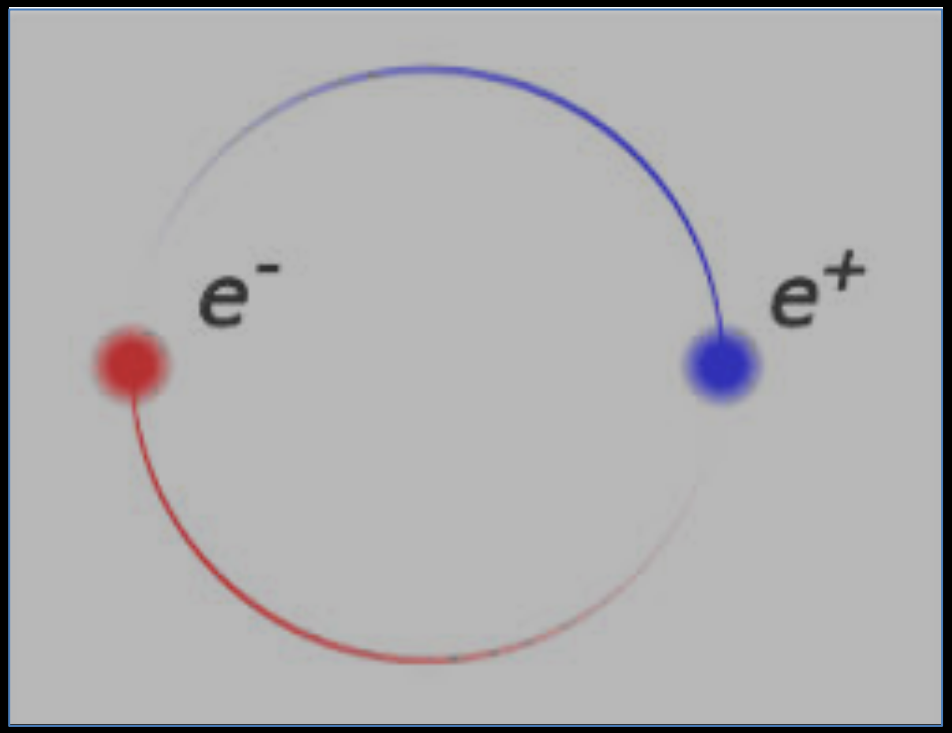
Higher Order Corrections to Positronium Energy Levels



psas 2018
International Conference on
Precision Physics of Simple Atomic Systems

Greg Adkins
Franklin & Marshall College
16 May 2018

Higher Order Corrections to Positronium Energy Levels

Motivation
Experimental Situation
Theoretical Status
Method of Calculation
- NRQED
- Method of Regions
- Hydrogen Atom in $D=3-2\varepsilon$ Dimensions

Summary

Collaborators

Lam Tran
Xuan Zhang
Ben Akers
Faisal Alam

Acknowledgments

NSF PHY-1707489
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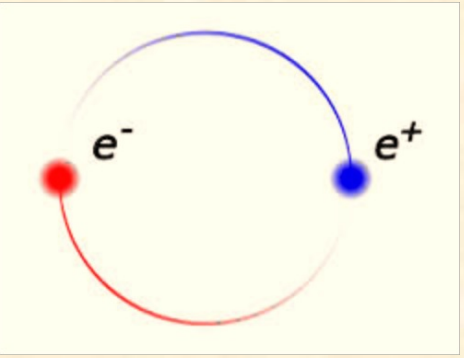
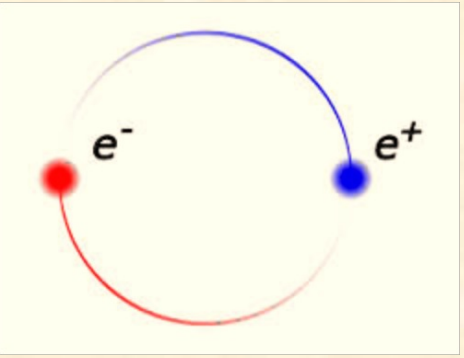


Why Study Positronium?

Positronium is intrinsically interesting. It is the simplest bound system. It's constituents are structureless pointlike particles. Many fundamental aspects of quantum field theory enter into its description. It differs from other exotic atoms in having large recoil effects, little sensitivity to hadronic physics, and is subject to real and virtual annihilation.

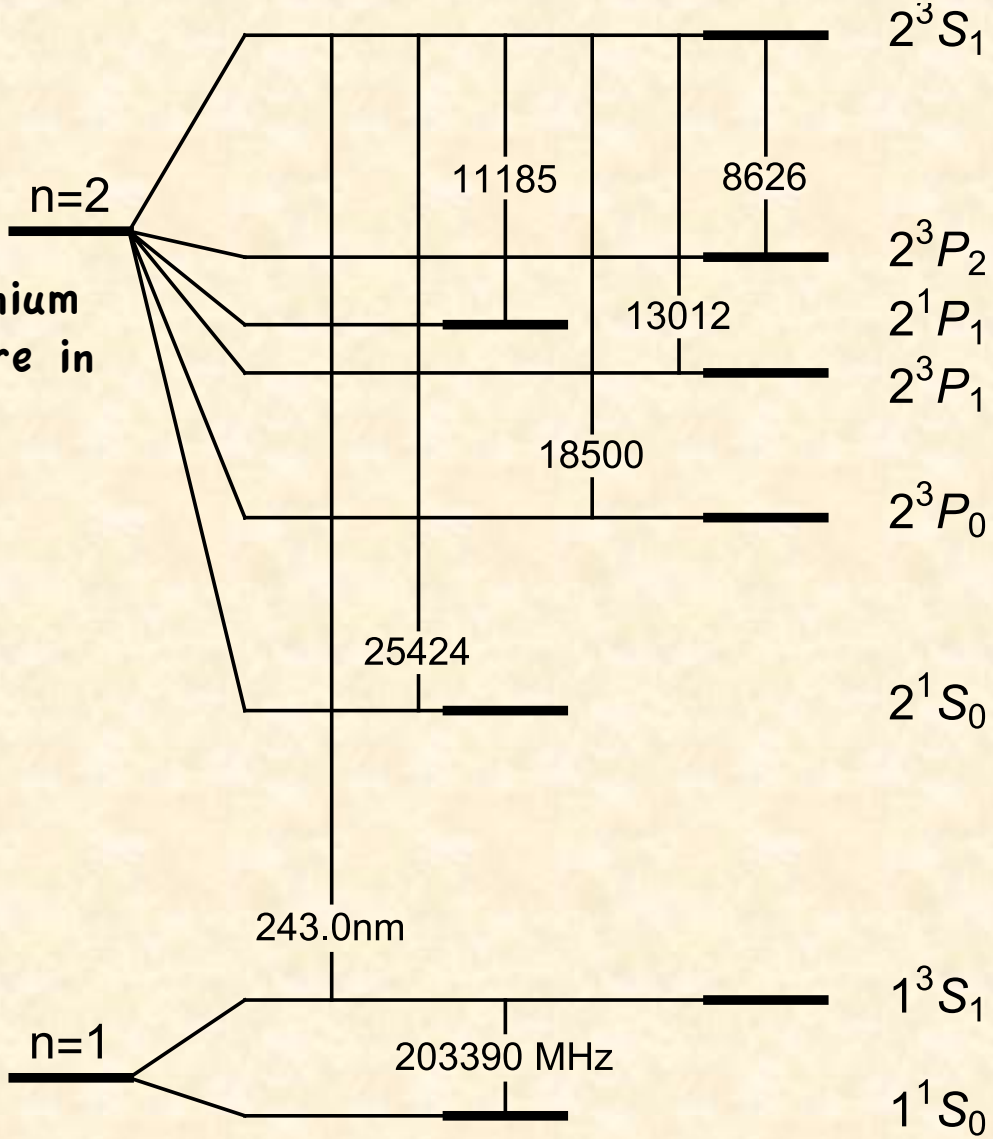
Positronium is accessible both to high precision experiments and to detailed calculations, so its study allows for a stringent test of the theory of bound states in QED (quantum electrodynamics) and quantum field theory generally.

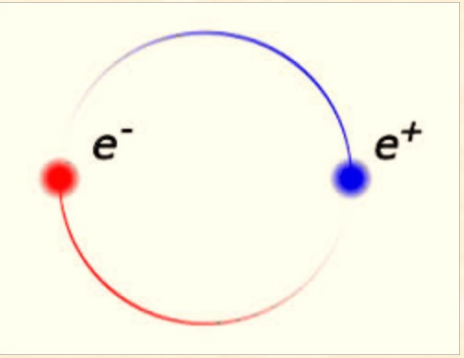
Positronium is ideal for tests of fundamental symmetries and is useful in searches for "new physics".



Positronium Spectrum

The n=1 and n=2 levels of positronium are shown. (Transition energies are in units of MHz.)





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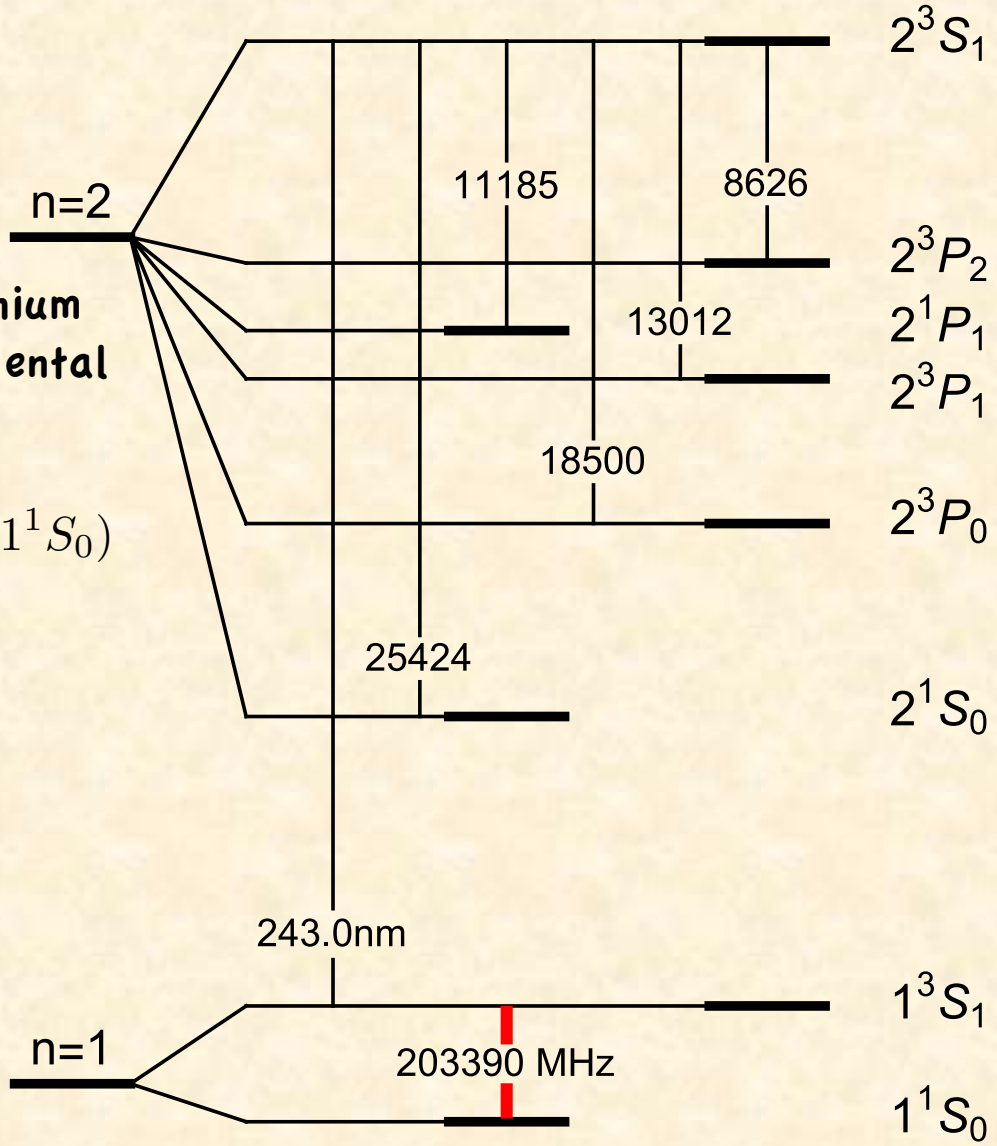

