OR:
Putting Goldstone's theorem to the test

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Night of 12th of June – biggest party of the year in Lisbon

The Two-Higgs Doublet scalar potential

Most general SU(2) × U(1) scalar potential:

$$
\begin{aligned}
V = & \, m_{11}^2 |\Phi_1|^2 + m_{22}^2 |\Phi_2|^2 - \left(m_{12}^2 \Phi_1^\dagger \Phi_2 + h.c. \right) \\
& + \frac{1}{2}\lambda_1 |\Phi_1|^4 + \frac{1}{2}\lambda_2 |\Phi_2|^4 + \lambda_3 |\Phi_1|^2 |\Phi_2|^2 + \lambda_4 |\Phi_1^\dagger \Phi_2|^2 \\
& + \left[\frac{1}{2}\lambda_5 \left(\Phi_1^\dagger \Phi_2 \right)^2 + \lambda_6 |\Phi_1|^2 \left(\Phi_1^\dagger \Phi_2 \right) + \lambda_7 |\Phi_2|^2 \left(\Phi_1^\dagger \Phi_2 \right) + \text{h.c.} \right]
\end{aligned}
$$

m^2_{12}, λ_5, λ_6 and **λ_7 complex - seemingly 14 independent real parameters, in fact only 11.**

Most frequently studied model: softly broken theory with a Z_2 symmetry,

$\Phi_1 \rightarrow - \Phi_1$ and $\Phi_2 \rightarrow \Phi_2$, meaning $\lambda_6, \lambda_7 = 0$.

It avoids potentially large flavour-changing neutral currents (FCNC)

The U(1) and Z_3 symmetries

The *Peccei-Quinn U(1) symmetry* imposes invariance under a rephasing of one of the doublets,

$$\Phi_1 \to \Phi_1' = \Phi_1 \quad , \quad \Phi_2 \to \Phi_2' = e^{i\theta}\Phi_2 \, .$$

with an arbitrary phase θ. The scalar potential then becomes

$$V_0 = m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 + \tfrac{1}{2}\lambda_1 (\Phi_1^\dagger \Phi_1)^2 + \tfrac{1}{2}\lambda_2 (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \lambda_4 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1)$$

If instead one considers a *Z_3 symmetry*,

$$\Phi_1 \to \Phi_1' = \Phi_1 \quad , \quad \Phi_2 \to \Phi_2' = e^{2i\pi/3}\Phi_2$$

the scalar potential one obtains is exactly the U(1) potential.

This is an example of an *accidental symmetry* – since the potential only has quadratic+quartic terms, imposing a *discrete Z_3 symmetry* yields a *continuous U(1) symmetry*. This also holds for the gauge interactions, but NOT for the Yukawa sector…

R. D. Peccei and H. R. Quinn, *CP Conservation in the Presence of Instantons.* Phys. Rev. Lett. **38** (1977)

U(1) and Z_3 Yukawa sectors

U(1) and Z_3 symmetries can be extended to the Yukawa sector and yield very different Yukawa matrix textures. The 2HDM quark Yukawa lagrangian is

$$-\mathcal{L}_Y = \bar{Q}_L \Gamma_1 \Phi_1 n_R + \bar{Q}_L \Gamma_2 \Phi_2 n_R + \bar{Q}_L \Delta_1 \tilde{\Phi}_1 p_R + \bar{Q}_L \Delta_2 \tilde{\Phi}_2 p_R + \text{h.c.}$$

with complex 3×3 matrices Γ_a and Δ_a. The left and right handed quark fields transform under the symmetry as

$$\begin{aligned} Q_L &\to Q'_L = \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3})\, Q_L\,, \\ n_R &\to n'_R = \text{diag}(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3})\, n_R\,, \\ p_R &\to p'_R = \text{diag}(e^{i\gamma_1}, e^{i\gamma_2}, e^{i\gamma_3})\, p_R\,, \end{aligned}$$

so that the Yukawa matrices transform as

$$\begin{aligned} (\Gamma_a)_{ij} &= e^{i(\alpha_i - \beta_j - \theta_a)} (\Gamma_a)_{ij} \\ (\Delta_a)_{ij} &= e^{i(\alpha_i - \gamma_j + \theta_a)} (\Delta_a)_{ij} \end{aligned}$$

where $\theta_1 = 0$, $\theta_2 = \theta$ are the phases affecting the doublets:

$$\Phi_1 \to \Phi'_1 = \Phi_1 \quad , \quad \Phi_2 \to \Phi'_2 = e^{i\theta} \Phi_2 \,.$$

The complete list of possible Yukawa textures for abelian symmetries was obtained in [1] *Phys. Rev. D 83 (2011) 065026, arXiv:1012.2874.*

U(1) Yukawa textures

U(1) textures will be those that can be obtained for an arbitrary value of the phase θ. There are many possible examples, for instance:

1.

