

Dark pions at next-to-leading order

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Beyond-the-Standard-Model scenarios propose new QCD-like, strongly interacting sectors, e.g:

- Composite pionic dark matter, e.g. Strongly Interacting Massive Particles (**SIMPs**),
- Composite Higgs models.

At low energies, we assume the non-zero condensate which breaks the flavor symmetry and Chiral perturbation theory is used to describe the dynamics of pseudo-Goldstone bosons from spontaneous symmetry breaking.

$$\langle \bar{q}q \rangle \neq 0 \quad \begin{array}{c} G \\ \downarrow \\ H \end{array}$$

Flavour symmetries

Depending on the representation of the N_F fermionic flavors the chiral symmetry breaking pattern differs:

Representation of the fermions	Complex	Pseudoreal	Real
Example: fundamental of	$SU(N_c)$	$Sp(N_c)$	$O(N_c)$
Chiral symmetry breaking pattern	$SU(N_F) \times SU(N_F) / SU(N_F)$	$SU(2N_F) / Sp(2N_F)$	$SU(2N_F) / SO(2N_F)$
Number of pions	$N_F^2 - 1$	$(2N_F + 1)(N_F - 1)$	$N_F(2N_F + 1) - 1$

N_F flavors of left-handed fermions and N_F flavors of right-handed fermions in the fundamental representation of $Sp(N_c)$ or $SO(N_c)$:

$$\mathcal{L} = \bar{q}_{Li} i\gamma^\mu D_\mu q_{Li} + \bar{q}_{Ri} i\gamma^\mu D_\mu q_{Ri} - \bar{q}_{Ri} \mathcal{M}_{ij} q_{Lj} - \bar{q}_{Li} \mathcal{M}_{ij}^\dagger q_{Rj}.$$

(Pseudo-)real representation: \exists a unitary matrix ϵ such that

$$-T^{a*} = \epsilon^{-1} T^a \epsilon, \quad \epsilon^T = \eta \epsilon, \quad \begin{cases} \eta = 1, & \text{real fermions} \\ \eta = -1, & \text{pseudoreal fermions} \end{cases}$$

With the definition

$$\hat{q} = \left(\begin{array}{c} q_R \\ \epsilon C \bar{q}_L^T \end{array} \right) \Big\}^{2N_F}, \quad \hat{\mathcal{M}} = \begin{pmatrix} 0 & \eta \mathcal{M} \\ \mathcal{M}^T & 0 \end{pmatrix}, \quad C = i\gamma^2 \gamma^0$$

we can rewrite the Lagrangian as

$$\mathcal{L} = \bar{\hat{q}} i\gamma^\mu D_\mu \hat{q} - \frac{\eta}{2} \hat{q}^T \hat{\mathcal{M}}^\dagger \epsilon^* C \hat{q} - \frac{1}{2} \bar{\hat{q}} C \epsilon \hat{\mathcal{M}} \bar{\hat{q}}^T.$$

$$\mathcal{L} = \bar{q}i\gamma^\mu D_\mu \hat{q} - \frac{\eta}{2} \hat{q}^T \hat{\mathcal{M}}^\dagger \epsilon^* C \hat{q} - \frac{1}{2} \bar{q} C \epsilon \hat{\mathcal{M}} \bar{q}^T$$

- if $\hat{\mathcal{M}} \rightarrow 0$, global $U(2N_F)$ symmetry,
- axial anomaly breaks it down to $SU(2N_F)$,
- Chiral condensate breaks the symmetry spontaneously:

$$\sum_{i=1}^{N_F} \langle \bar{q}_i q_i \rangle = \langle \frac{1}{2} \hat{q}^T J \epsilon^* C \hat{q} + \frac{1}{2} \bar{q} C \epsilon J \bar{q}^T \rangle \neq 0, \quad J = \begin{pmatrix} 0 & \eta \mathbf{1}_{N_F} \\ \mathbf{1}_{N_F} & 0 \end{pmatrix},$$

The condensate is invariant under $H \subset SU(2N_F)$ such that

$$J = h^T J h, \quad h \in H \subset G = SU(2N_F), \quad \begin{cases} H = SO(2N_F), & \eta = 1 \\ H = Sp(2N_F), & \eta = -1 \end{cases}.$$