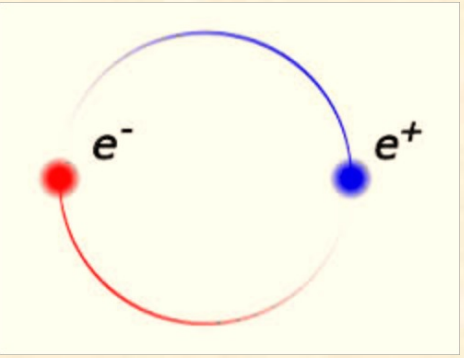

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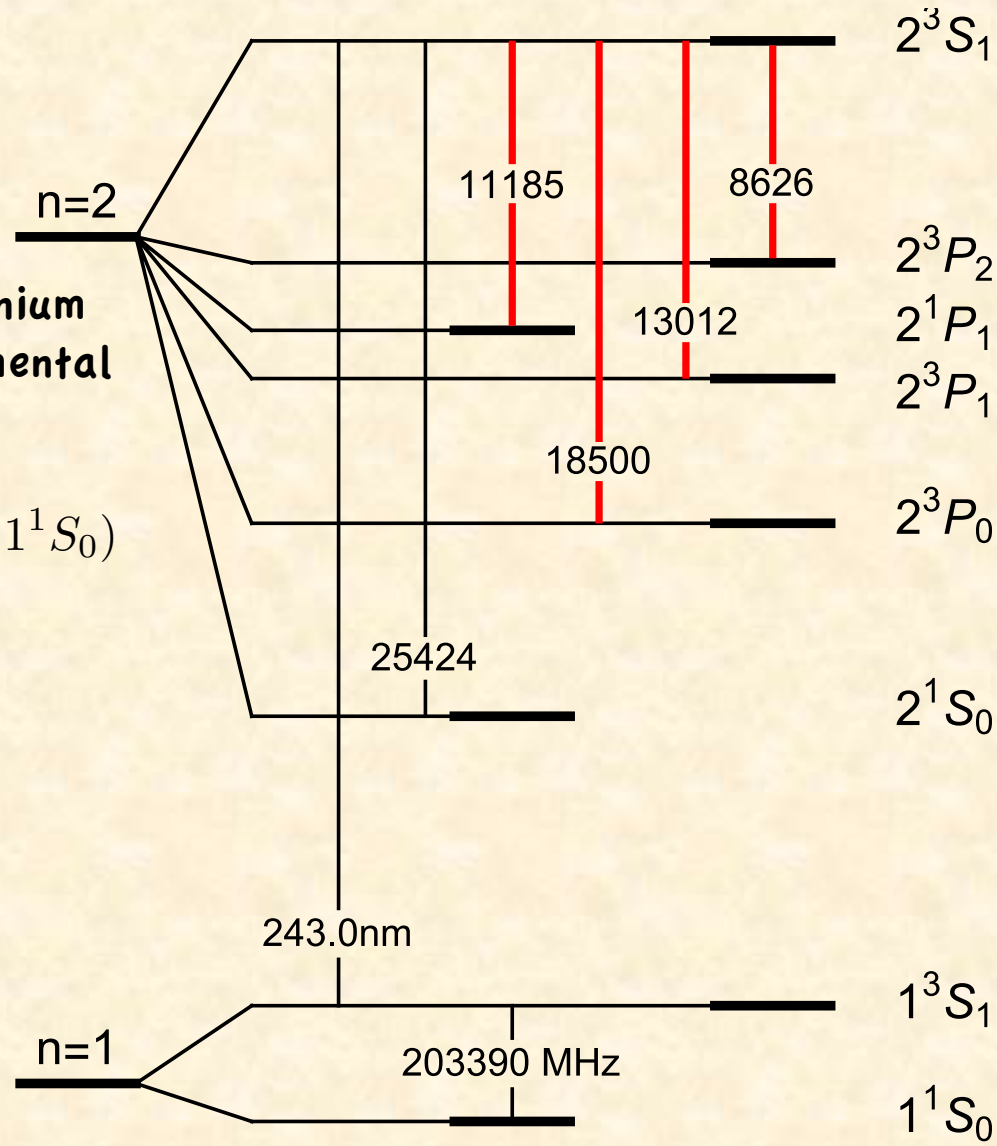
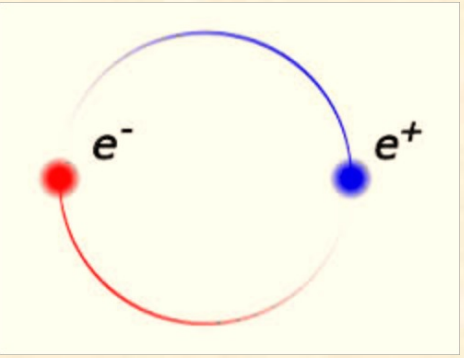
(2) n=2 fine structure $(2^3S_1 - 2P)$





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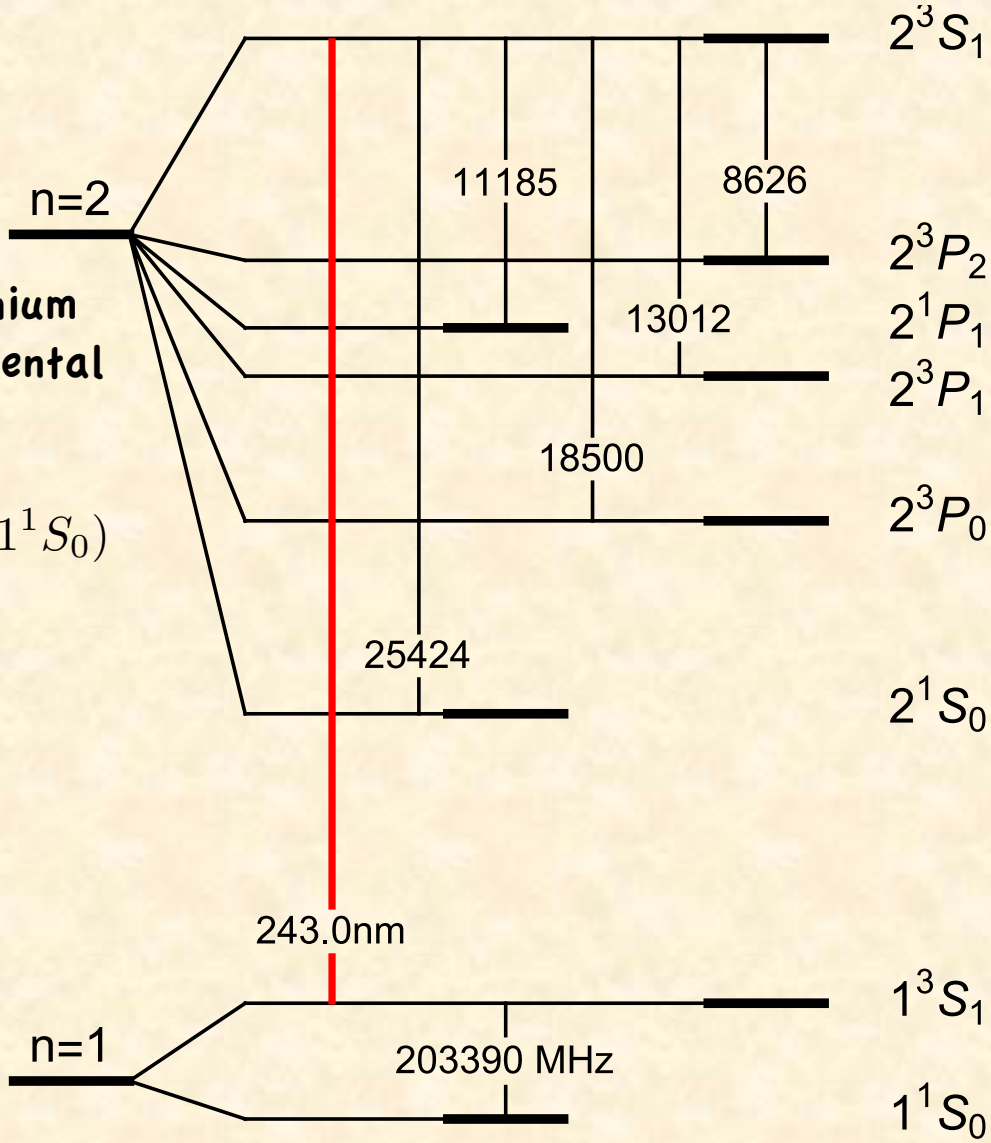