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \Gamma_2 = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}$$

$$\Delta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \Delta_2 = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}$$

Model Type-I: other flavour preserving models (Type II, Flipped, Lepton-specific) easily obtained as well.

2.

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \times \\ \times & \times & 0 \end{pmatrix} \quad , \quad \Gamma_2 = \begin{pmatrix} \times & \times & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}$$

$$\Delta_1 = \begin{pmatrix} \times & \times & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \times \end{pmatrix} \quad , \quad \Delta_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \times \\ \times & \times & 0 \end{pmatrix}$$

Unlike the previous case, this case has FCNCs.

3. … and many other possibilities.

Z_3 Yukawa textures

Z_3 textures will be those that can only be obtained if $\theta = 2\pi/3$. There are many possible examples, for instance:

1.

$$\Gamma_1 = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix} \quad , \quad \Gamma_2 = \begin{pmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{pmatrix}$$

$$\Delta_1 = \begin{pmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{pmatrix} \quad , \quad \Delta_2 = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}$$

Specific pattern of non-zero entries of Γ (Δ) matrices alone is enough to force $\theta = 2\pi/3$.

2.

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \times \\ \times & \times & 0 \end{pmatrix} \quad , \quad \Gamma_2 = \begin{pmatrix} \times & \times & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}$$

$$\Delta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \times \\ \times & \times & 0 \end{pmatrix} \quad , \quad \Delta_2 = \begin{pmatrix} 0 & 0 & \times \\ \times & \times & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Combined conditions for non-zero entries of Γ and Δ matrices necessary to obtain $\theta = 2\pi/3$.

P. M. Ferreira, L. Lavoura, and J. P. Silva, *A Soft origin for CKM-type CP violation.* Phys. Lett. B **704** (2011) 179–188, arXiv:1102.0784 [hep-ph].

3. Other possibilities listed in section III.C of reference [1].

The Pseudoscalar mass matrix

If both doublets acquire a vev,

$$\langle \Phi_1 \rangle = v_1/\sqrt{2} \ , \ \langle \Phi_2 \rangle = v_2/\sqrt{2} \ \text{ such that } v_1^2 + v_2^2 = v^2, \text{ with } v = 246 \text{ GeV}$$

the U(1)/Z_3 symmetries are spontaneously broken and the minimization conditions imply

$$\left(m_{11}^2 + \frac{1}{2}\lambda_1 v_1^2 + \frac{1}{2}\lambda_{34} v_2^2 \right) v_1 = 0$$

$$\left(m_{22}^2 + \frac{1}{2}\lambda_2 v_2^2 + \frac{1}{2}\lambda_{34} v_1^2 \right) v_2 = 0$$

The pseudoscalar mass matrix is found to be

$$[M_A^2] = \begin{pmatrix} m_{11}^2 + \frac{1}{2}\lambda_1 v_1^2 + \frac{1}{2}\lambda_{34} v_2^2 & 0 \\ 0 & m_{22}^2 + \frac{1}{2}\lambda_2 v_2^2 + \frac{1}{2}\lambda_{34} v_1^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Therefore, there are two massless scalars: the usual neutral Goldstone, G^0, and a massless axion A.

Goldstone's Theorem

It is not surprising that we are obtaining a massless pseudoscalar:

- **Spontaneous breaking of a continuous symmetry necessarily produces a massless scalar mode (Goldstone's theorem).**

But for the Z_3 symmetry, we have a curious situation:

- **The scalar and gauge sector of the Z_3-symmetric model are indistinguishable from the U(1) model. Therefore, if only those sectors are considered, a massless pesudoscalar is the be expected.**

- **BUT the Yukawa sector can distinguish between Z_3 and U(1) symmetries.**

Therefore, when fermions are considered, the *Lagrangian* of the Z_3 model is that of a DISCRETE symmetry, and no massless pseudoscalar should occur.

- **Since only fermions distinguish between both models, *it should be the Yukawa contributions to the pseudoscalar mass that render it massive*. Which leads us to a 1-loop calculation of the pseudoscalar mass…**

J. Goldstone, *Field Theories with Superconductor Solutions.* Nuovo Cim. **19** (1961) 154–164.
J. Goldstone, A. Salam, and S. Weinberg, *Broken Symmetries.* Phys. Rev. **127** (1962) 965–970.

