新物理の発見に向けて：ミュー粒子異常磁気能率と格子QCD計算
Standard Model on a test: Muon g-2 and lattice QCD calculation

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RIKEN BNL Research Center
→ Boston University

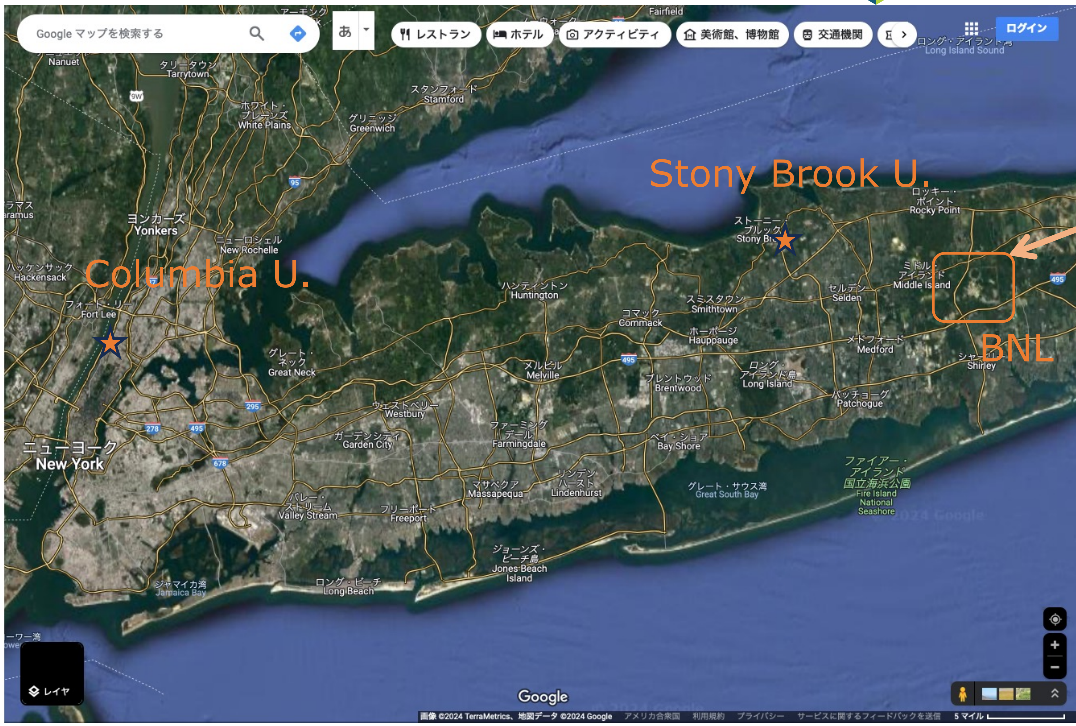
02/18/2024
@ The 30th ICEPP symposium

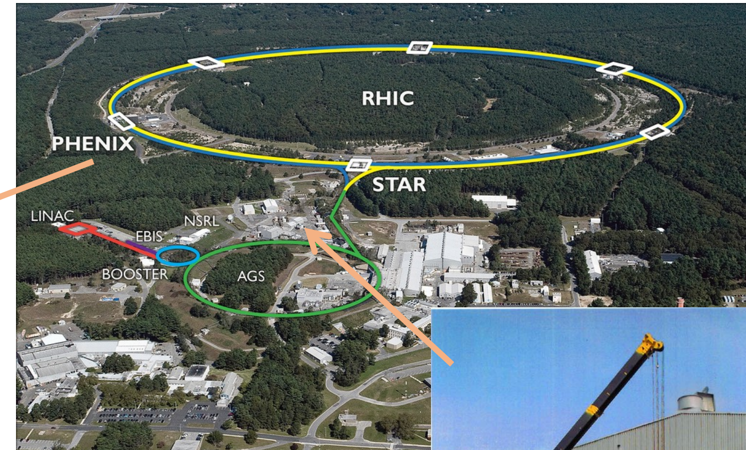
RBC/UKQCD Collaboration
PRD 108 no.5, 054507 (2023) [2301.08696]

See also PRL 2018, RBC/UKQCD [1801.07224]

(SPDR)

"A passion for discovery" = Motto of Brookhaven National Laboratory



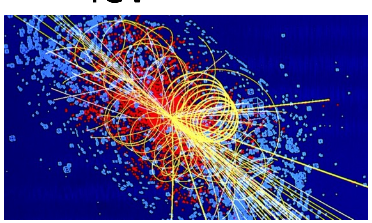


www.g-2.bnl.gov

- Contemporary understanding of particle physics
  = The Standard Model described in terms of Quantum Field Theory

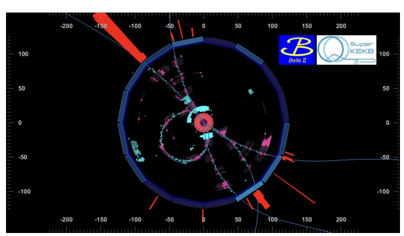
- It mostly consistently describes the physics of nature up to the scale of 10 TeV

TeV



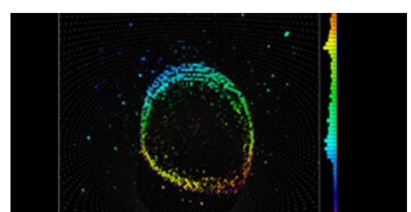
www.cms.cern

GeV



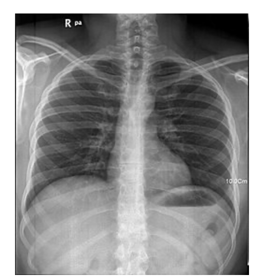
KEK/Belle II

MeV



Super-Kamiokande

keV



Wikipedia

eV

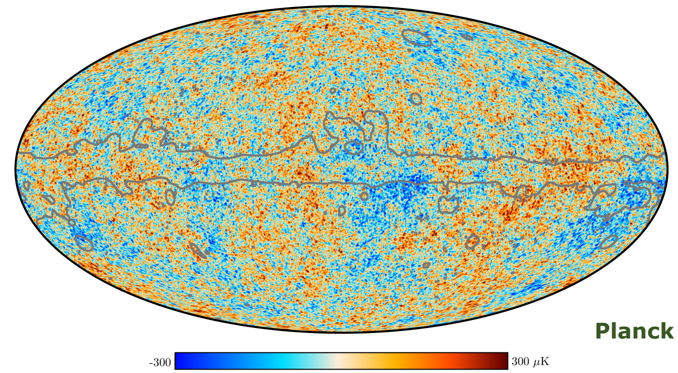
From BU

Ultraviolet (UV) Infrared (IR)

- Clues for Beyond Standard Model (BSM)
  - Dark matter **Sunday PM, Tuesday AM**
  - Strong CP problem
  - Hierarchy problem/Naturalness
  - Early universe
  - Quantum gravity

  ⋮

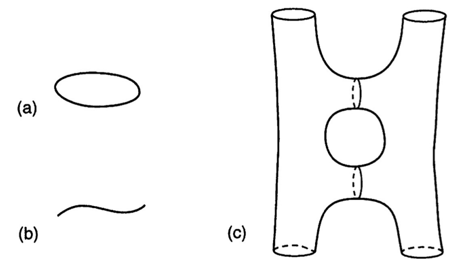
New theory?



Planck 2018

New framework?

string theory?
matrix model?
loop gravity?
…

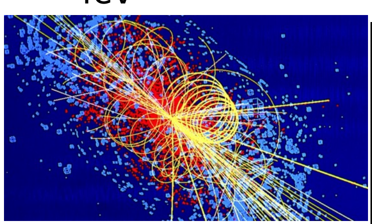


Polchinski

- It is almost a consensus that the Standard Model is an *effective theory*; *i.e.*, there is a grand theory that contains the Standard Model at low energy

$M_{\rm pl}$?

TeV



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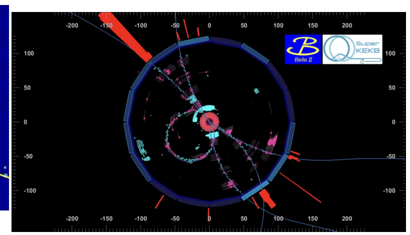
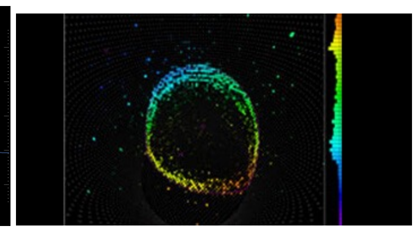
GeV



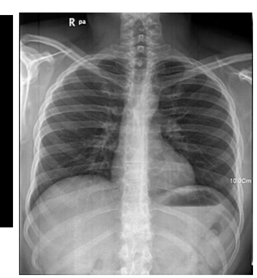
KEK/Belle II

MeV



Super-Kamiokande

keV



Wikipedia

eV

From BU

$\times 10^{16}$

$\times 10^{12}$

GUT? SUSY?

ATLAS: Sunday PM, Wednesday AM

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"A passion for discovery"

Frontiers

Precision

High energy

Rare events

New physics

Cosmology

Gravitational waves

Extreme conditions

INGRID for T2K

SK/T2K: Sunday evening

KOTO: Tuesday evening

Prof. Hajime Nanjo (Kyoto→Osaka)
(when I was an undergrad)

「知らないことがあったら
その場でやっつける」

"We learn what we don't know
when we encounter it"

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Frontiers

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@J-PARC

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KOTO: Tuesday evening

So please raise questions and comments!

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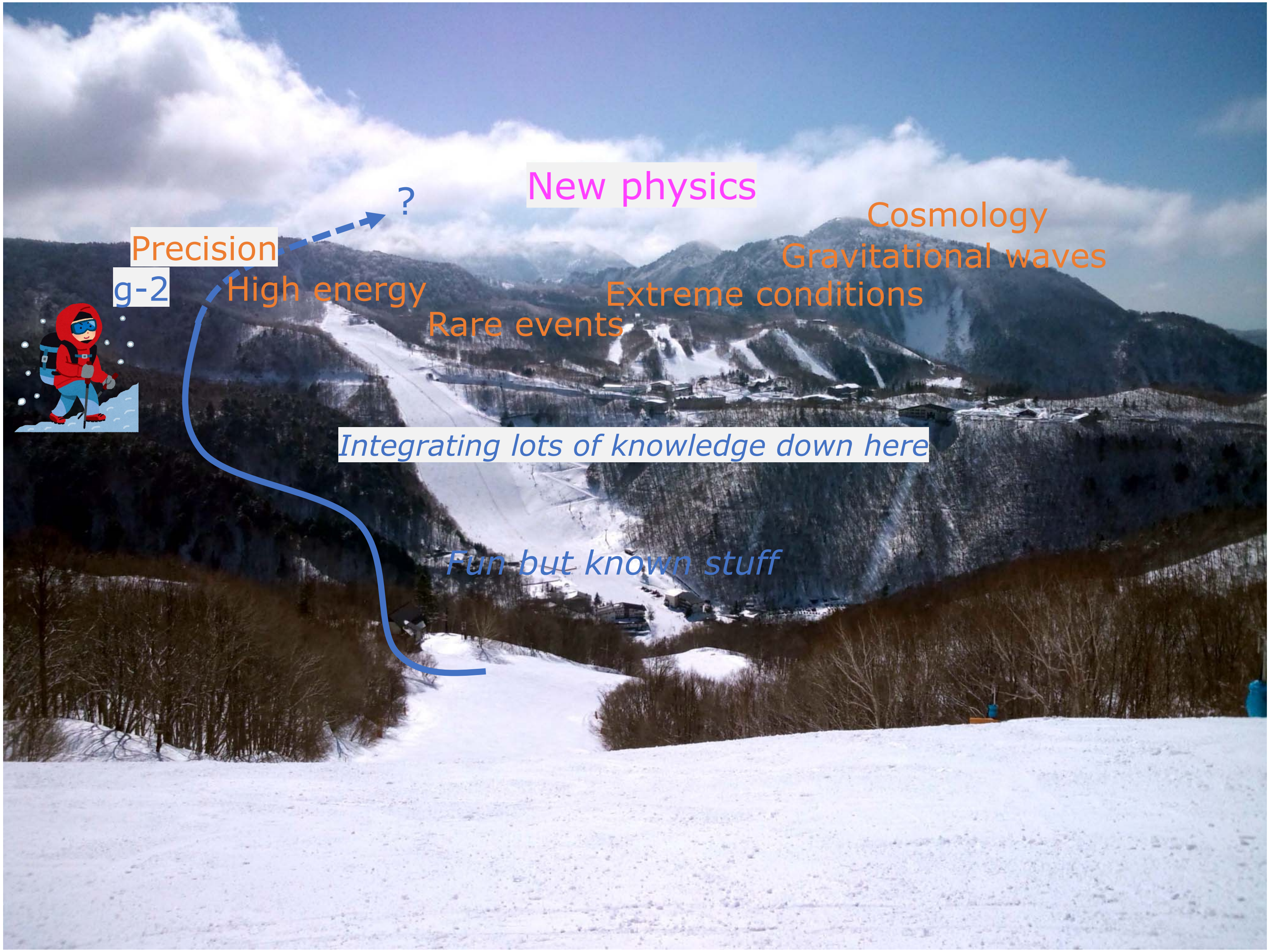

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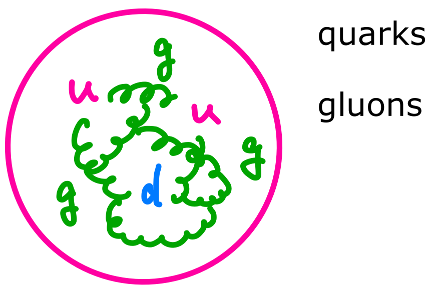
Table of contents

Table of contents

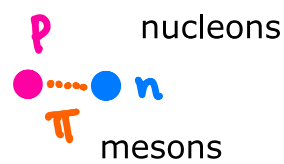
- As mentioned, the Standard Model is most likely a low-energy effective theory.
- By saying "low-energy effective theory", the notion of *scale separation* is in mind:

At large scales, it often happens that the details of small scales do not matter, the information gets reduced, and rather new structures take place.

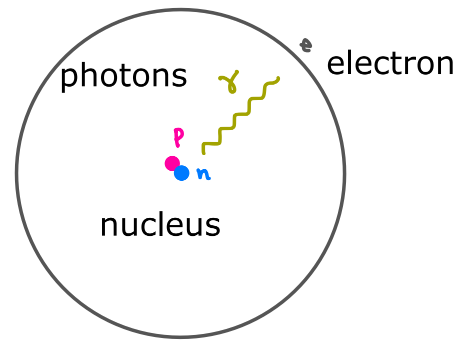
Partons — QCD



Nucleus — nuclear force



Atom — electromagnetic force



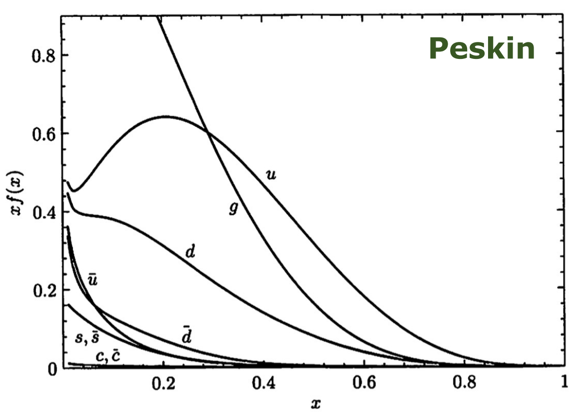
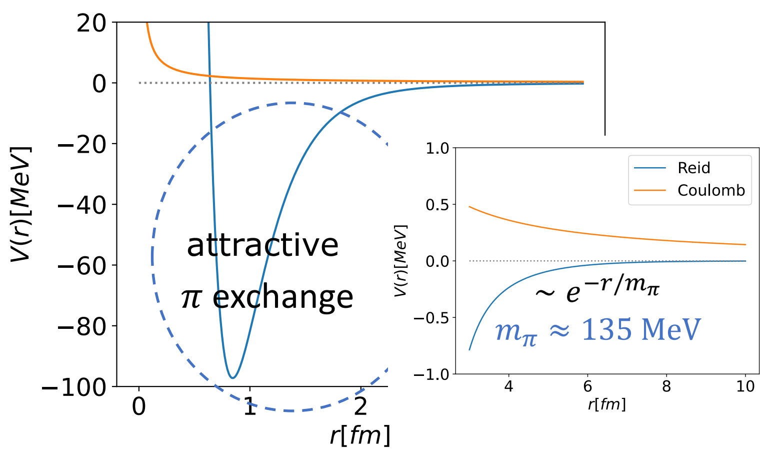


Figure 17.6. Parton distribution functions $xf_f(x)$ for quarks, antiquarks, and gluons in the proton, at $Q^2 = 4$ GeV^2. These distributions are obtained from a fit to deep inelastic scattering data performed by the CTEQ collaboration (CTEQ2L), described in J. Botts, et. al., *Phys. Lett.* **B304**, 159 (1993).

DIS with $Q^2 = 4\,\mathrm{GeV}^2$



$\Lambda_{QCD} \approx 200\mathrm{MeV}$

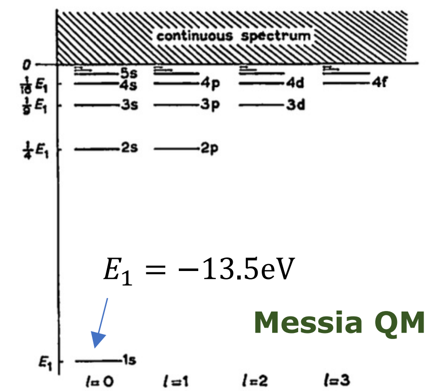


Fig. XI.1. Spectrum of the hydrogen atom.

Atomic spectroscopy: Tuesday evening

- The concept of scale separation is crystalized in Thermodynamics and Hydrodynamics:
  - Thermodynamical systems can be described with global quantities such as P, T by the equation of state:

    ideal gas

    $$PV = nRT$$

    van der Waals

    $$\left(P + \frac{an}{V}\right)(V - bn) = nRT$$

    (a encodes info of interaction
    b encodes particle size)

  - Nonrelativistic fluids can be described similarly with ρ and P by the Euler equation:

    ideal fluid

    $$\rho\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v} = -\boldsymbol{\nabla}P$$

    viscous fluid (Navier-Stokes)

    (η, ζ: describes viscosity)

    $$\rho\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v} = -\boldsymbol{\nabla}P + \eta\boldsymbol{\nabla}^2\mathbf{v} + \left(\zeta + \frac{1}{3}\eta\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v})$$

When writing down these equations, we do not care what exactly the microscopic theory is.

Here, information of UV theory is reduced to a few variables and parameters in IR; in turn, equations can be messy when adding corrections to describe the details.

Key point

Infinitely many degrees of freedom, with symmetry, interacting locally

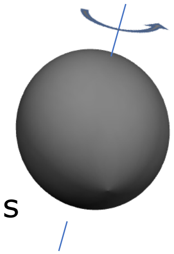
➡ At large scales, theory becomes less sensitive to the tiny structures. *Universality*

- QFT is based on the same mechanism, further empowered by the *renormalization group*.

- We would like to consider the field variables over space-time $x = (t, \mathbf{x})$

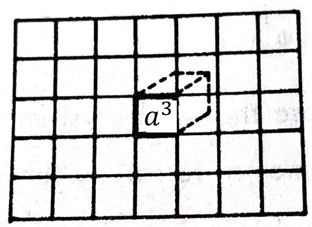
$\phi(x)$
scalars
Higgs

$\psi(x)$
spinors
quarks, leptons



$A_\mu(x)$
vectors
gauge bosons

- Just as in hydrodynamics, let us treat
  the average field value in a small cube as a single effective variable. *coarse-graining*



Ω = Na^3

T. D. Lee 1981

a: lattice cutoff

Fig. 2.1 Division of Ω into N tiny cubes, each of size a^3.