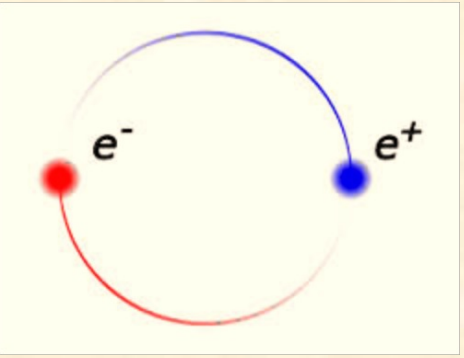

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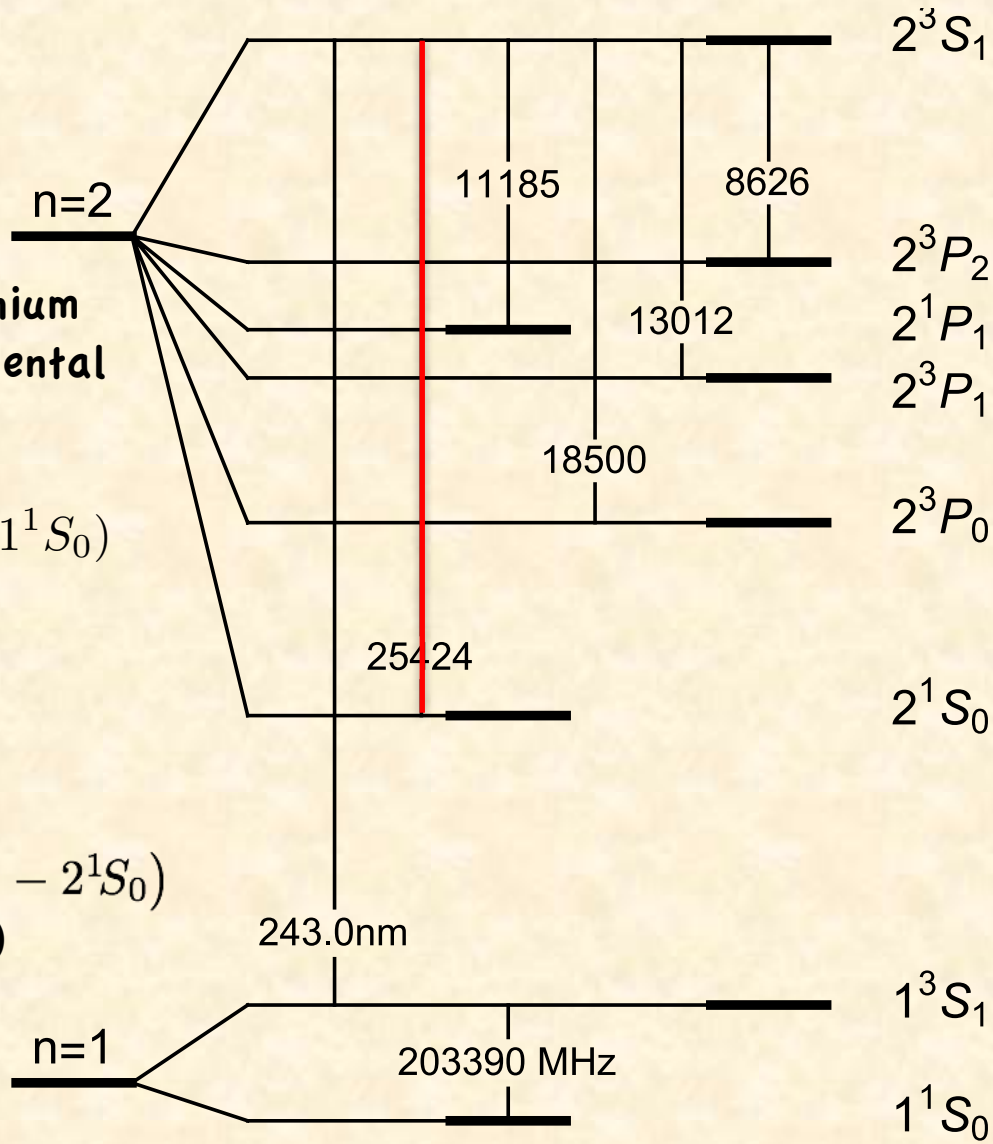
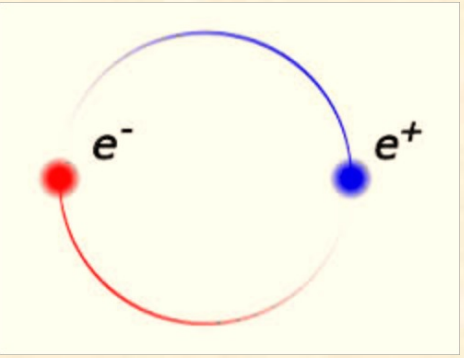
(also: n=2 hyperfine splitting: $(2^3S_1 - 2^1S_0)$
see the talk by Michael Heiss)





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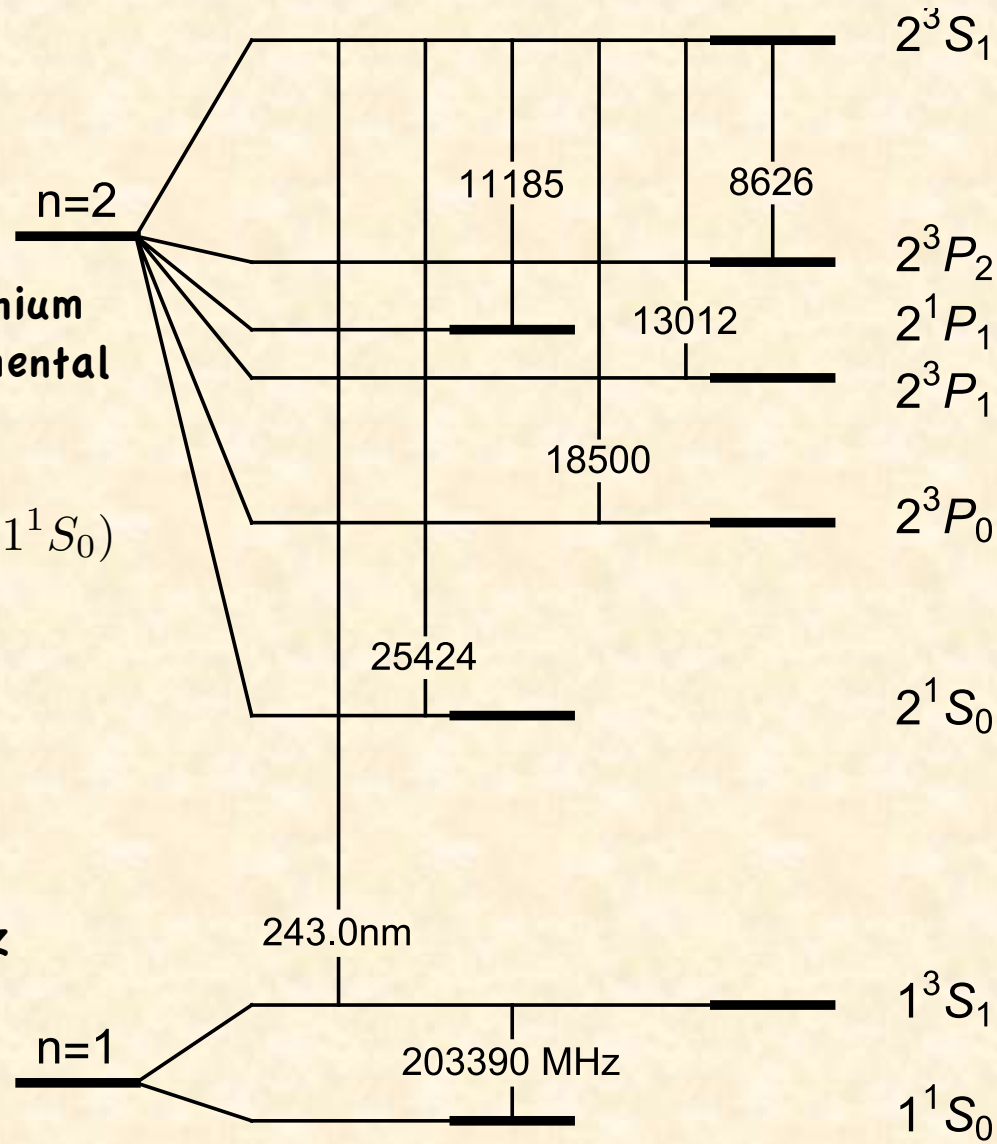
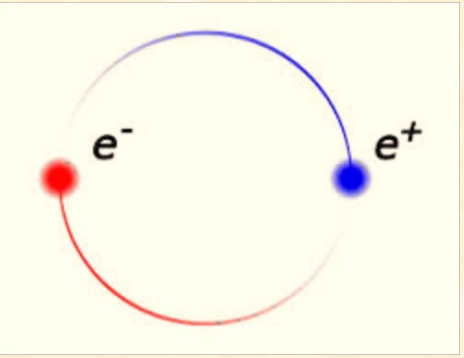
All of the measurements have uncertainties on the order of 1MHz





Hyperfine Structure
Experiment vs. Theory

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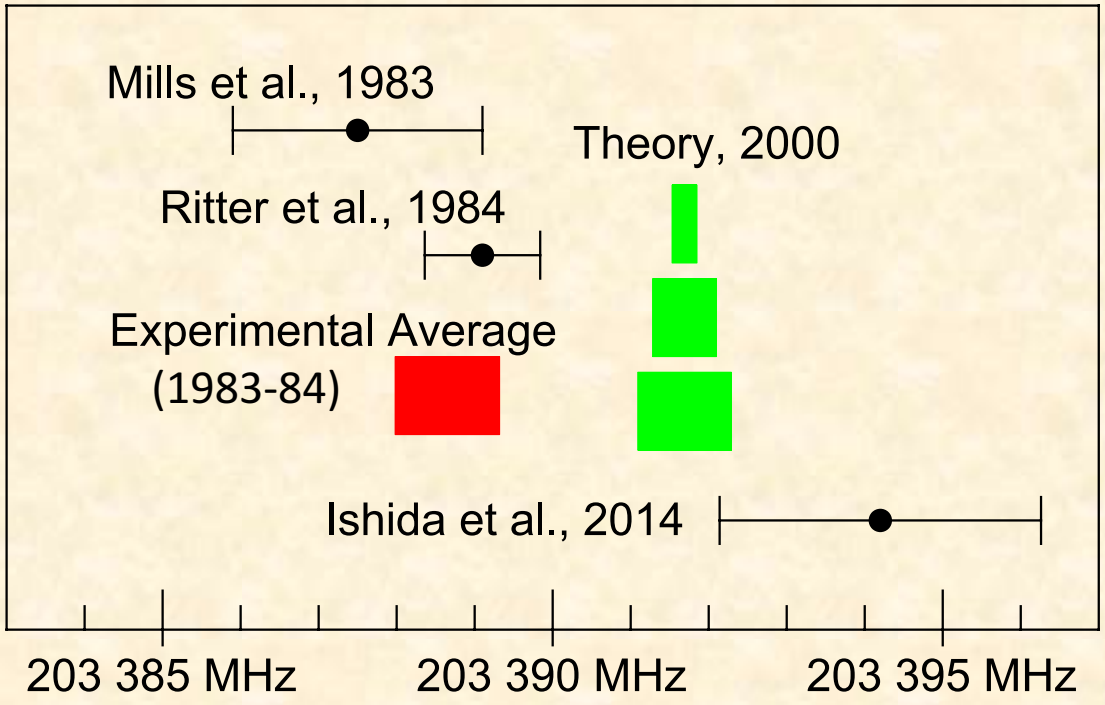
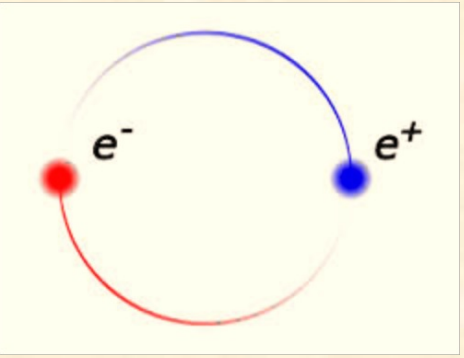
The situation for the hyperfine splitting has recently been clarified. Earlier suggestions of a problem between theory and experiment have probably disappeared with a new measurement from the Tokyo group. Additional experimental work with the promise of new ppm measurements is ongoing.





