Fractional topological charge in lattice Abelian gauge theory

Based on Abe, Morikawa, Suzuki, arXiv:2210.12967 (PTEP to appear)

Kagoshima Workshop on Particles, Fields and Strings 2023 @ サンプラザ天文館

阿部　元一
九州大学素粒子理論研究室
共同研究：鈴木博（九大）、森川億人（阪大）

対称性とアノマリー①

- 古典論：保存則 ⟷ 対称性（Noetherの定理）
- 量子論：古典論で成立していた保存則が破れることがある（アノマリー）
  - ➢ 分配関数に注目

$$Z[A] = \int [\mathcal{D}(\text{field})]\ e^{S[\text{field},A]}$$

  - ➢ アノマリーの判別：Zが変換のもとで不変か？

$$Z' \stackrel{?}{=} Z$$

$$\rightarrow Z'[A+\partial\theta] = \int [\mathcal{D}(\text{field})]\ e^{S[\text{field},A+\partial\theta]}$$

$$= \underbrace{e^{\mathcal{A}[\theta,A]}}_{\text{'t Hooft anomaly}} \underbrace{\int [\mathcal{D}(\text{field})]\ e^{S[\text{field},A]}}_{=Z}$$

対称性とアノマリー②

- ゲージ理論の低エネルギー物理を予言できる

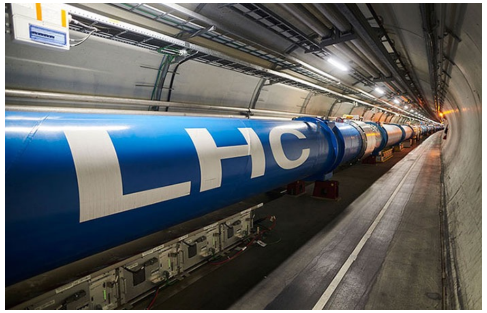
  ※ゲージ理論：素粒子の標準模型を記述する理論

- ✓ 例：理論と実験がうまく整合していることから、強い力を記述する理論（$SU(3)$ゲージ理論）を決定

素粒子理論

予言

素粒子実験



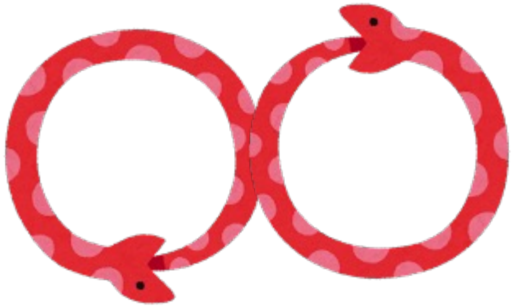
https://www.icepp.s.u-tokyo.ac.jp/information/20220426.html

高　　　　低

エネルギー

アノマリーと場の量子論

- アノマリーは場の量子論で特有の現象
  - ➢ 場の量子論は無限大の自由度をもつ
  - ➢ 場の量子論を定義するために無限大をまず有限にする（正則化）
  - ➢ 正則化が対称性を破る（アノマリー）



無限大の量

アノマリー！

正則化

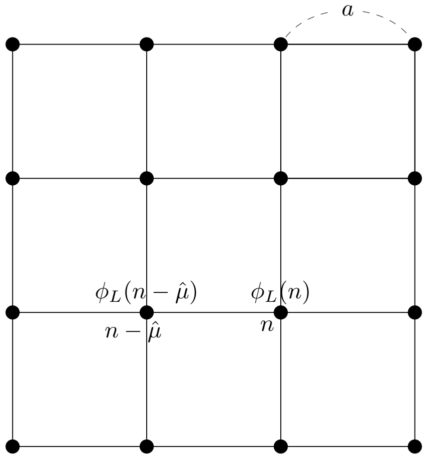
有限の量

アノマリーに関する近年の進展

- Gaiottoらにより、対称性の概念が拡張された：高次対称性(higher form symmetry)
  (Gaiotto, Kapustin, Seiberg, Willett, arXiv:1412.5148)
  - 高次対称性（と離散的な対称性）に付随するアノマリーに基づいて、ゲージ理論の低エネルギー物理が議論された
    (Gaiotto, Kapustin, Komargodski, Seiberg, arXiv:1703.00501)
  - 多くの新しいアノマリーが発見され、関連する研究が行われている
    - ✓ Yamaguchi, arXiv:1811.09390
    - ✓ Hidaka, Hirono, Nitta, Tanizaki, Yokokura, arXiv:1903.06389
    - ✓ Honda, Tanizaki, arXiv:2009.10183
    - ✓ etc.

☆正則化をうまく取り扱える理論
（格子ゲージ理論）で理解し、応用したい！！

格子ゲージ理論



θ 項を持つ$SU(N)$ゲージ理論でのアノマリー

- θ 項を持つ$SU(N)$ゲージ理論は$\theta = \pi$のとき時間反転($\mathcal{T}$)対称性をもつ

$$Z = \int \mathcal{D}a\ e^{S[a]} = \int \mathcal{D}a\ e^{S_{SU(N)}[a]} e^{i\theta Q[a]}, \quad Q \in \mathbb{Z}$$

$$\xrightarrow[\theta=\pi,\ \mathcal{T}\ \text{trans.}]{} Z' = \int \mathcal{D}a\ e^{S_{SU(N)}[a]} e^{i\pi(-Q[a])} = \int \mathcal{D}a\ e^{S_{SU(N)}[a]} e^{i\pi(+Q[a])} \underbrace{e^{-i2\pi Q[a]}}_{=1} = Z$$

- 高次対称性（$\mathbb{Z}_N$-one formゲージ対称性）をもつ$SU(N)$ゲージ理論を構築
  - トポロジカル電荷が分数になり、$\mathcal{T}$変換のもとで不変ではなくなる

注目！！

$$e^{-i2\pi Q} \neq 1$$

  - $\theta = \pi$のとき$\mathbb{Z}_N$-one formゲージ対称性と$\mathcal{T}$対称性の間にアノマリーをもつ

ファイバー束

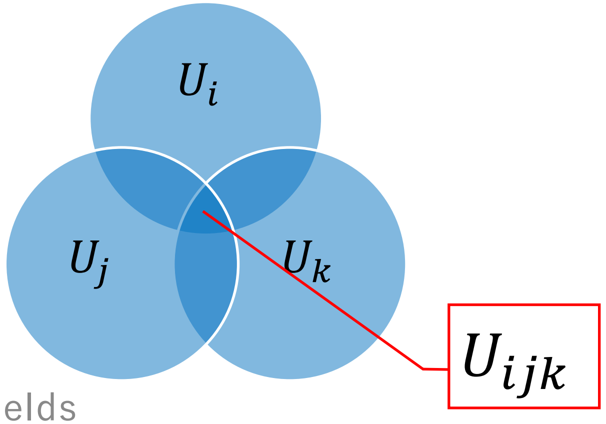
- ゲージ理論を記述する
  - 多様体MをパッチU_iで覆って、各パッチに $SU(N)$ゲージ場a_i、既約表現ρを持つ物質場ϕ_iがある
- $U_{ij} = U_i \cap U_j$でゲージ変換関数g_{ij}で結びついている

$$a_j = g_{ij}{}^{-1} a_i g_{ij} - i g_{ij}{}^{-1} \mathrm{d} g_{ij}$$

$$\phi_j = \rho(g_{ij}{}^{-1}) \phi_i$$

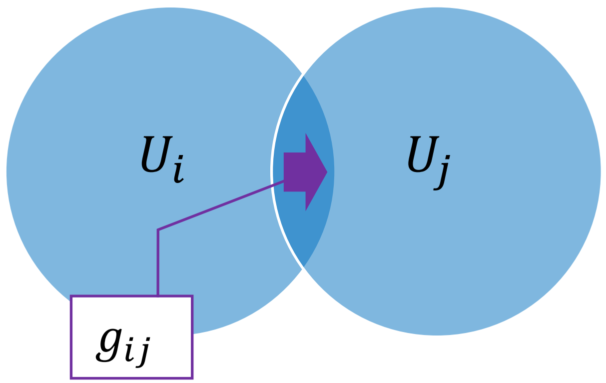
- 3つのpatchが重なる部分$U_{ijk} = U_i \cap U_j \cap U_k$でcocycle conditionが成り立つ

$$g_{ij} g_{jk} g_{ki} = 1$$



ファイバー束と分数トポロジカル電荷

- トポロジカル電荷が分数となるファイバー束を構成できる
  ('t Hooft, Nucl. Phys. B 153 (1979), van Baal, Commun. Math. Phys. 85 (1982))



cocycle condition: $g_{ij}g_{jk}g_{ki} = \exp\left(\frac{2\pi i}{N} n_{ijk}\right)$

$\underbrace{\qquad}_{\in \mathbb{Z}_N}$

(非自明な transition function) $\sim \omega_\mu \times$ ($SU(N)$ transition function)

分数にする要素

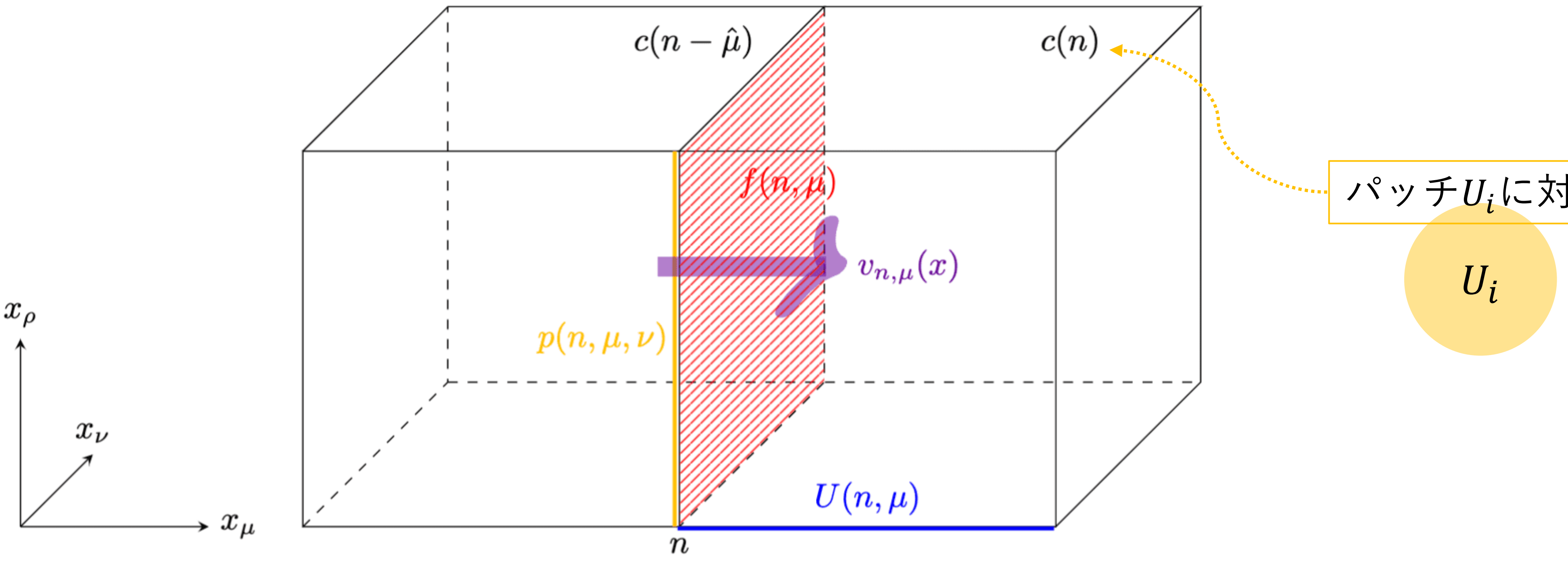
☆格子ゲージ理論で分数トポロジカル電荷を定式化する！！

- ➢ 格子$SU(N)$ ゲージ理論における整数トポロジカル電荷の定式化を利用する
  (Lüscher, Commun. Math. Phys. 85 (1982))

➢本研究では格子$U(1)$ゲージ理論において定式化した
(Fujiwara, Suzuki, Wu, arXiv:0001029 を応用)