1-loop potential and scalar self-energies

The 1-loop effective potential is

$$V_1 = V_0 + \Delta V_1,$$

where the 1-loop contribution is

$$\Delta V_1 = \frac{1}{64\pi^2} \sum_\alpha n_\alpha m_\alpha^4 \left[\log\left(\frac{m_\alpha^2}{\mu^2}\right) - \frac{3}{2} \right]$$

where n_α counts the number of spin, charge, colour, degrees of freedom of each particle of tree-level mass m_α, and μ is the renormalization scale (we are using the Landau gauge and dimensional reduction for simplification). The minimization conditions now become

$$\left(m_{11}^2 + \frac{1}{2}\lambda_1 v_1^2 + \frac{1}{2}\lambda_{34} v_2^2\right) v_1 + \frac{1}{32\pi^2} \sum_\alpha n_\alpha m_\alpha^2 \frac{\partial m_\alpha^2}{\partial v_1} \left[\log\left(\frac{m_\alpha^2}{\mu^2}\right) - 1 \right] = 0,$$

$$\left(m_{22}^2 + \frac{1}{2}\lambda_2 v_2^2 + \frac{1}{2}\lambda_{34} v_1^2\right) v_2 + \frac{1}{32\pi^2} \sum_\alpha n_\alpha m_\alpha^2 \frac{\partial m_\alpha^2}{\partial v_2} \left[\log\left(\frac{m_\alpha^2}{\mu^2}\right) - 1 \right] = 0.$$

We will use the formalism developed by Stephen Martin to compute the pseudoscalar self-energies, a mass-independent renormalization scheme for which masses or cross sections are outputs, and inputs are the renormalized running couplings and massive parameters. Physical 1-loop masses are the $m(1)^2$ solutions of the equation,

$$\mathrm{Det}\left[\left(m(0)_i^2 - s\right)\delta_{ij} + \frac{1}{16\pi^2}\,\Pi_{ij}\left(m(0)_k^2\right)\right] = 0$$

where $m(0)$ are the tree-level masses and $s = m(1)^2 = p^2$, "p" being the external momentum and Π_{ij} are the scalar self-energies. In this formalism, the interaction lagrangians between scalars (S), fermions (F) and gauge fields (G) are expressed in terms of mass eigenstates as

$$\begin{aligned}
\mathcal{L}_{\mathrm{S}} &= -\frac{1}{6}\lambda^{ijk} R_i R_j R_k - \frac{1}{24}\lambda^{ijkl} R_i R_j R_k R_l\,, \\
\mathcal{L}_{\mathrm{SF}} &= -\frac{1}{2} y^{IJk} \psi_I \psi_J R_k + \mathrm{h.c.}\,, \\
\mathcal{L}_{\mathrm{SG}} &= -g^{aij} A_a^\mu R_i \partial_\mu R_j - \frac{1}{4} g^{abij} A_a^\mu A_{b\mu} R_i R_j - \frac{1}{2} g^{abi} A_a^\mu A_{b\mu} R_i
\end{aligned}$$

with tensors λ^{ijk}, λ^{ijkl}, y^{IJk}, g^{aij}, g^{abij} and g^{abi} containing the couplings associated with each vertex.

S. P. Martin, *Two loop scalar self energies in a general renormalizable theory at leading order in gauge couplings.* Phys. Rev. D **70** (2004) 016005, arXiv:hep-ph/0312092.

S. P. Martin, *Evaluation of two loop selfenergy basis integrals using differential equations.* Phys. Rev. D **68** (2003) 075002, arXiv:hep-ph/0307101.

The self-energies are then given by

$$\begin{aligned}\Pi_{ij} &= \frac{1}{2}\lambda^{ijkk}A(m_k^2) - \frac{1}{2}\lambda^{ikl}\lambda^{jkl}B(m_k^2, m_l^2) \\ &+ g^{aik}g^{ajk}B_{SG}(m_k^2, m_a^2) + \frac{1}{2}g^{aaij}A_G(m_a^2) + \frac{1}{2}g^{abi}g^{abj}B_{GG}(m_a^2, m_b^2) \\ &+ \mathrm{Re}\left[y^{KLi}y_{KLj}\right]B_{FF}(m_K^2, m_L^2) + \mathrm{Re}\left[y^{KLi}y^{K'L'j}M_{KK'}M_{LL'}\right]B_{\bar{F}\bar{F}}(m_K^2, m_L^2)\end{aligned}$$

in terms of 1-loop Passarino-Veltman *A* and *B* functions. The pseudoscalar mass matrix then becomes

$$[M_A^2] = \begin{pmatrix} m_{11}^2 + \frac{1}{2}\lambda_1 v_1^2 + \frac{1}{2}\lambda_{34}v_2^2 + \frac{1}{16\pi^2}\Pi_{G^0G^0} & \frac{1}{16\pi^2}\Pi_{G^0A} \\ \frac{1}{16\pi^2}\Pi_{AG^0} & m_{22}^2 + \frac{1}{2}\lambda_2 v_2^2 + \frac{1}{2}\lambda_{34}v_1^2 + \frac{1}{16\pi^2}\Pi_{AA} \end{pmatrix}$$

which is then evaluated at the 1-loop minimum using the expressions shown earlier.