Measured Transitions

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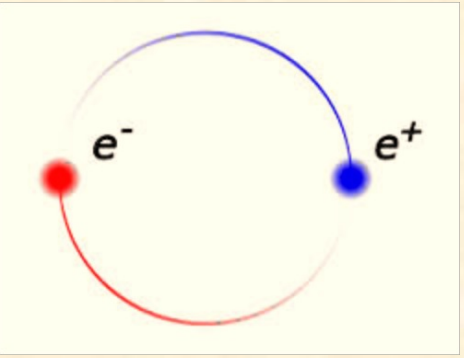
Transition	Common Name	Natural Linewidth	Expt. Uncert.
$1^3S_1 - 1^1S_0$	n=1 hyperfine	≈1300MHz	≈1MHz
$2^3S_1 - 2P$	n=2 fine structure	≈50MHz	≈2MHz
$1^3S_1 - 2^3S_1$	1S-2S	≈1.3MHz	≈3MHz



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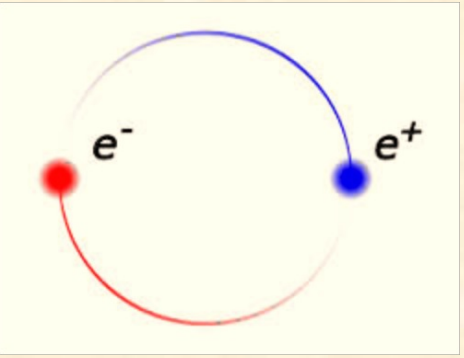
1S-2S ΔE=1233607216.4(3.2) MHz (2.6ppb)
seems to have the greatest potential for improvement.



Status of Energy Level Theory

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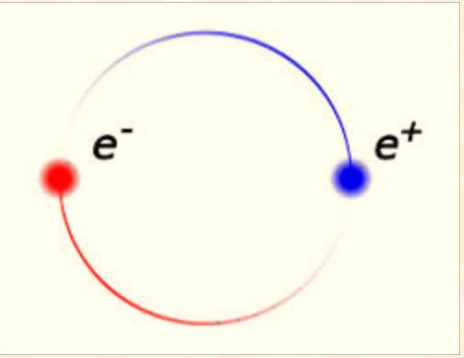
The theoretical formula for the energy levels of positronium states can be written as

$$\Delta E = m\Big\{C_{20}\alpha^2 + C_{40}\alpha^4 + C_{51}\alpha^5 L + C_{50}\alpha^5 + C_{61}\alpha^6 L + C_{60}\alpha^6 + C_{72}\alpha^7 L^2 + C_{71}\alpha^7 L + C_{70}\alpha^7 + \cdots\Big\}$$

where $L=\ln(1/\alpha)$. All terms through order $\alpha^7 \ln^2(1/\alpha)$ are known, as are the order $\alpha^7 \ln(1/\alpha)$ terms for the hyperfine interval. Estimates of the theoretical uncertainties are

0.6 MHz	1S-2S
0.5 MHz	hfs
0.1 MHz	n=2 fine structure



Status of Energy Level Theory

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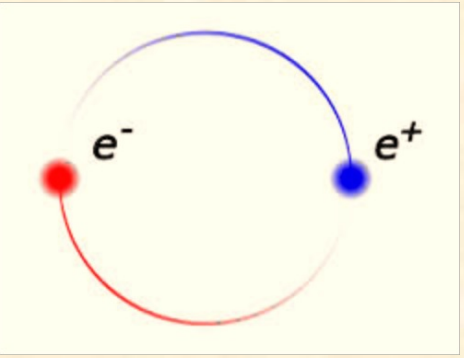
Contributions to positronium energy levels for n=1 (in MHz)

Order	Value	$E_{\rm av}\ (1S)$	$E_{\rm HFS}\ (1^3S_1{-}1^1S_0)$
$m\alpha^4$	350 377	38 322.493	204 386.630
$m\alpha^5 \ln(1/\alpha)$	12 580	3003.302	0
$m\alpha^5$	2557	−1018.784	−1005.497
$m\alpha^6 \ln(1/\alpha)$	91.8	2.869	19.125
$m\alpha^6$	18.7	3.000	−7.330
$m\alpha^7 \ln^2(1/\alpha)$	3.30	−1.091	−0.918
$m\alpha^7 \ln(1/\alpha)$	0.67	...	−0.323
$m\alpha^7$	0.14	...	...

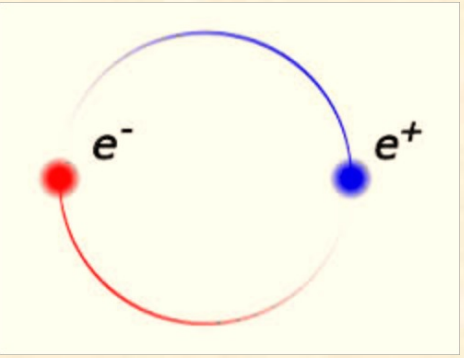


Lagrangian of the Effective Quantum Field Theory NRQED (Non-Relativistic QED)

We will replace the usual QED Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\gamma^{\mu} D_{\mu} - m\right)\psi + \text{photon terms}$$

by



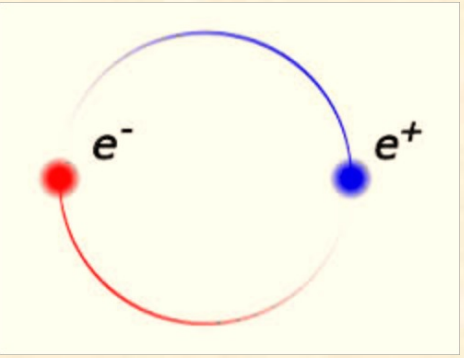
Lagrangian of the Effective Quantum Field Theory NRQED (Non-Relativistic QED)

$$\mathcal{L} = \psi^\dagger \Big\{ iD_t + \frac{\vec{D}^{\,2}}{2m} + \frac{\vec{D}^{\,4}}{8m^3} + c_F \frac{q}{2m} \vec{\sigma} \cdot \vec{B} + c_D \frac{q}{2m^2} [\vec{\nabla} \cdot \vec{E}]$$
$$+ c_S \frac{iq}{8m^2} \vec{\sigma} \cdot \Big(\vec{D} \times \vec{E} - \vec{E} \times \vec{D} \Big) + \cdots \Big\} \psi$$
$$+ \text{positron terms}$$
$$+ \text{four–fermion contact terms}$$
$$+ \text{photon terms}$$

with

$$D_t = \frac{\partial}{\partial t} + iqA^0, \quad \vec{D} = \vec{\nabla} - iq\vec{A}, \quad \vec{E} = -\vec{\nabla} A^0 - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

The electron charge and mass are q and m



NRQED Feynman Rules

Propagation Factors:

Fermion

$$\frac{i}{p_0 - \frac{\vec{p}^{\,2}}{2m} + i\epsilon}$$

Relativistic Kinetic Energy

$$\frac{i\,\vec{p}^{\,4}}{8\,m^3}$$

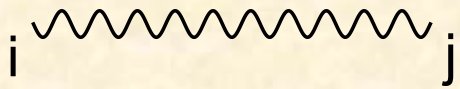
Coulomb Photon

$$\frac{i}{\vec{k}^{\,2}}$$

Coulomb Vacuum Polarization

$$c_{\mathrm{VP}}\,\frac{i\,\vec{k}^{\,2}\,k^2}{m_e^2}$$

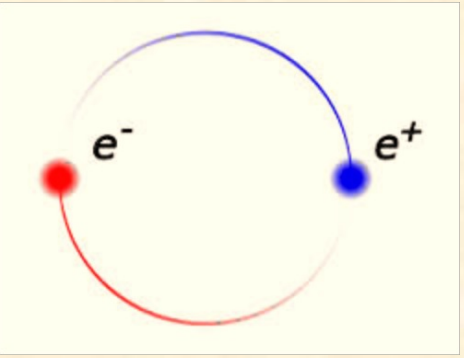
Transverse Photon



$$\frac{i\delta_{ij}^{T}(\vec{k})}{k^2 + i\epsilon}$$

Transverse Vacuum Polarization

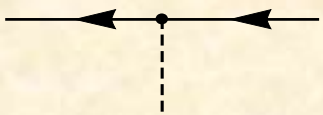
$$c_{\mathrm{VP}}\,\frac{ik^4}{m_e^2}\,\delta_{ij}$$



NRQED Feynman Rules

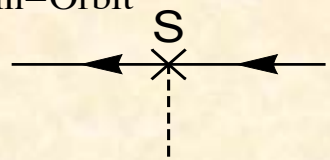
Interaction Vertices:

Coulomb



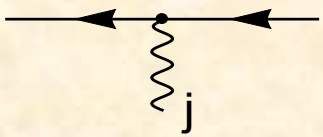
$$-iq$$

Spin–Orbit



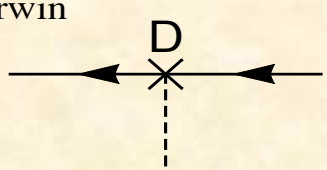
$$c_S \frac{q}{4\,m^2}(\vec{p}_2 \times \vec{p}_1)\cdot\vec{\sigma}$$

Convection



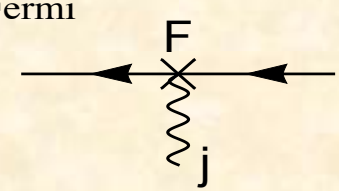
$$\frac{iq}{2\,m}(p_2+p_1)_j$$

Darwin



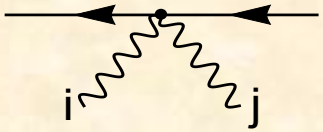
$$c_D \frac{iq}{8\,m^2}(\vec{p}_2-\vec{p}_1)^2$$

Fermi



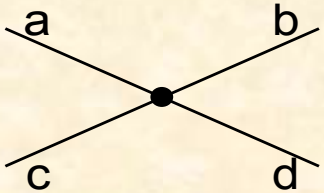
$$c_F \frac{q}{2\,m}[(\vec{p}_2-\vec{p}_1)\times\vec{\sigma}]_j$$

Seagull

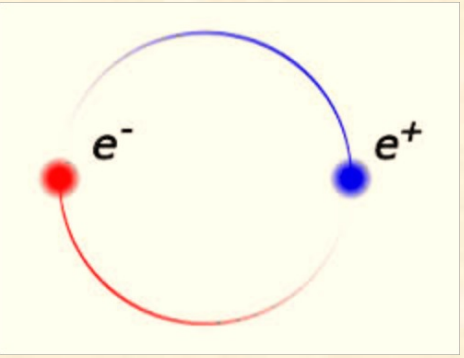


$$\frac{-iq^2}{m}\delta_{ij}$$

Contact



$$\frac{d_s}{m^2}\delta_{ab}\delta_{cd} + \frac{d_v}{m^2}\vec{\sigma}_{ab}\cdot\vec{\sigma}_{cd} \qquad + \cdots$$



