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Turbulent dynamos

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Topics

- **Theory for inhomogeneous turbulence**

  Strong nonlinearity and inhomogeneity

- **Transports in strongly compressible MHD turbulence**

  Strong compressibility = Large $\langle \rho'^2 \rangle / \bar{\rho}^2$

  Deviations from the gradient-diffusion approximation

- **Dynamo coupled with large-scale flow**

  Cross helicity

- **Global flow generation due to helicities**

  Helicity and cross helicity

Background

Equation of fluctuating velocity $\mathbf{u} = \mathbf{U} + \mathbf{u}', \quad \mathbf{U} = \langle \mathbf{u} \rangle, \quad \mathbf{u}' = \mathbf{u} - \langle \mathbf{u} \rangle$

$$\frac{\partial u'_\alpha}{\partial t} + U_a \frac{\partial u'_\alpha}{\partial x_a} = -u'_a \frac{\partial U_\alpha}{\partial x_a} - u'_a \frac{\partial u'_\alpha}{\partial x_a} + \frac{\partial}{\partial x_a} \langle u'_a u'_\alpha \rangle - \frac{\partial p'}{\partial x_\alpha} + \nu \frac{\partial^2 u'_\alpha}{\partial x_a^2}$$

turbulence–mean velocity interaction

turbulence–turbulence interaction

Instability approach

$$\frac{\partial u'_\alpha}{\partial t} + U_a \frac{\partial u'_\alpha}{\partial x_a} = -u'_a \frac{\partial U_\alpha}{\partial x_a} - \frac{\partial p'^{(\mathrm{R})}}{\partial x_\alpha} + \nu \frac{\partial^2 u'_\alpha}{\partial x_a^2}$$

Linear in $\mathbf{u}'$ and $p'^{(\mathrm{R})}$, each (Fourier) mode evolves independently

Closure approach

$$\frac{\partial u'_\alpha}{\partial t} + U_a \frac{\partial u'_\alpha}{\partial x_a} = -u'_a \frac{\partial u'_\alpha}{\partial x_a} + \frac{\partial}{\partial x_a} \langle u'_a u'_\alpha \rangle - \frac{\partial p'^{(\mathrm{S})}}{\partial x_\alpha} + \nu \frac{\partial^2 u'_\alpha}{\partial x_a^2}$$

Homogeneous turbulence, no dependence on large-scale inhomogeneity

Homogeneous turbulence

Navier–Stokes equation in the wave-number space

$$ik_a\hat{u}_a(\mathbf{k};t) = 0$$

$$\frac{\partial \hat{u}_\alpha(\mathbf{k};t)}{\partial t} - ik_a \iint d\mathbf{p} d\mathbf{q} \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \hat{u}_a(\mathbf{p};t)\hat{u}_\alpha(\mathbf{q};t)$$
$$= ik_\alpha \hat{p}(\mathbf{k};t) - \nu k^2 \hat{u}_\alpha(\mathbf{k};t)$$

$$\hat{p}(\mathbf{k};t) = -\frac{k_a k_b}{k^2} \iint d\mathbf{p} d\mathbf{q} \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \hat{u}_a(\mathbf{p};t)\hat{u}_b(\mathbf{q};t)$$

$$\longrightarrow \quad \frac{\partial \hat{u}_\alpha(\mathbf{k};t)}{\partial t} = -\nu k^2 \hat{u}_\alpha(\mathbf{k};t) + iM_{\alpha ab}(\mathbf{k}) \iint d\mathbf{p} d\mathbf{q} \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \hat{u}_a(\mathbf{p};t)\hat{u}_b(\mathbf{q};t)$$

where $M_{\alpha ab}(\mathbf{k}) = \frac{1}{2}\left[k_b D_{\alpha a}(\mathbf{k}) + k_a D_{\alpha b}(\mathbf{k})\right]$

with the projection operator $D_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}$

Closure theory for homogeneous turbulence

Direct-Interaction Approximation (DIA)

Kraichnan, R. H. (1959)
"The structure of isotropic turbulence **at very high Reynolds number**,"
J. Fluid Mech. **5**, 497