- A theory can be specified by the Lagrangian:

Physics *should* be the same at the scales $\Delta x \gg a$

scalar ϕ^4 theory (Euclidean)

$$S = \int d^D x \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4\right]$$

latticize & express with dimensionless variables

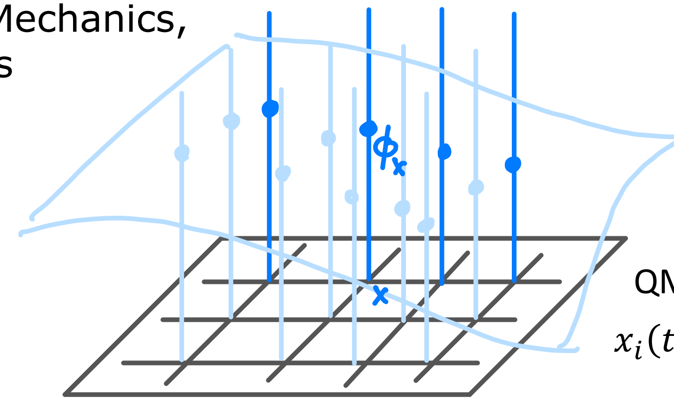
$$\hat{\phi}_x \equiv a^{\frac{D-2}{2}}\phi(x)$$
$$\hat{m} \equiv ma$$
$$\hat{\lambda} \equiv \lambda a^{4-D}$$

$$S_{\text{lat}} = \sum_x \left(\frac{1}{2}\sum_\mu (\hat{\phi}_{x+\hat{\mu}} - \hat{\phi}_x)^2 + \frac{\hat{m}^2}{2}\hat{\phi}_x^2 + \frac{\hat{\lambda}}{4}\hat{\phi}^4 \right)$$

(time direction also discretized)

- In a finite box, this is just a quantum mechanical system with finite (though many) variables

➡ Can be treated with conventional Quantum Mechanics, either in operator or path integral formalisms



QM QFT

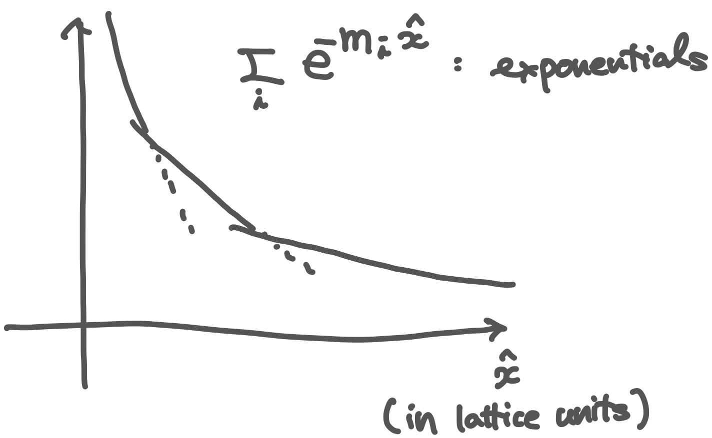
$x_i(t) \Leftrightarrow \phi_x(t)$

Two point function

$$\langle \hat{\phi}_x \hat{\phi}_y \rangle \equiv \frac{\int \left(\prod_x d\hat{\phi}_x\right) e^{iS_{\mathrm{lat}}(\hat{\phi})}\, \hat{\phi}_x\, \hat{\phi}_y}{\int \left(\prod_x d\hat{\phi}_x\right) e^{iS_{\mathrm{lat}}(\hat{\phi})}}$$

finite-dimensional integral;
no mathematical ambiguity once defined

- The correlation typically decreases at long distance



QFT as a cutoff theory (3/3)

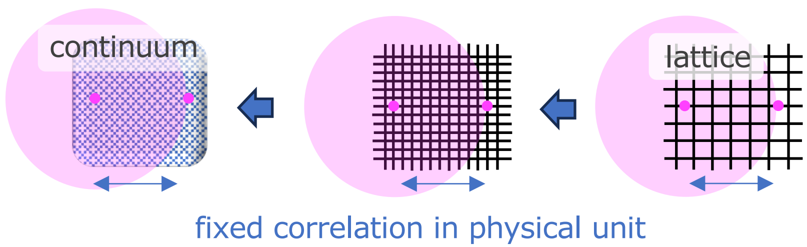
- We take the size of the cubes infinitesimally small by fixing <u>the emerging structures</u> to the target theory.
  ∵ Infinitely many DOF, with symmetry, interacting locally

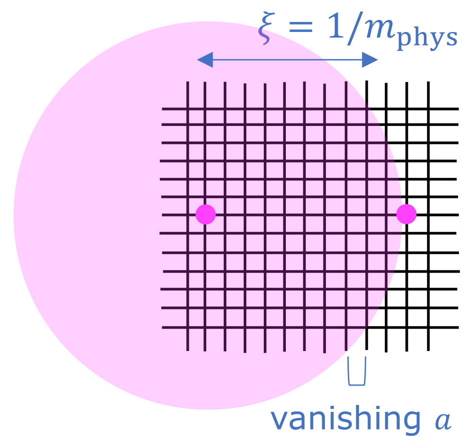
continuum limit

renormalization condition

<u>*"structures"*</u>

- ○ Correlation length ξ





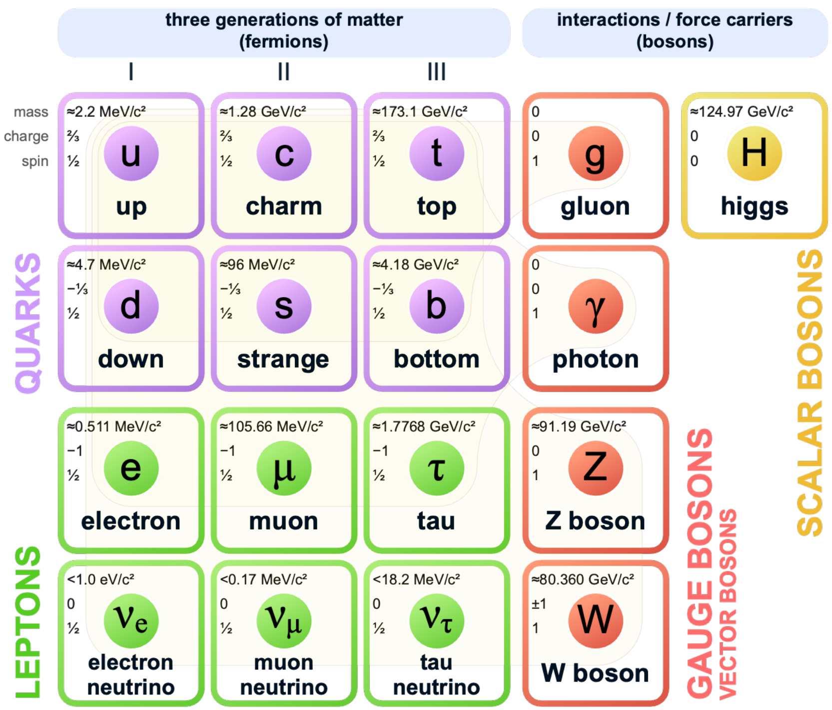
- ○ Low momentum behavior of the vertex functions describing interactions (specifically, *form factors;* to be described more in the $g-2$ section)

- Basically, we say that the parameter is *relevant* when its value affects the continuum limit; we often also consider parameters that can diminish logarithmically *(marginally irrelevant).*

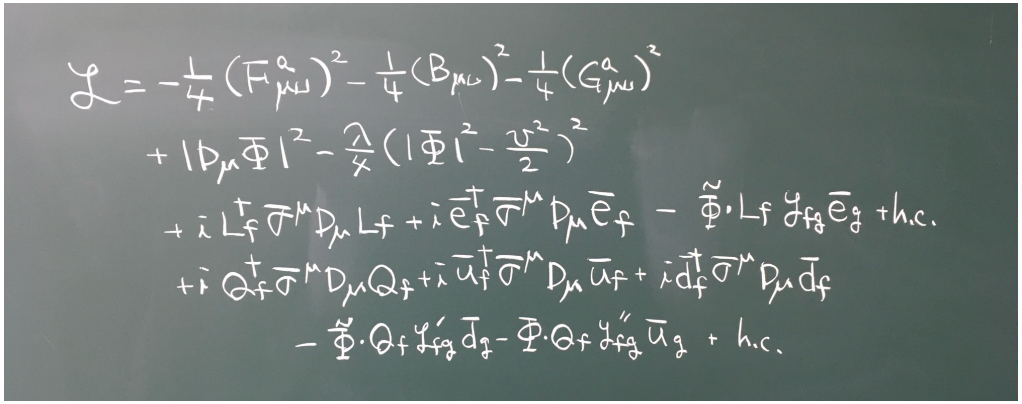
As a rule of thumb,
- ○ relevant: dimensionful ($[m] > 0$) and thus sets a mass scale
  e.g., masses of quarks and leptons
- ○ marginal: no mass scale ($[\lambda] = 0$)
  e.g., EM coupling e, strong coupling g

In short, *we fine-tune the relevant and marginal parameters* to satisfy the renormalization conditions.

Standard Model



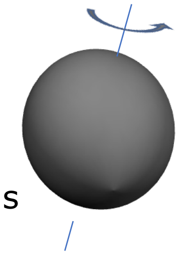
wikipedia



$Q_f, u_f, d_f,$
L_f, e_f, ν_f

spinors

quarks, leptons



A_μ^a, B_μ, G_μ^a

vectors

gauge bosons

Φ

scalar

Higgs

- Finite number of relevant/marginal parameters:
  18 + 1 (θ angle) + a few for ν mixing and masses

- We say SM is renormalizable $\Leftrightarrow$ finite # of parameters to be tuned
- Tuned parameters can then be seen as functions of the cutoff a:

$$\lambda = \lambda(a),\ m = m(a), \ldots$$

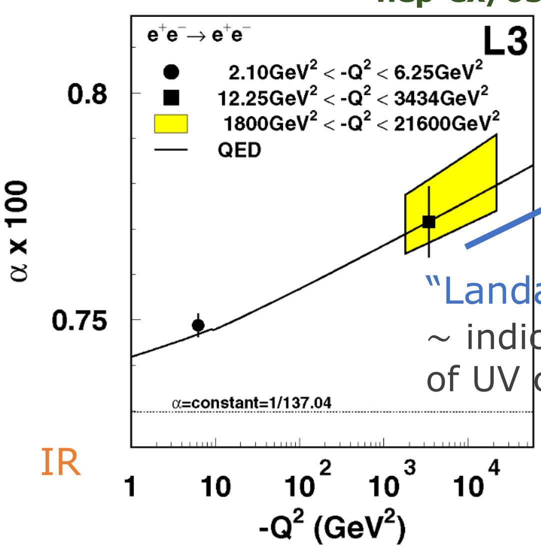
➡ draw a curve in the theory space

- The derived curve, in turn, sets a strength of coupling at the scales $\mu \Leftrightarrow \frac{1}{a}$
  (here assuming we are tweaking purely relevant/marginal parameters)

renormalization group flow

○ $\alpha_{QED} \equiv \frac{e^2}{4\pi}$

hep-ex/0507078



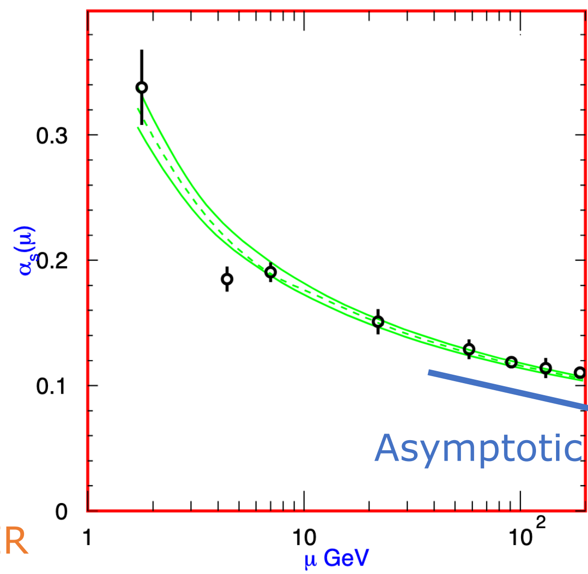
IR UV

"Landau pole"
~ indicating a theory
of UV completion (=EW)

Fig. 6. Evolution of the electromagnetic coupling with Q^2 determined from the present measurement of C for $1800\ \mathrm{GeV}^2 < -Q^2 < 21600\ \mathrm{GeV}^2$, yellow band, and from previous data for Bhabha scattering at $2.10\ \mathrm{GeV}^2 < -Q^2 < 6.25\ \mathrm{GeV}^2$ and $12.25\ \mathrm{GeV}^2 < -Q^2 < 3434\ \mathrm{GeV}^2$ [10], full symbols. The solid line represent the QED predictions [5].

○ $\alpha_{QCD} \equiv \frac{g^2}{4\pi}$

PDG



IR UV

Asymptotic freedom

Figure 9.2: Summary of the values of $\alpha_s(\mu)$ at the values of μ where they are measured. The lines show the central values and the $\pm1\sigma$ limits of our average. The figure clearly shows the decrease in $\alpha_s(\mu)$ with increasing μ. The data are, in increasing order of μ, τ width, Υ decays, deep inelastic scattering, e^+e^- event shapes at 22 GeV from the JADE data, shapes at TRISTAN at 58 GeV, Z width, and e^+e^- event shapes at 135 and 189 GeV.

- The renormalization group flow can terminate at UV and/or IR fixed points.

QCD-like $N_f = 2, N_c = 3$ $N_f = 12, N_c = 3$

Hasenfratz-Witzel 1911.11531

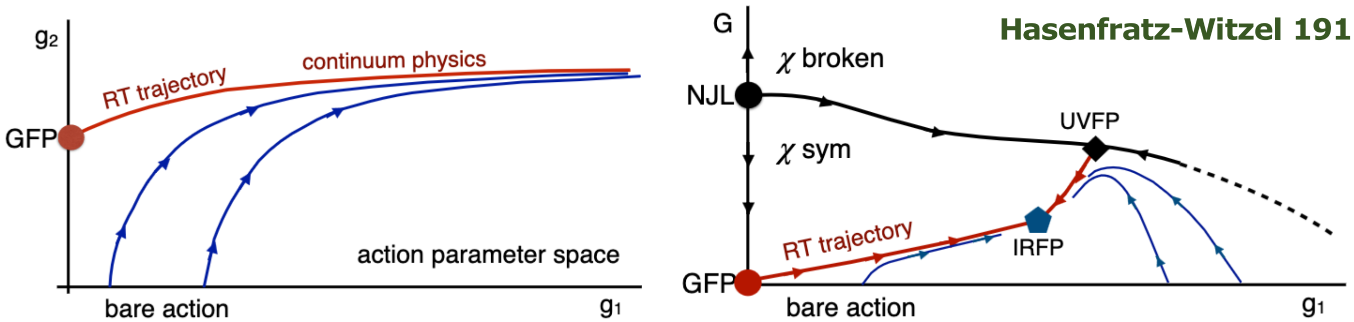


Figure 1: Sketch of possible phase diagrams and RG flows in the multi-dimensional action parameter space. Left panel: QCD-like gauge-fermion system. g_1 refers to the relevant gauge coupling, while g_2 indicates all other irrelevant couplings. The blue lines represent RG flow trajectories. Right panel: Infrared conformal system with 4-fermion interaction. G denotes the 4-fermion coupling and only the relevant g_1 coupling of the gauge-fermion system is shown. The black solid line denotes a 2nd order phase transition separating

- In that *very limit*, local relativistic theories become conformal field theories (CFTs), theories that have *no dynamical mass scale*.

- Correspondingly, all the masses m approaches zero towards these points: $m \to 0$, which makes the correlation length in the system diverge: $\xi = 1/m \to \infty$.

- Since this is a generic property of 2nd order phase transition, we say *"we look for 2nd order phase transition to take the continuum limit"*

this is the case for QCD

Wilson 1974

We now make predictions that explain experiments!

- To completely define a theory requires inputs for each tuning parameter:

$$\lambda = \lambda(\langle O_1\rangle|_\mu, \langle O_2\rangle|_\mu),\ m = m(\langle O_1\rangle|_\mu, \langle O_2\rangle|_\mu)$$

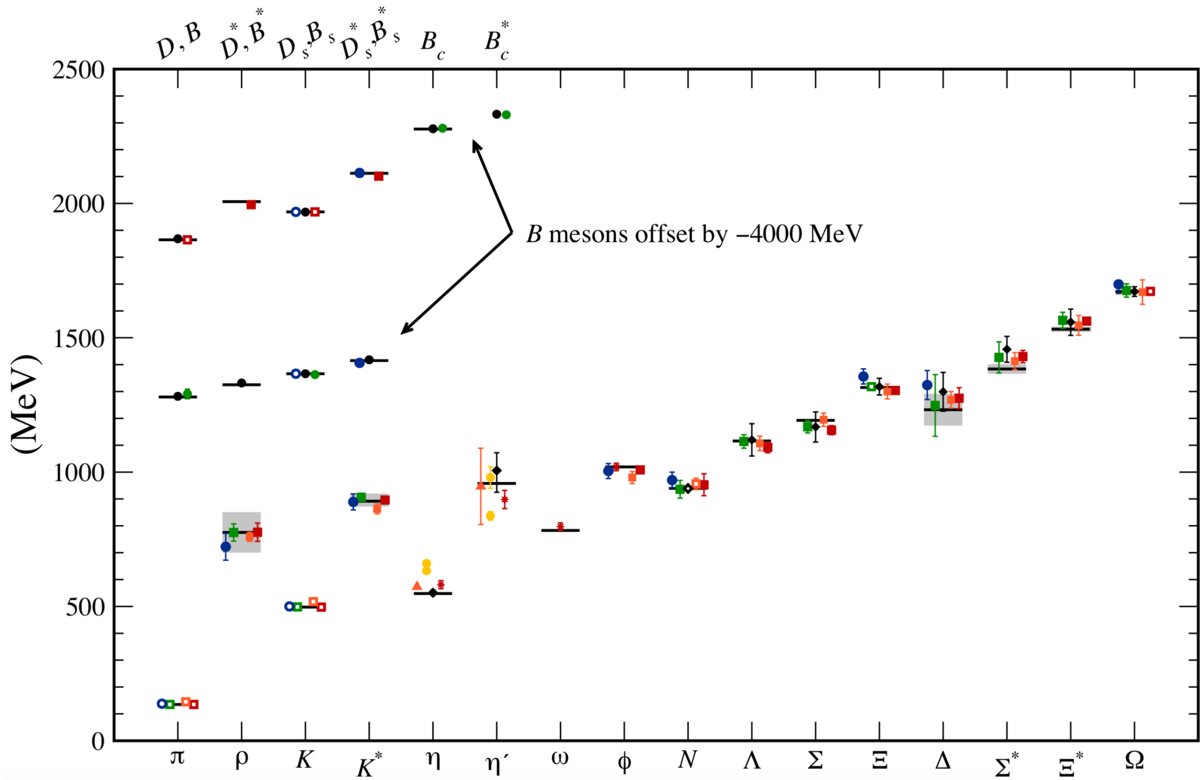
$\langle O_1\rangle|_\mu, \langle O_2\rangle|_\mu$: setting renormalization conditions, requires experimental values for the tuning

SM: 18 + 1 (θ angle) + a few for ν mixing and masses

- Theory prediction is made for quantities other than the inputs:

$$m' = m'(\lambda, m)$$

the theory serving as a function of the inputs to ∀observables



from PDG

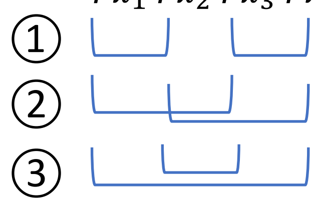
To make a theory prediction, we need a way of evaluating amplitudes/correlators.

- At the regimes where the couplings are small, one may consider a Taylor-expansion:

$$\int \left(\prod_x d\phi_x\right) e^{-\frac{1}{2}\sum \phi_x (\Delta^2+m^2)_{xy}\phi_y} e^{-\lambda \sum \phi_x^4}$$
$$= \int \left(\prod_x d\phi_x\right) e^{-\frac{1}{2}\sum \phi_x (\Delta^2+m^2)_{xy}\phi_y} \left(1 - \lambda \sum \phi_x^4 + \frac{\lambda^2}{2}\sum \phi_x^4 \phi_{x'}^4 + \cdots\right)$$

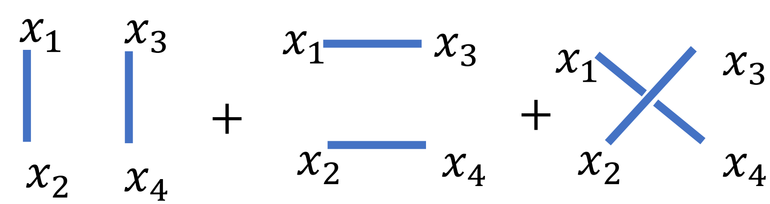
- Term by term, this is a Gaussian integral of the form:

$$\int \left(\prod_x d\phi_x\right) e^{-\frac{1}{2}\sum \phi_x M_{xy}\phi_y}\, \phi_{x_1}\phi_{x_2}\phi_{x_3}\phi_{x_4} \propto M^{-1}_{x_1x_2}M^{-1}_{x_3x_4} + M^{-1}_{x_1x_3}M^{-1}_{x_2x_4} + M^{-1}_{x_1x_4}M^{-1}_{x_2x_3}$$



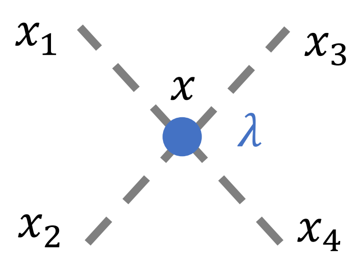
① ② ③

→ Feynman diagram



- The interaction term gives the diagram

$$\lambda \sum_x \phi_x^4 \cdot \phi_{x_1}\phi_{x_2}\phi_{x_3}\phi_{x_4} \quad \rightarrow \sum_x$$



Perturbative expansion is:

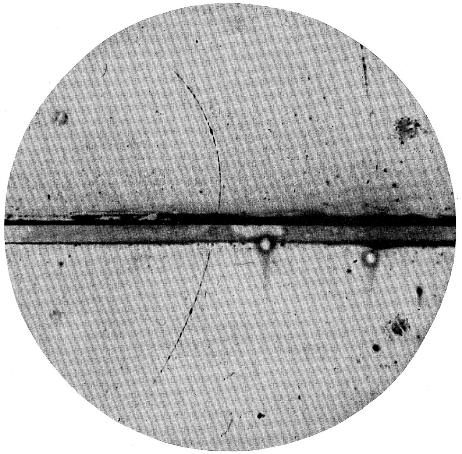
- not convergent (mathematically called *asymptotic series*)

  ∵ # digagram $\sim N! \sim N^N$
  power of coupling $\sim \alpha^N$ ⇒ N-th order: $O(\alpha N)^N$ ∴ Effective up to $N = O(1/\alpha)$

- useful approximation, but sometimes *too intuitive*; for example:
  - Perturbation theory corresponds to using the free basis $|k_1, \cdots\rangle_0$ to describe "particles".
  - However, generically,
    true "one-particle state" requires an infinite sum:

$$|k\rangle = |k\rangle_0 + \sum_{\{k\}} O(\alpha)\, |k_1, k_2, \cdots\rangle_0$$

and the exact coefficients are *not* accessible
solely by perturbation theory.