ファイバー束と格子理論

- 多様体を「超立方体」$c(n)$ (hyper cube)で分ける
- 例：3次元



格子上での新しいTransition Function

連続理論における分数要素を含むtransition function

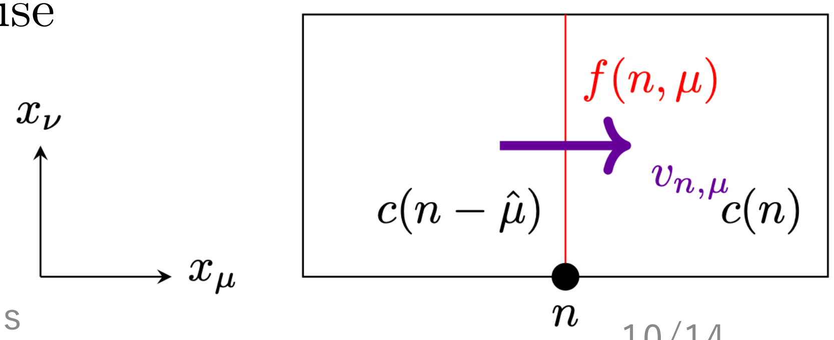
$$(\text{非自明な transition function}) \sim \omega_\mu \times (SU(N) \text{ transition function})$$

- $x \in f(n, \mu)$ においてtransition function $v_{n,\mu}$ を格子$(U(1)/\mathbb{Z}_q)$ゲージ理論で構成した
  - ω_μは格子上での分数を作る要素

$$v_{n,\mu}(x) = \omega_\mu(x)\check{v}_{n,\mu}(x)$$

$$\omega_\mu(x) \equiv \begin{cases} \exp\left(\frac{\pi i}{q}\sum_{\nu\neq\mu}\frac{z_{\mu\nu}x_\nu}{L}\right) & \text{for } x_\mu = 0 \bmod L \\ 1 & \text{otherwise} \end{cases}$$

  - $z_{\mu\nu} \in \mathbb{Z}$ and $z_{\mu\nu} = -z_{\nu\mu}$



格子上での分数トポロジカル電荷

- トポロジカル電荷は新しいtransition functionで計算される

$$Q = -\frac{1}{8\pi^2}\sum_n \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \int_{p(n,\mu,\nu)} \mathrm{d}^2x\, [v_{n,\mu}(x)\partial_\rho v_{n,\mu}(x)^{-1}][v_{n-\hat{\mu},\nu}(x)^{-1}\partial_\sigma v_{n-\hat{\mu},\nu}(x)]$$

- 新しいtransition function $v_{n,\mu}(x) = \omega_\mu(x)\check{v}_{n,\mu}(x)$ により

分数要素

$$\omega_\mu(x) \sim \exp\left(\frac{\pi i}{q}\sum_{\nu\neq\mu}\frac{z_{\mu\nu}x_\nu}{L}\right)$$

$$Q = \underbrace{\frac{1}{8q^2}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}}_{\text{分数 !!}} + \underbrace{\frac{1}{8\pi q}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu}\sum_{n_\mu=0}\check{F}_{\rho\sigma}(n)}_{\text{cross term}}$$

$$+ \underbrace{\frac{1}{32\pi^2}\sum_n\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}\check{F}_{\mu\nu}(n)\check{F}_{\rho\sigma}(n+\hat{\mu}+\hat{\nu})}_{\text{整数}}$$

アノマリー

- 格子上での作用：

$$S \equiv \overbrace{\frac{1}{4g_0^2}\sum_n\sum_{\mu,\nu}\check{F}_{\mu\nu}(n)\check{F}_{\mu\nu}(n) + S_{\text{matter}}}^{=S_0} \underbrace{-iq\theta Q}$$

Witten effect による (Honda, Tanizaki, arXiv:2009.10183)

- トポロジカル電荷：

$$qQ = \frac{1}{8q}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}z_{\mu\nu}z_{\rho\sigma} + \mathbb{Z} \equiv \text{frac}[z] + \text{int}[a, z]$$

✧ $\mathbb{Z}_q$ one-formゲージ変換の元で不変

✧ 格子での$\mathcal{T}$変換により符号を変える： $qQ \xrightarrow{\mathcal{T}} -qQ$

➢ $\theta = \pi$のとき$\mathbb{Z}_q$-one formゲージ対称性と$\mathcal{T}$対称性の間のアノマリーについて議論する

アノマリー

- （局所的なcounter termを含めると）$\theta=\pi$のとき分配関数は$\mathcal{T}$変換の元で

$$Z[z]=\int \mathcal{D}a\ e^{S[a,z]}=\int \mathcal{D}a\ e^{S_0[a,z]}e^{i\theta q Q[a,z]}$$

$$\xrightarrow[\theta=\pi,\ \mathcal{T}\ \text{trans.}]{} Z'=\int \mathcal{D}a\ e^{S_0[a,z]}e^{i\pi(-qQ[a,z])}=\int \mathcal{D}a\ e^{S_0[a,z]}e^{i\pi q Q[a,z]}\underbrace{e^{-i2\pi q Q[a,z]}}_{=e^{-i2\pi \mathrm{int}[a,z]}e^{-i2\pi \mathrm{frac}[z]}}$$

$$=e^{-i2\pi \mathrm{frac}[z]}\underbrace{\int \mathcal{D}a\ e^{S_0[a,z]}e^{i\pi q Q[a,z]}}_{=Z}\neq Z$$

$$\xrightarrow[\text{including counter term}]{} \exp\left[-\frac{2\pi i(2k+1)}{8q}\underbrace{\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}z_{\mu\nu}z_{\rho\sigma}}_{=0,\pm 8,\pm 16,\cdots,}\right]Z$$

$q\in 2\mathbb{Z}$のときアノマリーがある！

Conclusion and Future work

☆Conclusion

- 格子上の $U(1)$ ゲージ理論で分数トポロジカル電荷を定式化できた
- 得られたトポロジカル電荷によって$\theta = \pi$のとき$\mathbb{Z}_q$-one formゲージ対称性と時間反転($\mathcal{T}$)対称性の間のアノマリーが示せた

☆Future work

- 格子上の $SU(N)$ ゲージ理論で分数トポロジカル電荷を定式化する
- Witten effectを格子上で確認する

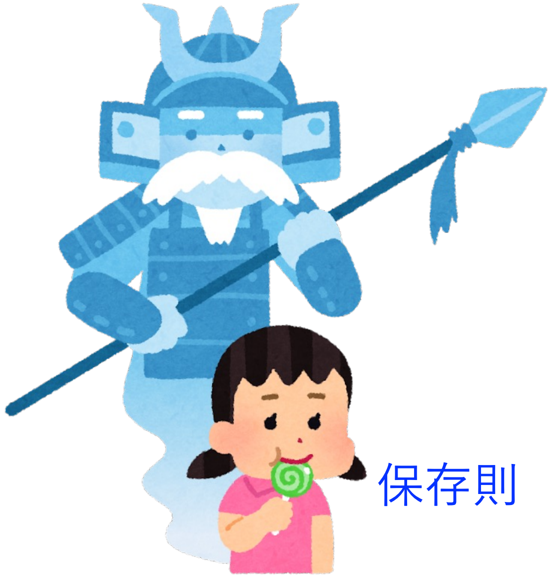
Back Up

保存則と対称性

- 保存則：自然界で成立する事実
  - ✓ 例：エネルギー保存則、電荷保存則、etc.
- 対称性：物理学での基本的な概念の一つ
  - ➢ 物理を変えない：ラグランジアンを不変に
- 古典論：保存則 ⟷ 対称性（Noetherの定理）
  - ✓ 例： 電荷保存則 ⟷ 大域的$U(1)$対称性
  - ➢ ラグランジアン：$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$

$$\frac{\mathrm{d}(\text{電荷})}{\mathrm{d}t} = 0 \quad \longleftrightarrow \quad \psi \to e^{i\alpha}\psi$$

Noetherの定理



't Hooft Anomaly

- 't Hooft anomaly：
  Couple a background gauge field A_μ with the preserved current j_μ related to the symmetry

$$Z[A_\mu] = \left\langle \exp\left(i \int A_\mu j^\mu\right)\right\rangle \qquad Z[A_\mu + \partial_\mu\theta] = Z[A_\mu]\exp(i\mathcal{A}(\theta, A_\mu))$$

Phase Gap

- 't Hooft anomaly matching:
  The property of matching the 't Hooft anomaly calculated respectively in both UV and IR theory

➢ Using the prediction of the low-energy physics of gauge theories

’t Hooft Anomaly Matching Condition

➢ Application example

✓ Restricting the low-energy effective theory of QCD, this condition requires lagrangian to have the Wess-Zumino-Witten term.

✓ Since a part of the background gauge field exists as the gauge field in Electro-Weak gauge theory, ’t Hooft anomaly can be observed in the collapse of neutral π meson. To match the experiment with this theory, the strong field theory is detected to $SU(3)$ gauge theory.

トポロジカル電荷

• トポロジー：

• 通常の量子力学：電子の電荷という量子数で波動関数が特徴づけられる

➢ 「トポロジカル電荷」という量子数で特徴づけられる物理

✓ 例：超伝導体、トポロジカルソリトン、etc.

Gaiotto, Kapustin, Seiberg, Willett, arXiv:1412.5148

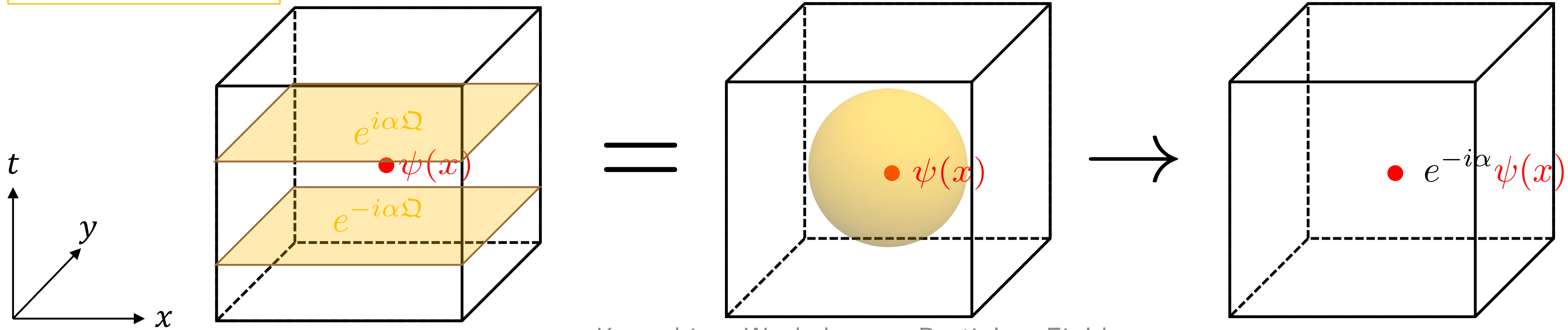
Higher form symmetry

- まず通常の対称性（zero-form対称性）を空間の広がりで捉え直す。

➢ (2+1)次元で具体的に見る。
場 $\psi(x)$の変換を考えると、charge $\mathfrak{Q}$は2次元の広がりを持つ。

$$e^{i\alpha\mathfrak{Q}}\psi(x)e^{-i\alpha\mathfrak{Q}} = e^{-i\alpha}\psi(x), \quad \mathfrak{Q} = \int_{M^2} \mathrm{d}^2x\, j^0(x), \quad j^\mu(x) = i\bar{\psi}(x)\gamma^\mu\psi(x)$$

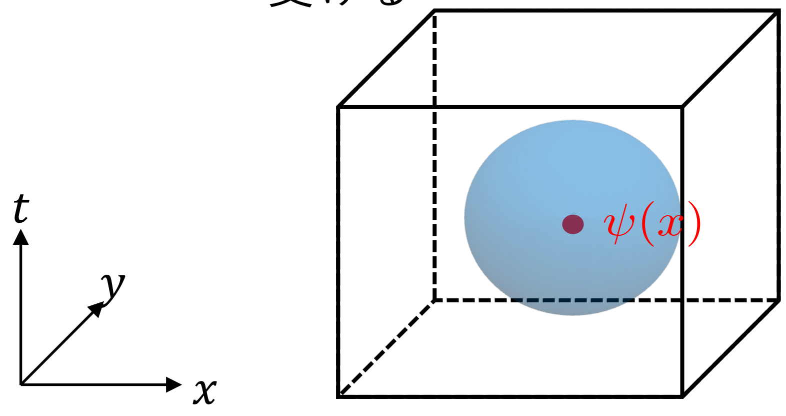
graphicalな見方



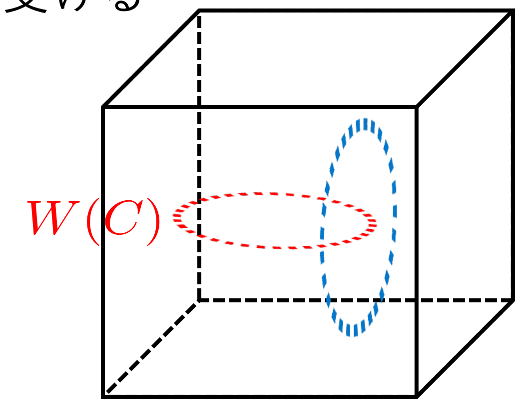
高次対称性

- 従来の対称性（zero form対称性）：点ψ(x)が変換を受ける
  - ✓例：大域的$U(1)$対称性 $\psi(x) \to e^{i\alpha}\psi(x)$
- 変換を受ける対象が広がりを持った物体へ拡張