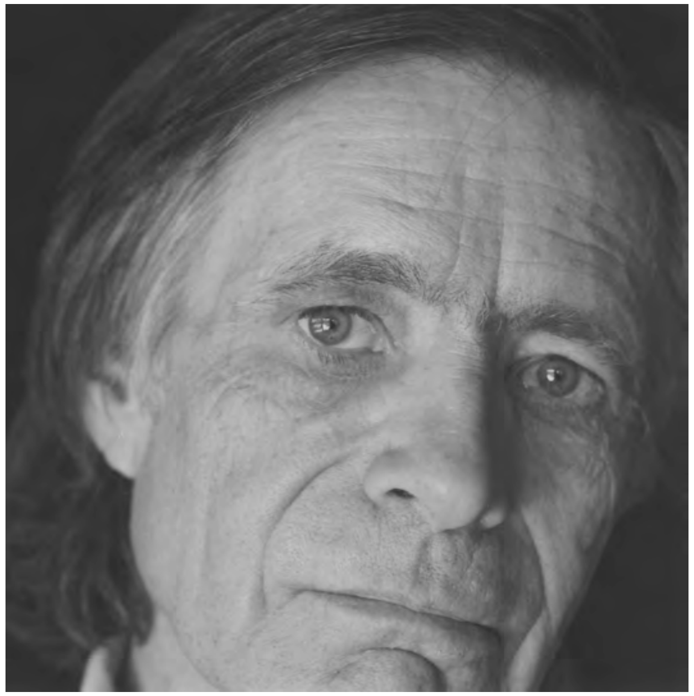


Photo provided by JMK

Navier–Stokes equation

⟶ Correlation function $Q_{\alpha\beta}(\mathbf{k},\mathbf{k}';t,t')$

$$\left[\frac{Q_{\alpha\beta}(\mathbf{k},\mathbf{k}';t,t')}{\delta(\mathbf{k}+\mathbf{k}')} = \frac{1}{(2\pi)^3}\int d\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}}\tilde{R}_{\alpha\beta}(\mathbf{r};t,t')\right]$$

⟶ Response function $G_{\alpha\beta}(\mathbf{k},\mathbf{k}';t,t')$

Equations for $Q_{\alpha\beta}(\mathbf{k},\mathbf{k}';t,t')$ and $G_{\alpha\beta}(\mathbf{k},\mathbf{k}';t,t')$

⟶ A closed system of equations

DIA = Lowest-order line (propagator not vertex) renormalization

$$f(x) = 1 + x + x^2 + x^3 + \cdots$$

$$f_{\mathrm{ex}}(x) = \frac{1}{1-x}$$

$$f(x) = 1 + x(1 + x + x^2 + x^3 + \cdots)$$

$$f_{\mathrm{ex}}(x) = 1 + x f_{\mathrm{ex}}(x)$$

(i) Perturbation expansion

Non-perturbed (linear) equations

$$\mathcal{L}\hat{u}_\alpha(\mathbf{k};t) = 0$$

$$\mathcal{L}G'_{\alpha\beta}(\mathbf{k};t,t') = D_{\alpha\beta}(\mathbf{k})\delta(t-t')$$

Linear solutions $\mathbf{u}^{(\mathrm{L})}(\mathbf{k};t)$ $G^{(\mathrm{L})}_{\alpha\beta}(\mathbf{k};t,t')$

Linear Green's function

$$G^{(\mathrm{L})}_{\alpha\beta}(\mathbf{k};\tau,\tau') = D_{\alpha\beta}(\mathbf{k})\Xi(\tau-\tau')\exp\left[-\nu k^2(\tau-\tau')\right]$$

Ξ : Heaviside step function

Velocity

$$u_\alpha(\mathbf{k};t) = u^{(\mathrm{L})}_\alpha(\mathbf{k};t) + iM_{cab}(\mathbf{k})\iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q} \times \int_{-\infty}^{t} dt_1 G^{(\mathrm{L})}_{\alpha c}(\mathbf{k};t,t_1)u_a(\mathbf{p};t_1)u_b(\mathbf{q};t_1)$$