This calculation was analytical in the scalar and gauge sectors, but had to proceed numerically in the fermion sector.

And the scalar contributions to the 1-loop self-energies:

$$\Pi^S_{AG^0} = \lambda_4\, s_\beta c_\beta \left[A(m^2_{G^\pm}) - A(m^2_{H^\pm}) + \left(m^2_{G^\pm} - m^2_{H^\pm}\right) \frac{A(m^2_{H^\pm}) - A(m^2_{G^\pm})}{m^2_{G^\pm} - m^2_{H^\pm}} \right] = 0\,.$$

$$\begin{aligned}
\Pi^S_{G^0G^0} &= \frac{1}{2}\Big[\lambda_{34} A(m^2_A) + 3\lambda_1 A(m^2_{G^0}) + \left(\lambda_{34} c^2_\alpha + \lambda_1 s^2_\alpha\right) A(m^2_H) + \left(\lambda_1 c^2_\alpha + \lambda_{34} s^2_\alpha\right) A(m^2_h) \Big] \\
&\quad + \left(\lambda_3 c^2_\beta + \lambda_1 s^2_\beta\right) A(m^2_{G^\pm}) + \left(\lambda_1 c^2_\beta + \lambda_3 s^2_\beta\right) A(m^2_{H^\pm}) - \frac{v^2_2 \lambda^2_4}{2} B(m^2_{G^\pm}, m^2_{H^\pm}) \\
&\quad - \left(\lambda_1 v_1 s_\alpha + \lambda_{34} v_2 c_\alpha\right)^2 B(m^2_{G^0}, m^2_H) - \left(\lambda_1 v_1 c_\alpha - \lambda_{34} v_2 s_\alpha\right)^2 B(m^2_{G^0}, m^2_h)\,, \\
\Pi^S_{AA} &= \frac{1}{2}\Big[3\lambda_2 A(m^2_A) + \lambda_{34} A(m^2_{G^0}) + \left(\lambda_2 c^2_\alpha + \lambda_{34} s^2_\alpha\right) A(m^2_H) + \left(\lambda_{34} c^2_\alpha + \lambda_2 s^2_\alpha\right) A(m^2_h) \Big] \\
&\quad + \left(\lambda_2 c^2_\beta + \lambda_3 s^2_\beta\right) A(m^2_{G^\pm}) + \left(\lambda_3 c^2_\beta + \lambda_2 s^2_\beta\right) A(m^2_{H^\pm}) - \frac{v^2_1 \lambda^2_4}{2} B(m^2_{G^\pm}, m_{H^\pm}) \\
&\quad - \left(\lambda_{34} v_1 s_\alpha + \lambda_2 v_2 c_\alpha\right)^2 B(m^2_A, m^2_H) - \left(\lambda_{34} v_1 c_\alpha - \lambda_2 v_2 s_\alpha\right)^2 B(m^2_A, m^2_h)\,.
\end{aligned}$$

where α and β are the diagonalization angles of the tree-level neutral CP-even and charged mass matrices.

Results for the scalar and gauge sectors

Though far from easy, it can be shown analytically that:

- **At 1-loop the Goldstone boson G^0 has zero mass (a good check).**
- **Both for the U(1) and Z_3 models, the pseudoscalar A remains massless at 1-loop.**
- **This is also an importante test, since for those sectors both models are identical (the Z_3 model acuires accidentally the U(1) symmetry), therefore spontaneous breaking of a continuous symmetry explains $m_A = 0$.**

The final test to distinguish both models is therefore to consider the fermionic contributions to the 1-loop masses: we should, for both models (a) preserve the masslessness of the Goldstone boson; (b) have $m_A = 0$ for the U(1) model, even including fermions; and (c) have $m_A \neq 0$ in the Z_3 model when fermions are included, since that lagrangian has a *discrete* symmetry.

Self-energy Yukawa contributions

Fermion contributions to 1-loop masses require diagonalization of quark mass matrices obtained for several quark Yukawa textures (we only considered the quark sector, leptons would be computed in the same way).

$$\Pi_{ij} = \frac{1}{2}\lambda^{ijkk}A(m_k^2) - \frac{1}{2}\lambda^{ikl}\lambda^{jkl}B(m_k^2, m_l^2)$$
$$+g^{aik}g^{ajk}B_{SG}(m_k^2, m_a^2) + \frac{1}{2}g^{aaij}A_G(m_a^2) + \frac{1}{2}g^{abi}g^{abj}B_{GG}(m_a^2, m_b^2)$$
$$+\mathrm{Re}\left[y^{KLi}y_{KLj}\right]B_{FF}(m_K^2, m_L^2) + \mathrm{Re}\left[y^{KLi}y^{K'L'j}M_{KK'}M_{LL'}\right]B_{\bar{F}\bar{F}}(m_K^2, m_L^2)$$

$$\mathcal{L}_{\mathrm{SF}} = -\frac{1}{2}y^{IJk}\psi_I\psi_J R_k + \mathrm{h.c.}$$