- zero form対称性
  - ➢点（0次元）ψ(x)が変換を受ける

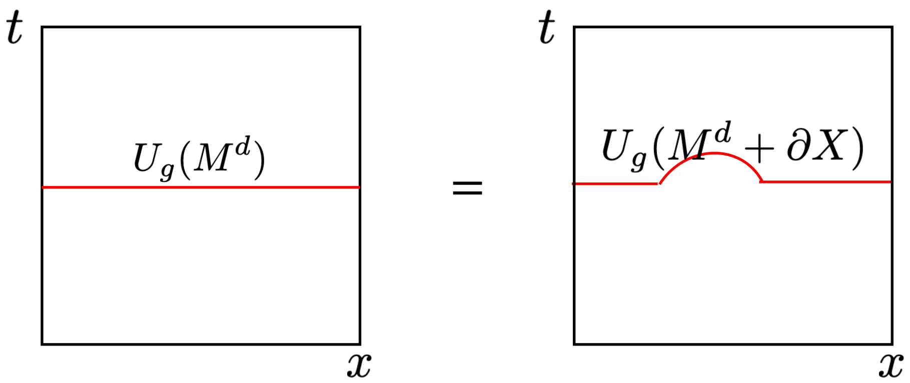


- one form対称性
  - ➢loop（1次元） $W(C)$が変換を受ける



Symmetry Operator's Topological Invariance

- Infinitisimal transformation of M^d,



$$\delta Q = \int_{M^d+\delta M^d} j - \int_{M^d} j = \int_{\partial X^d} j = \int_{X^d} \mathrm{d}j = 0$$

$\mathbb{Z}_N$ one-form ゲージ対称性

$\mathbb{Z}_N$ zero-form ゲージ対称性	$\mathbb{Z}_N$ one-form ゲージ対称性

$\mathbb{Z}_N$ zero-form ゲージ対称性

- $U(1)$ゲージ場 A_μとscalar場 ϕのペア(A_μ, ϕ)で$\mathbb{Z}_N$ one-formゲージ場
- 拘束条件 $NA_\mu = \partial_\mu \phi$
- $\mathbb{Z}_N$ zero-formゲージ変換

$$\phi \mapsto \phi + N\lambda$$
$$A_\mu \mapsto A_\mu + \partial_\mu \lambda$$

$\mathbb{Z}_N$ one-form ゲージ対称性

- $U(1)$ two-formゲージ場 $B_{\mu\nu}$と$U(1)$ゲージ場 C_μのペア$(B_{\mu\nu}, C_\mu)$で$\mathbb{Z}_N$ two-formゲージ場
- 拘束条件

$$NB_{\mu\nu} = \partial_\mu C_\nu$$

今後は簡単のため
$$NB = \mathrm{d}C$$
と書く。

- $\mathbb{Z}_N$ one-formゲージ変換

$$C_\mu \mapsto C_\mu + N\lambda_\mu$$
$$B_{\mu\nu} \mapsto B_{\mu\nu} + \partial_\mu \lambda_\nu$$

$\mathbb{Z}_N$ Zero-form Gauge Symmetry

- Introducing the $U(1)$ gauge field A_μ,

$$S = \int \mathrm{d}^4x\, D_\mu H^\dagger D_\mu H + \cdots\,, \quad D_\mu H = \partial_\mu H - iNA_\mu H$$

- Condense the Higgs H. ϕ is a scalar field.

$$H = he^{i\phi}, \quad \phi \sim \phi + 2\pi$$

$$S = \int \mathrm{d}^4x\, h^2(\partial_\mu\phi - NA_\mu)^2 + \cdots$$

- VEV $h \to \infty$, we get the constraint,

$$\partial_\mu\phi - NA_\mu = 0$$

$\mathbb{Z}_N$ Zero-form Gauge Symmetry

- Constraint: $\partial_\mu \phi = N A_\mu$
- If $N = 1$, A_μ is pure gauge by the constraint, $U(1)$ symmetry is broken completely. On the other hand, if $N > 1$, $\mathbb{Z}_N$ symmetry is remained. Wilson loop is

$$W^N = \left[\exp\left(i \int A_\mu\right)\right]^N = \exp\left(i \int \partial_\mu \phi\right) = 1$$

- By this constraint, a pair, (A_μ, ϕ), $U(1)$ gauge field A_μ and a scalar field ϕ, constructs $\mathbb{Z}_N$ one-form gauge field.
- This pair, (A_μ, ϕ), has the $\mathbb{Z}_N$ zero-form gauge symmetry, and the transformation is

$$\phi \mapsto \phi + N\lambda$$
$$A_\mu \mapsto A_\mu + \partial_\mu \lambda$$

Gaiotto, Kapustin, Komargodski, Seiberg, arXiv:1703.00501

$\mathbb{Z}_N$ One-form Gauge Symmetry

- An example of higher form symmetries, $\mathbb{Z}_N$ one-form gauge symmetry, is not familiar.

➢ Rough method of making $\mathbb{Z}_N$ one-form gauge symmetry

✓ Consider $\mathbb{Z}_N$ zero-form gauge symmetry

✓ Raise the rank of the derivative

✓ Consider $\mathbb{Z}_N$ one-form gauge symmetry

Couple with $SU(N)$ Gauge Theory with θ Term

- Action: $S = -\frac{1}{2g^2}\int \mathrm{tr}\left[(\mathcal{F}-\mathbb{1}B)\wedge\star(\mathcal{F}-\mathbb{1}B)\right] + \frac{\theta}{8\pi^2}\int \mathrm{tr}\left[(\mathcal{F}-\mathbb{1}B)\wedge(\mathcal{F}-\mathbb{1}B)\right] + \frac{1}{2\pi}\int u\wedge(\mathrm{tr}\,\mathcal{F}-NB)$

  $S = -\frac{1}{2g^2}\int \mathrm{tr}(F\wedge\star F) + \frac{\theta}{8\pi^2}\int \mathrm{tr}(F\wedge F)$
- Couple the pair, $(B_{\mu\nu}, C_{\mu})$, $\mathbb{Z}_N$ two-form gauge field, with $SU(N)$ gauge theory

➢ Extend the $SU(N)$ gauge theory to the $U(N)$ gauge theory

➢ $\mathcal{A}$: $U(N)$ gauge field, whose traceless part is $SU(N)$ gauge field A.

➢ Eliminate the trace-part by one-form gauge symmetry,

$$\mathcal{A} \mapsto \mathcal{A} + \lambda\mathbb{1}$$

➢ Imposing the constraint,

$$\mathrm{tr}(\mathcal{F}) = NB$$

➢ With the gauge transformation of a pair (B, C), $\mathbb{Z}_N$ two-form gauge field, $F = \mathcal{F} - \mathbb{1}B$ becomes λ gauge invariant.

➢ By this F, we obtain the $SU(N)$ gauge action coupling with the $\mathbb{Z}_N$ two-form gauge field.

$\mathbb{Z}_N$ One-form Gauge Transformation

$$C \mapsto C + N\lambda$$

$$B \mapsto B + \mathrm{d}\lambda$$

θ 項を持つ$SU(N)$ゲージ理論でのアノマリー

- 作用：

$$S=-\frac{1}{2g^2}\int \mathrm{tr}(f\wedge\star f)+\frac{\theta}{8\pi^2}\int \mathrm{tr}(f\wedge f)$$

θ = 0, πでT対称性を持つ

➢ $F=\mathcal{F}-B1$と置き換えて、 $\mathbb{Z}_N$ two-formゲージ場と$SU(N)$ゲージ理論をcoupleさせる。

$$S=-\frac{1}{2g^2}\int \mathrm{tr}\left[(\mathcal{F}-\mathbb{1}B)\wedge(\mathcal{F}-\mathbb{1}B)\right]+\frac{\theta}{8\pi^2}\int \mathrm{tr}\left[(\mathcal{F}-\mathbb{1}B)\wedge(\mathcal{F}-\mathbb{1}B)\right]+\frac{1}{2\pi}\int u\wedge(\mathrm{tr}\,\mathcal{F}-NB)$$

➢ $\mathbb{Z}_N$ one-form gauge対称性を保つときに、 T変換を施すと

$$Z[B]\xrightarrow{\mathcal{T}}Z[B]\exp\left[i\frac{-1+N+2p}{4\pi N}\int NB\wedge NB\right]$$

2πi×(fractional)になっている

➢ $\mathbb{Z}_N$ one-form gauge対称性と時間反転対称性の間にアノマリーを持つ。

➢ 完全に正則化された理論で理解したい。（格子理論）← motivation

$\mathbb{Z}_N$ one-form ゲージ対称性とfiber bundle

- 既約表現ρがadjointなときにcocycle conditionがrelaxされる。

$$g_{ij}g_{jk}g_{ki} = \exp\left(\frac{2\pi i}{N} n_{ijk}\right)$$

- ここで$\{n_{ijk}\}$はmod Nでantisymmetric。

$\in \mathbb{Z}_N$

➢ $\{n_{ijk}\}$はgauge redundancyを持つ。

➢ transition functionの変換

$$g_{ij} \mapsto \exp\left(\frac{2\pi i}{N}\lambda_{ij}\right) g_{ij}$$

Transition Function in $SU(N)/\mathbb{Z}_N$ Gauge Theory

の元でcocycle conditionがinvariantであるために

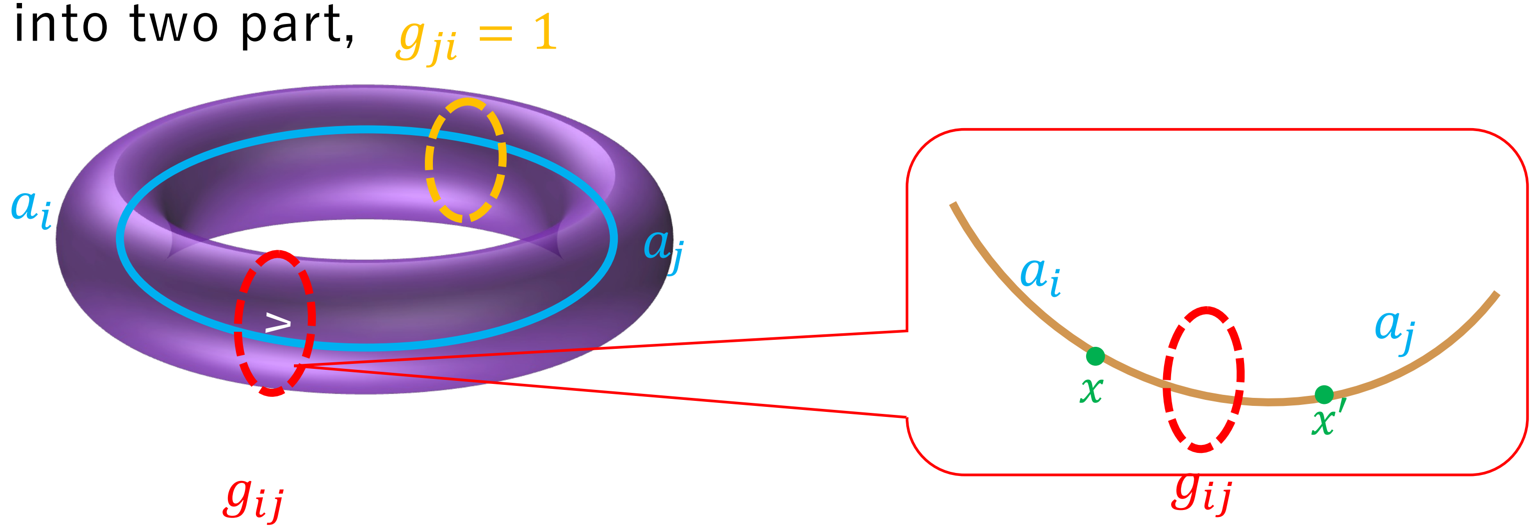
$$n_{ijk} \mapsto n_{ijk} + (\delta\lambda)_{ijk}$$

$$(\delta\lambda)_{ijk} \equiv \lambda_{ij} - \lambda_{ik} + \lambda_{jk}$$

➢ この変換を$\mathbb{Z}_N$ one-form ゲージ変換、 $\{n_{ijk}\}$は$\mathbb{Z}_N$ two-formゲージ場と言う。