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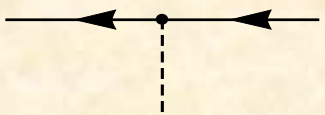

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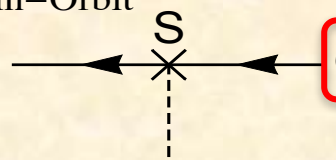

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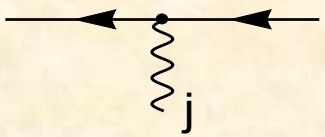
Convection



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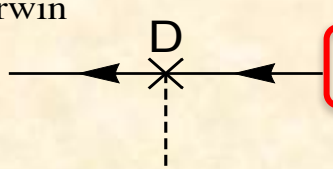
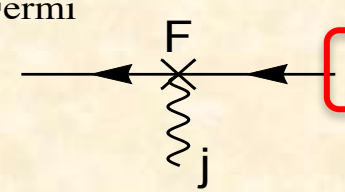
Darwin



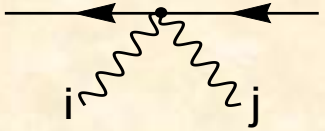
$$c_D \frac{iq}{8\,m^2}(\vec{p}_2-\vec{p}_1)^2$$

Fermi



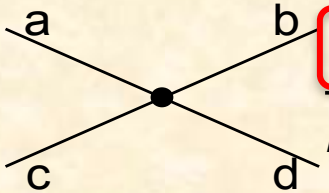
$$c_F \frac{q}{2\,m}[(\vec{p}_2-\vec{p}_1)\times\vec{\sigma}]_j$$

Seagull

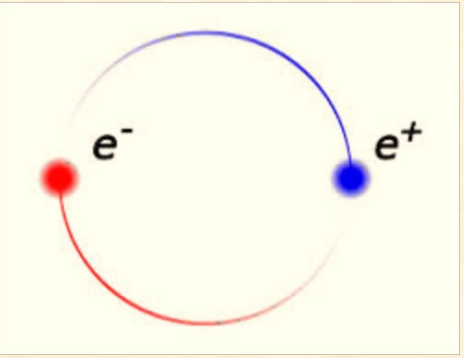


$$\frac{-iq^2}{m}\delta_{ij}$$

Contact



$$\frac{d_s}{m^2}\delta_{ab}\delta_{cd} + \frac{d_v}{m^2}\vec{\sigma}_{ab}\cdot\vec{\sigma}_{cd} \qquad + \cdots$$



Lagrangian of NRQED

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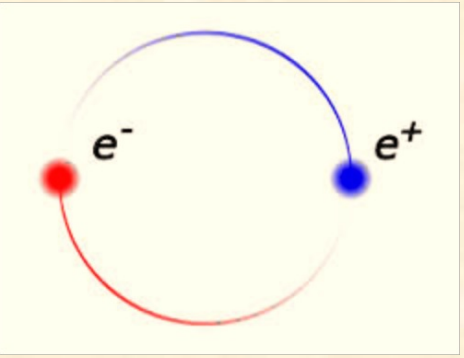
$$\mathcal{L} = \psi^\dagger \Big\{ iD_t + \frac{\vec{D}^{\,2}}{2m} + \frac{\vec{D}^{\,4}}{8m^3} + c_F \frac{q}{2m} \vec{\sigma} \cdot \vec{B} + c_D \frac{q}{2m^2} [\vec{\nabla} \cdot \vec{E}]$$

$$+ c_S \frac{iq}{8m^2} \vec{\sigma} \cdot \Big(\vec{D} \times \vec{E} - \vec{E} \times \vec{D} \Big) + \cdots \Big\} \psi$$

$+$ positron terms

$+$ four–fermion contact terms

$+$ photon terms



Lagrangian of NRQED

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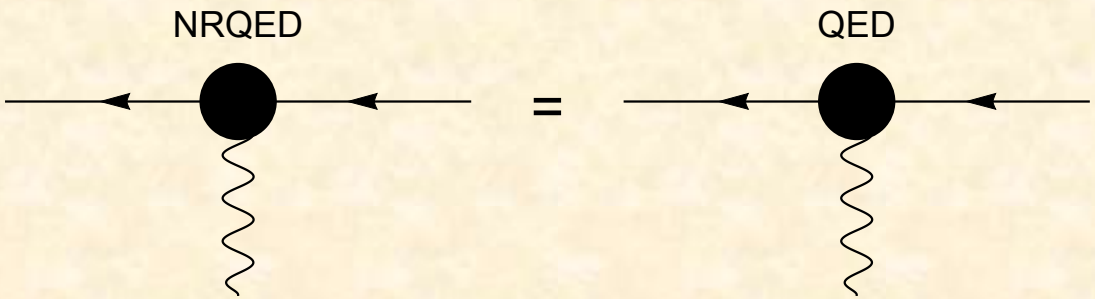


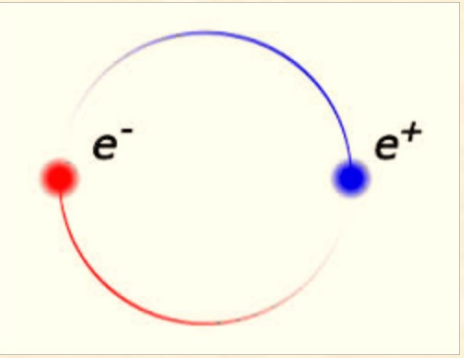
The matching coefficients have the values

(using dimensional regularization in $d = 4 - 2\epsilon$ dimensions)

$$c_F = 1 + \frac{\alpha}{2\pi} + \cdots$$

$$c_D = 1 + \frac{8\alpha}{3\pi}\left\{\frac{-1}{2\epsilon} + \ln\frac{m}{\mu}\right\} + \cdots$$

$$c_S = 1 + \frac{\alpha}{\pi} + \cdots$$



NRQED Lagrangian

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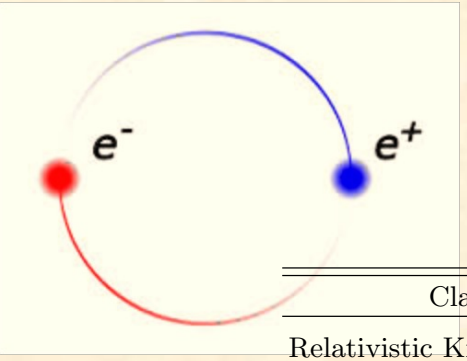
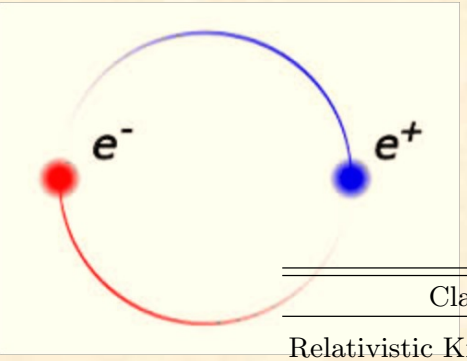
$$
\begin{aligned}
\mathcal{L} = \psi^\dagger \Big\{ & iD_t + c_2 \frac{\boldsymbol{D}^2}{2M} + c_4 \frac{\boldsymbol{D}^4}{8M^3} + c_F g \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2M} + c_D g \frac{[\boldsymbol{\partial}\cdot\boldsymbol{E}]}{8M^2} + i c_S g \frac{\boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E}-\boldsymbol{E}\times\boldsymbol{D})}{8M^2} + c_{W1} g \frac{\{\boldsymbol{D}^2, \boldsymbol{\sigma}\cdot\boldsymbol{B}\}}{8M^3} \\
& - c_{W2} g \frac{D^i \boldsymbol{\sigma}\cdot\boldsymbol{B} D^i}{4M^3} + c_{p'p} g \frac{\boldsymbol{\sigma}\cdot\boldsymbol{D}\boldsymbol{B}\cdot\boldsymbol{D} + \boldsymbol{D}\cdot\boldsymbol{B}\boldsymbol{\sigma}\cdot\boldsymbol{D}}{8M^3} + i c_M g \frac{\{\boldsymbol{D}^i, [\boldsymbol{\partial}\times\boldsymbol{B}]^i\}}{8M^3} + c_{A1} g^2 \frac{\boldsymbol{B}^2 - \boldsymbol{E}^2}{8M^3} - c_{A2} g^2 \frac{\boldsymbol{E}^2}{16M^3} \\
& + c_{X1} g \frac{[\boldsymbol{D}^2, \boldsymbol{D}\cdot\boldsymbol{E} + \boldsymbol{E}\cdot\boldsymbol{D}]}{M^4} + c_{X2} g \frac{\{\boldsymbol{D}^2, [\boldsymbol{\partial}\cdot\boldsymbol{E}]\}}{M^4} + c_{X3} g \frac{[\boldsymbol{\partial}^2 \boldsymbol{\partial}\cdot\boldsymbol{E}]}{M^4} + i c_{X4} g^2 \frac{\{\boldsymbol{D}^i, [\boldsymbol{E}\times\boldsymbol{B}]^i\}}{M^4} \\
& + i c_{X5} g \frac{D^i \boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E}-\boldsymbol{E}\times\boldsymbol{D}) D^i}{M^4} + i c_{X6} g \frac{\epsilon^{ijk}\sigma^i D^j [\boldsymbol{\partial}\cdot\boldsymbol{E}] D^k}{M^4} + c_{X7} g^2 \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}[\boldsymbol{\partial}\cdot\boldsymbol{E}]}{M^4} + c_{X8} g^2 \frac{[\boldsymbol{E}\cdot\boldsymbol{\partial}\boldsymbol{\sigma}\cdot\boldsymbol{B}]}{M^4} \\
& + c_{X9} g^2 \frac{[\boldsymbol{B}\cdot\boldsymbol{\partial}\boldsymbol{\sigma}\cdot\boldsymbol{E}]}{M^4} + c_{X10} g^2 \frac{[E^i \boldsymbol{\sigma}\cdot\boldsymbol{\partial} B^i]}{M^4} + c_{X11} g^2 \frac{[B^i \boldsymbol{\sigma}\cdot\boldsymbol{\partial} E^i]}{M^4} + c_{X12} g^2 \frac{\boldsymbol{\sigma}\cdot\boldsymbol{E}\times[\partial_t \boldsymbol{E} - \boldsymbol{\partial}\times\boldsymbol{B}]}{M^4} + \mathcal{O}(1/M^5) \Big\} \psi .
\end{aligned}
$$

From R. J. Hill, G. Lee, G. Paz, and M. P. Solon
NRQED Lagrangian at order $1/M^4$
Phys. Rev. D 87, 053017 (2013)



