Cvitanović, Nucl. Phys. B 127, 176 (1977))

Summary

- New experimental results are coming; anomalies are becoming significant.
- Calculations of the QED contributions to the lepton magnetic moments remain relevant.
- The computational discrepancy at the 10-th order remains. However, independent calculations are coming.
- My method is based on a deep understanding the structure of divergences in Feynman diagrams. It is the fastest method of high-order calculations in quantum field theory, but the problem of extending it beyond QED and lepton magnetic moments remains open.
- The method gives a possibility to obtain the 10-th order contribution on a relatively small supercomputer (it was demonstrated in 2019).
- Also, it gives a possibility to obtain the individual contributions of small gauge-invariant classes. It is useful for independent checking.
- In 2023 I obtained the contributions of 234 small gauge-invariant classes of 8-th order to the electron and muon anomalous magnetic moments; the corresponding *large class* contributions are in good agreement with known results.

Thank you for your attention!