Wilson Loop and Transition Function

- Divided the torus into two part,



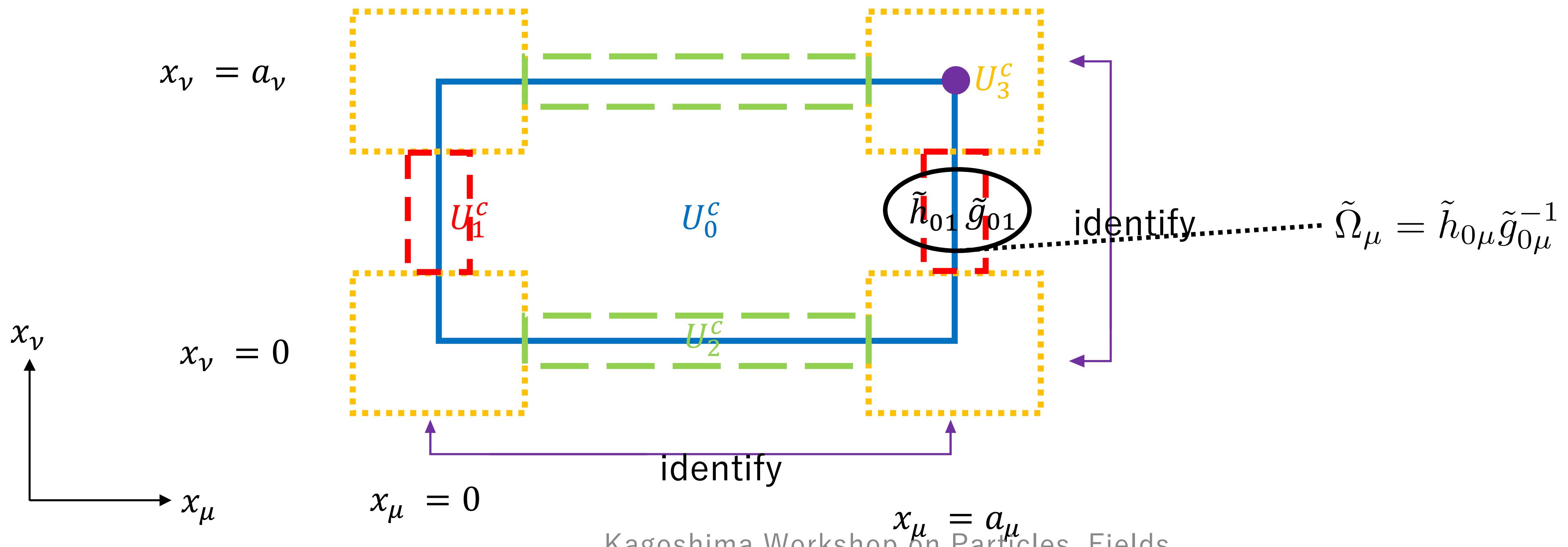
$$W(C) = e^{i\int_{y'}^{x'} a_j} e^{i\int_{y'}^{y} a_j} e^{i\int_{y}^{x} a_i} e^{i\int_{x}^{x'} a_j}$$

$$\xrightarrow[x \to x',\, y \to y']{} g_{ji} e^{i\int_{x}^{y} a_i} g_{ij} e^{i\int_{y}^{x} a_i}$$

$$= g_{ij} e^{i\int_{C} a_i}$$

Transition Function in $SU(N)$ Gauge Theory

- Transition function is defined in nontrivial patches.
- In $2d$, the manifold T^2 is divided by four patches



Transition Function in $SU(N)$ Gauge Theory

- By the transition function $\tilde{\Omega}_\mu$, the cocycle condition is

$$\tilde{\Omega}_\mu(x_\nu = a_\nu)\tilde{\Omega}_\nu(x_\mu = 0)\tilde{\Omega}_\mu^{-1}(x_\nu = 0)\tilde{\Omega}_\nu^{-1}(x_\mu = a_\mu) = 1$$

- To consider the fractional topological charge, we redefine the transition function Ω_μ. (Making $SU(N)/\mathbb{Z}_N$ bundle)

$$\Omega_\mu = \tilde{h}_{0\mu}\omega_\mu\tilde{g}_{0\mu}^{-1}$$

factor of fractionality

$$\omega_\mu = \exp\left(\frac{\pi i}{N}\sum_\nu \frac{n_{\mu\nu}x_\nu}{a_\nu}T_1\right)$$

$SU(N)$'s generator

➢ The cocycle condition is relaxed,

$$\Omega_\mu(x_\nu = a_\nu)\Omega_\nu(x_\mu = 0)\Omega_\mu^{-1}(x_\nu = 0)\Omega_\nu^{-1}(x_\mu = a_\mu) = \exp\left(\frac{2\pi i}{N}n_{\mu\nu}\right)$$

$\mathbb{Z}_N$ one-form ゲージ対称性とファイバー束

- 既約表現ρがadjointなときにcocycle conditionがrelaxされる。

$$g_{ij}g_{jk}g_{ki} = \exp\left(\frac{2\pi i}{N} n_{ijk}\right)$$

$\in \mathbb{Z}_N$

- ここで$\{n_{ijk}\}$はmod Nでantisymmetric。
  - $\{n_{ijk}\}$はgauge redundancyを持つ。
  - transition functionの変換

$$g_{ij} \mapsto \exp\left(\frac{2\pi i}{N}\lambda_{ij}\right) g_{ij}$$

Transition Function in $SU(N)/\mathbb{Z}_N$ Gauge Theory

の元でcocycle conditionがinvariantであるために

$$n_{ijk} \mapsto n_{ijk} + (\delta\lambda)_{ijk}$$

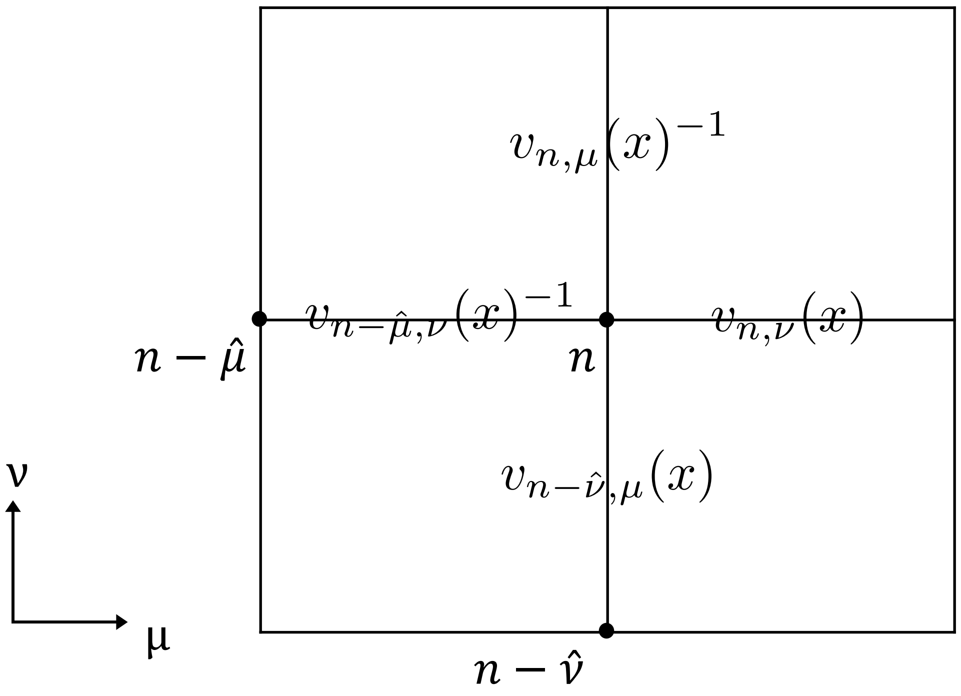
$$(\delta\lambda)_{ijk} \equiv \lambda_{ij} - \lambda_{ik} + \lambda_{jk}$$

  - この変換を$\mathbb{Z}_N$ one-form ゲージ変換、$\{n_{ijk}\}$は$\mathbb{Z}_N$ two-formゲージ場と言う

Cocycle Condition on the Lattice

$$(\text{new transition function}) \sim \omega_\mu \times (\text{normal transition function})$$

- By the new transition function, the cocycle condition is



ordinary

$$\check{v}_{n-\hat{\mu},\nu}(x)\check{v}_{n,\mu}(x)\check{v}_{n,\nu}(x)^{-1}\check{v}_{n-\hat{\nu},\mu}(x)^{-1} = 1$$

new

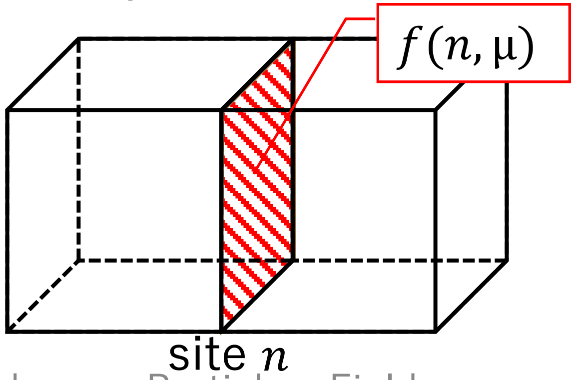
$$v_{n-\hat{\mu},\nu}(x)v_{n,\mu}(x)v_{n,\nu}(x)^{-1}v_{n-\hat{\nu},\mu}(x)^{-1} = \exp\left(\frac{2\pi i}{N} z_{\mu\nu}\right)$$

Lüscher's Idea

- Topological charge is defined by the continuum function: transition function $v_{n,\mu}$,

$$Q(v_{n,\mu}) = -\frac{1}{24\pi^2}\sum_{n}\sum_{\mu,\nu,\rho,\sigma}\int_{f(n,\mu)} \mathrm{d}^3x\,\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\Big((v_{n,\mu}^{-1}\partial_\nu v_{n,\mu})(v_{n,\mu}^{-1}\partial_\rho v_{n,\mu})(v_{n,\mu}^{-1}\partial_\sigma v_{n,\mu})\Big)$$
$$+\frac{1}{8\pi^2}\sum_{n}\sum_{\mu,\nu,\rho,\sigma}\int_{p(n,\mu,\nu)} \mathrm{d}^2x\,\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\Big((v_{n,\mu}\partial_\rho v_{n,\mu}^{-1})(v_{n-\hat{\mu},\nu}^{-1}\partial_\sigma v_{n-\hat{\mu},\nu})\Big)$$

- By the interpolate function: "Parallel transporter", he defined the transition function $v_{n,\mu}$ on the face $f(n,\mu)$.



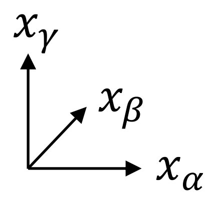
Lüscher, Commun. Math. Phys. 85 (1982)

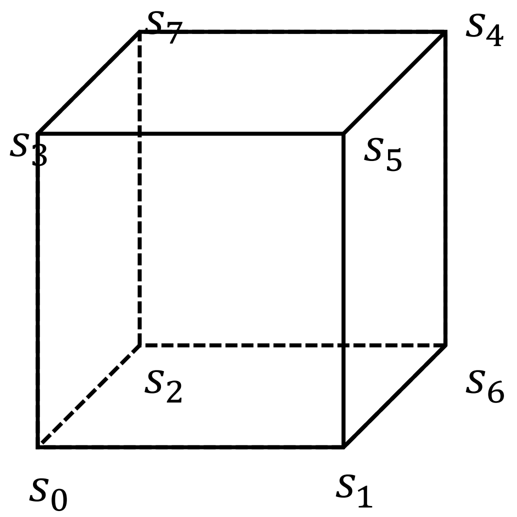
Interpolate Function in $SU(N)$ Gauge Theory

- In $x \in f(n, \mu)$,

$$
\begin{aligned}
f^m_{n,\mu}(x_\gamma) &= (u_{30})^{y_\gamma} (u^m_{03} u^m_{37} u^m_{72} u^m_{20})^{y_\gamma} u^m_{02} (u^m_{27})^{y_\gamma} \\
g^m_{n,\mu}(x_\gamma) &= (u_{51})^{y_\gamma} (u^m_{15} u^m_{54} u^m_{46} u^m_{61})^{y_\gamma} u^m_{16} (u^m_{64})^{y_\gamma} \\
h^m_{n,\mu}(x_\gamma) &= (u_{30})^{y_\gamma} (u^m_{03} u^m_{35} u^m_{51} u^m_{10})^{y_\gamma} u^m_{01} (u^m_{15})^{y_\gamma} \\
k^m_{n,\mu}(x_\gamma) &= (u_{72})^{y_\gamma} (u^m_{27} u^m_{74} u^m_{46} u^m_{62})^{y_\gamma} u^m_{26} (u^m_{64})^{y_\gamma} \\
l^m_{n,\mu}(x_\beta, x_\gamma) &= \left[f^m_{n,\mu}(x_\gamma)^{-1}\right]^{y_\beta} \left[f^m_{n,\mu}(x_\gamma) k^m_{n,\mu}(x_\gamma) g^m_{n,\mu}(x_\gamma)^{-1} h^m_{n,\mu}(x_\gamma)^{-1}\right]^{y_\beta} \\
&\quad \cdot h^m_{n,\mu}(x_\gamma) \left[g^m_{n,\mu}(x_\gamma)\right]^{y_\beta} \\
S^m_{n,\mu}(x_\alpha, x_\beta, x_\gamma) &= (u^m_{03})^{y_\gamma} \left[f^m_{n,\mu}(x_\gamma)\right]^{y_\beta} \left[l^m_{n,\mu}(x_\beta, x_\gamma)\right]^{y_\alpha}
\end{aligned}
$$

Difficult!!