Feynman Rules for NRQED

Class	Type	Label	Rule
Relativistic Kinetic Energy	$\psi^\dagger\psi$	$K4$	$\frac{i\vec{p}^{\,4}}{8m^3}$
		$K6$	$\frac{-i\vec{p}^{\,6}}{16m^5}$
Coulomb	$\psi^\dagger A^0\psi$		$-iq$
		D (Darwin)	$c_D\,\frac{iq}{8m^2}\vec{k}^{\,2}$
		C$X1$	$c_{X1}\,\frac{-iq}{m^4}\left(\vec{p}^{\,\prime 2}-\vec{p}^{\,2}\right)^2$
		C$X2$	$c_{X2}\,\frac{-iq}{m^4}\left(\vec{p}^{\,\prime 2}+\vec{p}^{\,2}\right)\vec{k}^{\,2}$
		C$X3$	$c_{X3}\,\frac{-iq}{m^4}\vec{k}^{\,4}$
Spin-orbit	$\psi^\dagger A^0\psi$	S (Spin-orbit)	$c_S\,\frac{q}{4m^2}\sigma_{ab}p'^a p^b$
		S$X5$	$c_{X5}\,\frac{-2q}{m^4}\,\vec{p}\,'\cdot\vec{p}\,\sigma_{ab}p'^a p^b$
		S$X6$	$c_{X6}\,\frac{q}{m^4}\,\vec{k}^{\,2}\sigma_{ab}p'^a p^b$
Convection	$\psi^\dagger A^i\psi$		$\frac{iq}{2m}P^i$
		con$K4$	$\frac{-iq}{8m^3}\left(\vec{p}^{\,\prime 2}+\vec{p}^{\,2}\right)P^i$
		con$M$	$c_M\,\frac{-iq}{8m^3}\vec{k}^{\,2}P^i$
		con$X1$	$c_{X1}\,\frac{iq}{m^4}\left(\vec{p}^{\,\prime 2}-\vec{p}^{\,2}\right)k^0P^i$
Fermi	$\psi^\dagger A^i\psi$	F (Fermi)	$c_F\,\frac{q}{2m}\sigma_{ij}k^j$
		F$W1$	$c_{W1}\,\frac{-q}{8m^3}\left(\vec{p}^{\,\prime 2}+\vec{p}^{\,2}\right)\sigma_{ij}k^j$
		F$W2$	$c_{W2}\,\frac{q}{4m^3}\vec{p}\,'\cdot\vec{p}\,\sigma_{ij}k^j$
		F$p'p$	$c_{p'p}\,\frac{-q}{16m^3}\vec{P}^{\,2}\,\sigma_{ij}k^j$
P-Fermi	$\psi^\dagger A^i\psi$	PFS	$c_S\,\frac{-q}{8m^2}k^0\,\sigma_{ij}P^j$
		PF$p'p$	$c_{p'p}\,\frac{q}{16m^3}\left(\vec{p}^{\,\prime 2}-\vec{p}^{\,2}\right)\sigma_{ij}P^j$
		PF$X5$	$c_{X5}\,\frac{q}{m^4}\vec{p}\,'\cdot\vec{p}\,k^0\,\sigma_{ij}P^j$
Spin-convection	$\psi^\dagger A^i\psi$	sc$p'p$	$c_{p'p}\,\frac{-q}{8m^3}\sigma_{ab}p'^a p^b P^i$



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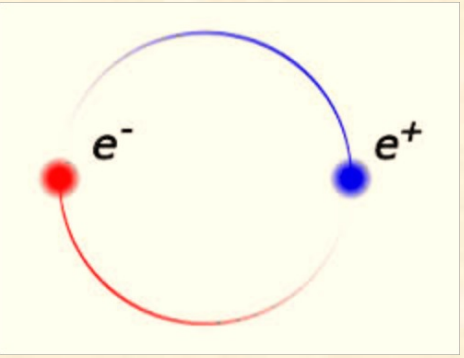
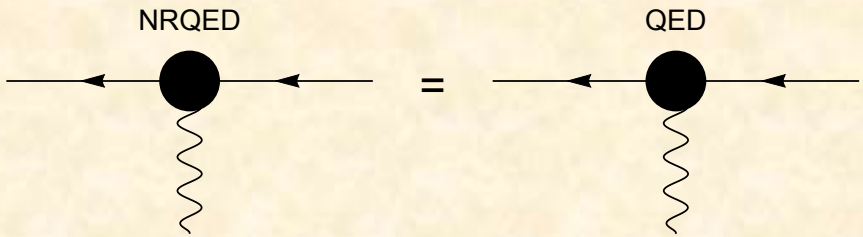
		C$X1$	$c_{X1}\,\frac{-iq}{m^4}\left(\vec{p}^{\,\prime 2}-\vec{p}^{\,2}\right)^2$
		C$X2$	$c_{X2}\,\frac{-iq}{m^4}\left(\vec{p}^{\,\prime 2}+\vec{p}^{\,2}\right)\vec{k}^{\,2}$
		C$X3$	$c_{X3}\,\frac{-iq}{m^4}\vec{k}^{\,4}$
Spin-orbit	$\psi^\dagger A^0\psi$	S (Spin-orbit)	$c_S\,\frac{q}{4m^2}\sigma_{ab}p'^a p^b$
		S$X5$	$c_{X5}\,\frac{-2q}{m^4}\,\vec{p}\,'\cdot\vec{p}\,\sigma_{ab}p'^a p^b$
		S$X6$	$c_{X6}\,\frac{q}{m^4}\,\vec{k}^{\,2}\sigma_{ab}p'^a p^b$
Convection	$\psi^\dagger A^i\psi$		$\frac{iq}{2m}P^i$
		con$K4$	$\frac{-iq}{8m^3}\left(\vec{p}^{\,\prime 2}+\vec{p}^{\,2}\right)P^i$
		con$M$	$c_M\,\frac{-iq}{8m^3}\vec{k}^{\,2}P^i$
		con$X1$	$c_{X1}\,\frac{iq}{m^4}\left(\vec{p}^{\,\prime 2}-\vec{p}^{\,2}\right)k^0P^i$
Fermi	$\psi^\dagger A^i\psi$	F (Fermi)	$c_F\,\frac{q}{2m}\sigma_{ij}k^j$
		F$W1$	$c_{W1}\,\frac{-q}{8m^3}\left(\vec{p}^{\,\prime 2}+\vec{p}^{\,2}\right)\sigma_{ij}k^j$
		F$W2$	$c_{W2}\,\frac{q}{4m^3}\vec{p}\,'\cdot\vec{p}\,\sigma_{ij}k^j$
		F$p'p$	$c_{p'p}\,\frac{-q}{16m^3}\vec{P}^{\,2}\,\sigma_{ij}k^j$
P-Fermi	$\psi^\dagger A^i\psi$	PFS	$c_S\,\frac{-q}{8m^2}k^0\,\sigma_{ij}P^j$
		PF$p'p$	$c_{p'p}\,\frac{q}{16m^3}\left(\vec{p}^{\,\prime 2}-\vec{p}^{\,2}\right)\sigma_{ij}P^j$
		PF$X5$	$c_{X5}\,\frac{q}{m^4}\vec{p}\,'\cdot\vec{p}\,k^0\,\sigma_{ij}P^j$
Spin-convection	$\psi^\dagger A^i\psi$	sc$p'p$	$c_{p'p}\,\frac{-q}{8m^3}\sigma_{ab}p'^a p^b P^i$



Matching Coefficient Expansions



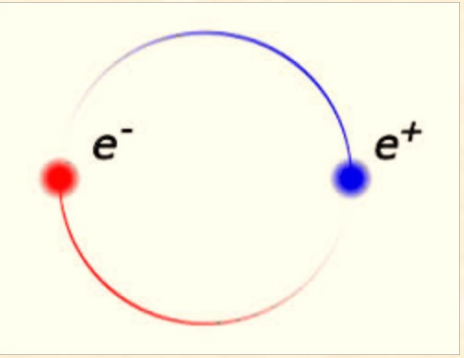
$$\Gamma_\mu^{\rm QED}(p_2,p_1) = \bar{u}(p_2)\left\{\gamma_\mu F_1(k^2) + \frac{i\sigma_{\mu\nu}k^\nu}{2m}F_2(k^2)\right\}u(p_1)$$

$$k^\mu = p_2^\mu - p_1^\mu$$

$$\overline{F}_i \equiv F_i(0), \qquad \overline{F}_i' \equiv \frac{dF_i}{d(q^2/m^2)}|_{q^2=0},$$

$$\overline{F}_i'' \equiv \frac{d^2F_i}{d(q^2/m^2)^2}|_{q^2=0}.$$

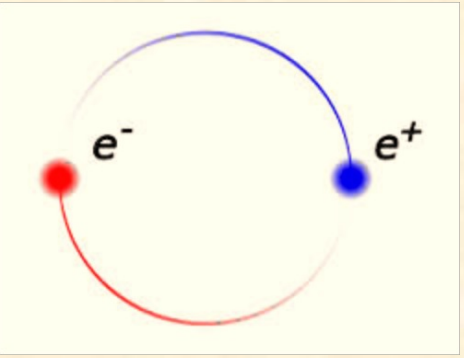
$$\begin{aligned}
c_F &= \overline{F}_1 + \overline{F}_2, \\
c_D &= \overline{F}_1 + 2\overline{F}_2 + 8\overline{F}_1', \\
c_S &= \overline{F}_1 + 2\overline{F}_2, \\
c_{W1} &= \overline{F}_1 + \frac{1}{2}\overline{F}_2 + 4\overline{F}_1' + 4\overline{F}_2', \\
c_{W2} &= \frac{1}{2}\overline{F}_2 + 4\overline{F}_1' + 4\overline{F}_2', \\
c_{p'p} &= \overline{F}_2, \\
c_M &= \frac{1}{2}\overline{F}_2 + 4\overline{F}_1', \\
c_{X1} &= \frac{5}{128}\overline{F}_1 + \frac{1}{32}\overline{F}_2 + \frac{1}{4}\overline{F}_1', \\
c_{X2} &= \frac{3}{64}\overline{F}_1 + \frac{1}{16}\overline{F}_2, \\
c_{X3} &= \frac{1}{8}\overline{F}_1' + \frac{1}{4}\overline{F}_2' + \frac{1}{2}\overline{F}_1'', \\
c_{X5} &= \frac{3}{32}\overline{F}_1 + \frac{1}{8}\overline{F}_2, \\
c_{X6} &= -\frac{3}{32}\overline{F}_1 - \frac{1}{8}\overline{F}_2 - \frac{1}{4}\overline{F}_1' - \frac{1}{2}\overline{F}_2'
\end{aligned}$$