Generators split into broken (X^a) and unbroken (Q^a) parts:

$$JQ^a + Q^{aT}J = 0, \quad JX^a - X^{aT}J = 0.$$

External vector as sources

Non-zero chiral condensate \Rightarrow spontaneous symmetry breaking.
Broken currents $A_\mu^a = \bar{\hat{q}}\gamma_\mu X^a \hat{q}$ have divergence

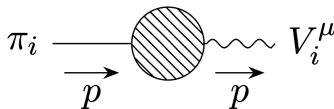
$$\partial^\mu A_\mu^a \neq 0.$$

To generate the correlation functions with the insertions of j_μ^a we enhance the Lagrangian with external vector fields

$$\mathcal{L} = \bar{\hat{q}} i \gamma^\mu D_\mu \hat{q} - \frac{\eta}{2} \hat{q}^T \hat{\mathcal{M}}^\dagger \epsilon^* C \hat{q} - \frac{1}{2} \bar{\hat{q}} C \epsilon \hat{\mathcal{M}} \bar{\hat{q}}^T + \bar{\hat{q}} \gamma^\mu V_\mu \hat{q},$$

where $V_\mu = V_\mu^a X^a$.

$$\langle 0 | A^a(0)_\mu | \pi^a(p) \rangle = i\sqrt{2} p_\mu F_{\pi,a} \Rightarrow$$



Effective theory

The EFT is written in terms of the matrix (CCWZ construction)

$$u = e^{\frac{i}{\sqrt{2}F}\pi^a X^a} \in G/H, \quad F - \text{pion decay constant.}$$

Using u we can define, the object valued in the unbroken part of G :

$$u_\mu = i \left(u^\dagger (\partial_\mu - iV_\mu) u - u (\partial_\mu + iJV_\mu^T J^T) u^\dagger \right) \xrightarrow{G} hu_\mu h^\dagger.$$

Further building blocks including quark masses read:

$$\chi_\pm = u^\dagger \chi J^T u^\dagger \pm u J \chi^\dagger u \xrightarrow{G} h\chi_\pm h^\dagger, \quad \chi = 2B_0 \hat{M}.$$

The χ PT is an expansion in small pion momentum

$$\frac{p^2}{(4\pi F)^2} \sim \frac{M^2}{(4\pi F)^2} < 1.$$

We count $\partial \sim \mathcal{O}(p)$ and $m_q \sim \mathcal{O}(p^2)$.

LO and NLO Lagrangians

Chiral Lagrangian consists of terms systemized by number of derivatives and quark mass insertions:

$$\mathcal{L}_2 = \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle \supset_{N_F=2} -B_0 \pi^a \pi^b \left\langle \chi^a \chi^b \begin{pmatrix} m_u & 0 & 0 & 0 \\ 0 & m_d & 0 & 0 \\ 0 & 0 & m_u & 0 \\ 0 & 0 & 0 & m_d \end{pmatrix} \right\rangle,$$

$$\begin{aligned} \mathcal{L}_4 = & L_0 \langle u^\mu u^\nu u_\mu u_\nu \rangle + L_1 \langle u^\mu u_\mu \rangle \langle u^\nu u_\nu \rangle + L_2 \langle u^\mu u^\nu \rangle \langle u_\mu u_\nu \rangle \\ & + L_3 \langle u^\mu u_\mu u^\nu u_\nu \rangle + L_4 \langle u^\mu u_\mu \rangle \langle \chi_+ \rangle + L_5 \langle u^\mu u_\mu \chi_+ \rangle \\ & + L_6 \langle \chi_+ \rangle^2 + L_7 \langle \chi_- \rangle^2 + \frac{1}{2} L_8 \langle \chi_+^2 + \chi_-^2 \rangle + H_2 \langle \chi \chi^\dagger \rangle. \end{aligned}$$

The NLO LECs, L_i , are renormalized to cancel the loop divergencies coming from \mathcal{L}_2 :

$$L_i = L_i^r(\mu) + \frac{\Gamma_i}{32\pi^2} R, \quad R \sim \frac{1}{\epsilon}.$$

Higher-order corrections are important for the BSM scenarios with relatively large pion mass, M_π . These corrections are important for

- phenomenology of SIMP dark matter (pointed out by Hansen, Langæble and Sannino, **1507.01590**)
- interpolation between the limit of massless quarks and the finite-mass regime where lattice simulations can be performed (relevant for the composite Higgs models)

For theories with N_F (pseudo-)real fermions the NLO and NNLO results are presented in papers by Bijmans and Lu **0910.5424** and **1102.0172**.