Parallel Transporter in the Lattice $U(1)$ Gauge Theory

- By the parallel transporter $w^m(x)$, we obtain the transition function $v_{n,\mu}$ in the continuum point x: $v_{n,\mu}(x) = w^{n-\hat{\mu}}(x) w^n(x)^{-1}$

lattice

$$w^n(\bar{x}) = U(n,1)^{\sigma_1} U(n+\sigma_1\hat{1},2)^{\sigma_2} \cdots U(n+\sigma_1\hat{1}+\sigma_2\hat{2}+\cdots+\sigma_{D-1}\widehat{D-1}, D)^{\sigma_D}$$

$$\bar{x} = n + \sum_{\mu=1}^{D} \sigma_\mu \hat{\mu} \quad (\sigma_\mu = \{0,1\})$$

Interpolate

Continuum

$$w^m(x) = \prod_{\{\sigma_k=0,1\}_{k=1,\cdots,D-1}} w^m\left(n + \sum_{k=1}^{D-1} \sigma_k \hat{\mu}_k\right)^{\prod_{k=1}^{D-1}(\sigma_k y_k + (1-\sigma_k)(1-y_k))}$$

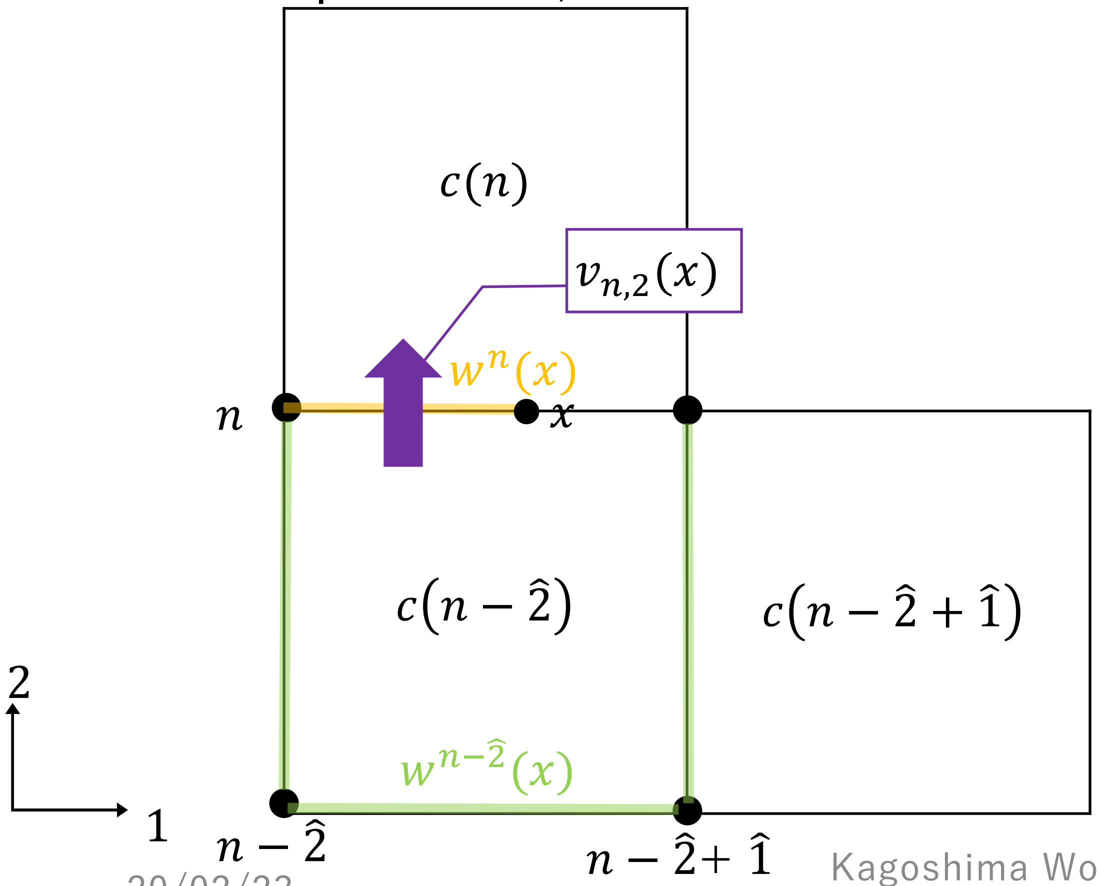
m: $n, n-\hat{\mu}$

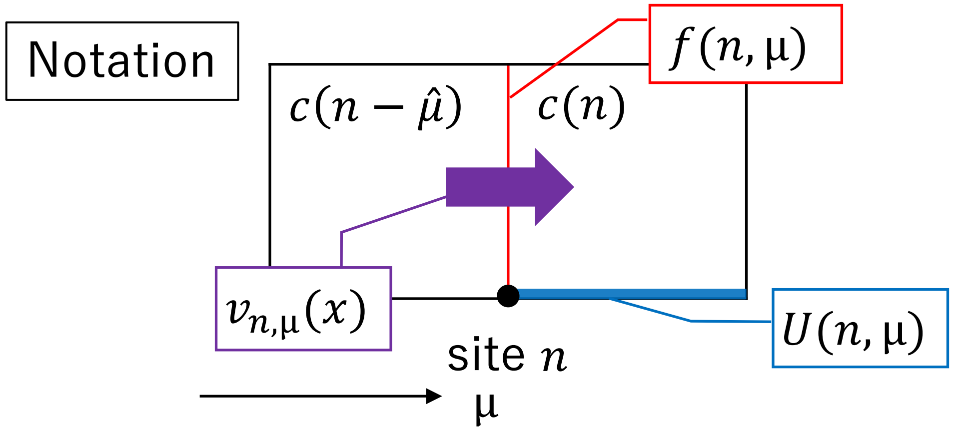
Interpolate Parameter y_k

$$x = n + \sum_{k=1}^{D-1} y_k \hat{\mu}_k, \quad 0 \le y_k \le 1$$

Image of Parallel Transporter

➢ Example: in $2d$,





Parallel Transporter

$0 \leq y_1 \leq 1$

$$w^n(x) = U(n,1)^{y_1}$$

$$w^{n-\hat{2}}(x) = \left[U(n-\hat{2},1)U(n-\hat{2}+\hat{1},2)\right]^{y_1} U(n-\hat{2},2)^{1-y_1}$$

Transition Function on the Lattice in $2d$

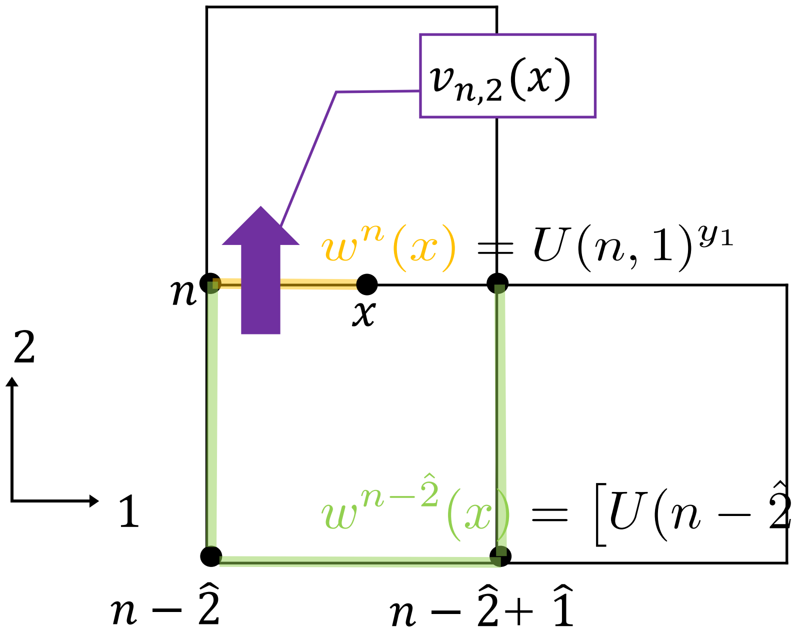
- By using the parallel transport function, the transition function is,

$$v_{n,\mu}(x) = w^{n-\hat{\mu}}(x) w^n(x)^{-1}$$

➢ Example: in $2d$,

$$v_{n,2}(x) = w^{n-\hat{2}}(x) w^n(x)^{-1}$$
$$= U(n-\hat{2},2)\left[U(n-\hat{2},1)U(n-\hat{2}+\hat{1},2)U(n,1)^{-1}U(n-\hat{2},2)^{-1}\right]^{y_1}$$
$$= U(n-\hat{2},2)\exp\left[iy_1 F_{12}(n-\hat{2})\right]$$

$$F_{\mu\nu}(n) = \frac{1}{i}\ln\left[U(n,\mu)U(n+\hat{\mu},\nu)U(n+\hat{\nu},\mu)^{-1}U(n,\nu)^{-1}\right]$$



$$w^n(x) = U(n,1)^{y_1}$$

$$w^{n-\hat{2}}(x) = \left[U(n-\hat{2},1)U(n-\hat{2}+\hat{1},2)\right]^{y_1} U(n-\hat{2},2)^{1-y_1}$$

Transition Function on the Lattice in $4d$

In $2d$

$$v_{n,2}(x) = U(n-\hat{2},2)\exp\left[iy_1F_{12}(n-\hat{2})\right]$$

- In $4d$,

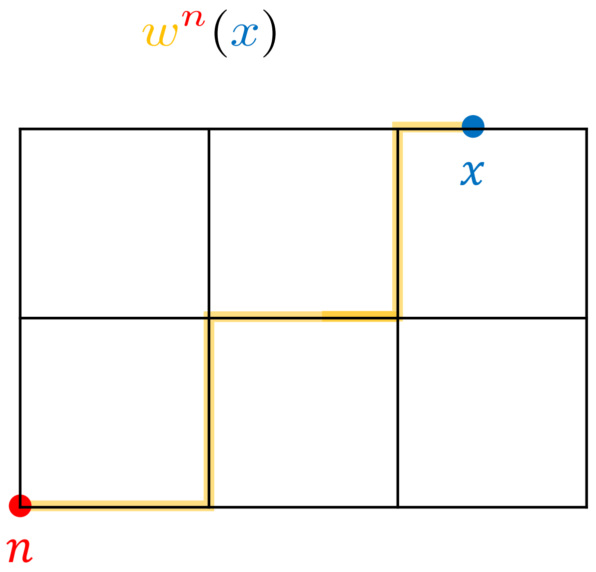
$$\begin{aligned} v_{n,1}(x) &= U(n-\hat{1},1) \\ &\times \exp\Big[iy_4F_{14}(n-\hat{1}) + iy_3y_4F_{13}(n-\hat{1}+\hat{4}) + iy_3(1-y_4)F_{13}(n-\hat{1}) \\ &\quad + iy_2y_3y_4F_{12}(n-\hat{1}+\hat{3}+\hat{4}) + iy_2y_3(1-y_4)F_{12}(n-\hat{1}+\hat{3}) \\ &\quad + iy_2(1-y_3)y_4F_{12}(n-\hat{1}+\hat{4}) + iy_2(1-y_3)(1-y_4)F_{12}(n-\hat{1})\Big], \\ v_{n,2}(x) &= U(n-\hat{2},2)\exp\left[iy_4F_{24}(n-\hat{2}) + iy_3y_4F_{23}(n-\hat{2}+\hat{4}) + iy_3(1-y_4)F_{23}(n-\hat{2})\right], \\ v_{n,3}(x) &= U(n-\hat{3},3)\exp\left[iy_4F_{34}(n-\hat{3})\right], \\ v_{n,4}(x) &= U(n-\hat{4},4) \end{aligned}$$

- Field strength is

$$F_{\mu\nu}(n) = \frac{1}{i}\ln\left[U(n,\mu)U(n+\hat{\mu},\nu)U(n+\hat{\nu},\mu)^{-1}U(n,\nu)^{-1}\right]$$

Parallel Transport Function

- Parallel transport function's image is "by the interpolate parameter y, the transition function is defined as the function on an arbitrarily point x on the link".