Weyl fermion mass eigenstates

Yukawa textures → Quark mass matrices → Rotation to mass basis → "Translation" to Weyl formalism

$$M_n = \frac{1}{\sqrt{2}}\left(\Gamma_1 v_1 + \Gamma_2 v_2\right) \quad , \quad M_p = \frac{1}{\sqrt{2}}\left(\Delta_1 v_1 + \Delta_2 v_2\right)$$

$$M_d = \mathrm{diag}(m_d\,,\, m_s\,,\, m_b) = U_{dL}^{\dagger}\, M_n\, U_{dR}$$

$$M_u = \mathrm{diag}(m_u\,,\, m_c\,,\, m_t) = U_{uL}^{\dagger}\, M_p\, U_{uR}$$

Fermionic contributions to 1-loop Pseudoscalar masses

The results for 1-loop pseudoscalar masses, including scalar, gauge AND fermionic contributions, are:

- **We tested ALL U(1)-invariant Yukawa textures from ref. [1]. We found a massless neutral Goldstone boson and a massless pseudoscalar as well, as was expected from Goldstone's theorem, which is thus verified for the U(1) models.**

- **For ALL Z_3-invariant Yukawa textures from ref. [1], we always found a massless neutral Goldstone boson…**

- **… but when we use these Z_3 Yukawa textures *the pseudoscalar mass becomes non-zero*!**

$$\Gamma_1 = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix} \quad , \quad \Gamma_2 = \begin{pmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{pmatrix}$$

- **Therefore Goldstone's theorem (obviously) holds! If the lagrangian has a DISCRETE symmetry, its spontaneous breaking will NOT generate a massless axion!**

$$\Delta_1 = \begin{pmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{pmatrix} \quad , \quad \Delta_2 = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}$$

- **But this calculation still yielded three surprises…**

SURPRISE # 1 – FAKE Z_3´s

For ALL BUT ONE of the Z_3-invariant Yukawa textures from ref. [1], *the pseudoscalar mass was still found to be zero*.

5. Special cases with $\theta = 2\pi/3$

We now turn to the last two cases where each α_{ij} can only take the values 0, θ, or $-\theta$ (mod2π). Because of Eq. (54), we can have $\alpha_{12} = \alpha_{23} = \alpha_{31} = \pm\theta$ if and only if $\theta = 2\pi/3$. The case $\alpha_{12} = \alpha_{23} = \alpha_{31} = -2\pi/3$ (or, which is the same, $4\pi/3$) is obtained by exchanging any two rows of the Yukawa matrices. We choose the case $\alpha_{12} = \alpha_{23} = \alpha_{31} = 2\pi/3$, implying that $\theta_{21} = \theta_{11} - 2\pi/3$, $\theta_{31} = \theta_{11} + 2\pi/3$, and

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{11} - 2\pi/3 & \theta_{12} - 2\pi/3 & \theta_{13} - 2\pi/3 \\ \theta_{11} + 2\pi/3 & \theta_{12} + 2\pi/3 & \theta_{13} + 2\pi/3 \end{bmatrix}, \quad (88)$$

implying that $\theta_{1j} \in \{0, 2\pi/3, -2\pi/3\}$. Recall that $\theta_{11} = \theta_{12} = \theta_{13}$ is excluded because it would lead to massless quarks.

Ignoring cases which differ only by permutation of the columns, we are left with the following structures:

(i) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, 0, 2\pi/3\}$

$$\Gamma_1 = \begin{bmatrix} x & x & \\ & & x \\ & & \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} & & x \\ & & \\ x & x & \end{bmatrix}, \quad \theta = 2\pi/3; \quad (89)$$

(ii) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, 0, -2\pi/3\}$

$$\Gamma_1 = \begin{bmatrix} x & x & \\ & & x \\ & & \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} & & x \\ & & \\ x & x & \end{bmatrix}, \quad \theta = 2\pi/3; \quad (90)$$

(iii) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, 2\pi/3, 2\pi/3\}$

$$\Gamma_1 = \begin{bmatrix} x & & \\ & x & x \\ & & \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} & x & x \\ & & \\ x & & \end{bmatrix}, \quad \theta = 2\pi/3; \quad (91)$$