We extend the $N_F = 2$ NLO results by assuming non-degenerate quark masses (specially relevant for SIMP phenomenology).

Example: masses for $SU(4)/Sp(4)$

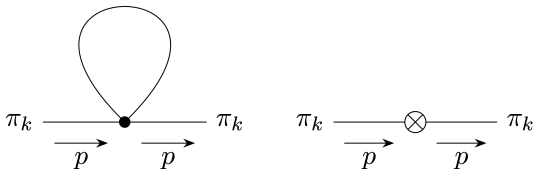
At LO: $M^2 \equiv M_{\text{LO},k}^2 = B_0(m_u + m_d)$, $k = 1 \dots 5$.

At NLO mass-splitting appears:

$$M_{\text{NLO},i \neq 3}^2 = M^2 \left(\frac{M^2}{F^2} \left[-32L_4^r - 8L_5^r + 64L_6^r + 16L_8^r + \frac{3}{64\pi^2} \log \left(\frac{M^2}{\mu^2} \right) \right] \right),$$

$$M_{\text{NLO},3}^2 = M^2 \left(\frac{M^2}{F^2} \left[-32L_4^r - 8L_5^r + 64L_6^r + 64L_7^r \left(\frac{1-r}{1+r} \right)^2 + 32L_8^r \frac{1+r^2}{(1+r)^2} + \frac{3}{64\pi^2} \log \left(\frac{M^2}{\mu^2} \right) \right] \right),$$

where $r \equiv m_u/m_d$.



Example: masses for $SU(4)/SO(4)$

At LO there are three different masses:

$$\begin{aligned} M_1^2 &\equiv M_{\text{LO},1}^2 = M_{\text{LO},2}^2 = M_{\text{LO},3}^2 = M_{\text{LO},4}^2 = M_{\text{LO},5}^2 = B_0 (m_u + m_d) , \\ M_6^2 &\equiv M_{\text{LO},6}^2 = M_{\text{LO},7}^2 = 2B_0 m_u, \\ M_8^2 &\equiv M_{\text{LO},8}^2 = M_{\text{LO},9}^2 = 2B_0 m_d. \end{aligned} \tag{1}$$

At NLO we observe four masses which confirms the group theoretical argument by Pomper and Kulkarni, **2402.04176v1**. The axial anomaly breaks the $U(4)$ down to $\mathbb{Z}_2 \times SU(4)$!

Example: masses for $SU(4)/SO(4)$

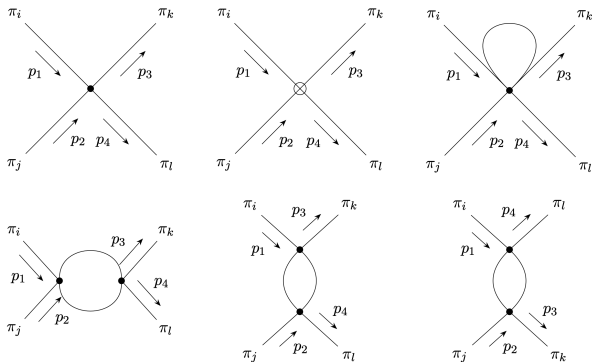
$$\begin{aligned}
 M_{\text{NLO},1}^2 &= M_{\text{NLO},2}^2 = M_{\text{NLO},4}^2 = M_{\text{NLO},5}^2 \\
 &= M_1^2 \left(\frac{M_1^2}{F^2} \left(-8(4L_4^r + L_5^r - 8L_6^r - 2L_8^r) - \frac{1}{64\pi^2} \ln \frac{M_1^2}{\mu^2} \right) \right), \\
 M_{\text{NLO},3}^2 &= -\frac{M_1^4}{F^2} 8(4L_4^r + L_5^r - 8L_6^r) + \frac{32B_0^2}{F^2} (2L_7^r(m_u - m_d)^2 + L_8^r(m_d^2 + m_u^2)) \\
 &\quad + \frac{1}{64\pi^2 F^2} \left(3M_1^4 \ln \frac{M_1^2}{\mu^2} - 2M_6^4 \ln \frac{M_6^2}{\mu^2} - 2M_8^4 \ln \frac{M_8^2}{\mu^2} \right), \\
 M_{\text{NLO},6}^2 &= M_{\text{NLO},7}^2 \\
 &= -\frac{M_6^2}{F^2} \left[32(L_4^r - 2L_6^r) M_1^2 + 8(L_5^r - 2L_8^r) M_6^2 + \frac{M_1^2}{64\pi^2} \ln \frac{M_1^2}{\mu^2} \right], \\
 M_{\text{NLO},8}^2 &= M_{\text{NLO},9}^2 \\
 &= -\frac{M_8^2}{F^2} \left[32(L_4^r - 2L_6^r) M_1^2 + 8(L_5^r - 2L_8^r) M_8^2 + \frac{M_1^2}{64\pi^2} \ln \frac{M_1^2}{\mu^2} \right].
 \end{aligned}$$