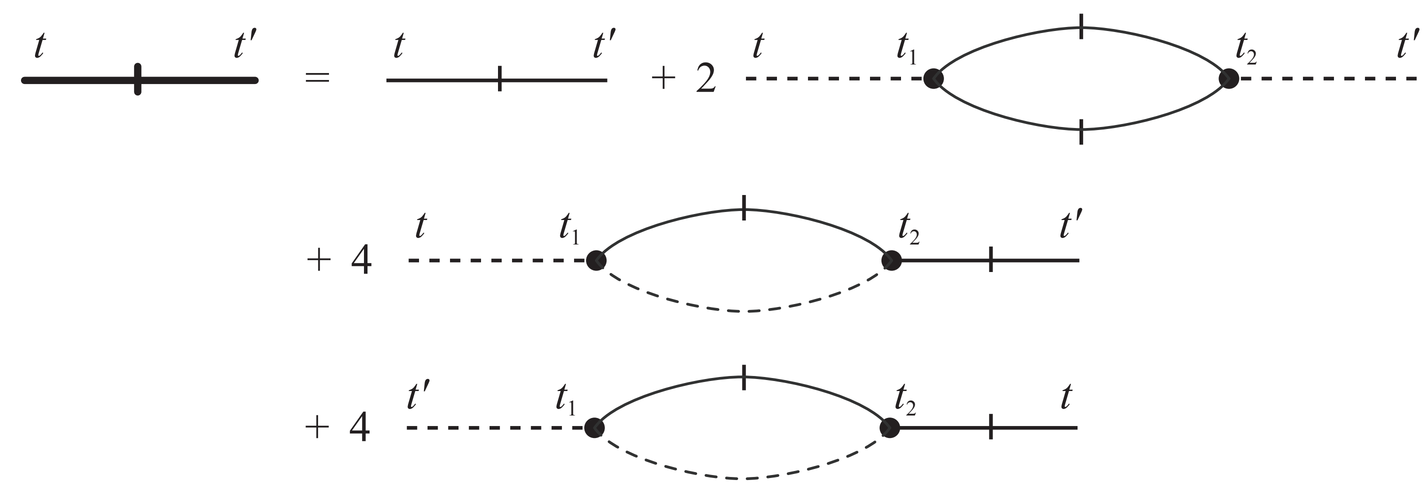
Green's function

$$G'_{\alpha\beta}(\mathbf{k};t,t') = G^{(\mathrm{L})}_{\alpha\beta}(\mathbf{k};t,t') + iM_{cab}(\mathbf{k})\iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q} \times \int_{t'}^{t} dt_1 G^{(\mathrm{L})}_{\alpha c}(\mathbf{k};t,t_1)u_a(\mathbf{p};t_1)G'_{b\beta}(\mathbf{q};t,t_1)$$

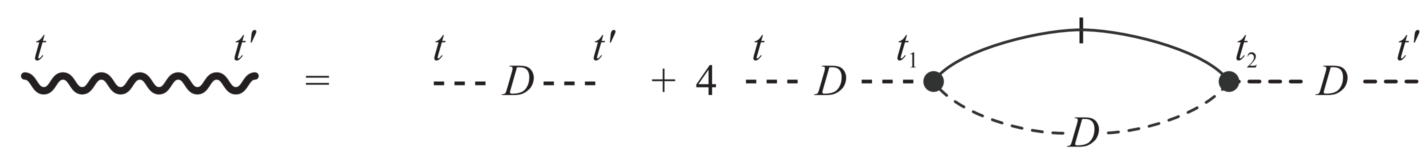
$$u_\alpha(\mathbf{k};t) = u^{(\mathrm{L})}_\alpha(\mathbf{k};t) \quad \text{at} \;\; t=-\infty$$

$$u_\alpha(\mathbf{k};t) = u_\alpha^{(\mathrm{L})}(\mathbf{k};t) + iM_{cab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}$$
$$\times \int_{-\infty}^{t} dt_1 G_{\alpha c}^{(\mathrm{L})}(\mathbf{k};t,t_1) u_a(\mathbf{p};t_1) u_b(\mathbf{q};t_1)$$

Correlation function $Q_{\alpha\beta}(\mathbf{k},\mathbf{k}';t,t')$



Response function $G_{\alpha\beta}(\mathbf{k},\mathbf{k}';t,t')$



(ii) Calculation based on the Gaussian statistics

Velocity correlation

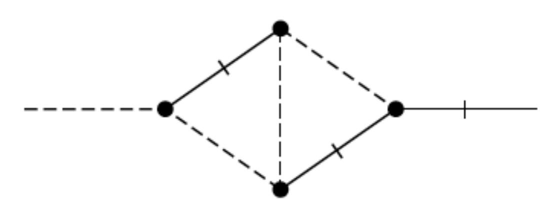
$$Q_{\alpha\beta}(\mathbf{k};t,t') = \frac{\langle u_\alpha(\mathbf{k};t)u_\beta(\mathbf{k}';t')\rangle}{\delta(\mathbf{k}+\mathbf{k}')}$$

Ensemble average of the Green's function

$$G_{\alpha\beta}(\mathbf{k};t,t') = \left\langle G'_{\alpha\beta}(\mathbf{k};t,t')\right\rangle$$

$$\left.\begin{aligned}&\mathcal{L}Q_{\alpha\beta}(\mathbf{k};t,t')\\ &\mathcal{L}G_{\alpha\beta}(\mathbf{k};t,t')\end{aligned}\right\} = \text{Functional of } \; Q^{(\mathrm{L})}_{\alpha\beta} \; \text{ and } \; G^{(\mathrm{L})}_{\alpha\beta}$$

(iii) Partial sum



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Truncate at the lowest order in M

⟶ Lowest-order line (not vertex) renormalization

(iv) Renormalization

$$Q^{(\mathrm{L})}_{\alpha\beta}(\mathbf{k};t,t') \longrightarrow Q_{\alpha\beta}(\mathbf{k};t,t')$$

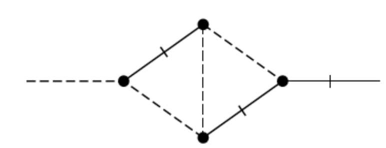
$$G^{(\mathrm{L})}_{\alpha\beta}(\mathbf{k};t,t') \longrightarrow G_{\alpha\beta}(\mathbf{k};t,t')$$

$$\left.\begin{array}{l}\mathcal{L}Q_{\alpha\beta}(\mathbf{k};t,t') \\ \\ \mathcal{L}G_{\alpha\beta}(\mathbf{k};t,t')\end{array}\right\} = \text{Functional of } \ Q_{\alpha\beta} \ \text{ and } \ G_{\alpha\beta}$$