(iv) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, 2\pi/3, -2\pi/3\}$

$$\Gamma_1 = \begin{bmatrix} x & & \\ & x & \\ & & x \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} & x & \\ & & x \\ x & & \end{bmatrix}, \quad \theta = 2\pi/3; \quad (92)$$

(v) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, -2\pi/3, -2\pi/3\}$

$$\Gamma_1 = \begin{bmatrix} x & & \\ & x & x \\ & & \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} & x & x \\ & & \\ x & & \end{bmatrix}, \quad \theta = 2\pi/3; \quad (93)$$

(vi) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{2\pi/3, 2\pi/3, -2\pi/3\}$

$$\Gamma_1 = \begin{bmatrix} x & x & \\ & & x \\ & & \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} x & x & \\ & & x \\ & & \end{bmatrix}, \quad \theta = 2\pi/3; \quad (94)$$

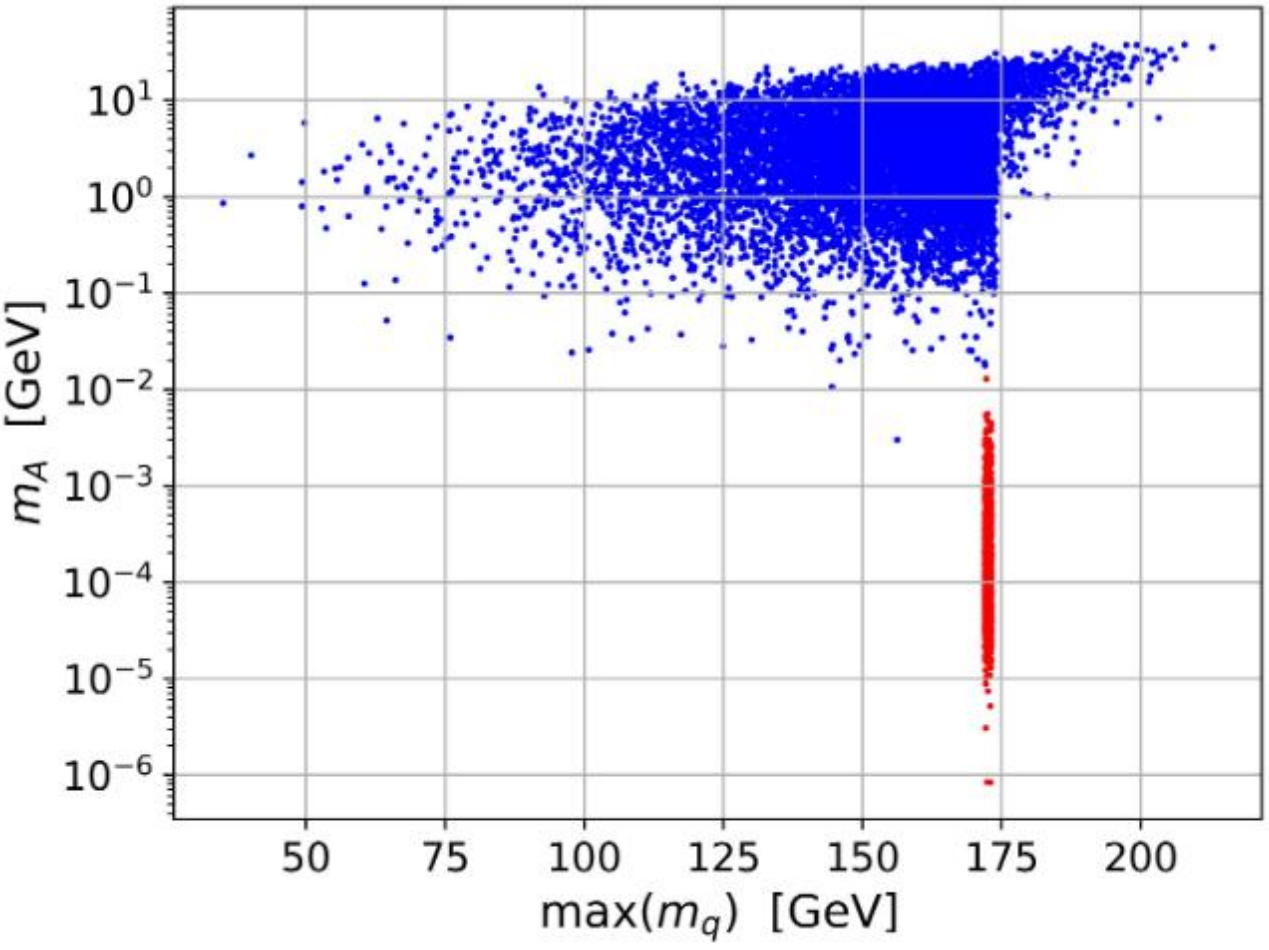
(vii) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{2\pi/3, -2\pi/3, -2\pi/3\}$

$$\Gamma_1 = \begin{bmatrix} x & & \\ & x & x \\ & & \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} x & & \\ & x & x \\ & & \end{bmatrix}, \quad \theta = 2\pi/3; \quad (95)$$

The reason is obvious, *a posteriori*: these apparently Z_3-invariant matrices can *also* be obtained by a U(1) symmetry, with arbitrary values of the θ phase, not necessarily $\theta = 2\pi/3$.