2 \rightarrow 2 scattering amplitude

$$\mathcal{M}^{ij \rightarrow kl} = \langle \pi_k(p_3) \pi_l(p_4) | \pi_i(p_1) \pi_j(p_2) \rangle$$

Less symmetric structure compared to the degenerate case, especially for $SU(4)/SO(4)$ theory!

See Mathematica functions publicly available at [Zotero repository](#).



We obtained the results for other observables, to be presented in the paper:

- decay constants,

$$\langle 0 | A^a(0)_\mu | \pi^a(p) \rangle = i\sqrt{2}p_\mu F_{\pi,a}, \quad a = 1 \dots N_\pi,$$

- condensates,

$$\langle \bar{u}u \rangle = \frac{\partial V_{\text{eff, min}}}{\partial m_u}, \quad \langle \bar{d}d \rangle = \frac{\partial V_{\text{eff, min}}}{\partial m_d}.$$

Fitting the lattice data

The LECs for $Sp(N_c = 4)$, $N_F = 2$ theory were not yet known. We perform a fit using the derived formulas of the lattice data:

- masses and decay constants for the non-degenerate quarks: Kulkarni et al., **2202.05191v2**. The lattice show splitting in the decay constants which can not be captured by the NLO formulas.
- scattering amplitude for the mass-degenerate quarks in the channel without s-channel vector mesons: Dengler, Maas, Zierler, **2405.06506v2**.

The papers use similar lattice ensembles. The continuum limit is not performed in both papers.

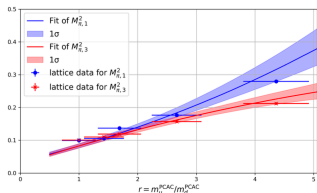
We can use scale-independent LECs:

$$L_i^r(\mu) = \frac{\Gamma_i}{32\pi^2} \left(\bar{L}_i + \log \frac{M_{\text{phys}}^2}{\mu^2} \right), \quad i \neq 7, 8,$$
$$L_i^r = \frac{1}{128\pi^2} \bar{L}_i, \quad i = 7, 8.$$

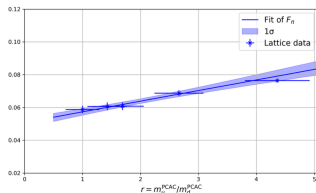
All the discussed observables depend on the following linear combinations:

$$\begin{aligned}\tilde{L}_1 &\equiv \bar{L}_4 + \bar{L}_5, \\ \tilde{L}_2 &\equiv \frac{1}{4}\bar{L}_0 - \frac{3}{4}\bar{L}_1 - \frac{3}{2}\bar{L}_2 - \bar{L}_3, \\ \tilde{L}_3 &\equiv \bar{L}_6 + \frac{8}{5}\bar{L}_8, \\ \tilde{L}_4 &\equiv \bar{L}_8 + 4\bar{L}_7.\end{aligned}$$