For isotropic turbulence $$Q_{\alpha\beta}(\mathbf{k};t,t') = D_{\alpha\beta}(\mathbf{k})Q(k;t,t')$$

$$G_{\alpha\beta}(\mathbf{k};t,t') = D_{\alpha\beta}(\mathbf{k})G(k;t,t')$$

DIA = line (propagator) renormalization (lowest-order in vertex)



not included

$$f_{\mathrm{ex}}(x) = 1 + x + x^2 + x^3 + \cdots$$

$$f(x) = 1 + x(1 + x + x^2 + \cdots)$$

$$f_{\mathrm{ex}}(x) = \frac{1}{1-x}$$

Truncation $f(x) = 1 + x(1+x) \neq f_{\mathrm{ex}}(x)$

Renormalization $f(x) = 1 + x f_{\mathrm{ex}}(x)$

A theoretical formulation for inhomogeneous turbulence

Two-Scale Direct-Interaction Approximation (TSDIA)

mirror-symmetric case: Yoshizawa, Phys. Fluids **27**, 1377 (1984)
non-mirror-symmetric case: Yokoi & Yoshizawa, Phys. Fluids A **5**, 464 (1993)

- DIA — An elaborate closure theory for homogeneous isotropic turbulence
- Multiple-scale analysis — Fast and slowly varying fields

- Introduction of two scales
- Fourier transform of the fast variables
- Scale-parameter expansion
- Introduction of the Green's function
- Statistical assumptions on the basic fields
- Calculation of the statistical quantities using the DIA

Introduction of two scales

Fast and slow variables

$$\boldsymbol{\xi} = \mathbf{x},\ \mathbf{X} = \delta \mathbf{x};\ \tau = t,\ T = \delta t$$

Slow variables ***X*** and *T* change only when ***x*** and *t* change much.

$$f = F\left(\mathbf{X}; T\right) + f'\left(\boldsymbol{\xi}, \mathbf{X}; \tau, T\right)$$

$$\nabla = \nabla_{\boldsymbol{\xi}} + \delta \nabla_{\mathbf{X}};\quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \delta \frac{\partial}{\partial T}$$

Velocity-fluctuation equation

$$\frac{\partial u'_\alpha}{\partial \tau} + U_a \frac{\partial u'_\alpha}{\partial \xi_a} + \frac{\partial}{\partial \xi_a} u'_a u'_\alpha + \frac{\partial p'}{\partial \xi_\alpha} - \nu \nabla^2_\xi u'_\alpha$$
$$= \delta \left(-u'_a \frac{\partial U_\alpha}{\partial X_a} - \frac{D u'_\alpha}{DT} - \frac{\partial p'}{\partial X_\alpha} - \frac{\partial}{\partial X_a} \left(u'_a u'_\alpha - R_{a\alpha} + 2\nu \frac{\partial^2 u'_\alpha}{\partial X_a \partial \xi_a} \right) \right)$$
$$+\delta^2 \left(\nu \nabla^2_X u'_\alpha \right)$$

$$\frac{\partial u'_a}{\partial \xi_a} + \delta \frac{\partial u'_a}{\partial X_a} = 0$$

where $$\frac{D}{DT} = \frac{\partial}{\partial T} + \mathbf{U} \cdot \nabla_X$$

Fourier transform of the fast variables

The fluctuation fields are homogeneous
with respect to the fast variables:

$$f'\left(\boldsymbol{\xi}, \mathbf{X}; \tau, T\right) = \int d\mathbf{k} f'\left(\mathbf{k}, \mathbf{X}; \tau, T\right) \exp\left(-i\mathbf{k} \cdot \left(\boldsymbol{\xi} - \mathbf{U}\tau\right)\right)$$

Velocity-fluctuation equation in the wave-number space:

$$\begin{aligned}
&\frac{\partial u'_\alpha(\mathbf{k};\tau)}{\partial \tau} + \nu k^2 u'_\alpha\left(\mathbf{k};\tau\right) - ik_\alpha p'(\mathbf{k};\tau) \\
&\quad -ik_a \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} u'_\alpha(\mathbf{p};\tau) u'_a(\mathbf{q};\tau) \\
&\quad = \delta\left(-u'_a(\mathbf{k};\tau) \frac{\partial U_\alpha}{\partial X_a} - \frac{D u'_\alpha\left(\mathbf{k};\tau\right)}{D T_{\mathrm{I}}} - \frac{\partial p'(\mathbf{k};\tau)}{\partial X_{\mathrm{I}\alpha}} \right. \\
&\quad \left. - \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \frac{\partial}{\partial X_{\mathrm{I}a}} \left(u'_\alpha(\mathbf{p};\tau) u'_a(\mathbf{q};\tau)\right) + \delta(\mathbf{k}) \frac{\partial R_{a\alpha}}{\partial X_a} \right)
\end{aligned}$$

Scale parameter expansion

$$f' = f'_0 + \delta f'_1 + \delta^2 f'_2 + \cdots = \sum_n \delta^n f'_n$$

zeroth-order field

$$\frac{\partial u'_{0\alpha}(\mathbf{k};\tau)}{\partial \tau} + \nu k^2 u'_{0\alpha}(\mathbf{k};\tau)$$
$$-iM_{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\, d\mathbf{p} d\mathbf{q} u'_{0a}(\mathbf{p};\tau)\, u'_{0b}(\mathbf{q};\tau) = 0$$