Thus a model with an apparent *discrete* symmetry actualy has a *continuous* one, and a massless pseudoscalar appears as the result of spontaneous symmetry breaking!

SURPRISE # 2 – THE ROLE OF QUARK MASS HIERARCHY



Blue points: allowing random values for all quark masses, which tend to have similar orders of magnitude.

Red points: enforcing measured values of the six quark masses. The highly hierarchical quark mass spectrum forces an upper bound of ~ 0.01 GeV for the pseudoscalar mass in this model.

WHY IS THIS HAPPENING? AN "APPROXIMATE" U(1)!

The reason for this behaviour can be understood analysing the quark textures:

$$\Gamma_1 = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix} \quad , \quad \Gamma_2 = \begin{pmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{pmatrix}$$

$$\Delta_1 = \begin{pmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{pmatrix} \quad , \quad \Delta_2 = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}$$

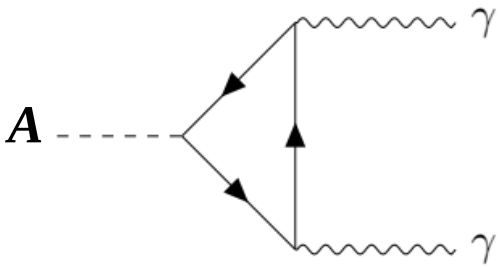
If any of the entries of these matrices are zero, the model has a U(1) symmetry, not a Z_3 one.

But because the quark masses are so different in orders of magnitude ($m_u \sim 10^{-3}$ GeV, $m_t \sim 10^2$ GeV), a fit of the quark spectrum with the above textures forces some of the entries of those matrices to be much smaller than others.

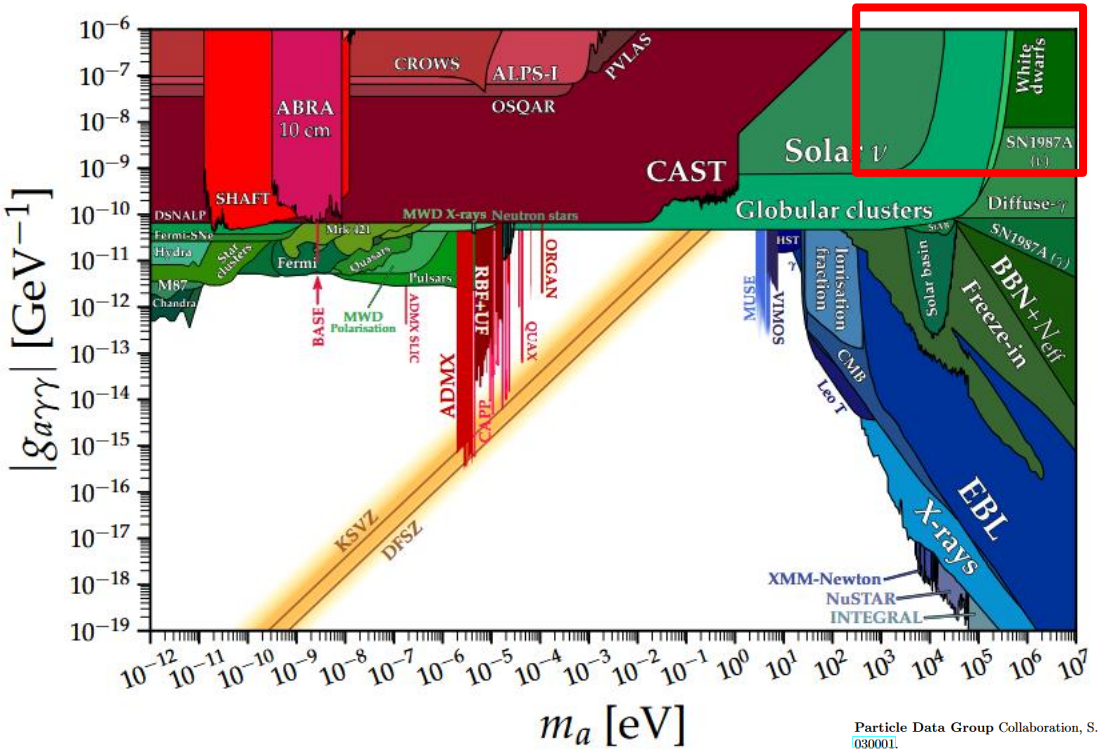
Therefore the Yukawa matrices become "approximate U(1)" textures, and produce a very low mass for the pseudoscalar ⇒ *"natural" way to generate low axion masses*.