Track of cosmic ray positron in cloud chamber
C. D. Anderson, Phys Rev 43 491 (1933)
For cosmic rays, see also GRAMS: Tuesday AM

Though the difference is formally only $O(\alpha)$,
at the same time, the two notions are fatally different.

Many theoretical branches: resummation, resurgence, Hamiltonian truncation, …

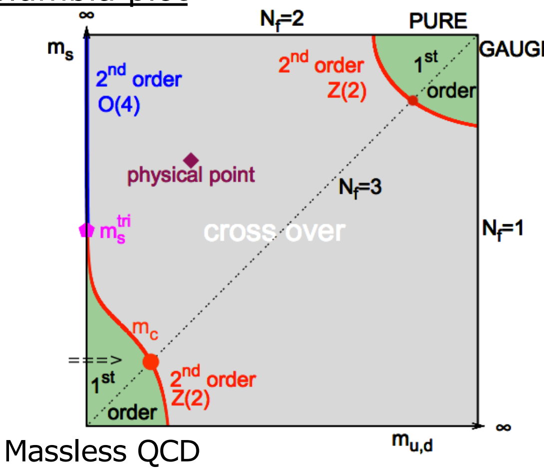
- This problem becomes more evident in a strongly-interacting system.

Examples of strongly-interacting physics

- confinement
  Quarks and gluons cannot be observed in an isolated state in IR, but only their color singlet states (hadrons, glueballs, …).
- bound state, resonance
  Poles of composite states can appear in propagators; even massless QCD and pure YM theory exhibit mass gaps.
  $1M Millennium Prize by Clay Institute

Such *collective phenomena* are possible because of infinitely many DOF (that's why it's so interesting!)

Columbia plot



Philippe de Forcrand, Massimo D'Elia
***PoS* LATTICE2016 (2017) 081**

- Note that:
  - The perturbative expansion is effective for terms only up to $O(1/\alpha)$.
    ➡ Obviously doesn't work at all when $\alpha = O(1) \Leftrightarrow \mu = \Lambda_{\rm QCD}$
  - The perturbative expansion may not exist in the first place because of an *essential singularity* at $g = 0$

    E.g., instanton amplitude $\sim e^{-\frac{8\pi|n|}{g^2}}$ ($n \in \mathbb{Z}$: winding#)
    Belavin-Polyakov-Schwartz-Tyupkin 1975
  - Formation of a new pole requires an infinite series: $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots$

    "search for analyticity"
    Wilson-Kogut 1973

- To describe the physics of hadrons with QCD, nonperturbative methods are actually *required* at the practical level.

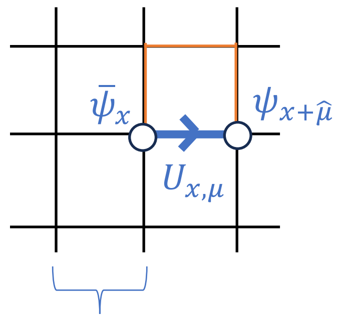
Lattice calculation has been most successful in this regard.

Other directions: functional renormalization group, exact diagonalization, …

Lattice calculation (1/2) **Wilson 1974**

Lattice QCD uses exactly the construction aforementioned, without perturbation theory!

- Assuming the analyticity of the amplitudes/correlators, we formulate field theories on *Euclidean lattice*:



ψ_x: quarks

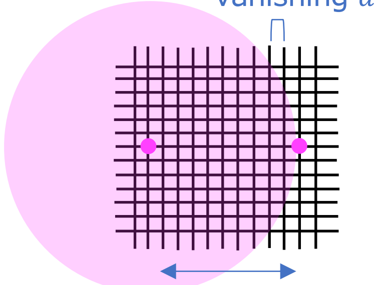
$U_{x,\mu}$: gauge field

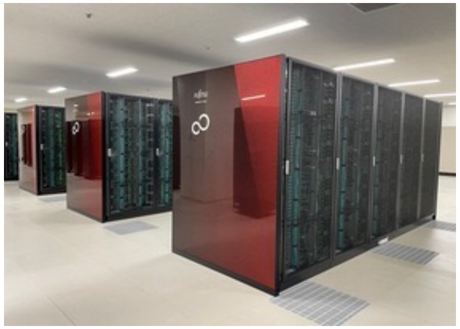
$U_{x,\mu} \sim e^{iaA_\mu(x)}$ (*Wilson line*)

$$S_{\mathrm{lat}}(U) \equiv -\frac{\beta}{6}\sum_{x,\mu<\nu} \mathrm{Re\,tr}\left[U_{x,\mu}U_{x+\mu,\nu}U^\dagger_{x+\nu,\mu}U^\dagger_{x,\nu}\right]$$

$$+\sum_x \left[\begin{array}{l} \frac{1}{2}\sum_\mu \bar{\psi}_x\gamma_\mu(U_{x,\mu}\psi_{x+\hat{\mu}} - U^\dagger_{x-\mu,\mu}\psi_{x-\hat{\mu}}) + m\bar{\psi}_x\psi_x \\ -\frac{r}{2}\sum_\mu \bar{\psi}_x(U_{x,\mu}\psi_{x+\hat{\mu}} + U^\dagger_{x-\mu,\mu}\psi_{x-\hat{\mu}} - 2\psi_x) \end{array}\right]$$

- Evaluate path integral with computers using Monte Carlo integration with several lattice spacings and take the continuum limit.



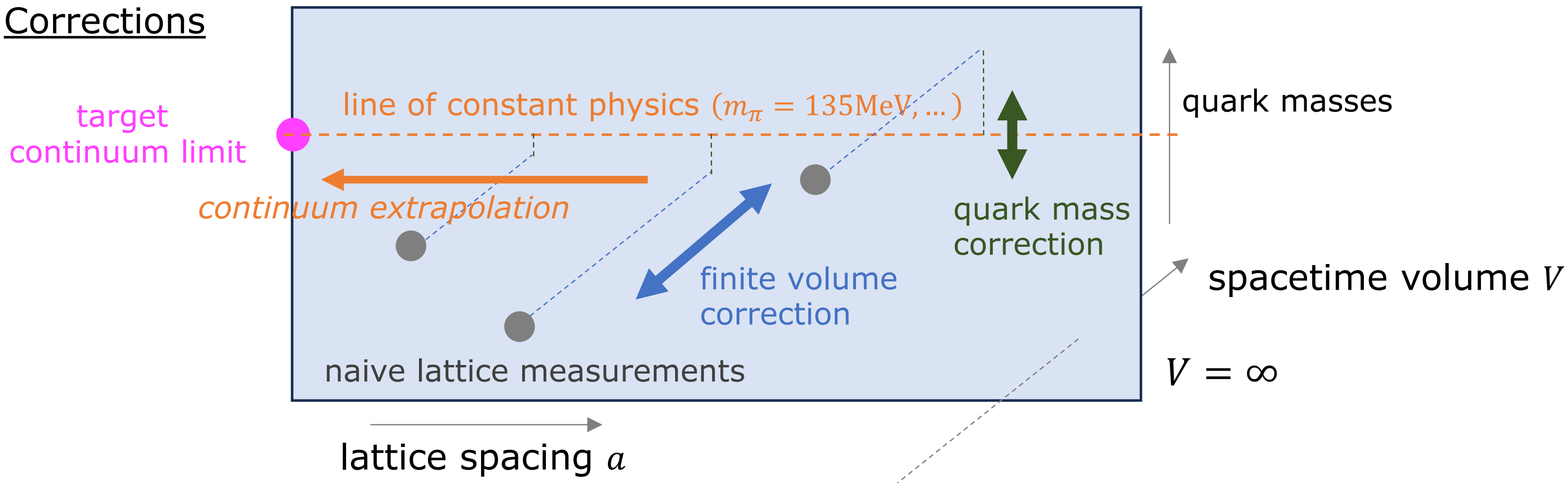


Wisteria @ U. Tokyo

Fugaku @ RIKEN R-CCS

- Practicalities:
  - What we want = continuum theory in infinitely large spacetime describing the nature
  - What can be put on computer = lattice QCD on finite volume with hand-tuned couplings

Corrections



- We cope with errors accordingly:
  - Lattice field ensembles are generated by Monte Carlo methods
    → statistical error
  - Model/ansatz dependent uncertainties of continuum extrapolation and corrections
    → systematic error

Pros

Everything is defined nonperturbatively, well-sorted, systematically improvable.

Cons

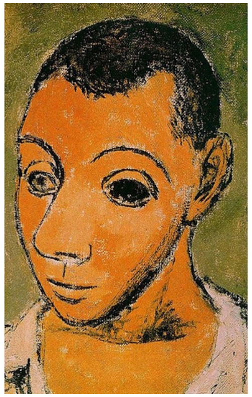
Sometimes faces numerical restrictions
(Euclidean correlator, sign problem, signal to noise, critical slowing down, …)

- Many islands of “well-known facts” from various points of view

Picasso

wikipedia

self-portraits; *in the state of the art*



perturbation theory

lattice calculation

Semi-classical analysis
(instanton, large-N, …)

- Every direction has its own decoration of mathematics
  that often makes it hard for us to learn anything

- The baseline however seems simple:

  Machineries for calculating physical observables have been well-developed,
  and *large efforts* are made to understand theories nonperturbatively.

- It is fascinating that $g-2$ gathers the major ingredients and puts them under a test.

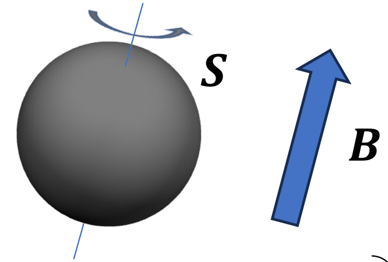
Muon $g-2$

Magnetic moment: coupling of a particle to the magnetic field

$$\Delta H = -\mu_B\, \mathbf{B} \cdot (\mathbf{L} + g\mathbf{S})$$

S: spin $(\pm 1/2)$

gyromagnetic ratio *or g-factor*



$\mu_B \equiv \frac{|e|}{2m}$ "Bohr magneton" $(e < 0)$

Dirac theory (classical field theory of fermions) **Dirac 1928**

$$\mathcal{L} \equiv \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi$$

$$H_{\text{Dirac}} \equiv \gamma^0 \boldsymbol{\gamma} \cdot (-i\boldsymbol{\partial} + e\mathbf{A}) + m\,\gamma^0 - e\phi$$

$$= m + \frac{\mathbf{p}^2}{2m} + \frac{e^2}{2m}\mathbf{A}^2 - e\phi - \mu_B\, \mathbf{B} \cdot (\mathbf{L} + 2\mathbf{S}) + O\left(\frac{1}{c^2}\right)$$

$g = 2$

$$S^i \equiv \frac{\sigma^i}{2}$$

Quantum Field theory

Radiative corrections shifts g from 2

"anomalous magnetic moment" $a \equiv \frac{g-2}{2}$

- $a_{\ell=e,\mu}$ can be determined precisely both theoretically and experimentally.

cf. a_e giving the most precise determination of α_{QED}:

$$a_e(\text{theory}) = 1159652182.032(720) \times 10^{-12}$$

Aoyama-Kinoshita-Nio 1712.06060

up to $O(\alpha_{\text{QED}}^5)$

$$\Rightarrow \alpha_{QED}^{-1} = 137.\underbrace{0359991491}_{\text{ten decimal places}}(331)$$

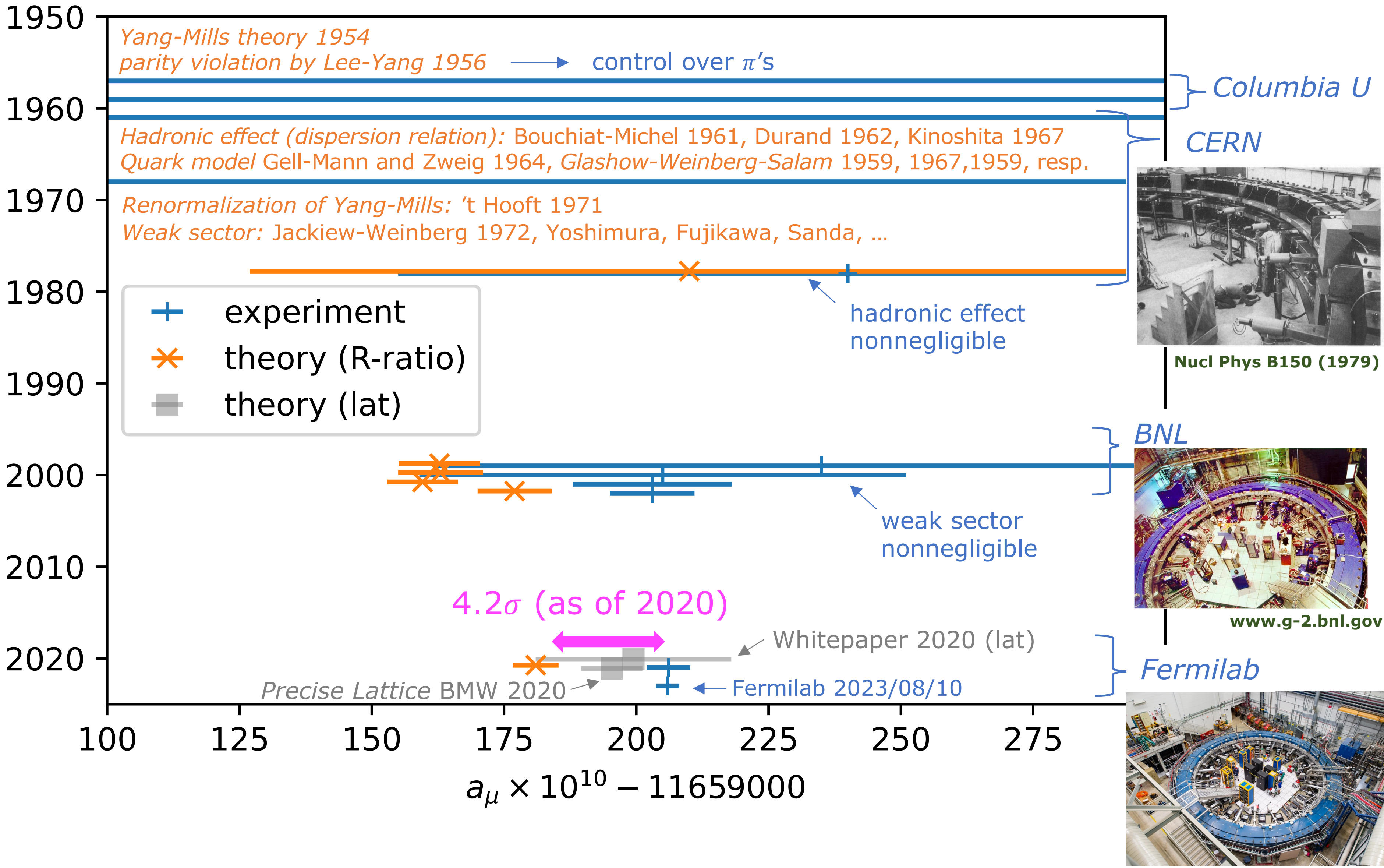
- $m_\mu = 106\,\text{MeV}$ → much more sensible to strong and weak interaction than $m_e = 511\,\text{keV}$

Berestetskii, Krokhin, Klebnikov 1956

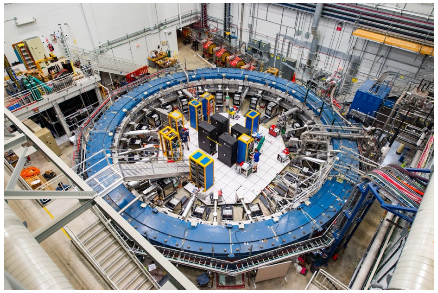
For $\mu \to e\gamma$, see MEG-II: Sunday PM

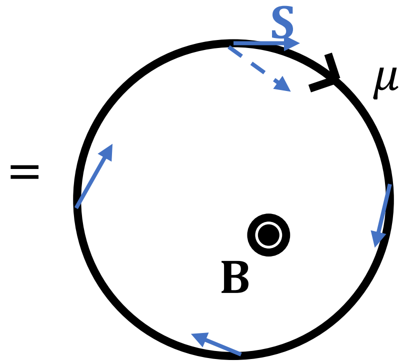
a_μ: Good ground for precision test of the Standard Model

Target Physics (3/3)



Comparison is now at the order of 10^{-10}





Orbital motion

$$\omega_c \equiv \frac{|e|B}{m} \qquad \text{(cyclotron frequency)}$$

Spin precession

- Magnetic moment $\Delta H_m \equiv -\mu_B g\, \mathbf{B}\cdot\mathbf{S}$

$\left(\mu_B = \dfrac{|e|}{2m}\right)$

Spin vector $\mathbf{S}$ rotates (*"Larmor Precession"*).

- In fact, quantum mechanics predicts:

$$\dot{\mathbf{S}} = \frac{1}{i}[\Delta H_m, \mathbf{S}] \Rightarrow \text{precession w/ } \omega_s \equiv \mu_B g B \text{ perp to } \mathbf{B}$$

∴ a_μ leads to anomalous precession with frequency:

$$\omega_a \equiv \omega_s - \omega_c = \mu_B(g-2)B = a_\mu \omega_c$$

NB Relativistic, sophisticated formula known:
Bargmann, Michael Telegdi 1959

$$\boldsymbol{\omega}_a = \frac{e}{m}\left[a_\mu \mathbf{B} - \left(a_\mu - \frac{1}{\gamma^2-1}\right)\boldsymbol{\beta}\times\mathbf{E}\right]$$

Magic momentum $p = 3.09$ GeV, $\gamma = 29.304$,
for which $\left(a_\mu - \frac{1}{\gamma^2-1}\right) = 0$ See, e.g., Miller, Roberts 1805.01944

BNL Muon (g – 2) Collaboration
Phys. Rev. Lett. 86, 2227 (2001)

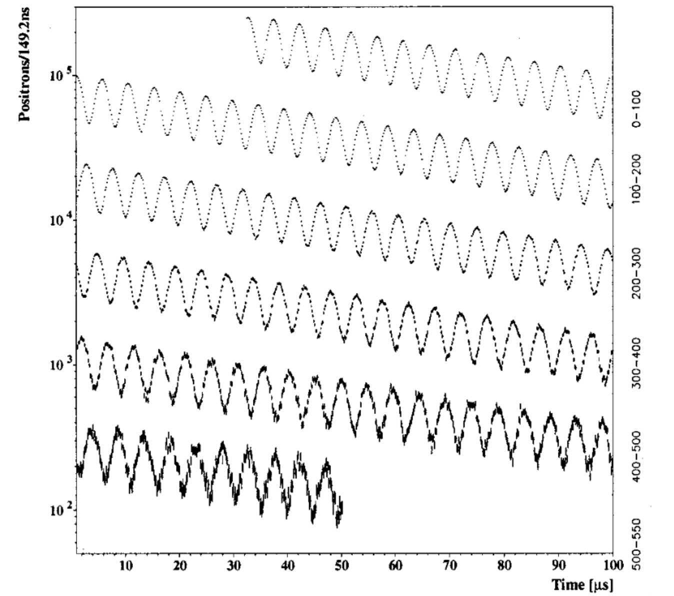
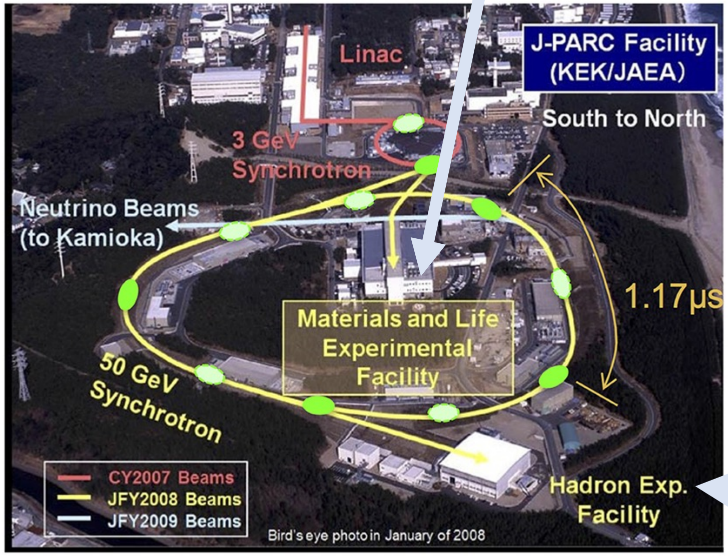


FIG. 2. The positron time spectrum obtained with muon injection for $E > 1.8$ GeV. These data represent 84 million positrons.

Direct measurement of $g-2$ (2/2)

- J-PARC E34 g-2/EDM: New μ trapping technique → different systematics; installation in progress

https://j-parc.jp/c/en/topics/2023/10/31001225.html



COMET: Sunday PM

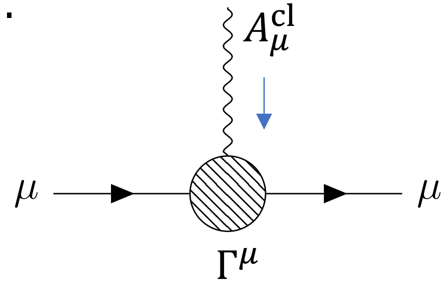
$g-2$ in QFT (1/2)

- Perturb $\mu \to \mu$ amplitude with static external field $A_\mu^{\rm cl}$.