Same as homogeneous turbulence

Introduction of the Green's functions

$$\frac{\partial G'_{\alpha\beta}(\mathbf{k};\tau,\tau')}{\partial \tau} + \nu k^2 G'_{\alpha\beta}(\mathbf{k};\tau,\tau')$$
$$-2iM^{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} u'_{0a}(\mathbf{p};\tau)\, G'_{b\beta}(\mathbf{q};\tau,\tau')$$
$$= D_{\alpha\beta}(\mathbf{k})\delta(\tau-\tau')$$

1st-order field

$$\frac{\partial u'_{1\alpha}(\mathbf{k};\tau)}{\partial \tau} + \nu k^2 u'_{1\alpha}(\mathbf{k};\tau)$$

$$-2iM_{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} u'_{0a}(\mathbf{p};\tau) u'_{S1b}(\mathbf{q};\tau)$$

$$= -D_{\alpha b}(\mathbf{k}) u'_{0a}(\mathbf{k};\tau) \frac{\partial U_b}{\partial X_a} - D_{\alpha a}(\mathbf{k}) \frac{D u'_{0a}(\mathbf{k};\tau)}{D T_{\mathrm{I}}}$$

$$+2M_{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \frac{q_b}{q^2} u'_{0a}(\mathbf{p};\tau) \frac{\partial u'_{0c}(\mathbf{q};\tau)}{\partial X_{\mathrm{I}c}}$$

$$-D_{\alpha d}(\mathbf{k}) M_{abcd}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \frac{\partial}{\partial X_{\mathrm{I}c}} \left(u'_{0a}(\mathbf{p};\tau) u'_{0b}(\mathbf{q};\tau) \right)$$

$$\mathbf{u}'_1(\mathbf{k};\tau) = \mathbf{u}'_{\mathrm{S}1}(\mathbf{k};\tau) - i \frac{\mathbf{k}}{k^2} \frac{\partial u'_{0a}}{\partial X_{\mathrm{I}a}}$$

$$\mathbf{k} \cdot \mathbf{u}'_{\mathrm{S}1}(\mathbf{k};\tau) = 0 \qquad M_{abcd}(\mathbf{k}) = \frac{1}{2}\delta_{ac}\delta_{bd} + \frac{1}{2}\delta_{ad}\delta_{bc} - \frac{k_a k_b}{k^2}\delta_{cd}$$

Green's function

$$\frac{\partial G'_{\alpha\beta}(\mathbf{k};\tau,\tau')}{\partial \tau} + \nu k^2 G'_{\alpha\beta}(\mathbf{k};\tau,\tau')$$

$$-2iM^{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} u'_{0a}(\mathbf{p};\tau) G'_{b\beta}(\mathbf{q};\tau,\tau')$$

$$= D_{\alpha\beta}(\mathbf{k}) \delta(\tau - \tau')$$

Formal solution in terms of $G'_{\alpha\beta}(\mathbf{k};\tau,\tau')$

$$
\begin{aligned}
u'_{\mathrm{S}1\alpha}(\mathbf{k};\tau) = &-\frac{\partial U_b}{\partial X_a}\int_{-\infty}^{\tau} d\tau_1 G'_{\alpha b}(\mathbf{k};\tau,\tau_1)u'_{0a}(\mathbf{k};\tau_1) \\
&-\int_{-\infty}^{\tau} d\tau_1 G'_{\alpha a}(\mathbf{k};\tau,\tau_1)\frac{Du'_{0a}(\mathbf{k};\tau_1)}{DT_{\mathrm{I}}} \\
&+2M_{dab}(\mathbf{k})\iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}\int_{-\infty}^{\tau} d\tau_1 G'_{\alpha d}(\mathbf{k};\tau,\tau_1) \\
&\qquad \times \frac{q_b}{q^2}u'_{0a}(\mathbf{p};\tau_1)\frac{\partial u'_{0c}(\mathbf{q};\tau_1)}{\partial X_{\mathrm{I}c}} \\
&-M_{abcd}(\mathbf{k})\iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}\int_{-\infty}^{\tau} d\tau_1 G'_{\alpha d}(\mathbf{k};\tau,\tau_1) \\
&\qquad \times \frac{\partial}{\partial X_{\mathrm{I}c}}\left(u'_{0a}(\mathbf{p};\tau_1)u'_{0b}(\mathbf{q};\tau_1)\right)
\end{aligned}
$$

1st-order field

$$\frac{\partial u'_{1\alpha}(\mathbf{k};\tau)}{\partial \tau} + \nu k^2 u'_{1\alpha}(\mathbf{k};\tau)$$

$$-2iM_{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} u'_{0a}(\mathbf{p};\tau) u'_{S1b}(\mathbf{q};\tau)$$

$$= -D_{\alpha b}(\mathbf{k}) u'_{0a}(\mathbf{k};\tau) \frac{\partial U_b}{\partial X_a} - D_{\alpha a}(\mathbf{k}) \frac{D u'_{0a}(\mathbf{k};\tau)}{D T_{\mathrm{I}}}$$

$$+2M_{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \frac{q_b}{q^2} u'_{0a}(\mathbf{p};\tau) \frac{\partial u'_{0c}(\mathbf{q};\tau)}{\partial X_{\mathrm{I}c}}$$

$$-D_{\alpha d}(\mathbf{k}) M_{abcd}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \frac{\partial}{\partial X_{\mathrm{I}c}} \left(u'_{0a}(\mathbf{p};\tau) u'_{0b}(\mathbf{q};\tau) \right)$$