Link Variables

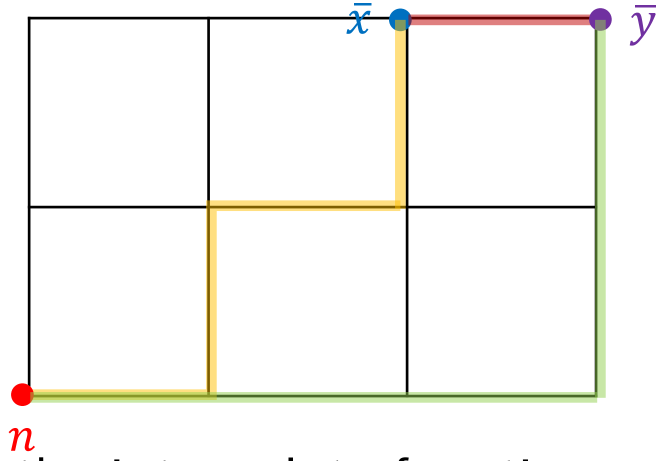
- In $SU(N)$ gauge field, this process is very complicated.

➢ By the parallel transport function, we defined the new link variable.

$$u^n_{xy} = w^n(\bar{x})U(\bar{x},\mu)w^n(\bar{y})^{-1} \qquad (\bar{y} = n + \hat{\mu})$$

$$u^n_{xy} = (u^n_{xy})^{-1} \qquad (\bar{y} = n - \hat{\mu})$$

Image of u^n_{xy}



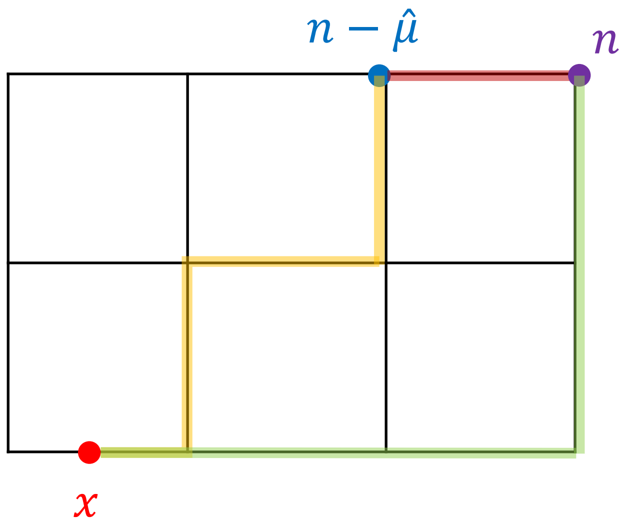
➢ By this link variable, we define the interpolate function.

Transition Function

- By the interpolate function made from the new link variable, we define the transition function as continuum function on the lattice .

$$\tilde{v}_{n,\mu}(x) = S^{n-\hat{\mu}}_{n,\mu}(x)^{-1}\tilde{v}_{n,\mu}(n)S^{n}_{n,\mu}(x)$$

Image of $\tilde{v}_{n,\mu}$



Cocycle Condition

- Check the cocycle condition by this new transition function

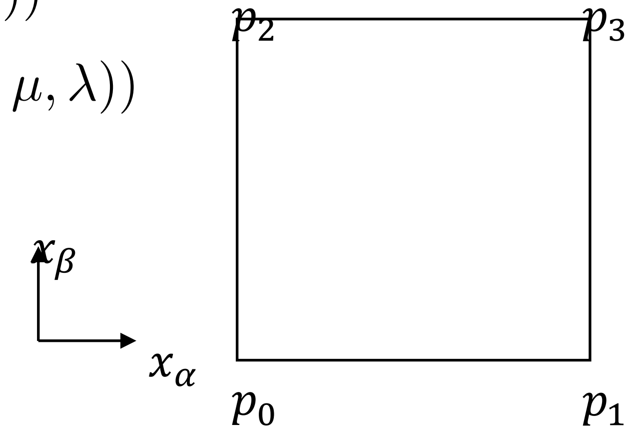
➢ In $x \in p(n, \mu, \nu)$, we define new function,

$$P^m_{n,\mu\nu}(x_\alpha, x_\beta) = (u^m_{p_0p_2})^{y_\beta} \left[(u^m_{p_2p_0})^{y_\beta} \left(u^m_{p_0p_2} u^m_{p_2p_3} u^m_{p_3p_1} u^m_{p_1p_0}\right)^{y_\beta} u^m_{p_0p_1} (u^m_{p_1p_3})^{y_\beta}\right]^{y_\alpha}$$

➢ The relation with $S^m_{n,\mu}(x)$ is

$$S^m_{n,\mu}(x) = P^m_{n,\mu\lambda}(x) \qquad (x \in p(n, \mu, \lambda))$$

$$S^m_{n,\mu}(x) = R^m_{n,\mu;\lambda} P^m_{n+\hat{\lambda},\mu\lambda}(x) \qquad (x \in p(n+\hat{\lambda}, \mu, \lambda))$$



格子でのCocycle Condition

新しいTransition Function

$$v_{n,\mu}(x) = \omega_\mu(x)\check{v}_{n,\mu}(x) \quad \text{at } x \in f(n,\mu)$$

- 元のtransition function $\check{v}_{n,\mu}$では、cocycle conditionは

$$\check{v}_{n-\hat{\mu},\nu}(x)\check{v}_{n,\mu}(x)\check{v}_{n,\nu}(x)^{-1}\check{v}_{n-\hat{\nu},\mu}(x)^{-1} = 1$$

格子版でのcocycle condition

$g_{ij}g_{jk}g_{ki} = 1$

- 新しいtransition function $v_{n,\mu}$では、ω_μにより

$$v_{n-\hat{\mu},\nu}(x)v_{n,\mu}(x)v_{n,\nu}(x)^{-1}v_{n-\hat{\nu},\mu}(x)^{-1}$$

$$= \begin{cases} \exp\left(\frac{2\pi i}{q} z_{\mu\nu}\right) & \text{for } x_\mu = x_\nu = 0 \text{ mod } L \\ 1 & \text{otherwise} \end{cases}$$

$\in \mathbb{Z}_q$

$\mathbb{Z}_q$の元

Cocycle Condition

- R^m is

$$
\begin{aligned}
R^m_{n,\mu;\alpha}(x_\beta, x_\gamma) =& \left[(u^m_{03}u^m_{37}u^m_{72}u^m_{20})^{y_\gamma}\, u^m_{02}\right.\\
&\cdot (u^m_{27}u^m_{74}u^m_{46}u^m_{62})^{y_\gamma}\, u^m_{26}u^m_{61}\,(u^m_{16}u^m_{64}u^m_{45}u^m_{51})^{y_\gamma}\\
&\cdot \left. u^m_{10}\,(u^m_{01}u^m_{15}u^m_{53}u^m_{30})^{y_\gamma}\right]^{y_\beta} (u^m_{03}u^m_{35}u^m_{51}u^m_{10})^{y_\gamma}\, u^m_{01}\\
R^m_{n,\mu;\beta}(x_\alpha, x_\gamma) =& (u^m_{03}u^m_{37}u^m_{72}u^m_{20})^{y_\gamma}\, u^m_{02}\\
R^m_{n,\mu;\gamma}(x_\alpha, x_\beta) =& u^m_{03}
\end{aligned}
$$

Cocycle Condition

- By the new interpolate function, in $x \in p(n, \mu, \nu)$, the cocycle condition is

$$\tilde{v}_{n-\hat{\mu},\nu}(x)\tilde{v}_{n,\mu}(x) = \left(P_{n,\mu\nu}^{n-\hat{\nu}-\hat{\mu}}(x)^{-1} v_{n-\hat{\mu},\nu}(n) P_{n,\mu\nu}^{n-\hat{\mu}}(x)\right)\left(P_{n,\mu\nu}^{n-\hat{\mu}}(x)^{-1} v_{n,\nu}(n) P_{n,\mu\nu}^{n}(x)\right)$$
$$= P_{n,\mu\nu}^{n-\hat{\nu}-\hat{\mu}}(x)^{-1} v_{n-\hat{\mu},\nu}(n) v_{n,\nu}(n) P_{n,\mu\nu}^{n}(x)$$

$$\tilde{v}_{n-\hat{\mu},\nu}(x)\tilde{v}_{n,\mu}(x) = P_{n,\mu\nu}^{n-\hat{\nu}-\hat{\mu}}(x)^{-1} v_{n-\hat{\nu},\mu}(n) v_{n,\mu}(n) P_{n,\mu\nu}^{n}(x)$$

- When (cocycle condition)=1 is satisfied at each site,

$$\tilde{v}_{n-\hat{\mu},\nu}(x)\tilde{v}_{n,\mu}(x)\tilde{v}_{n,\nu}(x)^{-1}\tilde{v}_{n-\hat{\mu},\nu}(x)^{-1} = 1$$

Topological Charge

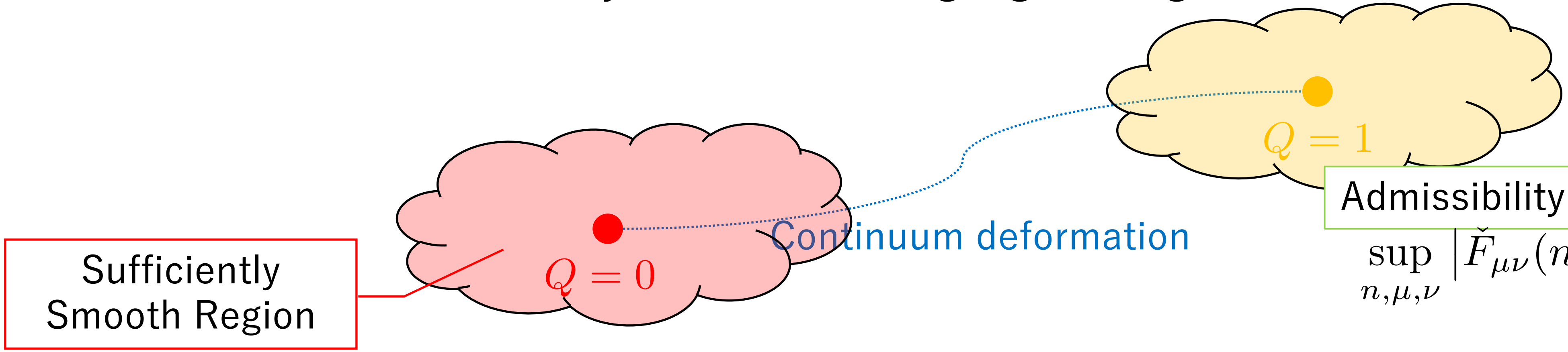
- By the new transition function, the topological charge is

$$
\begin{aligned}
P(\tilde{v}_{n,\mu}) =& \frac{1}{24\pi^2} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \Bigg\{ 3 \int_{p(n+\hat{\mu}+\hat{\nu},\mu,\nu)} \mathrm{d}^2 x \,\mathrm{Tr}\left[P^n_{n+\hat{\mu}+\hat{\nu},\mu\nu} \partial_\rho (P^n_{n+\hat{\mu}+\hat{\nu},\mu\nu})^{-1} (R^n_{n+\hat{\mu},\mu;\nu})^{-1} \partial_\sigma R^n_{n+\hat{\mu},\mu;\nu} \right] \\
& - 3 \int_{p(n+\hat{\nu},\mu,\nu)} \mathrm{d}^2 x \,\mathrm{Tr}\left[P^n_{n+\hat{\nu},\mu\nu} \partial_\rho (P^n_{n+\hat{\nu},\mu\nu})^{-1} (R^n_{n,\mu;\nu})^{-1} \partial_\sigma R^n_{n,\mu;\nu} \right] \\
& - \int_{f(n+\hat{\mu},\mu)} \mathrm{d}^3 x \,\mathrm{Tr}\left[S^n_{n+\hat{\mu},\mu} \partial_\nu (S^n_{n+\hat{\mu},\mu})^{-1} S^n_{n+\hat{\mu},\mu} \partial_\rho (S^n_{n+\hat{\mu},\mu})^{-1} S^n_{n+\hat{\mu},\mu} \partial_\sigma (S^n_{n+\hat{\mu},\mu})^{-1} \right] \\
& + \int_{f(n,\mu)} \mathrm{d}^3 x \,\mathrm{Tr}\left[S^n_{n,\mu} \partial_\nu (S^n_{n,\mu})^{-1} S^n_{n,\mu} \partial_\rho (S^n_{n,\mu})^{-1} S^n_{n,\mu} \partial_\sigma (S^n_{n,\mu})^{-1} \right] \Bigg\}
\end{aligned}
$$

Admissibility Condition

- It is impossible to define the topological charge which has intervals on the lattice.