  Linear response:

$$i\mathcal{M} = -ie\,(\bar{u}\,\Gamma^\mu u)\cdot A_\mu^{\rm cl}(\mathbf{q})$$

  vertex function (← Γ^μ)



  The fact that $i\mathcal{M}$ has $\mathbf{q}$ dependence suggests that the vertex Γ^μ has a structure.

- On-shell condition, Ward identity, Lorentz invariance:

$$\Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{\gamma^{\mu\nu}}{2m}F_2(q^2) \underbrace{- i\frac{\gamma^{\mu\nu}q_\nu}{2m}\gamma_5 F_3(q^2) - \frac{(q^2\delta^\mu_\nu - q^\mu q_\nu)\gamma^\nu}{m^2}\gamma_5 F_4(q^2)}_{\text{CP odd; from weak sector}}$$

  F_i: *"form factors"*

- Its nonrelativistic limit can be interpreted as the scattering by a potential:

$$i\mathcal{M} = -2mi\,\delta_{s_{\rm out}s_{\rm in}}\cdot\left\{\underbrace{-eF_1(0)\phi}_{\text{electric monopole moment = electrostatic potential}} \underbrace{- \mu_B\,\mathbf{B}\cdot(F_1(0)\mathbf{L} + 2[F_1(0)+F_2(0)]\mathbf{S})}_{\text{magnetic dipole moment = magnetic-field-angular momentum coupling}} \underbrace{+F_3(0)\frac{|e|\mathbf{S}}{m}\cdot\mathbf{E}}_{\text{electric dipole moment (EDM)}} \underbrace{+ F_4(0)\frac{2|e|}{m^2}\int d\mathbf{x}\,e^{-i\mathbf{q}\cdot\mathbf{x}}\,\mathbf{S}\cdot\nabla\times\boldsymbol{B}}_{\text{anapole moment (argued as nonphysical)}} + O\left(\frac{1}{c^2}\right)\right\}$$

electric monopole moment
= electrostatic potential

magnetic dipole moment
= magnetic-field-angular momentum coupling

electric dipole moment (EDM)

anapole moment
(argued as nonphysical)

Musolf, Holstein 1991

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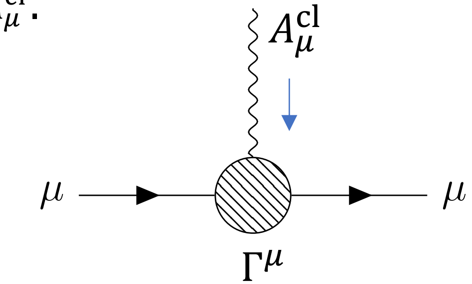
vertex function (→ Γ^μ)



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electric monopole moment
= electrostatic potential

magnetic dipole moment
= magnetic-field-angular momentum coupling

The electrostatic potential sets the unit of electric charge.

electric dipole moment (EDM)

anapole moment
(argued as nonphysical)

Musolf, Holstein 1991

Renormalization condition:

$$F_1(0) = 1$$

- The relevant part is thus:

$$i\mathcal{M} = -2mi\,\delta_{s_{\text{out}}s_{\text{in}}} \cdot \left\{ \begin{array}{l} -e\phi - \mu_B\,\mathbf{B}\cdot(\mathbf{L} + 2[1 + F_2(0)]\mathbf{S}) \\ + \text{CP ODD} + O\left(\frac{1}{c^2}\right) \end{array} \right\}$$

Desired correction to the Dirac theory!

$$H_{\text{Dirac}} = -e\phi - \mu_B\,\mathbf{B}\cdot(\mathbf{L} + 2\mathbf{S}) + \cdots$$

$$\therefore\ a_\mu = \frac{g_\mu - 2}{2} = F_2(0)$$

target physics

- $a_\mu = F_2(0)$ is completely determined by the vertex function Γ:

Barbieri, Mignaco, Remiddi 1972, Levine, Roskies 1974, Barbieri, Remiddi 1975

∃ projection operator P_μ s.t.

$$F_2(q^2) = \text{tr}\left[P_\mu \Gamma^\mu\right]$$

on-shell projectors

$$P_\mu \equiv \frac{1}{q^2(q^2+4)}(\not{p}_{\text{in}} - 1)\left[\gamma_\mu + 2\frac{q^2-2}{q^2+4}p_\mu\right](\not{p}_{\text{out}} - 1)$$

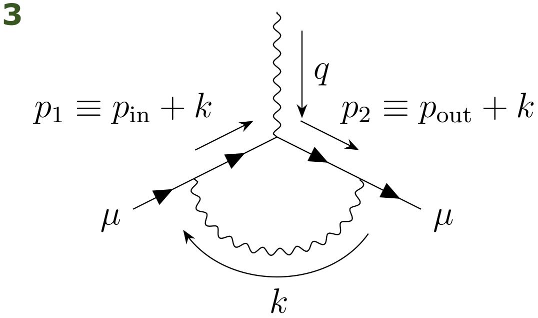
$(m = 1)$

∴ To do:

- Expand Γ^μ with α_{QED}
- Calculate: $a_\mu = F_2(0) = \text{tr}\left[P_\mu \Gamma^\mu\right]_{q\to 0}$

- One-loop (Schwinger contribution):

Levine, Roskies 1973



$$\Gamma^{\mu}_{(1)} \equiv e^2 \int_k \gamma_\alpha \frac{i}{\not p_2 + m - i\varepsilon} \gamma^\mu \frac{i}{\not p_1 + m - i\varepsilon} \gamma_\beta \cdot \frac{i\eta^{\alpha\beta}}{k^2 - i\varepsilon}$$

Apply the projector $(m = 1)$

$$F_2(q^2 = 0) = 4ie^2 \int_k \frac{1}{\left((p-k)^2 + 1 - i\varepsilon\right)} \cdot \frac{1}{k^2 - i\varepsilon} \cdot \left\{\frac{k^2}{3} + \frac{4}{3}(k\cdot p)^2 + k\cdot p\right\}$$

$$\left(p = \frac{p_{\rm out} + p_{\rm in}}{2} \quad q = p_{\rm out} - p_{\rm in}\right)$$

Analytical continuation (both internal & external) $\left\{ \begin{array}{l} k^0 \to ik_E^0,\ K \equiv |k_E| \\ p^0 \to ip_E^0,\ P \equiv |p_E| \end{array}\right.$

$$\to \frac{-4e^2\Omega_2}{(2\pi)^4} \int KdK \int \sin^2\chi\, d\chi \frac{1}{(K^2 + P^2 + 1 - 2KP\cos\chi)^2} \cdot \left\{\frac{K^2}{3} + \frac{4}{3}K^2P^2\cos^2\chi + KP\cos\chi\right\}$$

Angular integration, impose on-shell: $P^2 \to -m^2$ ("continuing back")

$$\to \int dK^2 \underbrace{\frac{e^2}{4\pi^2} \frac{K^2 Z^3 (1 - K^2 Z)}{1 + K^2 Z^2}}_{\equiv f(K^2)} \qquad \left(Z \equiv -\frac{K^2 - \sqrt{K^4 + 4K^2}}{2K^2} \right)$$

Function of <u>*spatial*</u> K^2*;*
K*: internal photon momentum*

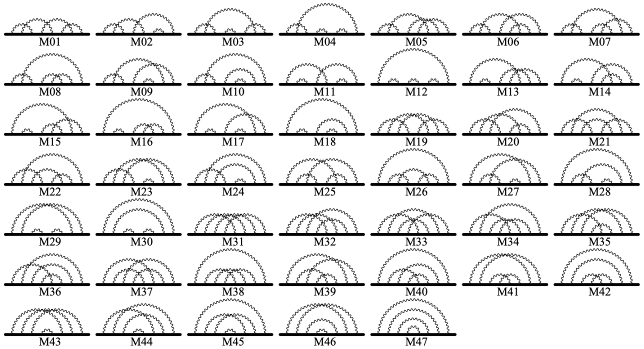
$$= \frac{\alpha}{2\pi}$$

Schwinger 1949

Lowest order does not require renormalization

Higher orders in α_{QED}

See whitepaper 2006.04822



<u>pure QED</u> $a_\mu^{\rm QED}/a_\mu = 99.994\ \%$

$O(\alpha_{\rm QED}^4)$: **Aoyama-Hayakawa-Kinoshita-Nio 0712.2607 (fig)**

$O(\alpha_{\rm QED}^5)$: **Aoyama-Kinoshita-Nio 1712.06060**

Perturbative QED only breaks down at $1/\alpha_{\rm QED} \approx 137$

<u>Weak sector</u> $a_\mu^{\rm EW}/a_\mu = 1.3{\times}10^{-4}\ \%$ **For detail: Czarnecki, Marciano, Vainshtein hep-ph/0212229**

E.g., 1-loop:

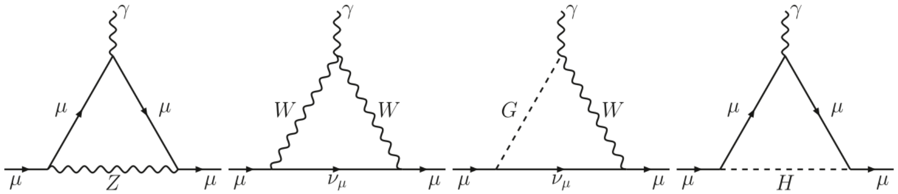
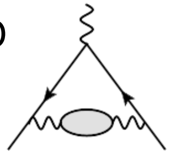


Fig: whitepaper

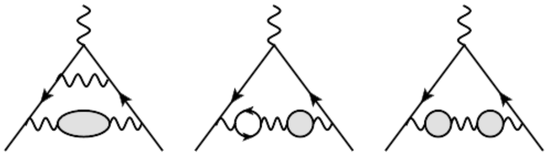
<u>Strong sector</u>

- HVP $a_\mu^{\rm HVP}/a_\mu = 5.87{\times}10^{-3}\%$

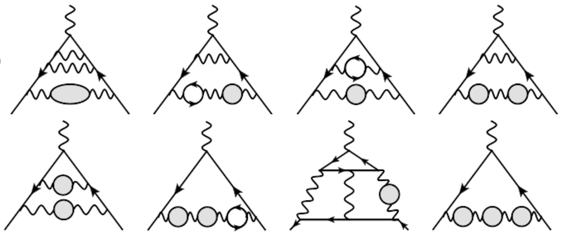
LO



NLO



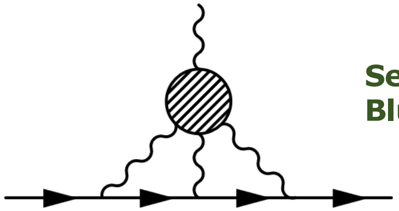
NNLO



$a_\mu^{\rm HVP\ LO}/a_\mu = 5.94{\times}10^{-3}\%$ $a_\mu^{\rm HVP\ NLO}/a_\mu = -0.08{\times}10^{-3}\%$ $a_\mu^{\rm HVP\ NNLO}/a_\mu = 0.01{\times}10^{-3}\%$

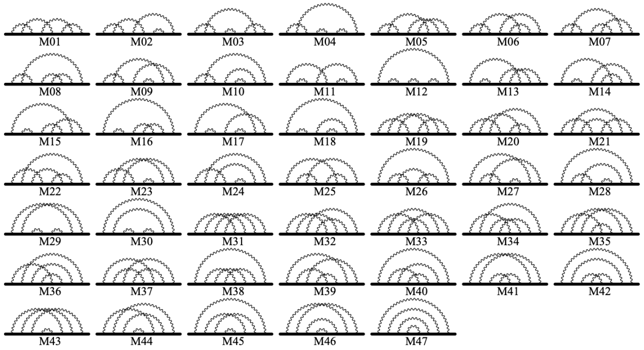
Fig: Kurz et al. 1403.6400

- Hadronic Light-by-Light (HLbL) $a_\mu^{\rm HLbL}/a_\mu = 7.9{\times}10^{-5}\%$



See, e.g.,:
Blum, Izubuchi, eat al. RBC/UKQCD 2015 (fig)

See whitepaper 2006.04822



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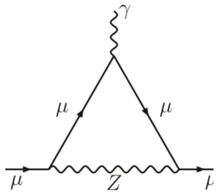

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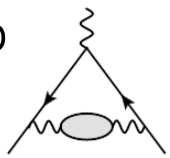
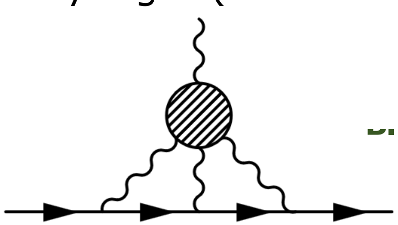
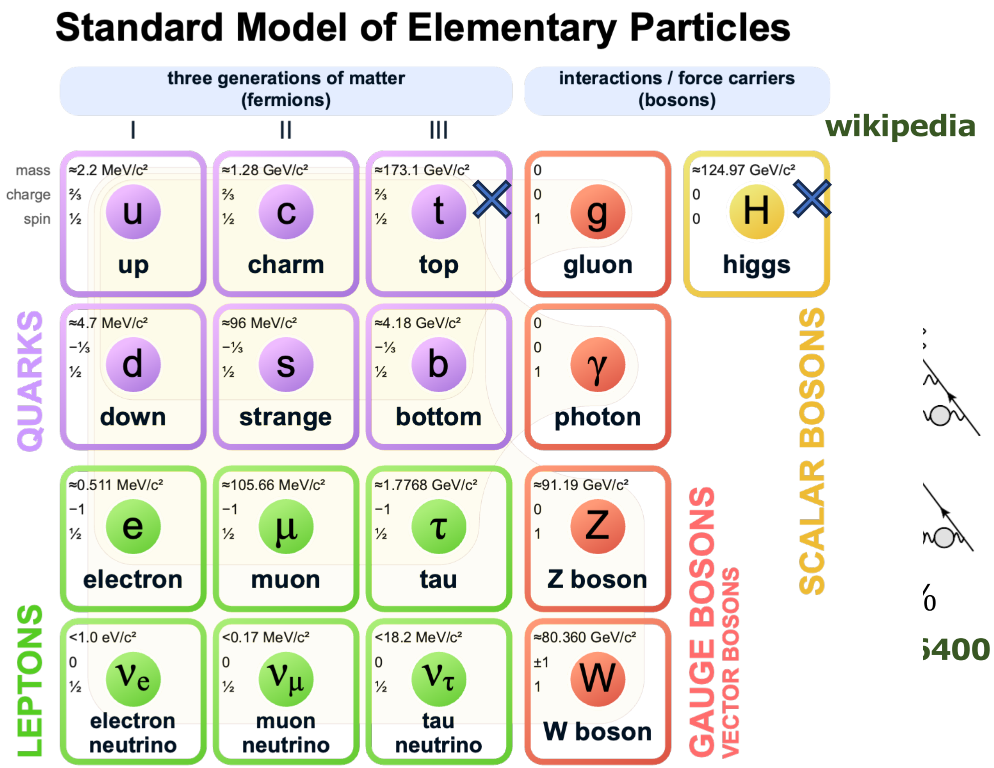


NLO

$a_\mu^{\text{HVP LO}}/a_\mu = 5.94\times10^{-3}\%$ a_μ^{HV}

- Hadronic Light-by-Light (HLbL

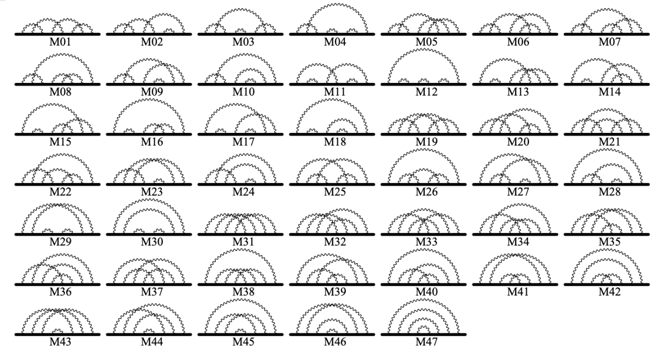




wikipedia

High sensitivity to BSM

See whitepaper 2006.04822



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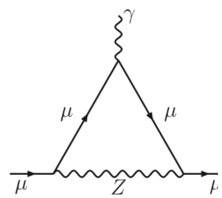

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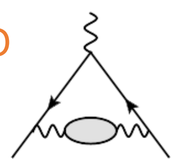
- HVP $a_\mu^{\mathrm{HVP}}/a_\mu = 5.87\times10^{-3}\%$

LO

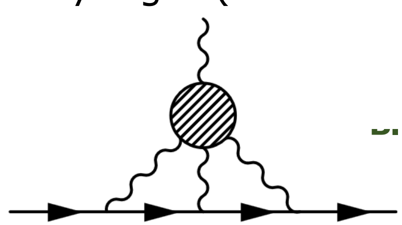


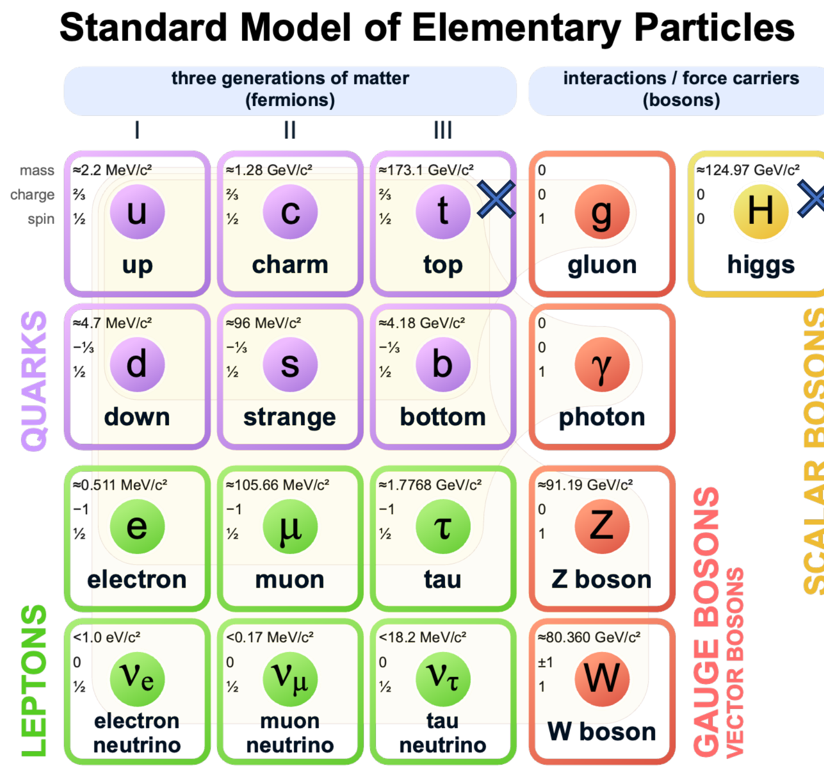
NLO

$a_\mu^{\mathrm{HVP\ LO}}/a_\mu = 5.94\times10^{-3}\%$

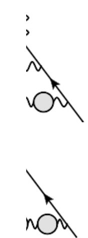
a_μ^{HV}

- Hadronic Light-by-Light (HLbl



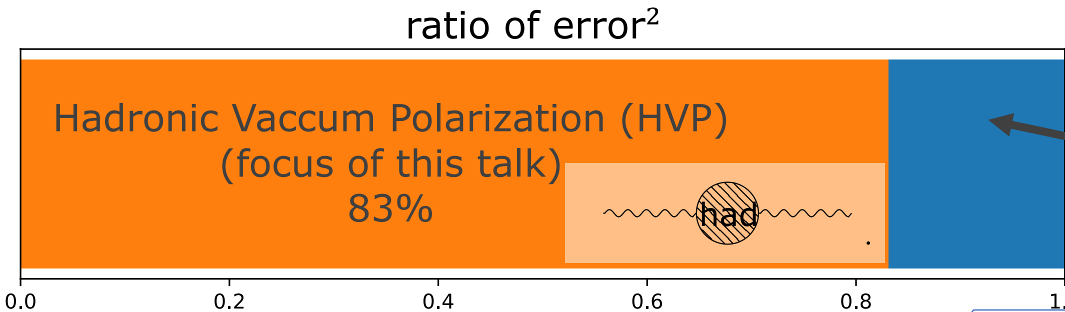


wikipedia

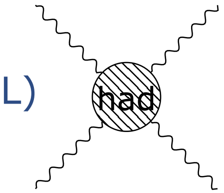


High sensitivity to BSM

g-2 Theory Whitepaper 2006.04822



Hadronic
Light by Light (HLbL)
17%



Electro-Weak(EW) 0.05%

QED 0.006%

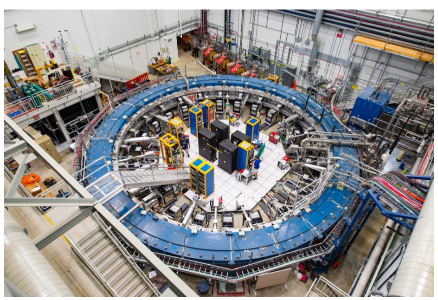
- Evaluation of the *hadronic vacuum polarization (HVP)* requires non-pertubative calculation of hadron dynamics.