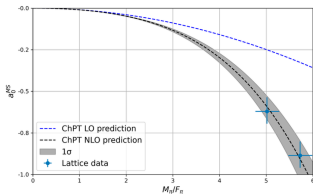
Results of the fit



(a) Masses

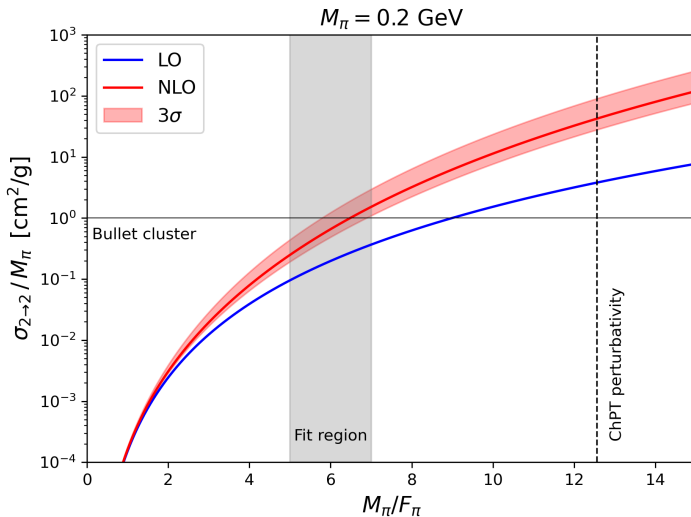


(b) Decay constant



\tilde{L}_1	\tilde{L}_2	\tilde{L}_3	\tilde{L}_4
$2.90^{+1.16}_{-0.76}$	$5.57^{+1.60}_{-1.40}$	$3.64^{+2.56}_{-1.38}$	$-1.25^{+0.51}_{-1.04}$

Self-scattering cross section

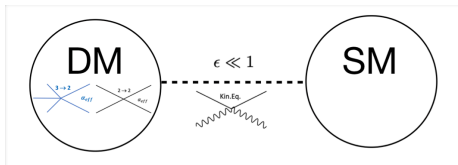


SIMP dark matter

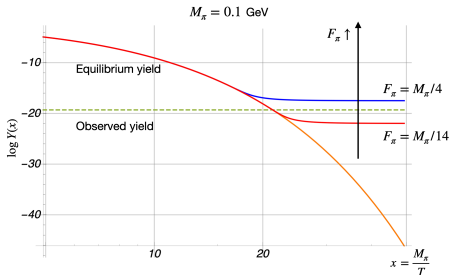
Consider a scenario in which the relic abundance is set by $3 \rightarrow 2$ self-interaction (Hochberg et al., **1402.5143v2**).

In χ PT there is 5-point interaction due to a topological **WZW term** (which gives $K^+ K^- \rightarrow \pi^+ \pi^- \pi^0$ in QCD):

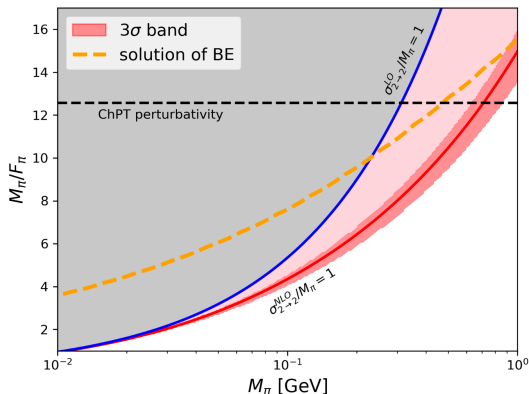
$$\mathcal{L}_{\text{WZW}} \supset \frac{N_c}{240\pi^2 F_\pi^5} \epsilon^{\mu\nu\rho\sigma} \langle \pi \partial_\mu \pi \partial_\nu \pi \partial_\rho \pi \partial_\sigma \pi \rangle$$



Reproduced from 1402.5143



SIMP validity



The result should be taken with care: vector mesons are light and should be taken into account in the EFT (Bernreuther et al., [2311.17157](#))
 \Rightarrow ongoing project.

Summary

- NLO formulas for masses, condensates, decay constants and scattering amplitudes for theory with $N_F = 2$ real and pseudoreal non-degenerate fermions will be presented in the paper,
- first estimate of the NLO LECs for $Sp(N_c = 4)$ theory using the lattice data,
- confirmation of the importance of the NLO calculations for the dark matter theories with relatively high M_π/F_π , with refined computation using LECs fitted from the lattice.

Thank you for attention!