➢ Under the “Admissibility condition”, the gauge configuration is sufficiently smooth.



Admissibility Condition

$$\sup_{n,\mu,\nu} \left| \check{F}_{\mu\nu}(n) \right| < \epsilon, \quad 0 < \epsilon < \frac{\pi}{3q}$$

- Field strength is

$$\check{F}_{\mu\nu}(n) = \frac{1}{iq} \ln \left[U(n,\mu) U(n+\hat{\mu},\nu) U(n+\hat{\nu},\mu)^{-1} U(n,\nu)^{-1} \right]^q$$

※ q is needed for the invariance under the $\mathbb{Z}_q$ one-form transformation.

Topological Charge in the $SU(N)$ Gauge Theory

- By the new transition function, we calculate topological charge $Q(v_{n,\mu})$.
- In $4d$ continuum theory, (van Baal, Commun. Math. Phys. 85 (1982))

$$Q(v_{n,\mu}) = \frac{1}{24\pi^2}\sum_{\mu}\int d_3\sigma_\mu \varepsilon_{\mu\nu\alpha\beta}\,\mathrm{Tr}\Big((v_{n,\mu}\partial_\nu v_{n,\mu}^{-1})(v_{n,\mu}\partial_\alpha v_{n,\mu}^{-1})(v_{n,\mu}\partial_\beta v_{n,\mu}^{-1})\Big)$$

$$+\frac{1}{8\pi^2}\sum_{\mu,\nu}\int d_2 S_{\mu\nu}\varepsilon_{\mu\nu\alpha\beta}\,\mathrm{Tr}\Big((v_{n,\nu}^{-1}\partial_\alpha v_{n,\nu})_{x_\mu=a_\mu}(v_{n,\mu}\partial_\beta v_{n,\mu}^{-1})_{x_\nu=0}\Big)$$

$$=\mathbb{Z}+\frac{N-1}{N}\cdot\frac{1}{8}\varepsilon_{\mu\nu\alpha\beta}z_{\mu\nu}z_{\alpha\beta}$$

integer ($\mathbb{Z}$) fractional ($\frac{N-1}{N}\cdot\frac{1}{8}\varepsilon_{\mu\nu\alpha\beta}z_{\mu\nu}z_{\alpha\beta}$)

Differential Calculus on the Lattice

- k-form function: $f(n) \equiv \frac{1}{k!} \sum_{\mu_1,\cdots,\mu_k} f_{\mu_1\cdots\mu_k}(n)\, \mathrm{d}x_{\mu_1} \cdots \mathrm{d}x_{\mu_k}$
- The definition of extra derivative: $\mathrm{d}x_\mu\, f_{\mu_1\cdots\mu_k}(n) = f_{\mu_1\cdots\mu_k}(n+\hat{\mu})\, \mathrm{d}x_\mu$

➢ By this extra derivative on the lattice, the Leibniz rule on the lattice is

$$\mathrm{d}[f(n)g(n)] = \mathrm{d}f(n)\cdot g(n) + (-1)^k f(n)\cdot \mathrm{d}g(n)$$

➢ Example:

$$f(n) = \frac{1}{2}\sum_{\mu,\nu} f_{\mu\nu}(n) dx_\mu dx_\nu \quad \Rightarrow \quad f(n)f(n) = \frac{1}{4}\sum_{\mu,\nu,\rho,\sigma} f_{\mu\nu}(n) f_{\rho\sigma}(n+\hat{\mu}+\hat{\nu}) dx_\mu dx_\nu dx_\rho dx_\sigma$$

$$= \frac{1}{4}\sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} f_{\mu\nu}(n) f_{\rho\sigma}(n+\hat{\mu}+\hat{\nu}) dx_1 dx_2 dx_3 dx_4$$

$\mathbb{Z}_q$ One-form Global Symmetry and Gauge Symmetry

- $\mathbb{Z}_q$ one-form symmetry is corresponding to multiplying the $\mathbb{Z}_q$ element by the transition function from the point of fiber bundle.

➢ Consider the transformation of the transition function on the lattice

➢ Firstly, consider the $\mathbb{Z}_q$ one-form global symmetry

Admissibility Condition

- Field strength is

$$\check{F}_{\mu\nu}(n) = \frac{1}{iq}\ln\left[U(n,\mu)U(n+\hat{\mu},\nu)U(n+\hat{\nu},\mu)^{-1}U(n,\nu)^{-1}\right]^q \quad |F_{\mu\nu}(n)| < \pi$$

- Invariant under the $\mathbb{Z}_q$ one-form gauge transformation
- We require the admissibility condition to the field strength,

$$\sup_{n,\mu,\nu}\left|\check{F}_{\mu\nu}(n)\right| < \epsilon, \quad 0 < \epsilon < \frac{\pi}{3q}$$

- Under this condition, the Bianchi identity is satisfied.

$$\sum_{\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma}\Delta_\nu \check{F}_{\rho\sigma}(n) = 0$$

Proof of Admissibility Condition

- Field strength is

$$
\begin{aligned}
F_{\mu\nu}(n) =& \frac{1}{iq} \ln \left[e^{i(a_\mu(n)+a_\nu(n)+\hat{\mu}-a_\mu(n+\hat{\nu})-a_\nu(n))} \right]^q \\
=& \frac{1}{iq} \left[i \left(a_\mu(n) + a_\nu(n) + \hat{\mu} - a_\mu(n+\hat{\nu}) - a_\nu(n) \right) \cdot q + 2\pi i N_{\mu\nu}(n) \right] \\
=& \Delta_\nu a_\mu(n) - \Delta_\mu a_\nu(n) + \frac{2\pi}{q} N_{\mu\nu}(n)
\end{aligned}
$$

➢ $N_{\mu\nu}$ is the function for taking $F_{\mu\nu}$ back to the range $[-\pi, \pi]$.

Proof of Admissibility Condition

- By the admissibility condition,

$$\sum_{\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \Delta_\nu F_{\mu\nu}(n) < 6\epsilon$$

➢ By definition,

$$\sum_{\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \Delta_\nu \left(\Delta_\rho a_\sigma(n) - \Delta_\rho a_\sigma(n) + \frac{2\pi}{q} N_{\rho\sigma}(n) \right) = \frac{2\pi}{q} \sum_{\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \Delta_\nu N_{\rho\sigma}(n)$$

➢ By $\varepsilon_{\mu\nu\rho\sigma} \Delta_\nu N_{\rho\sigma}(n) < 1$

$$0 < 6\epsilon < \frac{2\pi}{q} \quad \Rightarrow \quad 0 < \epsilon < \frac{\pi}{3q}$$

$\mathbb{Z}_q$ Two-form Gauge Field

- $\mathbb{Z}_q$ two-form gauge field is defined by

$$z_{\mu\nu}(n) = z_{\mu\nu}\delta_{n_\mu,L-1}\delta_{n_\nu,L-1} + \Delta_\mu z_\nu(n) - \Delta_\nu z_\mu(n) + qN_{\mu\nu}(n) \quad \in \mathbb{Z}$$

➢ To protect the antisymmetric value,

$$\begin{cases} 0 \le z_{\mu\nu}(n) < q & \text{for } \mu < \nu, \\ z_{\mu\nu}(n) \equiv -z_{\nu\mu}(n) & \text{for } \mu > \nu \end{cases}$$

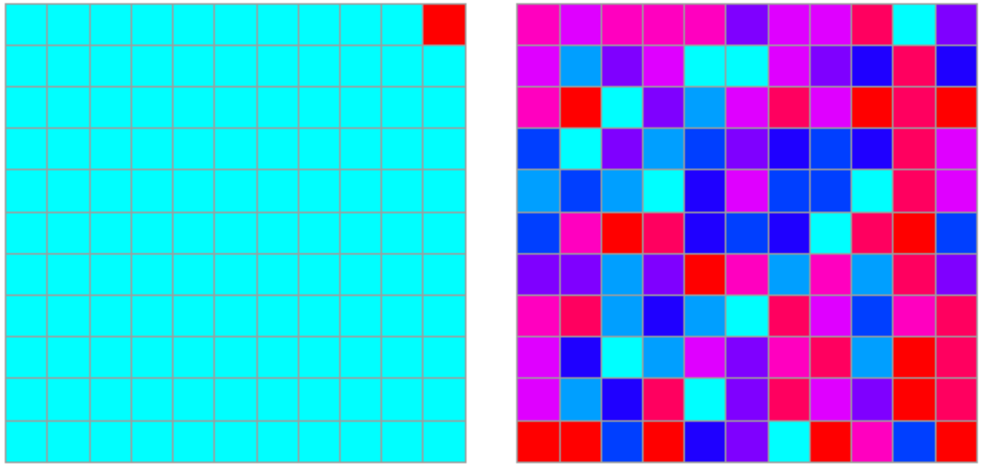
➢ Under the $\mathbb{Z}_q$ one-form gauge transformation, $\mathbb{Z}_q$ two-form field is

$$z_{\mu\nu}(n) \to z_{\mu\nu}(n) + \Delta_\mu z_\nu(n) - \Delta_\nu z_\mu(n) + qN_{\mu\nu}(n)$$

$\mathbb{Z}_q$ Two-form Gauge Field

- This $\mathbb{Z}_q$ two-form gauge field is connected to an arbitrary gauge configuration by the $\mathbb{Z}_q$ one-form gauge transformation.

$$z_{\mu\nu}(n) = z_{\mu\nu}\delta_{n_\mu,L-1}\delta_{n_\nu,L-1} + \Delta_\mu z_\nu(n) - \Delta_\nu z_\mu(n) + qN_{\mu\nu}(n) \quad \in \mathbb{Z}$$



Fractional Topological Charge by $\mathbb{Z}_q$ Two-form Gauge Field

$$Q = \frac{1}{32\pi^2} \sum_{n\in\Lambda} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \left[F_{\mu\nu}(n) + \frac{2\pi}{q} z_{\mu\nu}(n)\right] \left[F_{\rho\sigma}(n+\hat{\mu}+\hat{\nu}) + \frac{2\pi}{q} z_{\rho\sigma}(n+\hat{\mu}+\hat{\nu})\right]$$

$$z_{\mu\nu}(n) = z_{\mu\nu}\delta_{n_\mu,L-1}\delta_{n_\nu,L-1} + \Delta_\mu z_\nu(n) - \Delta_\nu z_\mu(n) + qN_{\mu\nu}(n) \quad \in \mathbb{Z}$$

$$Q = \frac{1}{8q^2} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma} + \frac{1}{8\pi q} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} \sum_{n_\mu=0} \check{F}_{\rho\sigma}(n)$$
$$+ \frac{1}{32\pi^2} \sum_{n} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \check{F}_{\mu\nu}(n) \check{F}_{\rho\sigma}(n+\hat{\mu}+\hat{\nu})$$

$\mathbb{Z}_q$ One-form Global Symmetry on the Lattice

- The factor of fractionality ω_μ is related to the $\mathbb{Z}_q$ one-form transform.