  Lattice QCD did not have enough precision until BMW 2020:

  Budapest-Marseille-Wuppertal (BMW)

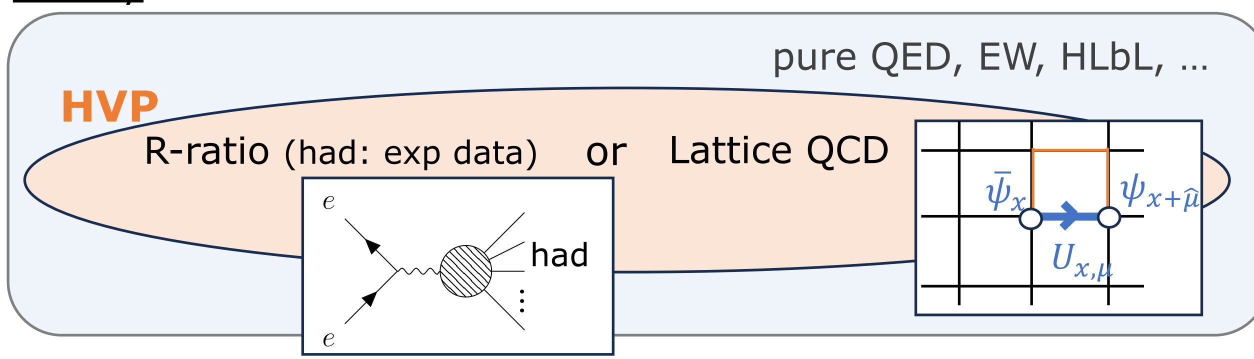
Nature 2021 [2002.12347]
Borsanyi, Miura, et al.

Experiment



fnal.gov

Theory



- Estimate with R-ratio → previously-mentioned tension
- Estimate with lattice → more consistent with the experiment

Difference between R-ratio and lattice has been the recent subject of scrutiny

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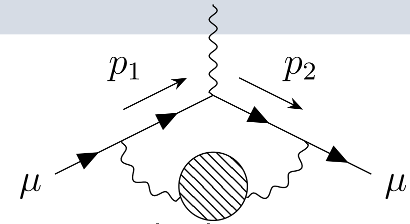


- LO HVP digram = HVP inserted to the one loop vertex

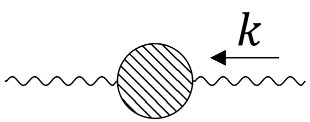
$$\Gamma^{\mu}_{(\mathrm{HVP})} \equiv -e^2 \int_k \gamma_\alpha \frac{i}{\not{p}_2 + m - i\varepsilon} \gamma^\mu \frac{i}{\not{p}_1 + m - i\varepsilon} \gamma_\beta \cdot \frac{i\eta^{\alpha\rho}}{k^2 - i\varepsilon} \left(ie^2 \sum_{j=1}^{N_f} Q_j^2 \hat{\Pi}_{\rho\sigma}(k) \right) \frac{i\eta^{\rho\beta}}{k^2 - i\varepsilon}$$

electric charge of quarks

e.g., $Q_u = \frac{2}{3}, Q_d = -\frac{1}{3}$

Hadronic vacuum polarization:

$$\Pi_{\rho\sigma}(k) \equiv \int e^{-ik\cdot x} \langle T\, j_\rho^{\mathrm{EM}}(x)\, j_\sigma^{\mathrm{EM}}(0) \rangle_{\mathrm{QCD}}$$
$$= \left(k^2\eta_{\rho\sigma} - k_\rho k_\sigma\right) \Pi(k^2)$$

$\Pi_{\rho\sigma}(k) =$ 

Wavefunction renormalization
(residue at the massless pole = 1)

$$\hat{\Pi}_{\rho\sigma}(k) \equiv \left(k^2\eta_{\rho\sigma} - k_\rho k_\sigma\right) \underline{\{\Pi(k^2) - \Pi(0)\}}$$

($\equiv \hat{\Pi}(K^2)$)

- With analytic continuation: **T. Blum 2002**

$$a_\mu^{\mathrm{HVP,LO}} = \int dK^2 f(K^2) \left(e^2 \sum_{j=1}^{N_f} Q_j^2 \right) \hat{\Pi}(K^2)$$

$$\left(f(K^2) = \frac{e^2}{4\pi^2} \frac{K^2 Z^3 (1 - K^2 Z)}{1 + K^2 Z^2} \right)$$

- Rewrite $a_\mu^{\mathrm{HVP,LO}}$ in a convenient form with correlator for lattice calculation:

$$\left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right)\Pi(k^2) = \int d^4x\, e^{-ik\cdot x}\,\langle j_\mu(x) j_\nu(0)\rangle$$

Euclidean

$k = (\omega, \mathbf{0}), \mu = \nu \equiv z$

$$\Pi(\omega^2) = \frac{1}{\omega^2}\int dt\, e^{-i\omega t} \underbrace{\int d^3\mathbf{x}\,\langle j_z(t,\mathbf{x}) j_z(0)\rangle}_{\equiv\, C(t)}$$

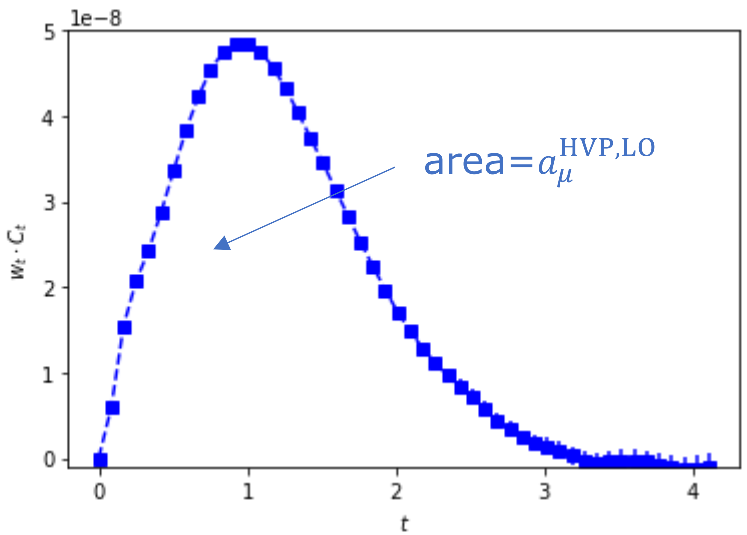
$$a_\mu^{\mathrm{HVP,LO}} = \int dK^2 f(K^2)\left(e^2\sum_{j=1}^{N_f} Q_j^2\right)\widehat{\Pi}(K^2)$$

$$a_\mu^{\mathrm{HVP,LO}} = \int_0^\infty dt\, w(t) C(t)$$

$$w(t) \equiv \left(e^2\sum_{j=1}^{N_f} Q_j^2\right)\int dK^2 f(K^2)\frac{1}{K^2}\left[K^2t^2 - 4\sin^2\left(\frac{Kt}{2}\right)\right]$$

information of loop structure

$$f(K^2) = \frac{e^2}{4\pi^2}\frac{K^2Z^3(1-K^2Z)}{1+K^2Z^2}$$



$a_\mu^{HVP,LO}$ from *Euclidean correlator*!

Isospin limit

$m_u = m_d$

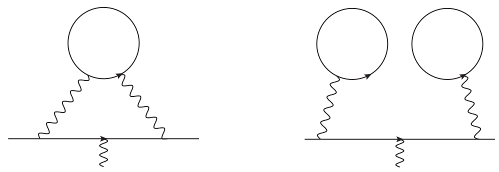
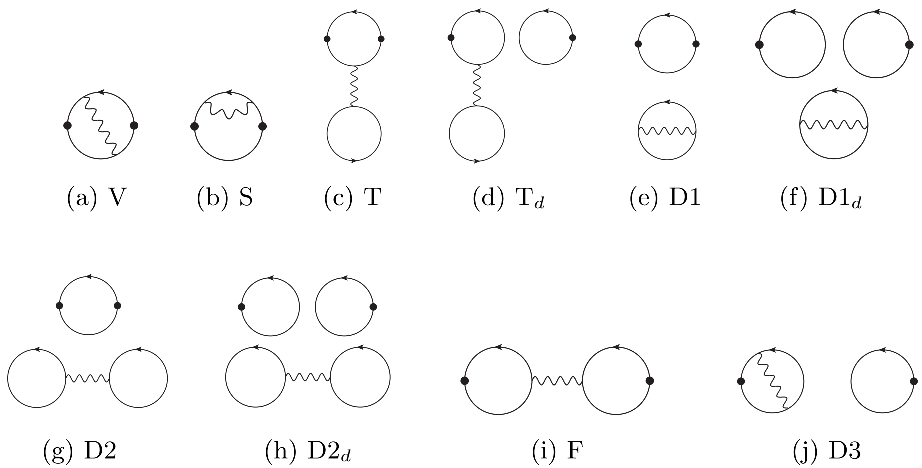


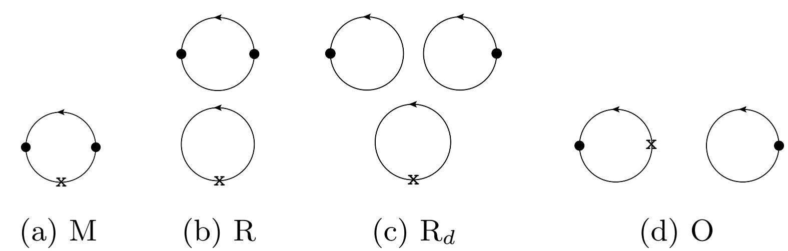
Fig: RBC/UKQCD [2301.08696]
See also RBC/UKQCD PRL [1801.07224]

QED corrections

$O(\alpha_{\text{QED}})$ relative



Strong isospin breaking



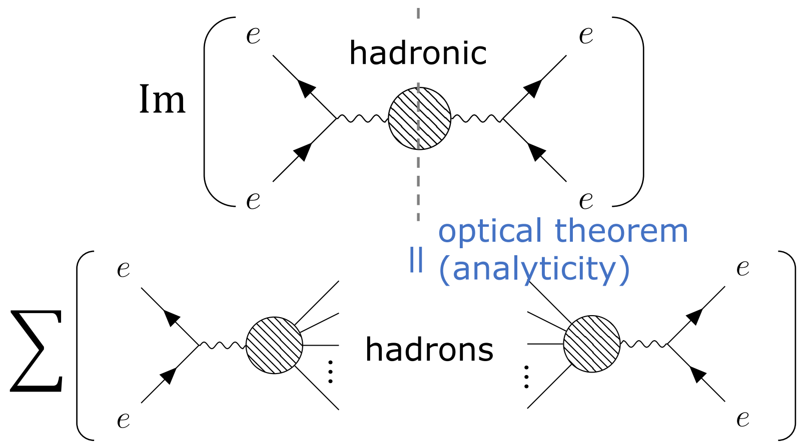
mass splitting $\Delta m \equiv m_u - m_d$ as perturbation

See, e.g., Jehgerlener et al. 0902.3360

- Cutting the hadronic bubble:

$$\hat{\Pi}(k^2) = \Pi(k^2) - \Pi(0) \overset{\text{analyticity}}{=} \frac{k^2}{\pi}\int_0^\infty ds \frac{\operatorname{Im}\Pi(s)}{s(s-k^2-i\varepsilon)}$$

$$\operatorname{Im}\Pi(s) \overset{\text{optical theorem}}{=} \frac{s}{4\pi\alpha}\sigma(e^+e^- \to \text{hadrons})$$



$a_\mu^{\text{HVP,LO}}$ can be related to e^+e^- cross sections since:

$$a_\mu^{\text{HVP,LO}} = \int dK^2 f(K^2)\left(e^2\sum_{j=1}^{N_f} Q_j^2\right)\hat{\Pi}(K^2)$$

- Arranging in a convenient form:

optical theorem

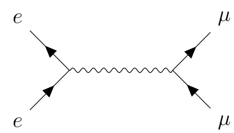
$$\mathrm{Im}\,\Pi(s) = \frac{s}{4\pi\alpha}\sigma(e^+e^- \to \text{hadrons}) \equiv \frac{\alpha}{3}R(s)$$

R-ratio

$$R(s) \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma_0}$$

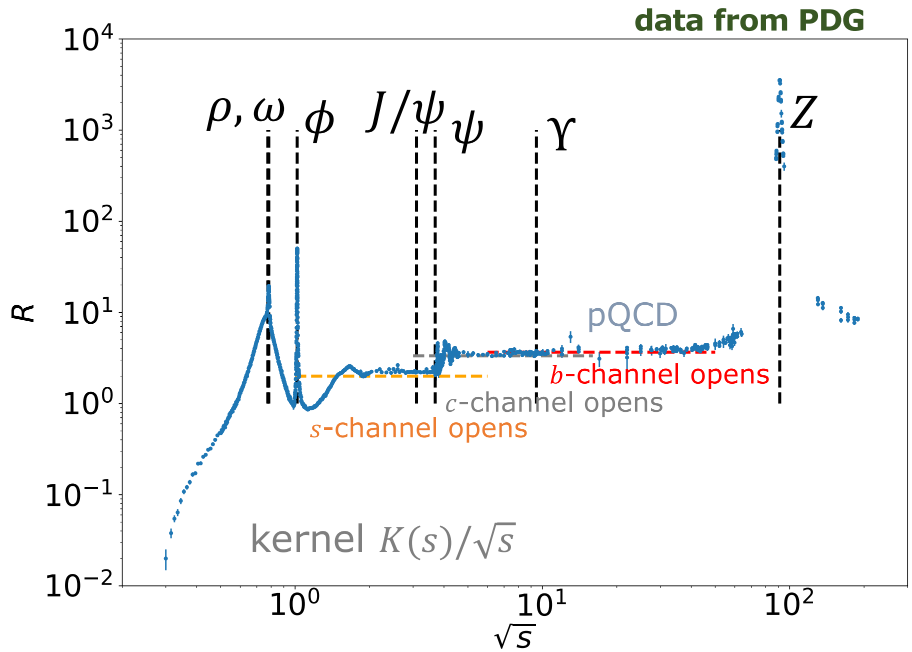
$$\left(\sigma_0 \equiv \frac{4\pi\alpha^2}{3s} = \sigma\begin{pmatrix} e^+e^- \to \mu^+\mu^- \\ \text{LO, massless limit} \end{pmatrix} \right)$$

(kinematic normalization factor)



$R(s)$ roughly counts #particles created in e^+e^- scattering (weighted by Q^2)

∴ Naive quark model: $R(s) \simeq 3 \sum_{2m_q<\sqrt{s}} Q_q^2$



- At the end: **Lautrup, de Rafael 1968**

$$a_\mu^{\text{HVP,LO}} = \frac{\alpha^2}{3\pi^2}\int_{m_\pi^2}^{\infty} ds\, \frac{K(s)}{s} R(s)$$

$$\left(K(s) \equiv \int_0^1 dy\, \frac{y^2(1-y)}{y^2 + (1-y)(s/m_\mu^2)} \right)$$

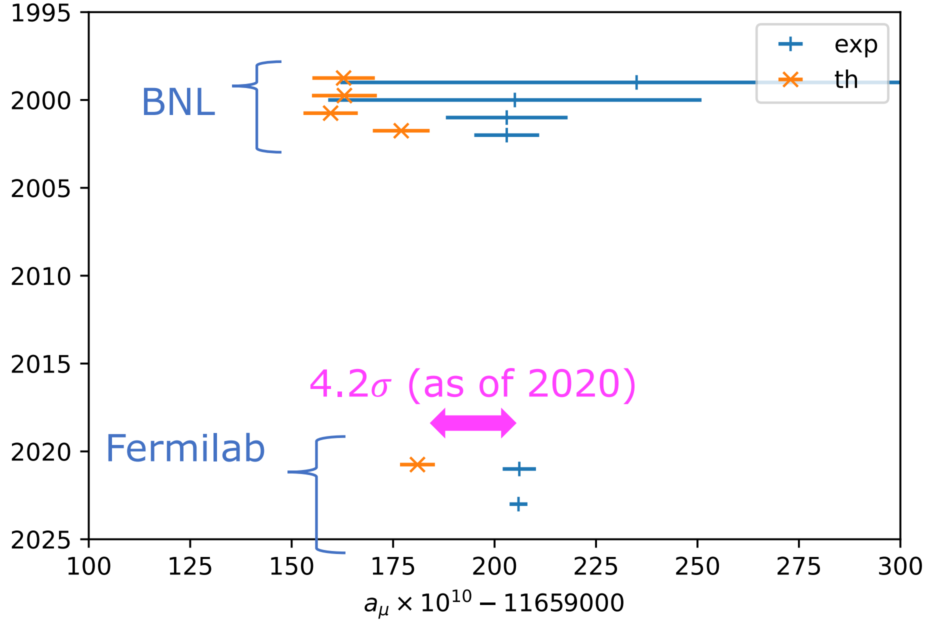
	Ref. [21]	Ref. [22]	Difference
$\pi^+\pi^-$	507.85(0.83)(3.23)(0.55)	504.23(1.90)	3.62
$\pi^+\pi^-\pi^0$	46.21(0.40)(1.10)(0.86)	46.63(94)	−0.42
$\pi^+\pi^-\pi^+\pi^-$	13.68(0.03)(0.27)(0.14)	13.99(19)	−0.31
$\pi^+\pi^-\pi^0\pi^0$	18.03(0.06)(0.48)(0.26)	18.15(74)	−0.12
K^+K^-	23.08(0.20)(0.33)(0.21)	23.00(22)	0.08
K_SK_L	12.82(0.06)(0.18)(0.15)	13.04(19)	−0.22
$\pi^0\gamma$	4.41(0.06)(0.04)(0.07)	4.58(10)	−0.17
Sum of the above	626.08(0.95)(3.48)(1.47)	623.62(2.27)	2.46
[1.8, 3.7] GeV (without $c\bar{c}$)	33.45(71)	34.45(56)	−1.00
J/ψ, $\psi(2S)$	7.76(12)	7.84(19)	−0.08
[3.7, ∞) GeV	17.15(31)	16.95(19)	0.20
Total $a_\mu^{\text{HVP, LO}}$	694.0(1.0)(3.5)(1.6)(0.1)$_\psi$(0.7)$_{\text{DV+QCD}}$	692.8(2.4)	1.2

Table 2: Comparison of selected exclusive-mode contributions to $a_\mu^{\text{HVP, LO}}$ from Refs. [21, 22], for the energy range ≤ 1.8 GeV, in units of 10^{-10}, see Ref. [6] for details.