Matching Coefficient Expansions

Coefficient	$O(\alpha^0)$	$O(\alpha^1)$	$O(\alpha^2)$	$O(\alpha^3)$
c_F	1	$\frac{1}{2}$	$\frac{3}{4}\zeta_3 - 3\zeta_2 \ln 2 + \frac{1}{2}\zeta_2 + \frac{197}{144}$	$-\frac{215}{24}\zeta_5 + \frac{83}{12}\zeta_3\zeta_2 - \frac{239}{24}\zeta_4 - \frac{25}{3}\zeta_2 \ln^2 2 + \frac{100}{3}a_4$ $+\frac{25}{18}\ln^4 2 + \frac{139}{18}\zeta_3 - \frac{596}{3}\zeta_2 \ln 2 + \frac{17101}{135}\zeta_2 + \frac{28259}{5184}$
c_D	1	$-\frac{8}{3}\Xi$	$-\frac{9}{2}\zeta_3 + 18\zeta_2 \ln 2 - \frac{40}{9}\zeta_2 - \frac{1523}{324}$	$\frac{85}{12}\zeta_5 - \frac{121}{6}\zeta_3\zeta_2 + \frac{1591}{18}\zeta_4 - \frac{956}{45}\zeta_2 \ln^2 2 - \frac{1136}{9}a_4$ $-\frac{142}{27}\ln^4 2 - \frac{791}{12}\zeta_3 + \frac{23791}{45}\zeta_2 \ln 2 - \frac{249767}{810}\zeta_2 + \frac{88409}{11664}$
c_S	1	1	$\frac{3}{2}\zeta_3 - 6\zeta_2 \ln 2 + \zeta_2 + \frac{197}{72}$	$-\frac{215}{12}\zeta_5 + \frac{83}{6}\zeta_3\zeta_2 - \frac{239}{12}\zeta_4 - \frac{50}{3}\zeta_2 \ln^2 2 + \frac{200}{3}a_4$ $+\frac{25}{9}\ln^4 2 + \frac{139}{9}\zeta_3 - \frac{1192}{3}\zeta_2 \ln 2 + \frac{34202}{135}\zeta_2 + \frac{28259}{2592}$
c_{W1}	1	$-\frac{4}{3}\Xi + \frac{1}{12}$		
c_{W2}	0	$-\frac{4}{3}\Xi + \frac{1}{12}$		
$c_{p'p}$	0	$\frac{1}{2}$		
c_M	0	$-\frac{4}{3}\Xi - \frac{1}{4}$		
c_{X1}	$\frac{5}{128}$	$-\frac{1}{12}\Xi - \frac{1}{64}$		
c_{X2}	$\frac{3}{64}$	$\frac{1}{32}$		
c_{X3}	0	$-\frac{11}{120}\Xi - \frac{13}{320}$		
c_{X5}	$\frac{3}{32}$	$\frac{1}{16}$		
c_{X6}	$-\frac{3}{32}$	$\frac{1}{12}\Xi - \frac{7}{96}$		

$$\Xi = \frac{1}{2\epsilon} - \ln\left(\frac{m}{\mu}\right)$$



Method of Calculation

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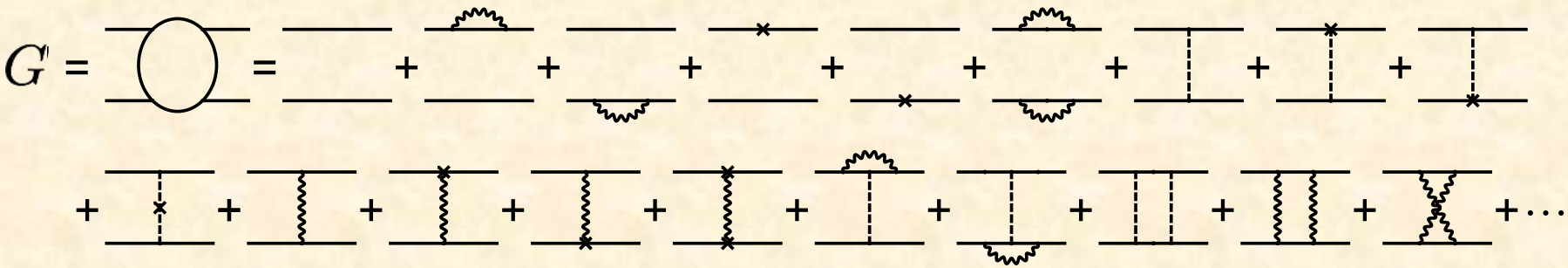
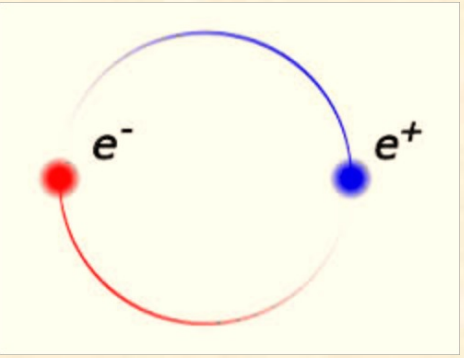
Energy levels are found as the positions of poles in the electron-positron 2-to-2 NRQED Green function G:

$$G(E;p_2,p_1) \to \frac{i\sum_k \Psi_{nak}(p_2)\bar{\Psi}_{nak}(p_1)}{E-E_{na}}$$


G =
=
+
+
+



Method of Calculation

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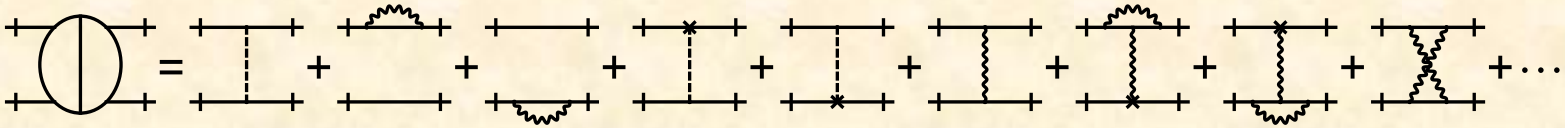
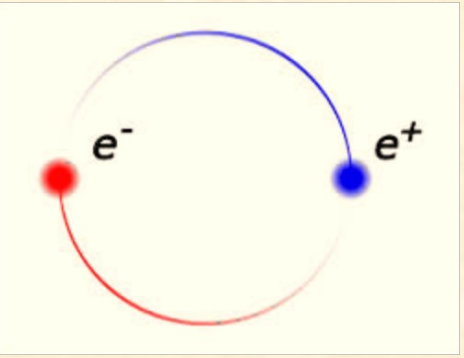
G satisfies the NRQED Bethe-Salpeter equation

$$G = S + SKG$$

Where S is the product of a free electron and a free positron propagator and K is the two-particle irreducible electron-positron to electron-positron kernel

$S =$

$K =$ 



Method of Calculation

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We want to base a perturbation scheme on an exactly-soluble lowest order problem:

$$G_0 = S + SK_0G_0$$

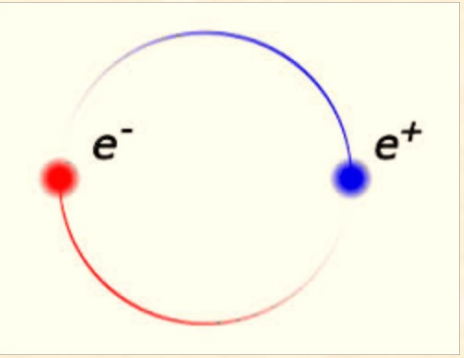
Where $K_0(E;p_2,p_1) = -iV(\vec{p}_2 - \vec{p}_1)$, $\qquad V(\vec{k}) = -\frac{4\pi\alpha\bar{\mu}^{2\epsilon}}{\vec{k}^2}$

represents the exchange of a Coulomb photon. The solution for G_0 is based on the usual Schrodinger-Coulomb (SC) equation (but in D=3-2ε dimensions). An exact solution of the SC equation is known—but only in D=3 dimensions.

The perturbative scheme uses the perturbing kernel $\delta K = K - K_0$

$$\Delta E = (\delta K) + (\delta K \hat{G}_0 \delta K) + \left(\frac{d}{dE}\delta K|_{E_0}\right) + \cdots$$

with $(\delta K) = \int \frac{d^dp_2}{(2\pi)^d}\frac{d^dp_1}{(2\pi)^d}\bar{\Psi}(E;p_2)\delta K(E;p_2,p_1)\Psi(E;p_1)$,etc.



Method of Calculation

Method of Regions

There are four regions of energy-momentum space that are important for Coulombic bound states calculations:

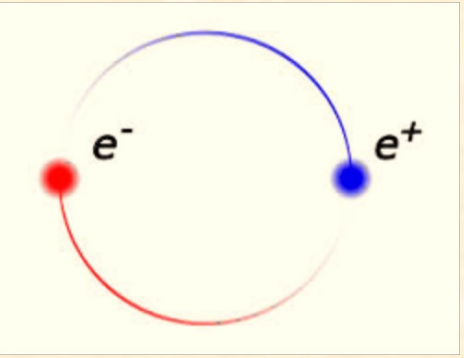
hard $E \sim mc^2, \ |\vec{p}| \sim mc$

soft $E \sim \alpha mc^2, \ |\vec{p}| \sim \alpha mc$

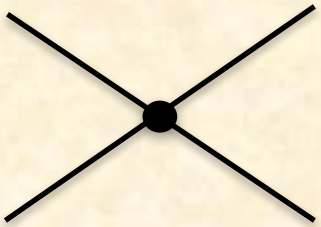
potential $E \sim \alpha^2 mc^2, \ |\vec{p}| \sim \alpha mc$

ultrasoft $E \sim \alpha^2 mc^2, \ |\vec{p}| \sim \alpha^2 mc$

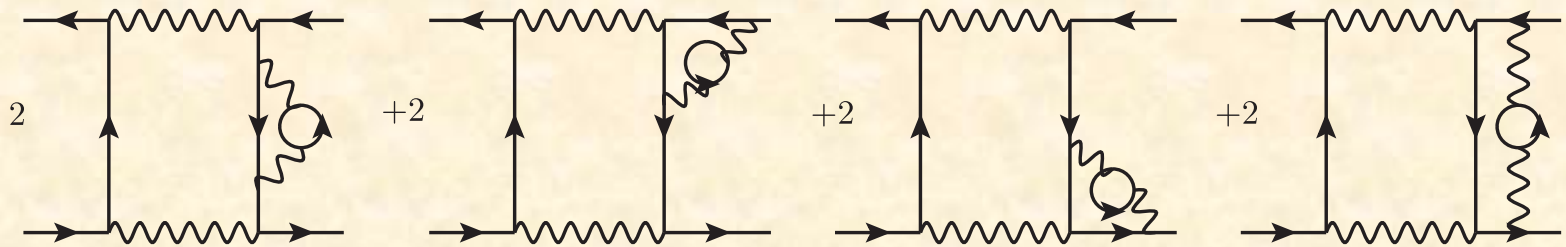
The procedure is to expand the integrand of an energy level contribution in powers of whatever variable is small in a given region, then integrate the simpler integrand over the full momentum space. Contributions from all regions are then summed.



Latest Contribution



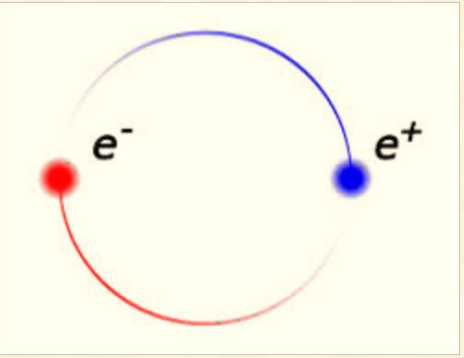
The most recently obtained contribution to the positronium energy levels is due to Eides and Shelyuto (2017). This is a "hard" contribution involving vacuum polarization corrections to two-photon virtual annihilation.



The contribution to the parapositronium n=1 energy is

$$\Delta E = 0.03297(2)\frac{m\alpha^7}{\pi^3} = 0.145kHz$$

(plus an imaginary part representing annihilation to two real photons).