➢ Link variable

$$U(n,\mu) \to \exp\left(\frac{2\pi i}{q} z_\mu\right) U(n,\mu) \qquad n_\mu = 0$$

$\in \mathbb{Z}_q$

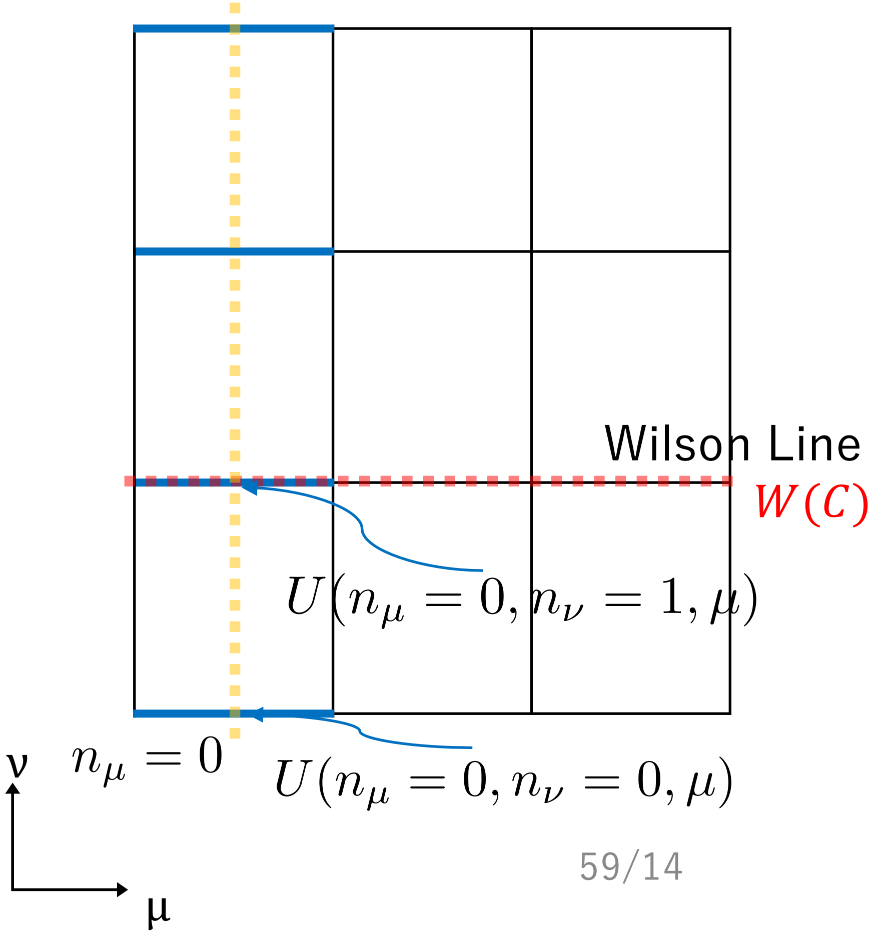
➢ Transition function

$$\check{v}_{n,\mu}(x) \to \begin{cases} \exp\left(\frac{2\pi i}{q} z_\mu\right) \check{v}_{n,\mu}(x) & \text{for } x_\mu = 1 \\ \check{v}_{n,\mu}(x) & \text{otherwise} \end{cases}$$

➢ Cocycle condition

Not $\mathbb{Z}_q$ "Relax"

$$\check{v}_{n-\hat{\nu},\mu}(x)\check{v}_{n,\nu}(x)\check{v}^{-1}_{n,\mu}(x)\check{v}^{-1}_{n-\hat{\mu},\nu}(x) = 1$$



$\mathbb{Z}_q$ One-form Gauge Symmetry on the Lattice

- The factor of fractionality ω_μ is related to the $\mathbb{Z}_q$ one-form transform.

➢ Link variable

$$U(n,\mu) \to \exp\left[\frac{2\pi i}{q} z_\mu(n)\right] U(n,\mu)$$

$\in \mathbb{Z}_q$

➢ Transition function

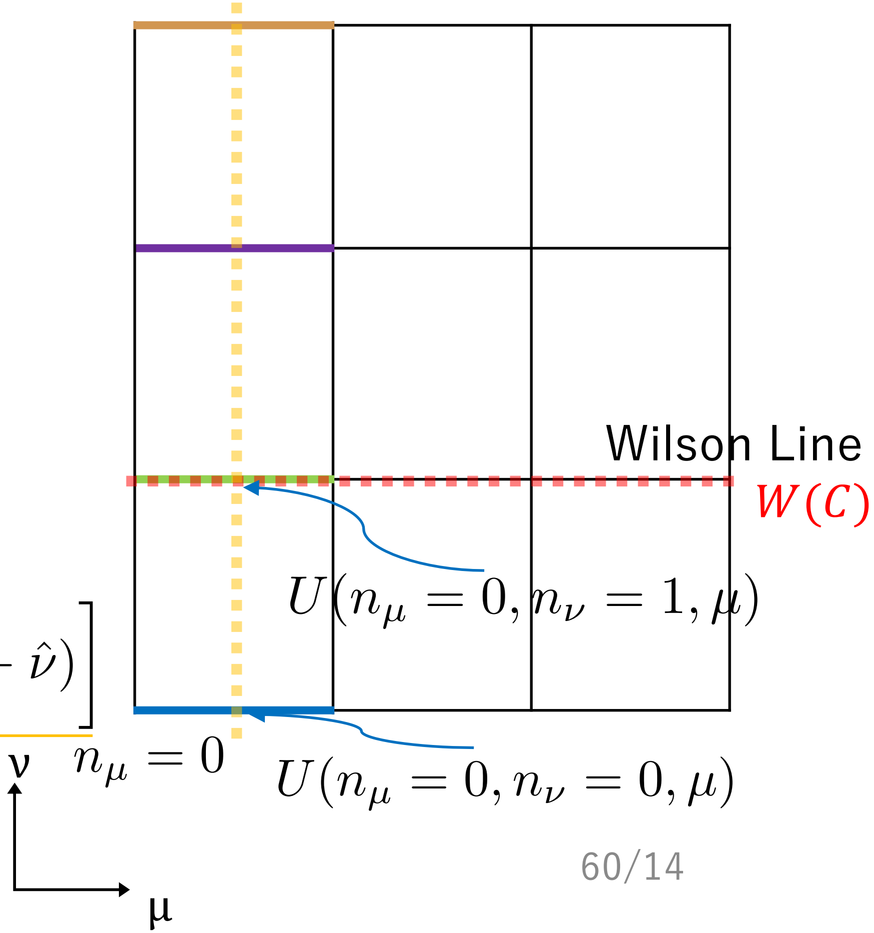
$$v_{n,\mu}(x) \to \exp\left[\frac{2\pi i}{q} z_\mu(n-\hat{\mu})\right] v_{n,\mu}(x) \qquad x \in f(n,\mu)$$

➢ Cocycle condition

$\mathbb{Z}_q$ "relax"

$$v_{n-\hat{\nu},\mu}(x) v_{n,\nu}(x) v_{n,\mu}(x)^{-1} v_{n-\hat{\mu},\nu}(x)^{-1} \equiv \exp\left[\frac{2\pi i}{q} z_{\mu\nu}(n-\hat{\mu}-\hat{\nu})\right]$$

$\in \mathbb{Z}_q$



Mixed 't Hooft Anomaly

- e^{iS} is ,under the $\mathcal{T}$-transformation,

$$e^{i\pi q Q} \overset{\mathcal{T}}{\to} e^{-i\pi q Q} = e^{-2\pi i q Q} \cdot e^{i\pi q Q}$$

$$= \exp\left(-\frac{2\pi i}{8q} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}\right) e^{i\pi q Q}$$

➢ Introducing a local counter term which is invariant under the $\mathbb{Z}_q$ one-form gauge transformation,

$$e^{-S_{\mathrm{counter}}} \equiv \exp\left[\frac{2\pi i k}{4q} \sum_{n} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu}(n) z_{\rho\sigma}(n+\hat{\mu}+\hat{\nu})\right]$$

$$= \exp\left(\frac{2\pi i k}{4q} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}\right)$$

Mixed 't Hooft Anomaly

- e^{iS} is ,under the $\mathcal{T}$-transformation, when $\theta = \pi$,

$$e^{i\pi qQ} \overset{\mathcal{T}}{\to} e^{-i\pi qQ} = e^{-2\pi iqQ} \cdot e^{i\pi qQ}$$
$$= \exp\left(-\frac{2\pi i}{8q} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}\right) e^{i\pi qQ}$$

➢ Introducing a local counter term which is invariant under the $\mathbb{Z}_q$ one-form gauge transformation,

$$e^{-S_{\text{counter}}} \equiv \exp\left(\frac{2\pi ik}{8q} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}\right)$$

Time Reversal Symmetry

$$U(n,\mu) \xrightarrow{\mathcal{T}} \begin{cases} U(\bar{n},\mu) & \text{for } \mu \neq 4, \\ U(\bar{n}-\hat{4},4)^{-1} & \text{for } \mu = 4, \end{cases}$$

$$\check{F}_{\mu\nu}(n) \xrightarrow{\mathcal{T}} \begin{cases} \check{F}_{\mu\nu}(\bar{n}) & \text{for } \mu \neq 4,\ \nu \neq 4, \\ -\check{F}_{4\nu}(\bar{n}-\hat{4}) & \text{for } \mu = 4, \\ -\check{F}_{\mu 4}(\bar{n}-\hat{4}) & \text{for } \nu = 4. \end{cases}$$

$$z_{\mu\nu}(n) \xrightarrow{\mathcal{T}} \begin{cases} z_{\mu\nu}(\bar{n}) & \text{for } \mu \neq 4,\ \nu \neq 4, \\ -z_{4\nu}(\bar{n}+\hat{4}) & \text{for } \mu = 4, \\ -z_{\mu 4}(\bar{n}+\hat{4}) & \text{for } \nu = 4, \end{cases}$$

$$z_{\mu\nu} \xrightarrow{\mathcal{T}} \begin{cases} z_{\mu\nu} & \text{for } \mu \neq 4,\ \nu \neq 4, \\ -z_{4\nu} & \text{for } \mu = 4, \\ -z_{\mu 4} & \text{for } \nu = 4. \end{cases}$$

Witten Effect

- Setting magnetic monopole with magnetic charge g, electric charge q is induced by θ term.

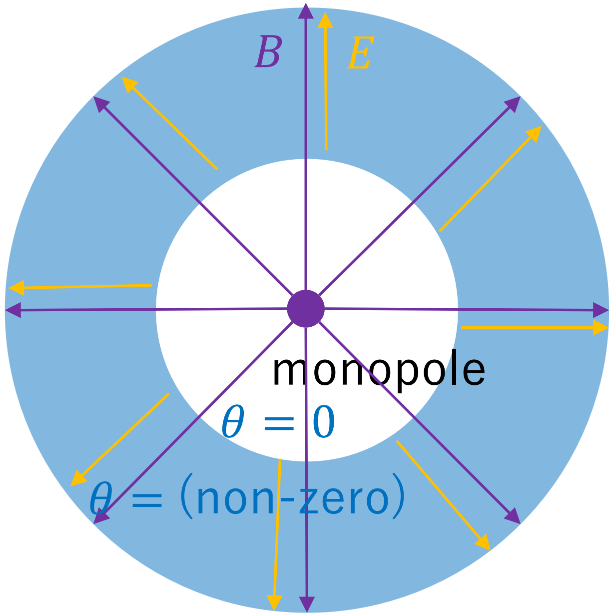
$$S = -\frac{1}{2g^2}\int \mathrm{tr}(f \wedge \star f) + \frac{\theta}{8\pi^2}\int \mathrm{tr}(f \wedge f)$$

➢ In the abelian gauge theory, EOM is

$$\partial_\mu F^{\mu\nu} = \frac{g^2}{4\pi^2}\varepsilon_{\mu\nu\rho\sigma}\partial_\mu\left(\theta\partial_\rho A_\sigma\right)$$

$$\nabla\cdot\mathbf{E} = -\frac{g^2}{4\pi^2\epsilon_0}\nabla\theta\cdot\mathbf{B}$$

ρ/ϵ_0



➢ Dirac quaternization is condition: $gq = \theta$

Cardy-Rabinovici model

$$S[\tilde{a}_\mu, s_{\mu\nu}, n_\mu] = S_{\mathrm{kin}}[\tilde{a}_\mu, s_{\mu\nu}] + S_{\mathrm{matter}}[\tilde{a}_\mu, s_{\mu\nu}, n_\mu]$$
$$= \frac{1}{2g^2} \sum_{(x,\mu,\nu)} f_{\mu\nu}(x)^2 + iN \sum_{(x,\mu)} \left(n_\mu(x) + \frac{\theta}{2\pi} \sum_{\tilde{x}} F(x-\tilde{x}) m_\mu(\tilde{x}) \right) \tilde{a}_\mu(x)$$