Such axion masses seem to be excluded, though…

Pseudoscalar would couple to two photons via fermion triangles, effective diphoton coupling found to be between 10^{-9} to 10^{-3} GeV^{-1} for 1-loop generated masses masses between 10^{2} to 10^{7} eV.



$$g_{A\gamma\gamma} = \frac{\alpha}{6\sqrt{2}\pi}\left|4\frac{(\bar{\Delta}_2)_{33}}{m_t}A^A_{1/2}(\tau_t) + \frac{(\bar{\Gamma}_2)_{33}}{m_b}A^A_{1/2}(\tau_b) + \cdots\right|$$



Particle Data Group Collaboration, S. Navas *et al.*, *Review of particle physics.* Phys. Rev. D **110** (2024) no. 3, 030001.

SURPRISE # 3 – DELAYED MASS GENERATION?

The following DEFINITELY Z_3 – cannot be obtained by a U(1) symmetry - textures still produced a 1-loop zero mass for the pseudoscalar!

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \times \\ \times & \times & 0 \end{pmatrix} \quad , \quad \Gamma_2 = \begin{pmatrix} \times & \times & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}$$

$$\Delta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \times \\ \times & \times & 0 \end{pmatrix} \quad , \quad \Delta_2 = \begin{pmatrix} 0 & 0 & \times \\ \times & \times & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, is this a counterexample of Goldstone's theorem? A lagrangian with a discrete symmetry producing a massless scalar?

We believe that is not the case – the issue seems to be that by themselves, up quark matrices can be generated by a U(1) symmetry, and the same for the down quark textures. It's only when we demand that up and down quarks have simultaneously these textures that we "fall" into a Z_3 symmetry.

P. M. Ferreira, L. Lavoura, and J. P. Silva, *A Soft origin for CKM-type CP violation.* Phys. Lett. B **704** (2011) 179–188, arXiv:1102.0784 [hep-ph].

At one loop, the up and down quarks contribute separately to the self energies – all contributions effective only involve products of the form $\Gamma\,\Gamma^{+}$ and $\Delta\,\Delta^{+}$.

$$\begin{aligned}\Pi_{ij} &= \frac{1}{2}\lambda^{ijkk}A(m_k^2) - \frac{1}{2}\lambda^{ikl}\lambda^{jkl}B(m_k^2, m_l^2) \\ &+ g^{aik}g^{ajk}B_{SG}(m_k^2, m_a^2) + \frac{1}{2}g^{aaij}A_G(m_a^2) + \frac{1}{2}g^{abi}g^{abj}B_{GG}(m_a^2, m_b^2) \\ &+ \mathrm{Re}\left[y^{KLi}y_{KLj}\right]B_{FF}(m_K^2, m_L^2) + \mathrm{Re}\left[y^{KLi}y^{K'L'j}M_{KK'}M_{LL'}\right]B_{\bar{F}\bar{F}}(m_K^2, m_L^2)\end{aligned}$$

Therefore, since these up and down quark textures can separately be generated by U(1) symmetries, the self energy "doesn't know" that they are, together, a Z_3 symmetry, and yields a massless pseudoscalar result consistent with a continuous symmetry!

Presumably a non-zero mass will be found at higher loops (maybe already at two loops) when, due to internal $W^{\pm}$ and $H^{\pm}$ lines, products of Γ and Δ matrices are expected, such as $\Delta\,\Gamma^{+}$ or $\Gamma\,\Delta^{+}$, which should be sensitive to the Z_3-ness of the model.

If confirmed, this would be a natural way of generating even smaller pseudoscalar masses…

CONCLUSIONS

- **Goldstone's theorem works! All U(1) fermion textures produces a massless scalar.**

- **Goldstone's theorem works! Fake Z_3 symmetries are revealed as having been U(1)'s after all, and yield a massless pseudoscalar!**

- **Goldstone's theorem works! A Z_3-symmetric texture yields a non-zero pseudoscalar mass, and quark mass hierarchy forces an upper bound of ~0.01 GeV, since large diferences in quark mass orders of magnitude make the quark textures become "approximately" U(1).**

- **Goldstone's theorem *maybe* works? A Z_3-symmetric texture yields a massless pseudoscalar at 1-loop! This is apparently due to those up and down Yukawa textures being *individually* analogous to U(1)-symmetric textures. Presumably at 2-loops, when products of up and down quark matrices contribute to self energies, a non-zero mass will emerge.**