$u'_{1\alpha}(\mathbf{k};t) = \cdots$ in terms of the force terms (r.h.s.) and Green's functions

Calculation of turbulent correlations with DIA

$$\langle f'(\mathbf{x};t) g'(\mathbf{x};\mathbf{t}) \rangle = \int d\mathbf{k} \left\langle f'(\mathbf{k};\tau) g'(\mathbf{k};\tau) \right\rangle / \delta(\mathbf{0})$$

$$= \int d\mathbf{k} \left(\langle f'_0 g'_0 \rangle + \langle f'_0 g'_1 \rangle + \langle f'_1 g'_0 \rangle + \cdots \right) / \delta(\mathbf{0})$$

Basic field: homogeneous isotropic but non-mirror-symmetric

$$\frac{\langle u'_{0\alpha}(\mathbf{k};\tau)u'_{0\beta}(\mathbf{k};\tau)\rangle}{\delta(\mathbf{k}+\mathbf{k}')} = D_{\alpha\beta}(\mathbf{k})Q_0(k;\tau,\tau') + \frac{i}{2}\frac{k_a}{k^2}\epsilon_{\alpha\beta a}H_0(k;\tau,\tau')$$

Calculation of the Reynolds stress

$$\left\langle {u'}^\alpha {u'}^\beta \right\rangle = \left\langle {u'_{\rm B}}^\alpha {u'_{\rm B}}^\beta \right\rangle + \left\langle {u'_{\rm B}}^\alpha {u'_{01}}^\beta \right\rangle + \left\langle {u'_{01}}^\alpha {u'_{\rm B}}^\beta \right\rangle + \cdots$$
$$+ \left\langle {u'_{\rm B}}^\alpha {u'_{10}}^\beta \right\rangle + \left\langle {u'_{10}}^\alpha {u'_{\rm B}}^\beta \right\rangle + \cdots$$

$$\left\langle {u'}^\alpha {u'}^\beta \right\rangle_{\rm D} = -\nu_{\rm T}\mathcal{S}^{\alpha\beta} + \left[\Gamma^\alpha\left(\Omega^\beta + 2\omega_{\rm F}^\beta\right) + \Gamma^\beta\left(\Omega^\alpha + 2\omega_{\rm F}^\alpha\right)\right]_{\rm D}$$

where $$\mathcal{S}^{\alpha\beta} = \frac{\partial U^\alpha}{\partial x^\beta} + \frac{\partial U^\beta}{\partial x^\alpha} - \frac{2}{3}\nabla\cdot\mathbf{U}\delta^{\alpha\beta}$$

mixing length
$$\nu_{\rm T} \sim \tau u^2 \sim u\ell$$

Eddy viscosity $$\nu_{\rm T} = \frac{7}{15}\int {\rm d}\mathbf{k}\int_{-\infty}^{t} d\tau_1\ G(k;\tau,\tau_1)Q(k;\tau,\tau_1)$$

Helicity-related coefficient $$\mathbf{\Gamma} = \frac{1}{30}\int k^{-2}{\rm d}\mathbf{k}\int_{-\infty}^{t} d\tau_1\ G(k;\tau,\tau_1)\nabla H(k;\tau,\tau_1)$$

helicity inhomogeneity is essential

Transport in strongly compressible MHD turbulence

Topics

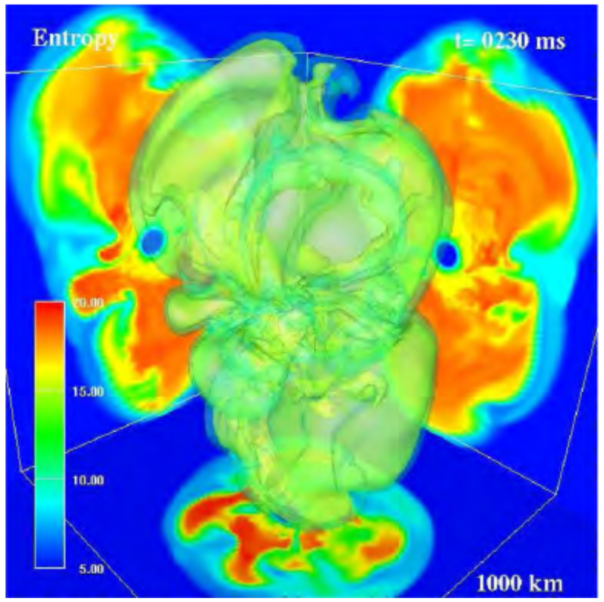
- **Theory for inhomogeneous turbulence**