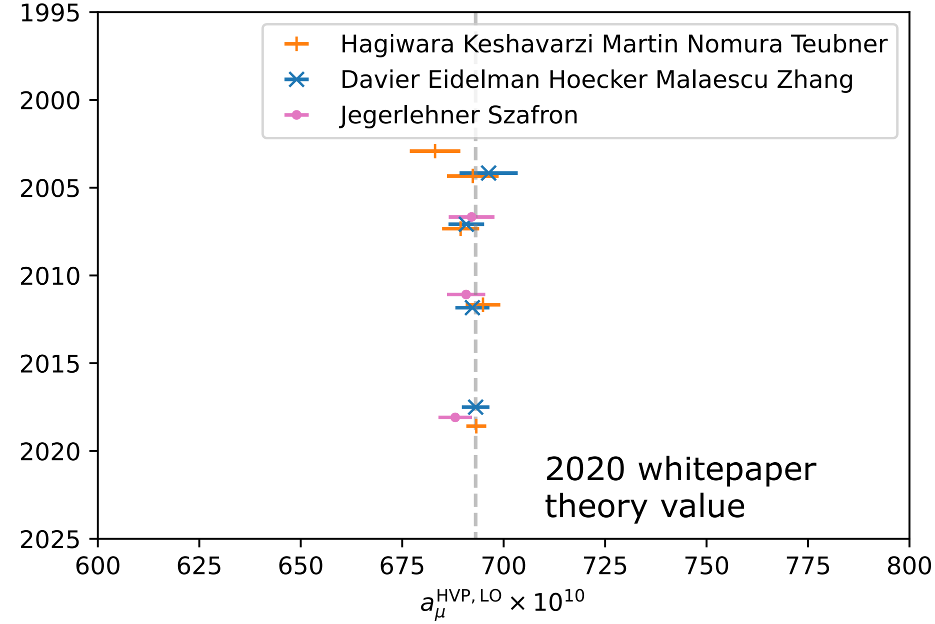
Snowmass 2021 [2203.15810]

- R-ratio has been used for the theory value of HVP historically:

a_μ (same as Introduction)



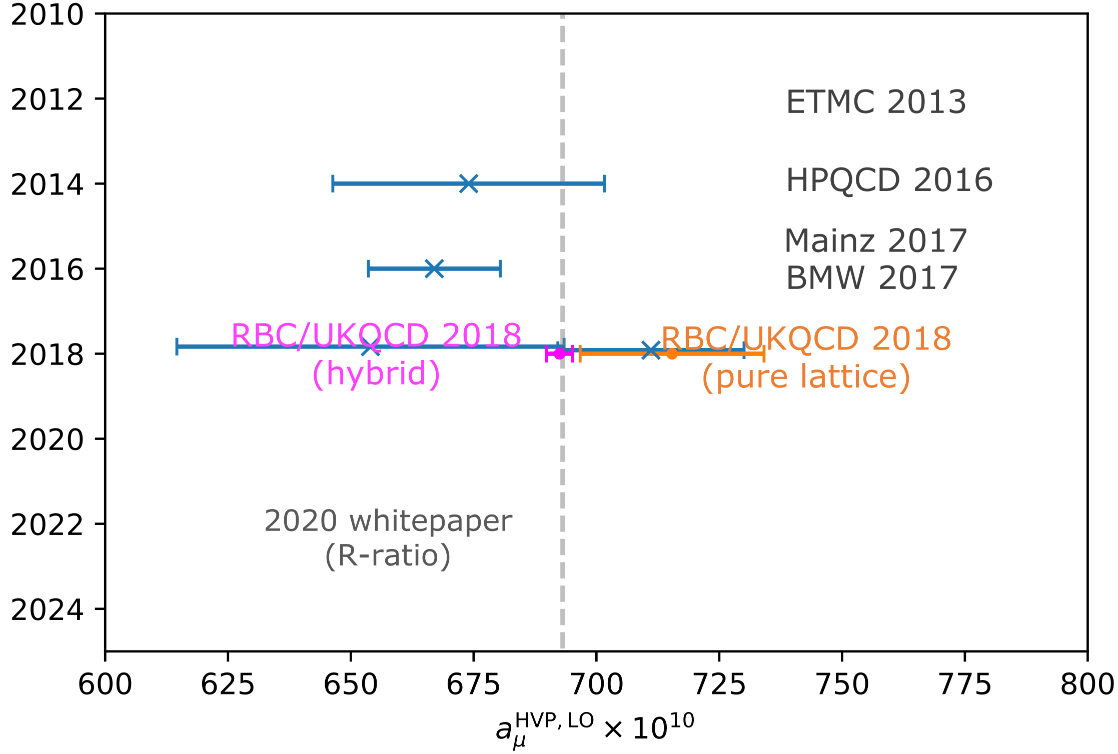
$a_\mu^{\text{HVP LO}}$ from R-ratio

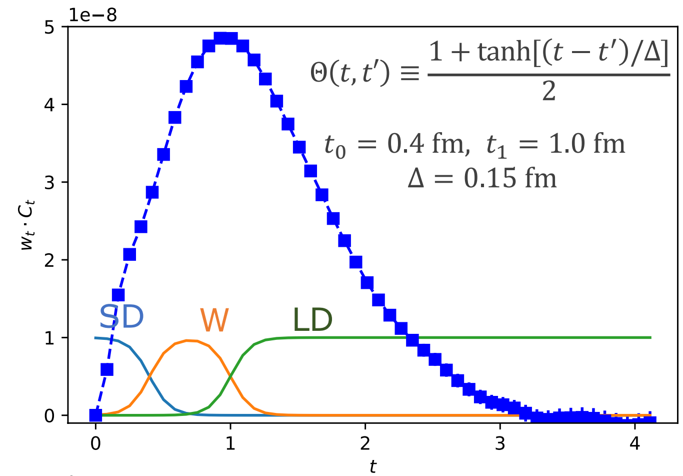


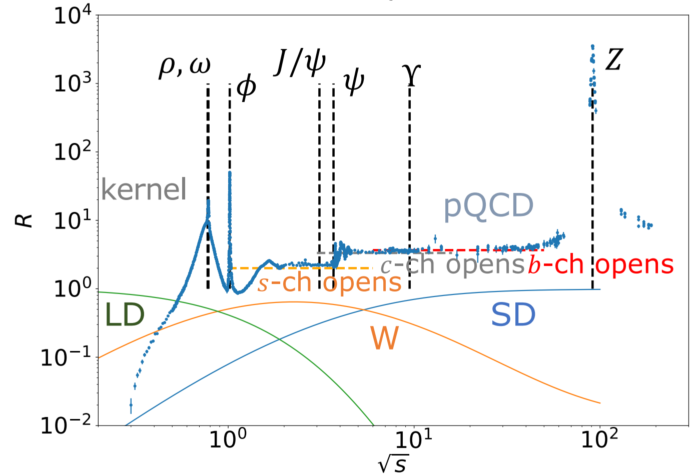
What about Lattice?

For details, e.g.,
Hagiwara, Liao, Martin, Nomura 1105.3149
Keshavarzi, Nomura, Teubner 1802.02995

 RBC/UKQCD PRL [1801.07224]







- Separation of scales:

Bernecker-Meyer 1107.4388
Lehner 1710.06874

$$a_\mu^{\text{HVP,LO}} = \int_0^\infty dt\, w(t) C(t)$$

$$= \int_0^\infty dt\, [1-\Theta(t,t_0)]\, w(t)C(t) \quad + \int_0^\infty dt\, [\Theta(t,t_0)-\Theta(t,t_1)]\, w(t)C(t) \quad + \int_0^\infty dt\, \Theta(t,t_1)\, w(t)C(t)$$

"short distance (SD)"
cutoff effects

"window (W)"
good region for lattice

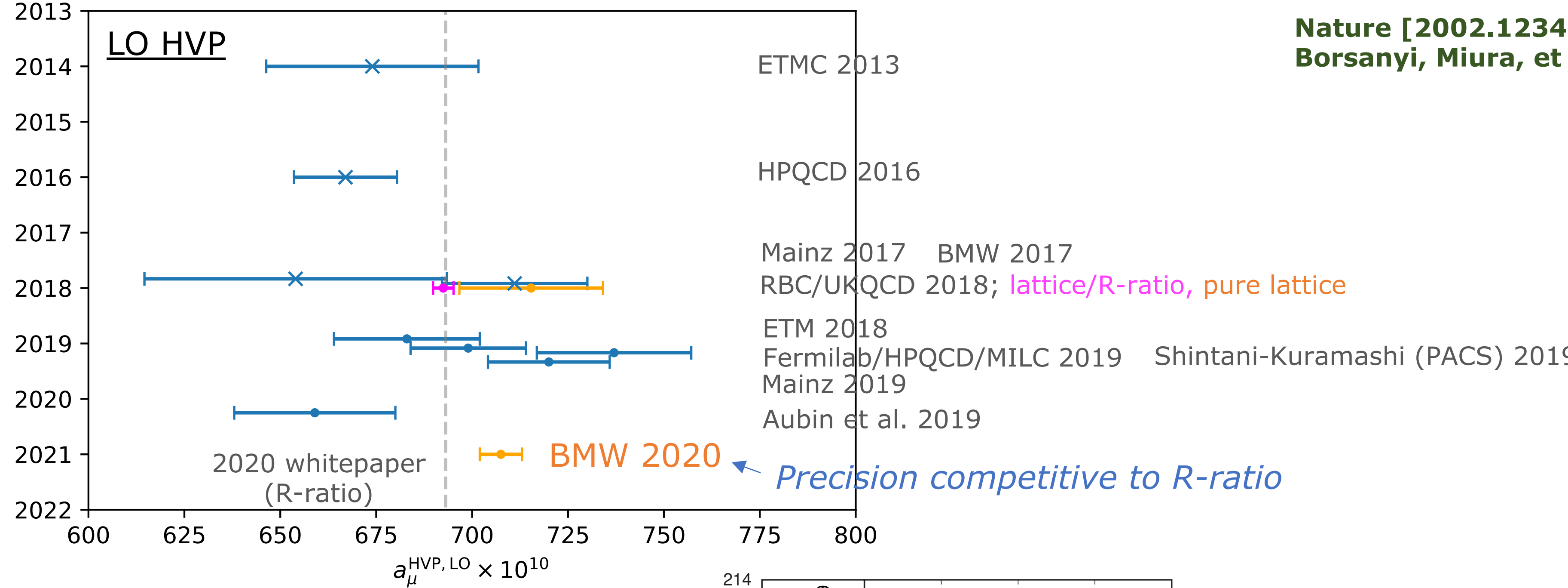
"long distance (LD)"
finite volume effect
& statistical error

- Pick the best parts of Lattice and R-ratio w/ the formula: $C(t) = \frac{1}{12\pi^2}\int_0^\infty d\sqrt{s}\ R(s) s\, e^{-\sqrt{s}t}$

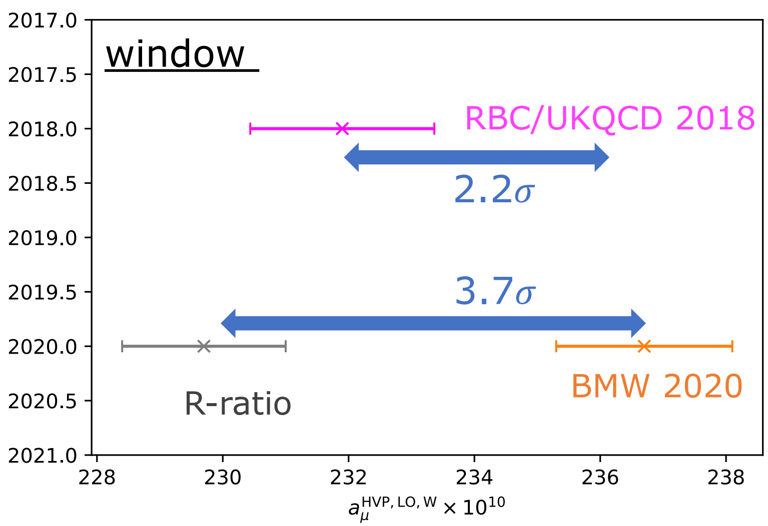
→ improved estimate

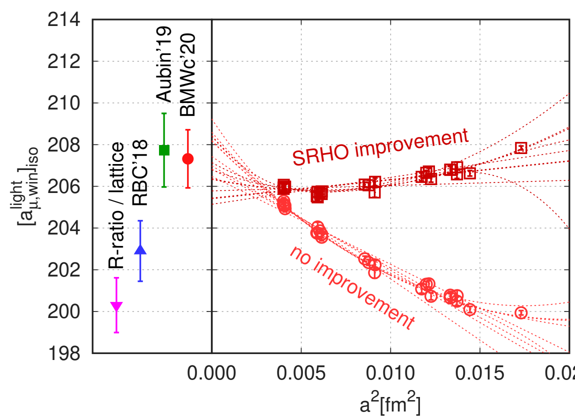
Bernecker-Meyer 1107.4388

Nature [2002.12347]
Borsanyi, Miura, et al.



Precision competitive to R-ratio

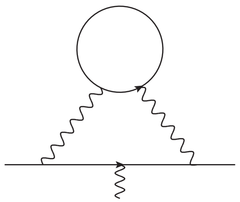




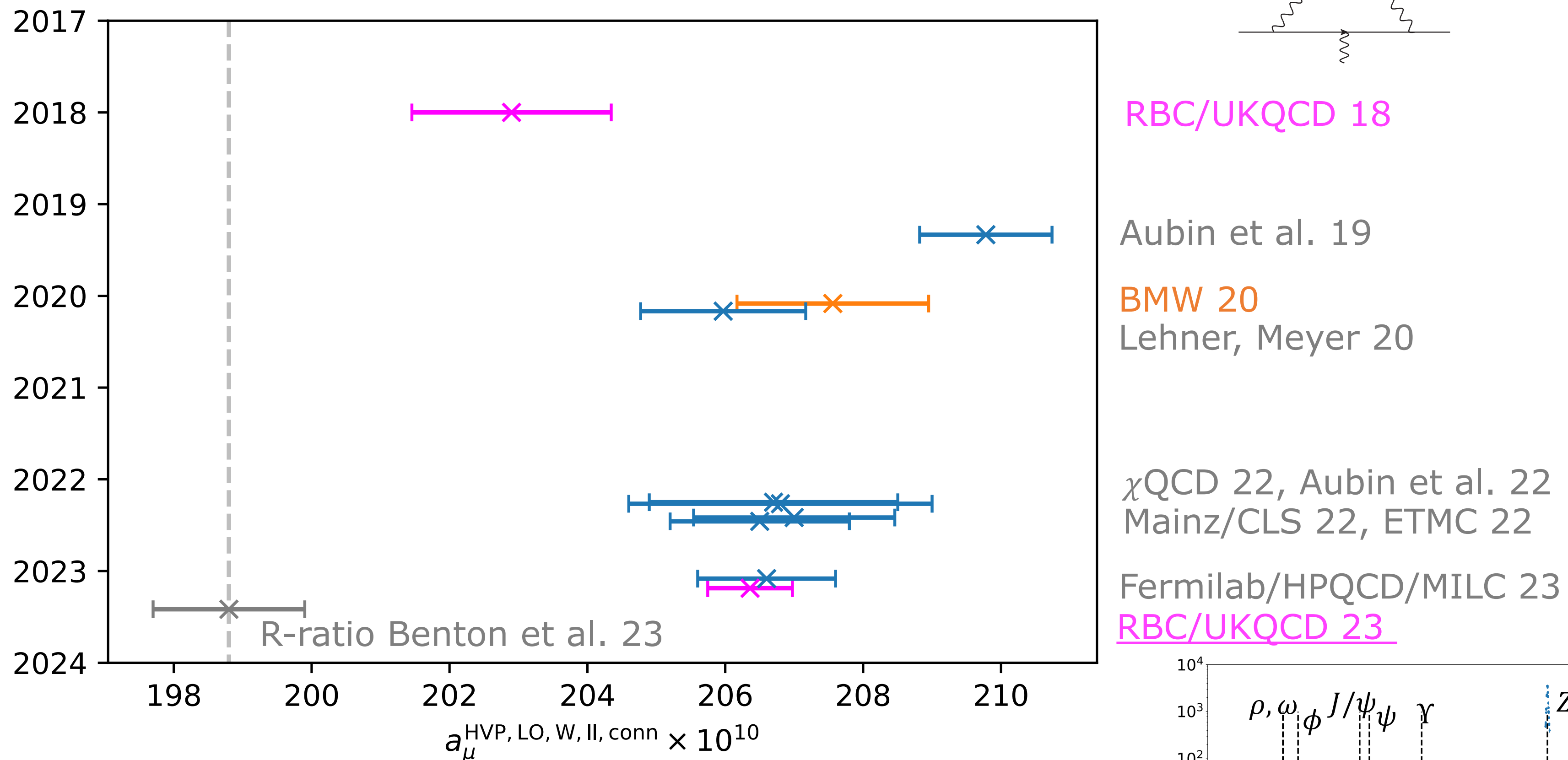
- Many ensembles w/ staggered quark (small cost)
- Systematic improvement w/ staggered eff theory
- Fit ansatze averaged w/ Akaike information criterion

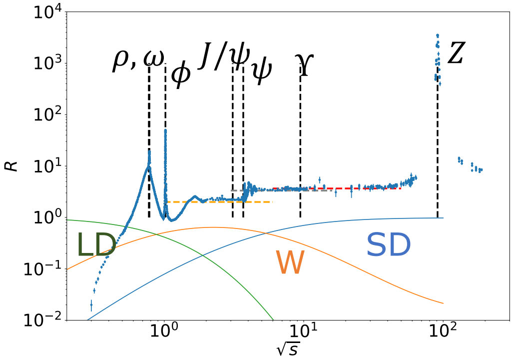
Akaike 1974

Lattice collaborations started to perform cross checks first on the window value to get a consensus.



- Restricting ourselves to the light quark connected component:





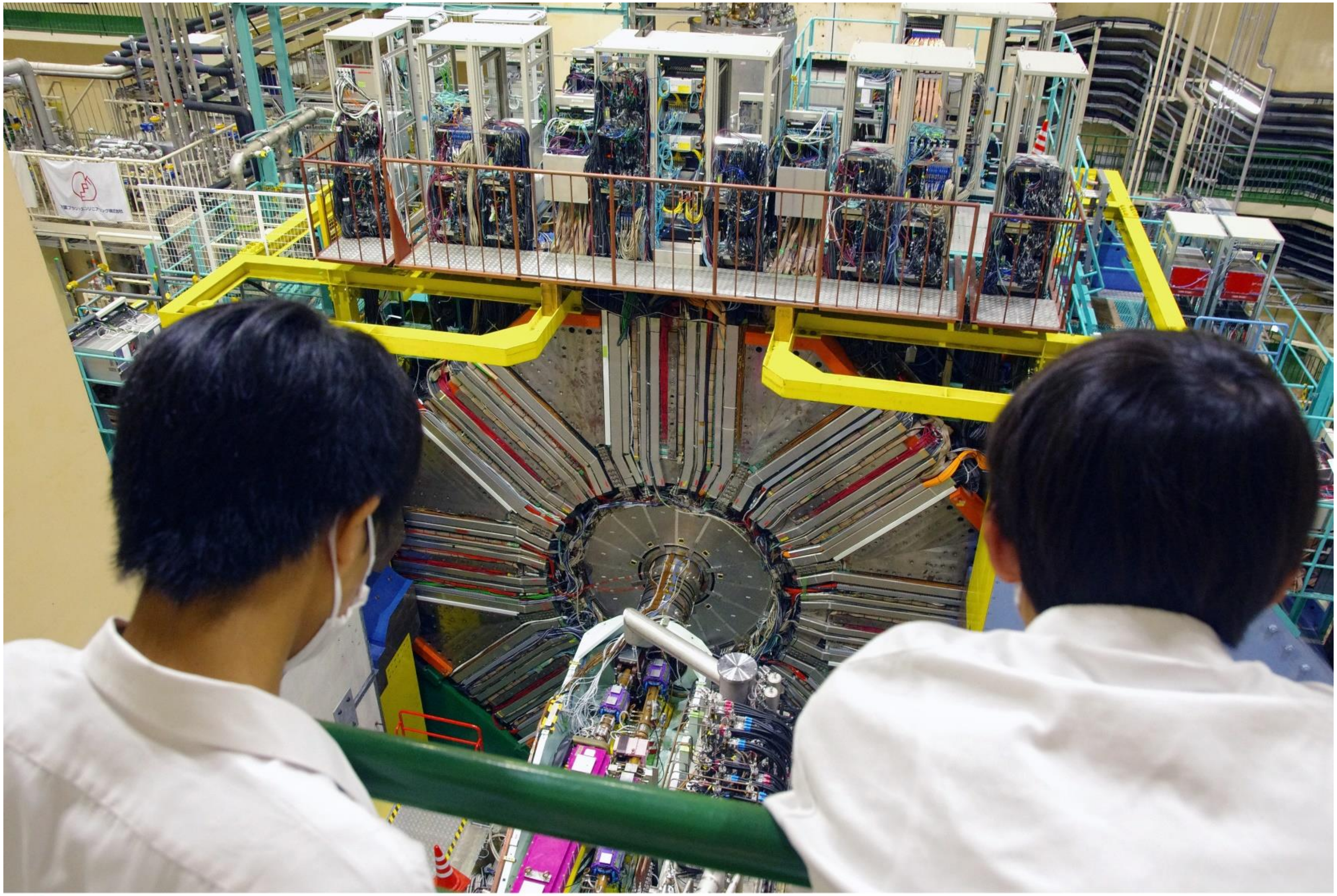
- Most recent lattice results agree for the intermediate window value.

- A detail R-ratio study **Benton et al. 2306.16808** shows that:

  "the discrepancy between data-driven (=R-ratio) and lattice-QCD results for $a_\mu^{\mathrm{HVP,LO\ W}}$
  is almost entirely due to the light-quark connected contribution,
  which, in turn, is strongly dominated by the 2π *channel …"*

See also: Tuesday AM

New analysis coming up for:
$$\begin{cases} e^+e^- \to \pi^+\pi^-\pi^0 \\ e^+e^- \to \pi^+\pi^-\gamma \end{cases}$$



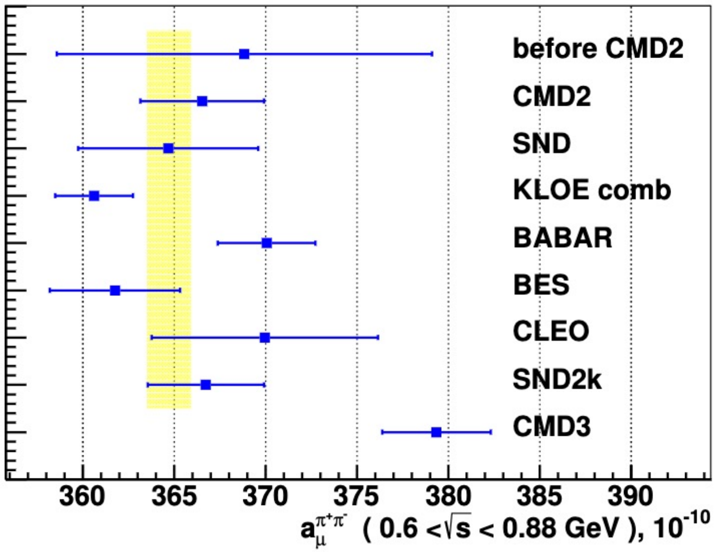
KEK Belle II

Visitors looking at the Belle II detector with the endyokes open

confluence.desy.de/display/BI/Official+Wall+Calendar

R-ratio

- CMD-3 **2302.08834**



cf. pion form factor:

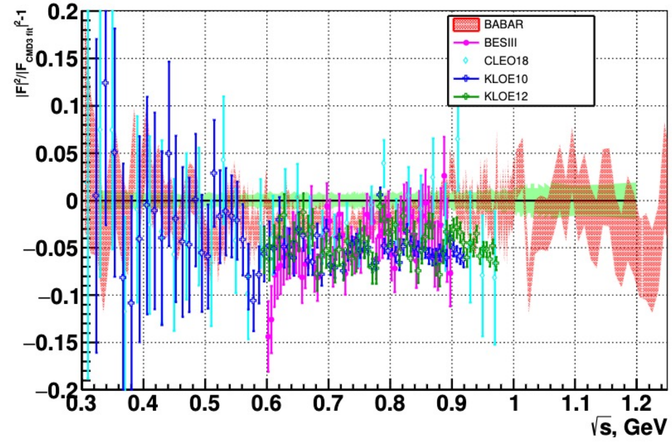
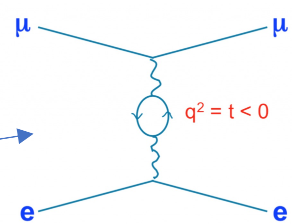


Figure 34: The relative differences of the pion form factors obtained in the ISR experiments (BABAR, BESIII, CLEO, KLOE) and the CMD-3 fit result.