Hydrogen Atom in D=3-2ε Dimensions

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The lowest order wave functions and energies satisfy the Schrödinger-Coulomb equation in D=3-2ε dimensions:

$$-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi(\vec{x}\,) + V(r)\psi(\vec{x}\,) = E\psi(\vec{x}\,)$$

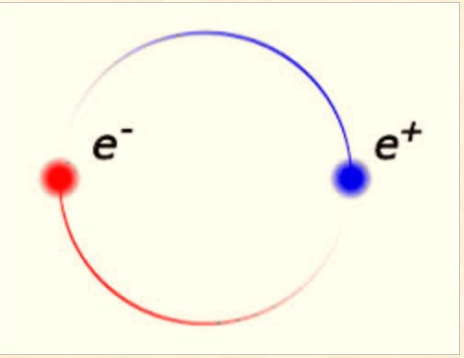
where V(r) is the D-dimensional Fourier transform of the Coulomb exchange interaction

$$V(r) = \int \frac{d^D k}{(2\pi)^D}\frac{-4\pi\alpha\bar{\mu}^{2\epsilon}}{\vec{k}^{\,2}} = -\frac{\Gamma(D/2-1)\bar{\mu}^{2\epsilon}\alpha}{\pi^{D/2-1}r^{D-2}}$$

and

$$\vec{\nabla}^2 = \left(\frac{\partial}{\partial r}\right)^2 + \frac{D-1}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+D-2)}{r^2}$$

when acting on spherical harmonics in D dimensions.



Hydrogen Atom in D=3-2ε Dimensions

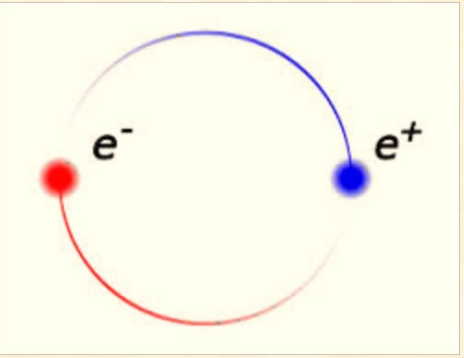
The radial function satisfies

$$\frac{1}{2m}\left\{-\left(\frac{\partial}{\partial r}\right)^2-\frac{D-1}{r}\frac{\partial}{\partial r}+\frac{\ell(\ell+D-2)}{r^2}\right\}R(r)+V(r)R(r)=ER(r)$$

The short and long distance behavior can be factored out as in three dimensions:

$$R(r)=\phi\,\Omega_{D-1}^{1/2}\left(\frac{(n+\ell)!}{n(n-\ell-1)!}\right)^{1/2}\frac{\rho^{\ell}e^{-\rho/2}}{(2\ell+1)!}L(\rho)$$

Where ϕ is the "wave function at the origin", Ω is a solid angle factor in D dimensions, $\rho=2\gamma r$ where $E=-\gamma^2/(2m)$ and $L(0)=1$. $L(\rho)$ generalizes the usual associated Laguerre polynomial.



Hydrogen Atom in D=3-2ε Dimensions

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The equation for L(ρ) is

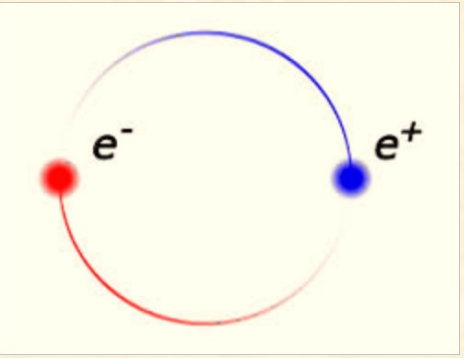
$$\left\{\left(\frac{\partial}{\partial\rho}\right)^2+\left(\frac{2(\ell+1-\epsilon)}{\rho}-1\right)\frac{\partial}{\partial\rho}-\frac{(\ell+1-\epsilon)}{\rho}+\frac{\bar{n}\rho^{2\epsilon}}{\rho}\right\}L(\rho)=0$$

with the series solution

$$\begin{aligned} L(\rho) &= \sum_{j=0}^{\infty}\sum_{k=0}^{j} a_{jk}\rho^j\left(\bar{n}\rho^{2\epsilon}\right)^k \\ &= 1+\rho\left(a_{10}+a_{11}\bar{n}\rho^{2\epsilon}\right)+\rho^2\left(a_{20}+a_{21}\bar{n}\rho^{2\epsilon}+a_{22}\bar{n}^2\rho^{4\epsilon}\right)+\cdots \end{aligned}$$

where

$$a_{jk}=\frac{a_{j-1,k}(j+\ell+\epsilon[2k-1])-a_{j-1,k-1}}{(j+2\epsilon k)(j+2\ell+1+2\epsilon[k-1])} \quad \text{with} \quad a_{0,0}=1,\ \ a_{j,k}=0 \text{ unless } 0\le \mathrm{k}\le \mathrm{j}$$



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Perturbation Theory

Hydrogen with the potential $V_0(r) = -\frac{\alpha}{r}$ can be solved exactly, even in D=3-2ε dimensions. Using $V_0(r)$ as the starting point of perturbation theory with

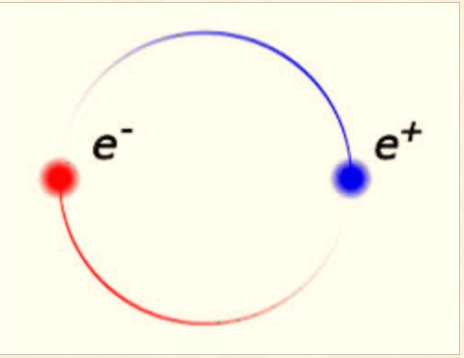
$$\delta V = V - V_0 = -\frac{\Gamma(D/2-1)\alpha\bar{\mu}^{2\epsilon}}{\pi^{D/2-1}r^{D-1}} + \frac{\alpha}{r} = -\frac{2\alpha}{r}\left(\ln(\mu r) + \gamma_E\right) + \cdots$$

one can find expansions for the energy levels and wave functions. These can be used to find expansions for the corrected principal quantum number $\bar{n}$ and the wave function at the origin ϕ. One finds

$$\bar{n} = n + \left(2\gamma_E - 2H_{n+\ell} - 1/n\right) n\epsilon + O(\epsilon^2) \qquad \left(H_n = \sum_{j=1}^{n} \frac{1}{j}\right)$$

for the state with quantum numbers n, ℓ; and (for the ground state)

$$\phi = \left(\frac{(m\alpha)^D}{\pi}\right)^{1/2} \left\{1 + \epsilon\left[3\ln\left(\frac{1}{\alpha}\right) - 2\zeta(2) + 4 + \frac{1}{2}\left(\ln\pi - \gamma_E\right)\right] + O(\epsilon^2)\right\}$$



Hydrogen Atom in D=3-2ε Dimensions

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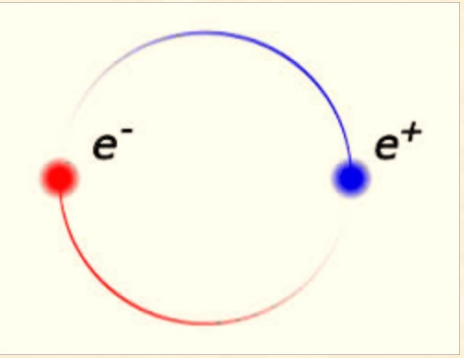
What is it good for? The analysis shown here allows for simple evaluation of (possibly divergent) expectation values. As an example, consider the following expectation value that contributes at $O(m\alpha^6)$:

$$\left\langle \left(\frac{dV}{dr}\right)^2 \right\rangle_{n\ell} = \int_0^\infty dr\, r^{D-1} \left[\frac{dV}{dr}\right]^2 R_{n\ell}^2(r) \propto \int_0^\infty dr\, r^{-2+2\epsilon} R_{n\ell}^2(r)$$

This expectation value is divergent for S states when D=3, but with D=3-2ε we can use the knowledge of the small r behavior of the wave function to find

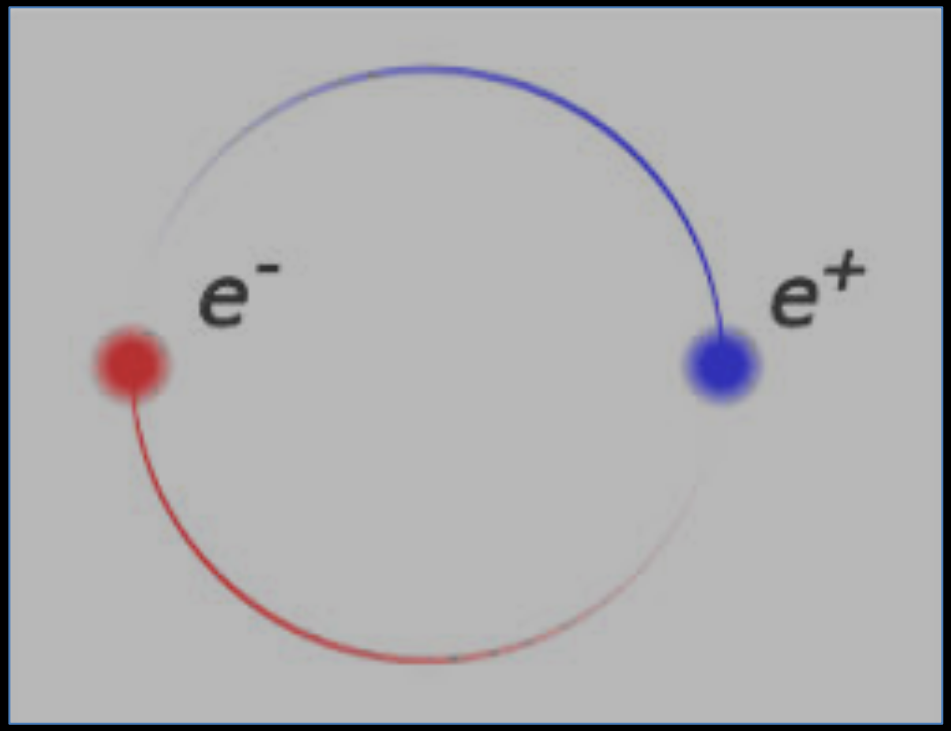
$$\left\langle \left(\frac{dV}{dr}\right)^2 \right\rangle_{n0} = \pi m \alpha^3 \phi^2 \bar{\mu}^{2\epsilon} \left\{ -\frac{2}{\epsilon} - 8\ln\left(\frac{\mu n}{2m\alpha}\right) + 8H_n + \frac{4}{3n^2} - \frac{4}{n} - \frac{16}{3} + O(\epsilon) \right\}$$



Summary

Methods are in place that are adequate for the calculation of order $m\alpha^7$ corrections to the energy levels of positronium. While many contributions at this order have already been obtained, much work remains to be done. In particular, a number of contributions to the four-fermion contact matching coefficients are still outstanding. This work is in progress.

Thank you!



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