  Strong nonlinearity and inhomogeneity

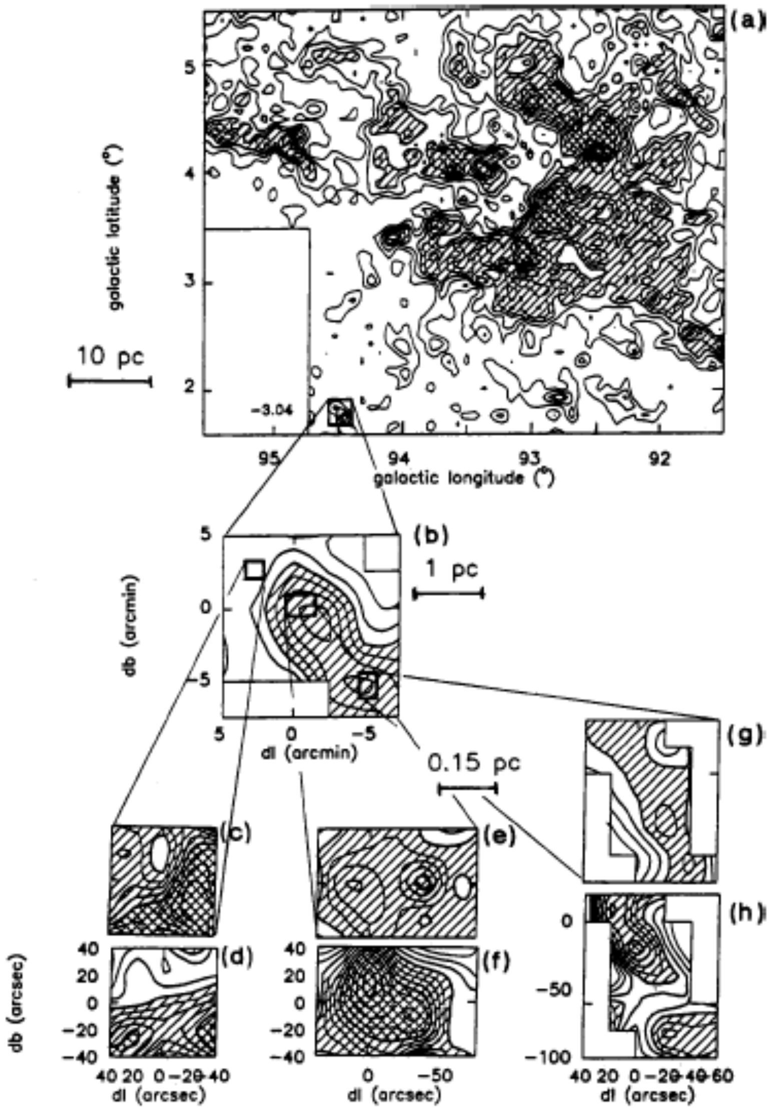
- **Transports in strongly compressible MHD turbulence**

  Strong compressibility = Large $\langle \rho'^2 \rangle / \overline{\rho}^2$

- **Deviations from the gradient-diffusion approximation model**



Supernova explosion
(Takiwaki, et al., 2014)



Molecular clouds in star forming region
(Falgarone, et al., 1992)

Transports in strongly compressible magnetohydrodynamic turbulence

Fundamental equations

Yokoi, N. J. Plasma Phys. **84**, 735840501 (2018)
Yokoi, N. J. Plasma Phys. **84**, 775840603 (2018)

Mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Momentum

$$\frac{\partial}{\partial t} \rho u^\alpha + \frac{\partial}{\partial x^a} \rho u^a u^\alpha = - \frac{\partial p}{\partial x^\alpha} + \frac{\partial}{\partial x^a} \mu s^{a\alpha} + (\mathbf{j} \times \mathbf{b})^\alpha + f_{\mathrm{ex}}^\alpha$$

$$s^{\alpha\beta} = \frac{\partial u^\beta}{\partial x^\alpha} + \frac{\partial u^\alpha}{\partial x^\beta} - \frac{2}{3} \nabla \cdot \mathbf{u} \delta^{\alpha\beta}$$

Internal energy

$$\frac{\partial}{\partial t} \rho q + \nabla \cdot (\rho \mathbf{u} q) = \nabla \cdot (\kappa \nabla \theta) - p \nabla \cdot \mathbf{u} + \phi$$

$$q = C_V(\theta) \theta$$

$$p = R \rho \theta = (\gamma_0 - 1) \rho q$$

Magnetic field

$$\frac{\partial \mathbf{b}}{\partial t} = - \nabla \times \mathbf{e}$$

$$\mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{b} = \sigma (\mathbf{e} + \mathbf{u} \times \mathbf{b})$$

Mean-field equations

Means and fluctuations

$$f = F + f', \quad F = \langle f \rangle$$

Density

$$\frac{\partial \overline{\rho}}{\partial t} + \nabla \cdot (\overline{\rho}\mathbf{U}) = -\nabla \cdot \langle \rho' \mathbf{u}' \rangle$$