- Belle II, BES III **PRL 2112.11728**

Full a_μ

- Fermilab E989
  - All 6 runs complete
  - Run 1 & 2,3 analyzed **2104.03281, 2308.06230**
- J-PARC E34 g-2/EDM
- MUonE @ CERN: $\mu e \to \mu e$ elastic **PRL 2309.14205**



https://web.infn.it/MUonE/

τ studies on isospin breaking corrections

- Lattice: **E.g., M. Bruno, Izubuchi, Lehner, Meyer 1811.00508**
- From e^+e^-: **Jegerlehner-Szafron 1101.2872**

R-ratio study also important in this regard

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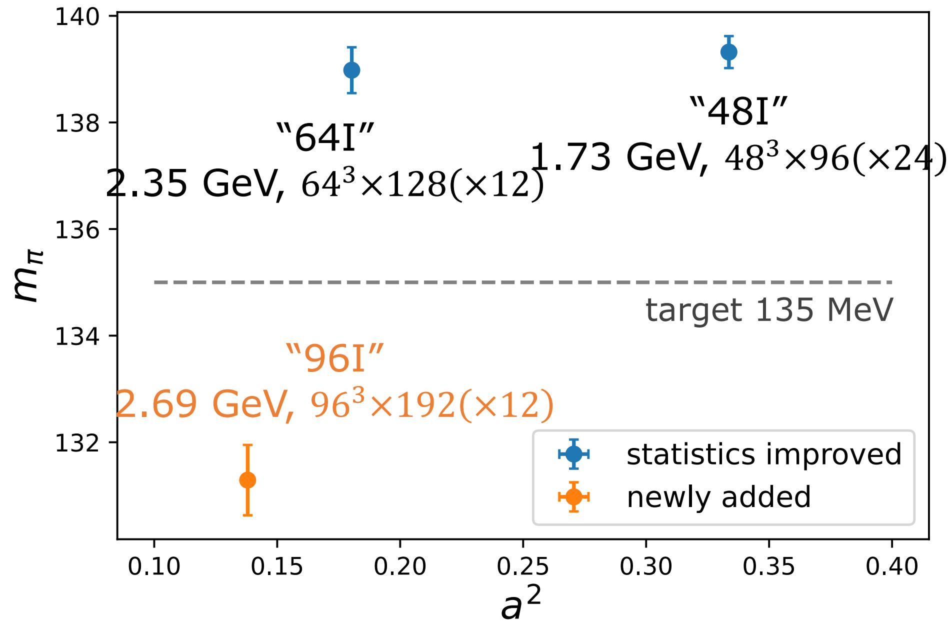
PRD 108 no.5, 054507 (2023) [RBC/UKQCD; 2301.08696]

clean but expensive

- (2+1)-flavor ensembles w/ Iwasaki gauge action & Mobius Domain wall fermion

  perturbatively improved — **Iwasaki 1985**

  chirality controlled by the auxiliary 5th dimension — **Shamir hep-lat/9303005**, **Furman, Shamir hep-lat/9405004**, **Brower, Neff, Orginos 1206.5214**



"ensemble name"
a^{-1}GeV, $L^3\times L_t(\times L_s)$

9 more supplementary ensembles
w/ fewer statistics for systematic corrections.

- Blind analysis w/ 5 groups:
  - Correlator data $C(t)$ distributed to each group w/ the blinding factor multiplied:

$$C_{\text{blind}}(t) = (b_0 + b_1 a^2 + b_2 a^4) C_{\text{orig}}(t)$$

  - Make estimates independently in each group developing their own methodology.
  - Perform relative unblinding when the groups become confident on their value. When a discrepancy arises, its source is studied until understood.
  - Final result given by the best method agreed among all groups.

 PRD 108 no.5, 054507 (2023) [RBC/UKQCD; 2301.08696]

HVP analysis

Regensburg:	D. Giusti, C. Lehner
Edinburg:	V. Gulpers, R.C. Hill
CERN:	A. Jüttner, J.T. Tsang
Millan:	M. Bruno
Connecticut:	T. Blum, L. Jin
Columbia:	Y.-C. Jang, R.D. Mawhinney
Berkeley:	A.S. Meyer
BNL:	P.A. Boyle, T. Izubuchi, C. Jung, C. Kelly, N. Matsumoto

(17 people, 5 groups)

Global fit

Group 1: Y.-C. Jung, N. Christ, B. Mawhinney (Columbia), C. Kelly (BNL)
Group 2: C. Lehner (Regensburg)

Cf. target isospin symmetric theories:

"RBC/UKQCD 18 world"

- $m_\pi = 0.135$ GeV
- $m_K = 0.4957$ GeV
- $m_\Omega = 1.67225$ GeV
- $m_{D_s} = 1.96847$ GeV

sea-charm correction studied

⟺ $m_{u,d}, m_s, a, m_c$

"BMW 20 world"

- $m_\pi = 0.13497$ GeV
- $m_{ss*} = 0.6898$ GeV
- $w_0 = 0.17236$ fm
- $m_{D_s} = 1.96847$ GeV

w_0: Wilson flow scale

BMW 1203.4469
cf. Lüscher 1006.4518

- Resources from:
  USQCD, HPCI, XSEDE, ANL BG/Q Mira (DOE, ALCC), Edinburgh BG/Q, JUWELS, Crasher (DOE), BNL BG/Q, RIKEN BG/Q and Cluster (RICC, HOKUSAI)
- Support from: RIKEN, JSPS, US DOE, BNL, DFG, Italy MUR, EU MSCA, UK STFC

The RBC & UKQCD collaborations

RBC=RIKEN-BNL-Columbia

University of Bern & Lund
Dan Hoying

BNL and BNL/RBRC
Peter Boyle (Edinburgh)
Taku Izubuchi
Yong-Chull Jang
Chulwoo Jung
Christopher Kelly
Meifeng Lin
Nobuyuki Matsumoto
Shigemi Ohta (KEK)
Amarjit Soni
Raza Sufian
Tianle Wang

CERN
Andreas Jüttner (Southampton)
Tobias Tsang

Columbia University
Norman Christ
Sarah Fields
Ceran Hu
Yikai Huo
Joseph Karpie (JLab)
Erik Lundstrum
Bob Mawhinney
Bigeng Wang (Kentucky)

University of Connecticut
Tom Blum
Luchang Jin (RBRC)
Douglas Stewart
Joshua Swaim
Masaaki Tomii

Edinburgh University
Matteo Di Carlo
Luigi Del Debbio
Felix Erben
Vera Gülpers
Maxwell T. Hansen
Tim Harris
Ryan Hill
Raoul Hodgson
Nelson Lachini
Zi Yan Li
Michael Marshall
Fionn Ó hÓgáin
Antonin Portelli
James Richings
Azusa Yamaguchi
Andrew Z.N. Yong

Liverpool Hope/Uni. of Liverpool
Nicolas Garron

LLNL
Aaron Meyer

University of Milano Bicocca
Mattia Bruno

Nara Women's University
Hiroshi Ohki

Peking University
Xu Feng

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Davide Giusti
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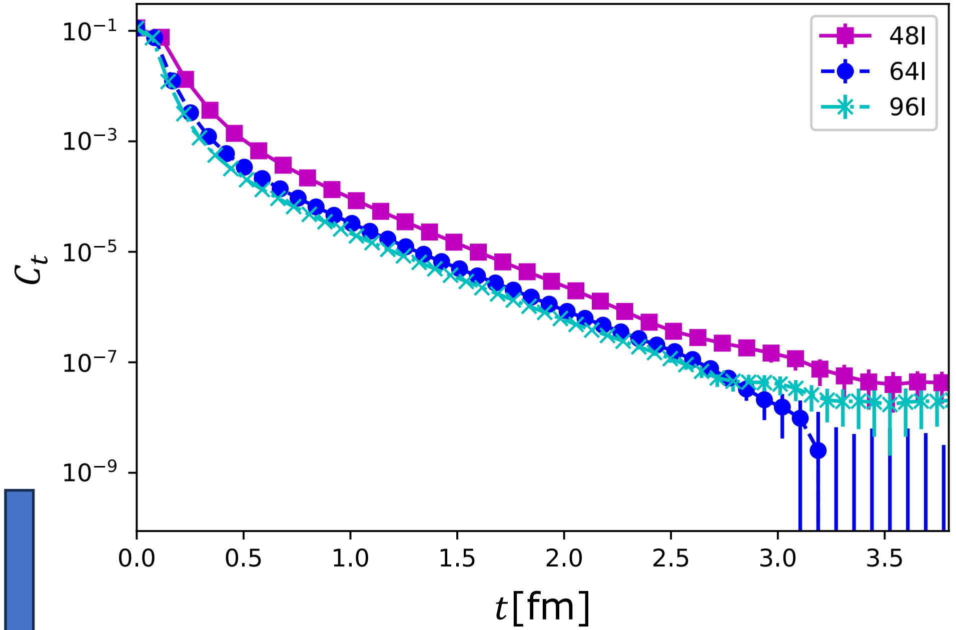
local-local correlator (blinded)

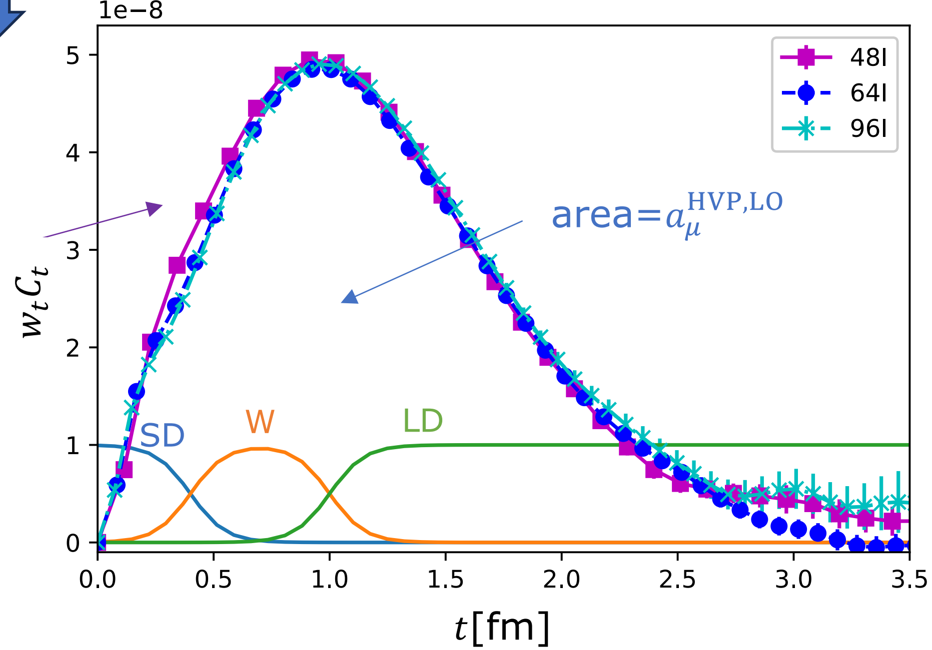


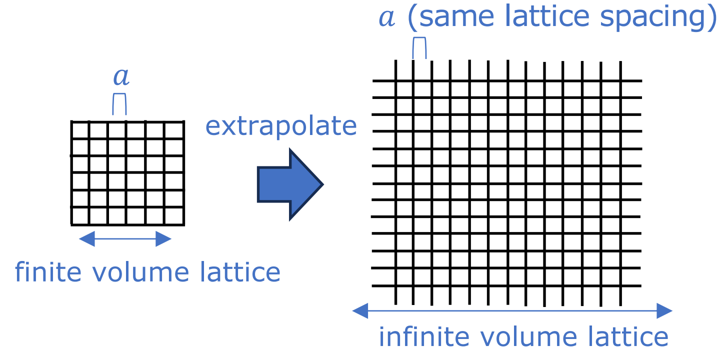
$$a_\mu^{\mathrm{HVP,LO}} = \int_0^\infty dt\, w(t) C(t)$$

$$\approx a_{\mathrm{lat}} \sum_{t\geq 0} w_t \cdot C_t$$

Even with simple Riemann sum, disc error = $O(a_{\mathrm{lat}}^2)$ thanks to: $w(0) = w(\infty) = 0$

Relatively large discretization effect for 48I; affects the window value





Two equivalent points of view:

- Discrete momentum (momentum view)

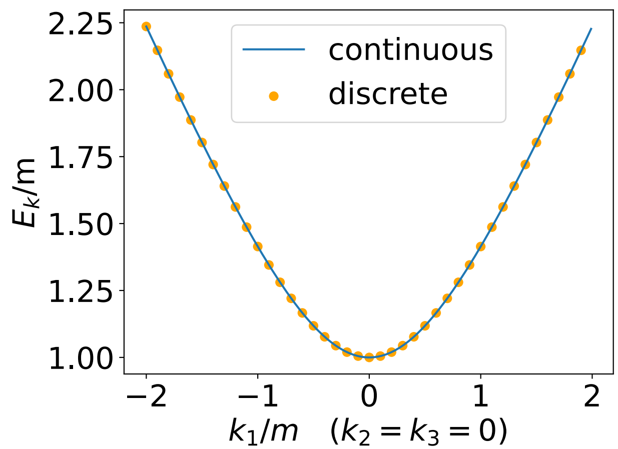
  For the free 2π case:

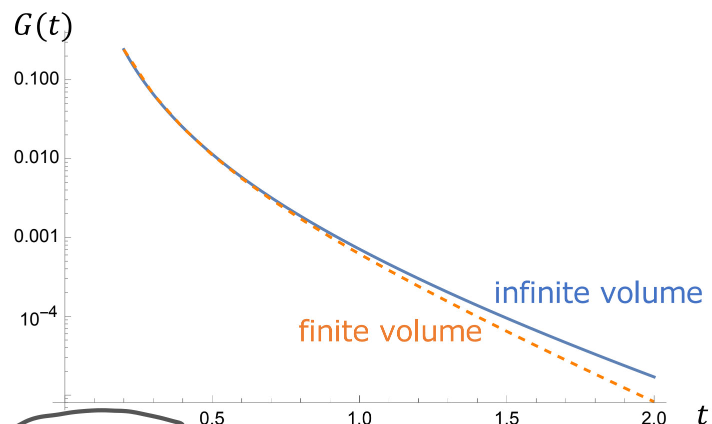
$$G(t)\delta_{ij}\delta^{ab} \equiv \int d\mathbf{x}\,\langle j_i^a(t,\mathbf{x}) j_k^b(0)\rangle$$

chiPT ⇒ $G_{\rm cont}(t) = \frac{1}{6}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\mathbf{k}^2 e^{-2E_\mathbf{k}t}\frac{1}{E_\mathbf{k}^2}$, $\quad G_{\rm disc}(t) = \frac{1}{6}\frac{1}{L^3}\sum_\mathbf{k} \mathbf{k}^2 e^{-2E_\mathbf{k}t}\frac{1}{E_\mathbf{k}^2}$

$$\left(= \frac{m^3}{12\pi^2}\sum_\mathbf{n}\int_0^\infty dr\,\frac{r^4}{1+r^2}\,\frac{\sin(mL|\mathbf{n}|r)}{mL|\mathbf{n}|r}\,e^{-2m\sqrt{1+r^2}\,t} \right)$$

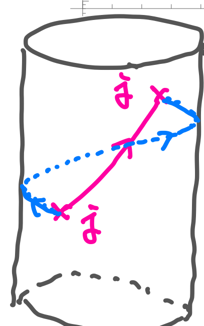
Francis-Jäger-Meyer-Wittig 1306.2532





- Wraparound effects (spatial view)

$$G_{\rm disc}(t) = G_{\rm cont}(t) + O(e^{-m_\pi L})$$



1. Meyer-Lellouch-Luescher-Gounaris-Sakurai model (momentum view)

dynamical hadronic information highlighted

Finite volume (FV)

Correlator from *discretized* $\pi\pi$ *spectrum*:

$$C_{\rm FV}(t) = \sum_n |A_n|^2 e^{-E_n t}$$

- Energy level $\sqrt{s} = E_n$ satisfies:

$$\delta(k) + \phi\left(\frac{kL}{2\pi}\right) = n\pi \quad (n \in \mathbb{Z})$$

Luescher 1991

Phase shifts in QM

Infinite volume (IV)

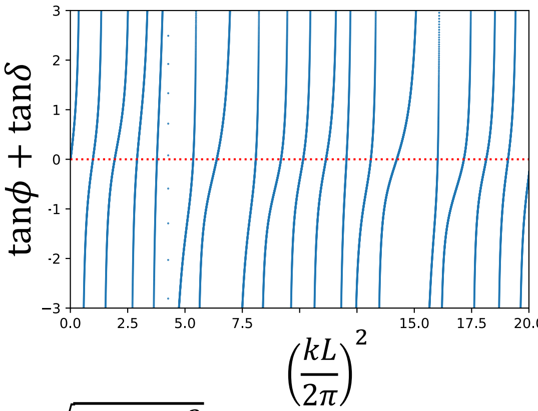
Correlator written w/ $R(s)$: **Bernecker, Meyer 1107.4388**

$$C_{\rm IV}(t) = \frac{1}{12\pi^2}\int_0^\infty d\sqrt{s}\; R(s) s\, e^{-\sqrt{s}t}$$

- Use the relation between F_π and $R(s)$:

$$R(s) = \frac{1}{4}\left(1 - \frac{4m_\pi^2}{s}\right)^{3/2} \cdot |F_\pi(s)|^2$$

$$|F_\pi(s)|^2 = \left[\frac{kL}{2\pi}\cdot\phi'\left(\frac{kL}{2\pi}\right) + k\delta'(k)\right]\frac{3\pi s}{2k^5}|A_n|^2$$



$k(s) \equiv \sqrt{s/4 - m_\pi^2}$

h: func of s

d: constant of m_ρ, k_ρ

$\phi(q) \equiv \frac{\pi^{3/2} q}{\mathcal{Z}_{00}(1;q^2)}$: kinematic function

$\mathcal{Z}_{00}(1;q^2)$: Luescher's zeta function

Lellouch-Luescher hep-lat/0003023

Gounaris-Sakurai model (based on vector meson dominance) **Gournaris, Sakurai 1968**

pion form factor

$$F_\pi(s) \approx \frac{m_\rho^2 + d\cdot m_\rho\Gamma_\rho}{(m_\rho^2 - s) + \Gamma_\rho\cdot(m_\rho^2/k_\rho^2)\{k^2[h(s) - h_\rho] + k_\rho^2 h'_\rho(m_\rho^2 - s)\}} - im_\rho\Gamma_\rho\left(\frac{k}{k_\rho}\right)^3\frac{m_\rho}{\sqrt{s}}$$

phase shift

$$\frac{k^3}{\sqrt{s}}\cot\delta \approx k^2 h(s) - k_\rho^2 h'(m_\rho^2) + 2bk_\rho k'_\rho$$

See also Chew, Mandelstam 1960

2. LO pion wraparound correction (spatial view)

$$\Delta C_t \approx A \cdot e^{-m_\pi L}$$

(L: spatial extent of the lattice)

↖ determined from the supplementary ensembles

3. Hansen-Patella formula (spatial view) **Hansen, Patella 2004.03935**

interacting pion effective theory

$$\Delta C_t \approx \sum_{\mathbf{n}\neq 0} \frac{1}{6\pi|\mathbf{n}|L} \mathrm{Im} \int_{\mathbb{R}+i\mu} \frac{dk_3}{2\pi} e^{ik_3|t|}(4m_\pi^2 + k_3^2) F_\pi(k_3^2) \times \left\{ \frac{e^{-|\mathbf{n}|L\sqrt{m_\pi^2+k_3^2/4}}}{4k_3} - i \int \frac{dp_3}{2\pi} \frac{e^{-|\mathbf{n}|L\sqrt{m_\pi^2+p_3^2}}}{k_3^2 - 4p_3^2} \right\}$$

pole part of the Compton scattering amplitude

($0 < \mu < 2m_\pi$)

$$+ \int \frac{dk_3}{2\pi} \cos(k_3 t)\, O(1/k_3^2) + O\left(e^{-\sqrt{2+\sqrt{3}}\, m_\pi t}\right)$$

regular part — higher-order exponentials → ignored

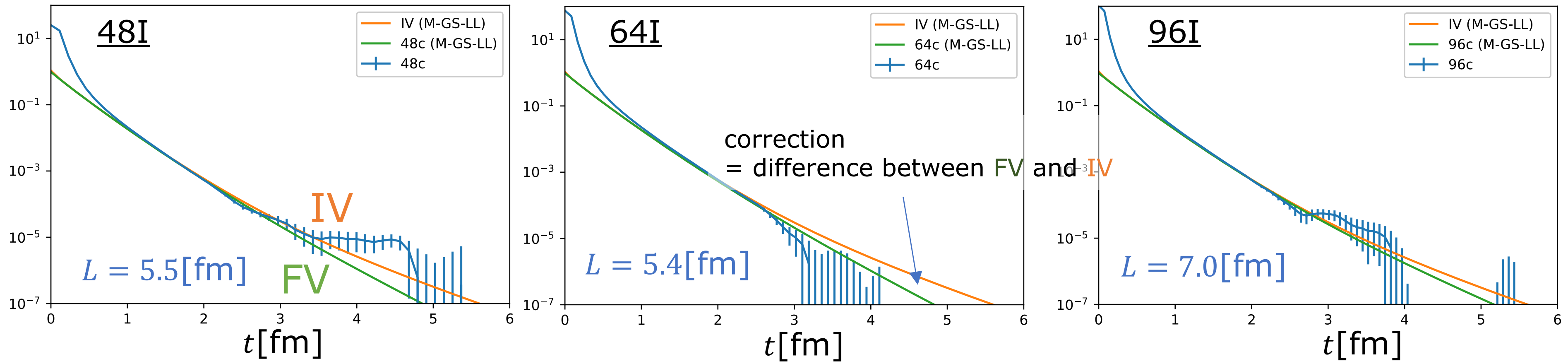
monopole model for F_π: $F_\pi(k^2) \approx \frac{1}{1+k^2/M^2}$ ($M \simeq 727$ MeV)

Phenomenologically model good for spacelike $k^2 (> 0)$
Brömmel, Nakamura, et al. [QCDSF/UKQCD] hep-lat 0608021

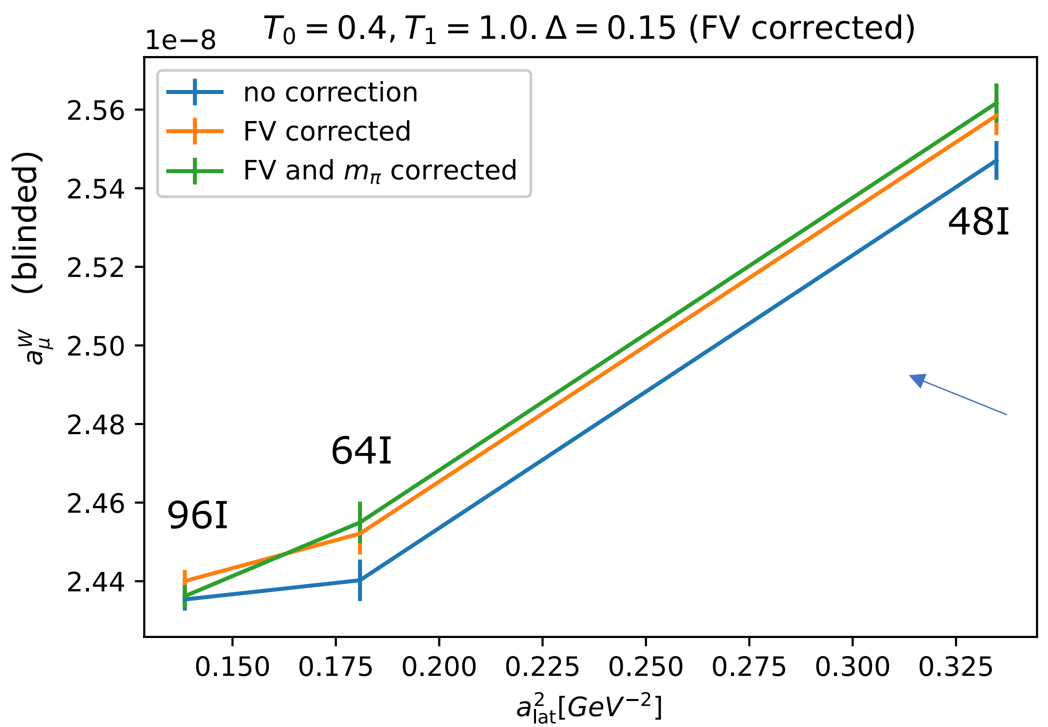
$$\approx \sum_{\mathbf{n}\neq 0} \frac{1}{6\pi|\mathbf{n}|L} \left\{ \mathrm{Im} \int_{\mathbb{R}+i\mu} \frac{dk_3}{2\pi} e^{ik_3|t|}(4m_\pi^2 + k_3^2) \frac{M^4}{(M^2+k_3^2)^2} \frac{e^{-|\mathbf{n}|L\sqrt{m_\pi^2+k_3^2/4}}}{4k_3} + \int \frac{dp_3}{2\pi} e^{-|\mathbf{n}|L\sqrt{m_\pi^2+p_3^2}} \frac{d}{dz}\left[\frac{e^{-z|t|}(z^2-4m_\pi^2)M^4}{(z+M^2)(z^2+4p_3^2)}\right]_{z=M} \right\}$$

Consistency among the models checked and confirmed

- Comparison of IV/FV correlators [w/ Meyer-Lellouch-Luescher-Gounaris-Sakurai]

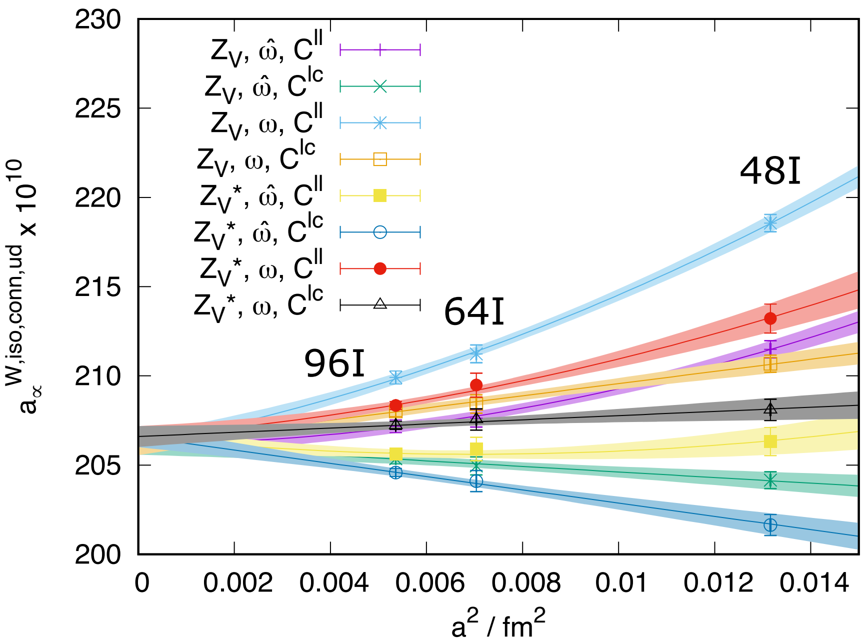


FV & m_π corrections on $a_\mu^{\text{HVP LO}}$ [w/ LO pion wraparound (data driven)]

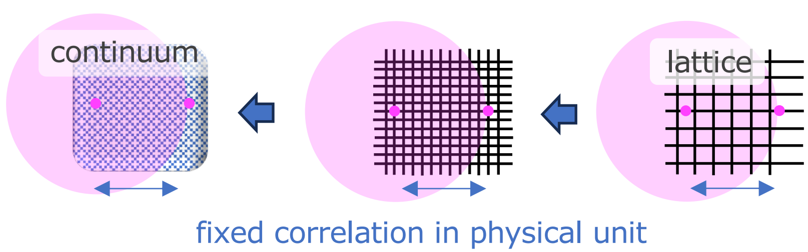


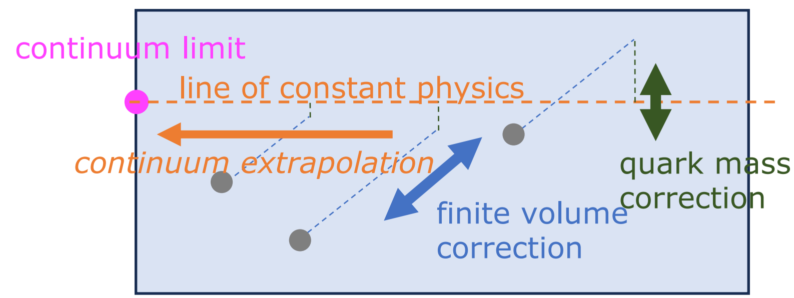
Better a^2 behavior after corrections

PRD 108 no.5, 054507 (2023) [RBC/UKQCD; 2301.08696]



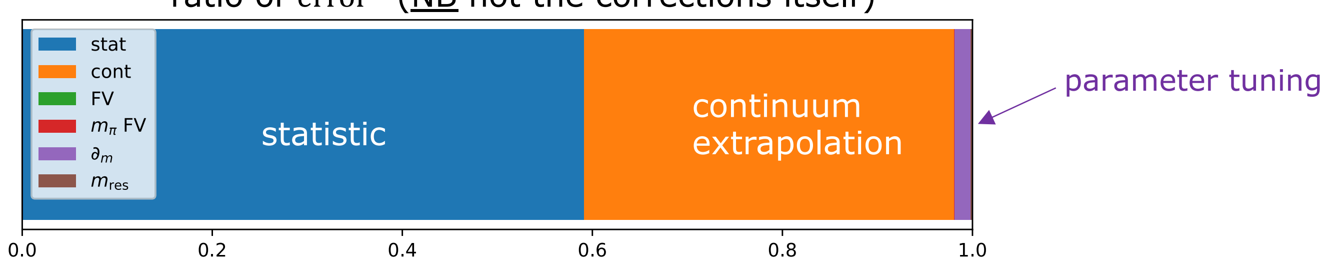
- Contrained fit w/ different a^2 effects
- Clean extrapolation w/ slight curvatures.
  Several Fitting ansatze to take nonlinear effects into account
  - $f(a^2) = c_0 + c_1 a^2$
  - $f(a^2) = c_0 + c_1 a^2 + c_2 a^4$
  - $f(a^2) = c_0 + c_l a^2 \log(a^2)$
  - $f(a^2) = c_0 + c_1 a^2 + c_l a^2 \log(a^2)$
- *Extrapolation of local-local (the most above) undershoots w/o 96I*





$$a_\mu^{\mathrm{W,iso,conn,ud}} = 206.46(53)_{\mathrm{S}}(43)_{\mathrm{C}}(01)_{\mathrm{FV}}(01)_{m_\pi\ \mathrm{FV}}(09)_{\partial_m\ \mathrm{C}}(00)_{\mathrm{WF\ order}}(03)_{m_{\mathrm{res}}} \times 10^{-10}$$ *(RBC/UKQCD 18 world)*

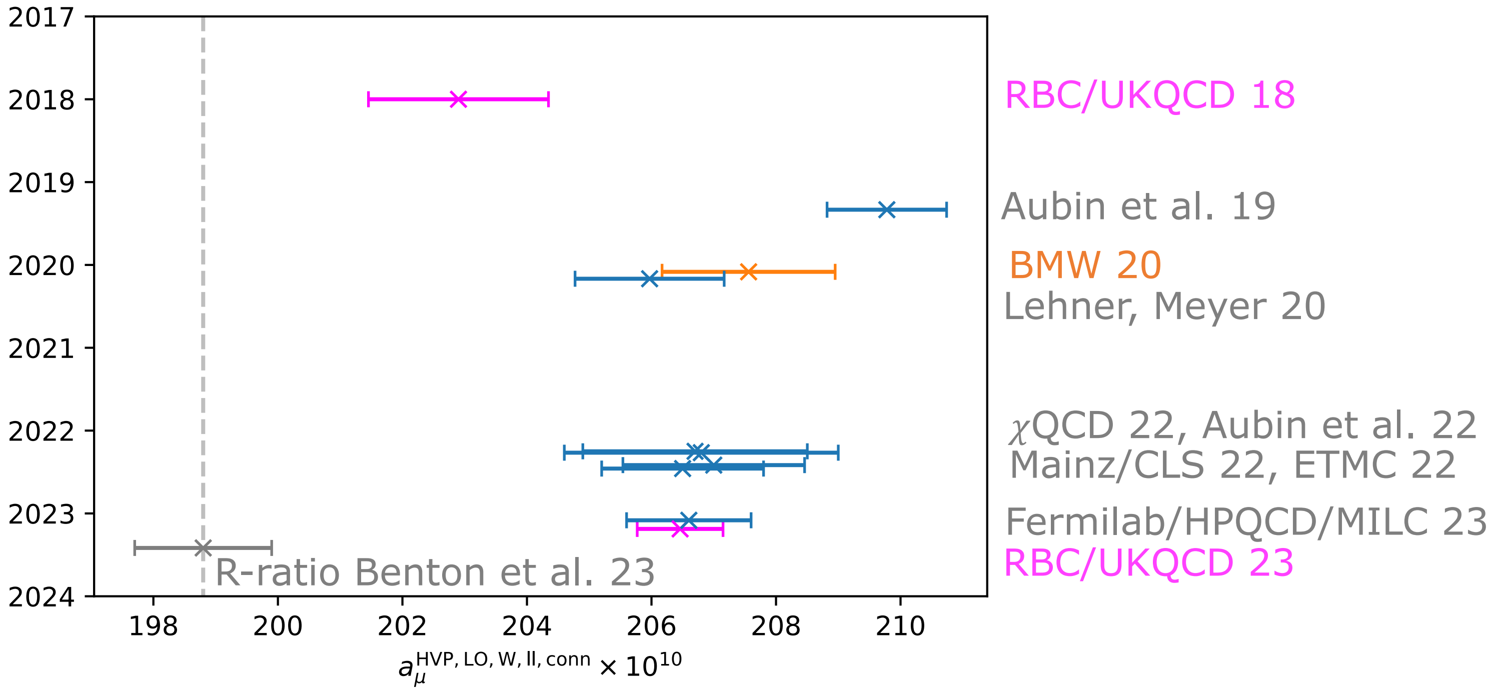
ratio of error^2 (NB not the corrections itself)



Summary & Outlook

- Lattice QCD giving precise first-principles estimates competitive to experiments
  ➡ *participating the precision frontier*

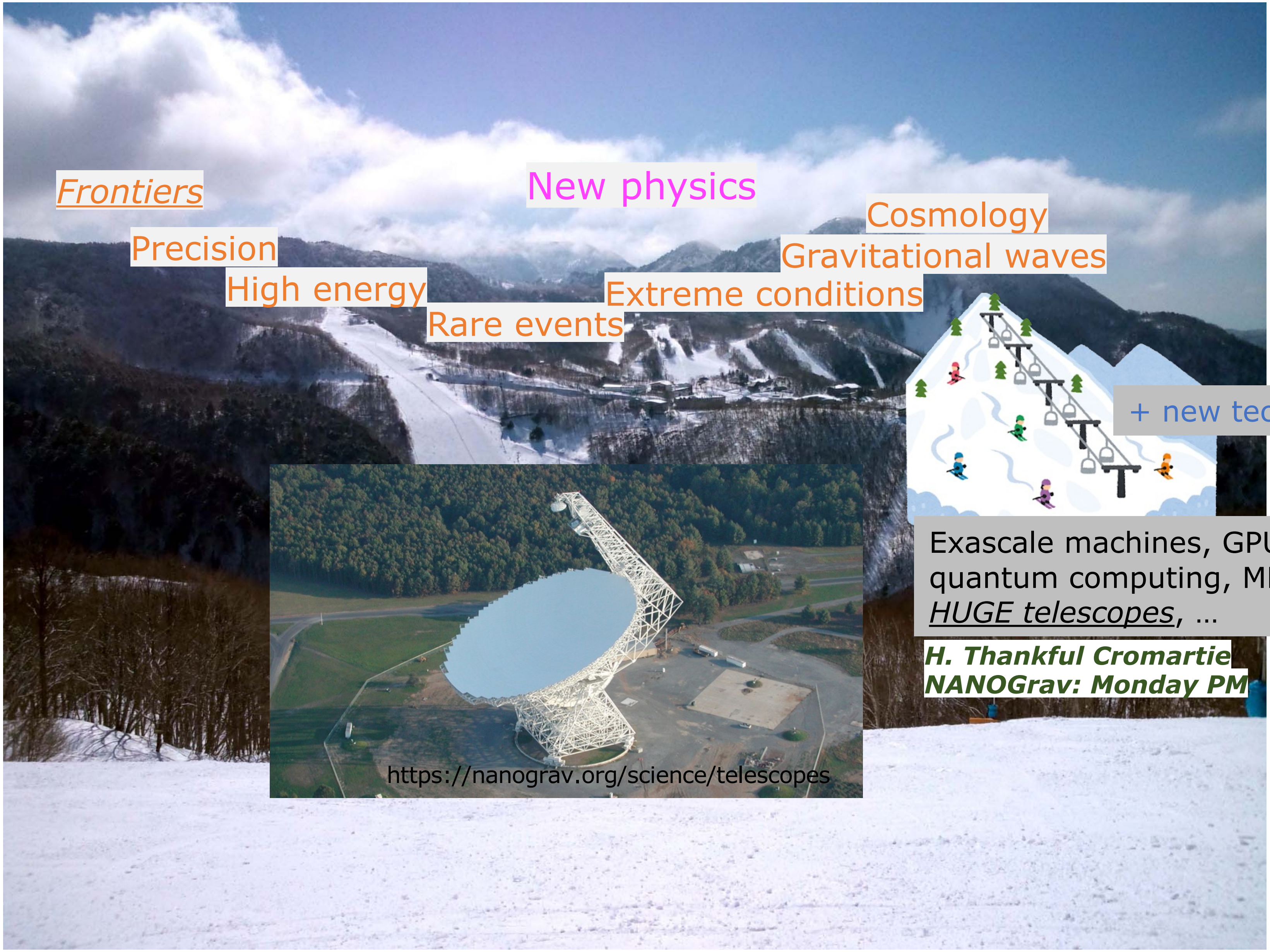
- Decomposing the problem into pieces,
  the understanding of the HVP puzzle is getting better and better. **Muon g-2 Theory Initiative**
  In particular, good agreement for the intermediate window among lattice collaborations:



- LD estimation and complete LO HVP **RBC/UKQCD work in progress**

- Experimental updates of both R-ratio and direct measurement
  - *CMD-3, Belle II, BES III*
  - *Fermilab Run 4,5,6, J-PARC E34 g-2/EDM, MUonE*

  ➡ *What remains after the smokes clears to be seen*



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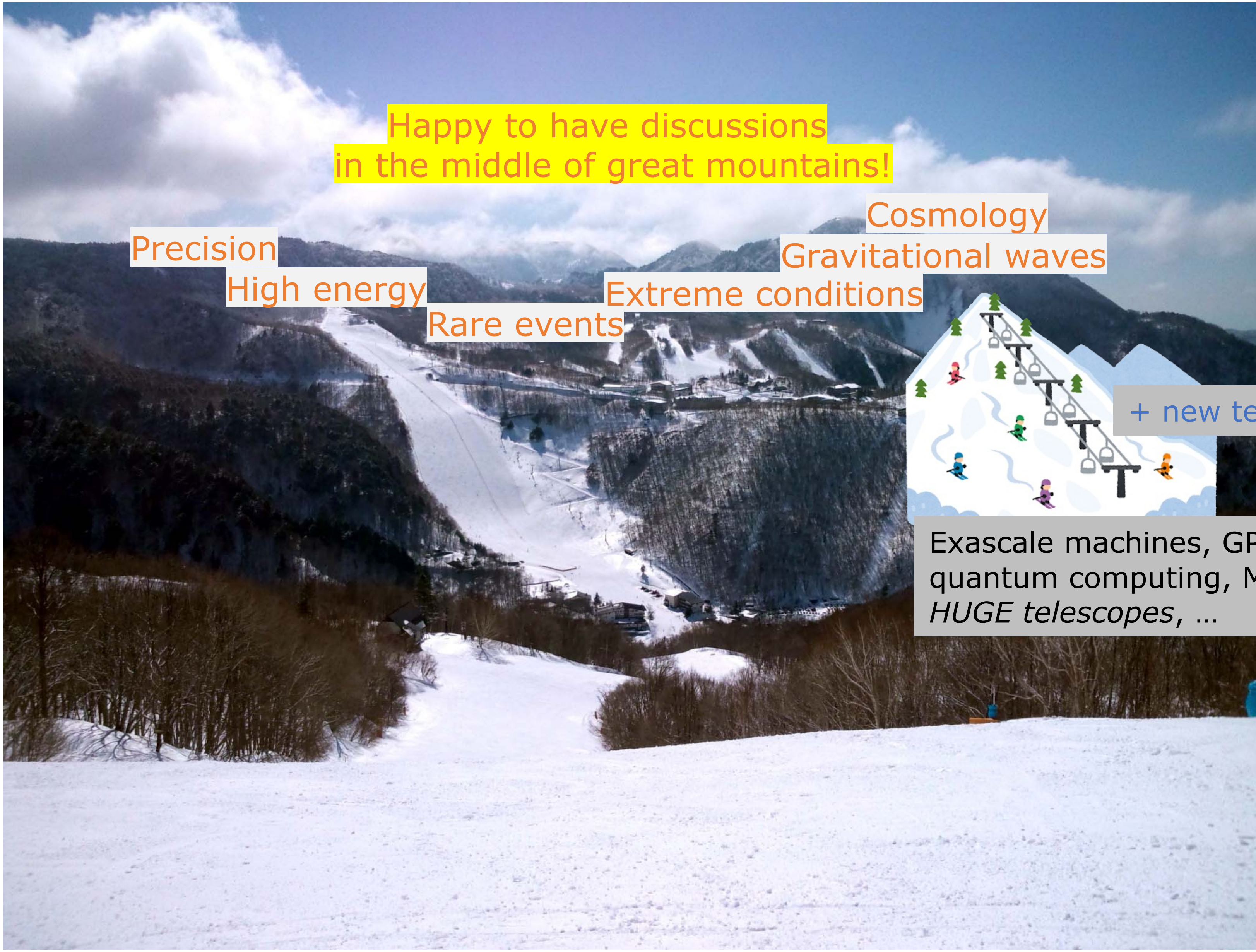

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Thank you