Momentum

$$\frac{\partial}{\partial t}\overline{\rho}U^\alpha + \frac{\partial}{\partial x^a}\overline{\rho}U^a U^\alpha$$
$$= -(\gamma_0 - 1)\frac{\partial}{\partial x^\alpha}\overline{\rho}Q + \frac{\partial}{\partial x^\alpha}\mu S^{a\alpha} + (\mathbf{J}\times\mathbf{B})^\alpha$$
$$- \frac{\partial}{\partial x^\alpha}\left(\overline{\rho}\langle u'^a u'^\alpha\rangle - \frac{1}{\mu_0}\langle b'^a b'^\alpha\rangle + U^a\langle \rho' u'^\alpha\rangle + U^\alpha \langle \rho' u'^a\rangle\right) + R_U^\alpha$$

Internal energy

$$\frac{\partial}{\partial t}\overline{\rho}Q + \nabla\cdot(\overline{\rho}\mathbf{U}Q) = \nabla\cdot\left(\frac{\kappa}{C_V}\nabla Q\right) - \nabla\cdot(\overline{\rho}\langle q'\mathbf{u}'\rangle + Q\langle\rho'\mathbf{u}'\rangle + \mathbf{U}\langle\rho' q'\rangle)$$
$$-(\gamma_0 - 1)\left(\overline{\rho}Q\nabla\cdot\mathbf{U} + \overline{\rho}\langle q'\nabla\cdot\mathbf{u}'\rangle + Q\langle\rho'\nabla\cdot\mathbf{u}'\rangle\right) + R_Q$$

Magnetic field

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{U}\times\mathbf{B} + \langle\mathbf{u}'\times\mathbf{b}'\rangle) + \eta\nabla^2\mathbf{B}$$

where

$$R_U^\alpha = -\frac{\partial}{\partial t}\langle\rho' u'^\alpha\rangle - \frac{\partial}{\partial x^a}\langle\rho' u'^a u'^\alpha\rangle$$
$$-(\gamma_0 - 1)\frac{\partial}{\partial x^\alpha}\langle\rho' q'\rangle - \frac{1}{2\mu_0}\frac{\partial}{\partial x^\alpha}\langle\mathbf{b}'^2\rangle$$

etc.

Statistical assumptions on the lowest-order (basic) fields

Basic fields are homogeneous isotropic

$$\frac{\langle {\rho'_{\mathrm{B}}}^{\alpha}(\mathbf{k};\tau){\rho'_{\mathrm{B}}}^{\beta}(\mathbf{k}';\tau')\rangle}{\delta(\mathbf{k}+\mathbf{k}')} = D^{\alpha\beta}(\mathbf{k})Q_{\rho}(k;\tau,\tau')$$

$$\frac{\langle {\vartheta'_{\mathrm{B}}}^{\alpha}(\mathbf{k};\tau){\chi'_{\mathrm{B}}}^{\beta}(\mathbf{k}';\tau')\rangle}{\delta(\mathbf{k}+\mathbf{k}')}$$

$$= D^{\alpha\beta}(\mathbf{k})Q_{\vartheta\chi\mathrm{S}}(k;\tau,\tau') + \Pi^{\alpha\beta}(\mathbf{k})Q_{\vartheta\chi\mathrm{C}}(k;\tau,\tau') + \frac{i}{2}\frac{k^c}{k^2}\epsilon^{\alpha\beta c}H_{\vartheta\chi}(k;\tau,\tau')$$

$$\frac{\langle {q'_{\mathrm{B}}}^{\alpha}(\mathbf{k};\tau){q'_{\mathrm{B}}}^{\beta}(\mathbf{k}';\tau')\rangle}{\delta(\mathbf{k}+\mathbf{k}')} = D^{\alpha\beta}(\mathbf{k})Q_{q}(k;\tau,\tau')$$

with solenoidal and dilatational projection operators

$$D^{\alpha\beta}(\mathbf{k}) = \delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2}, \quad \Pi^{\alpha\beta}(\mathbf{k}) = \frac{k^\alpha k^\beta}{k^2}$$

Turbulent electromotive force

$$\begin{aligned}
\langle \mathbf{u}' \times \mathbf{b}' \rangle^\alpha &= \epsilon^{\alpha ab} \langle u'^a b'^b \rangle \\
&= \epsilon^{\alpha ab} \left(\langle {u'_0}^a {b'_0}^b \rangle + \langle {u'_0}^a {b'_1}^b \rangle + \langle {u'_1}^a {b'_0}^b \rangle + \cdots \right) \\
&= \epsilon^{\alpha ab} \left(\langle {u'_\mathrm{B}}^a {b'_\mathrm{B}}^b \rangle + \langle {u'_\mathrm{B}}^a {b'_{01}}^b \rangle + \langle {u'_\mathrm{B}}^a {b'_{10}}^b \rangle + \cdots \right. \\
&\qquad \left. + \langle {u'_{01}}^a {b'_\mathrm{B}}^b \rangle + \cdots + \langle {u'_{10}}^a {b'_\mathrm{B}}^b \rangle + \langle {u'_{10}}^a {b'_{01}}^b \rangle + \cdots \right).
\end{aligned}$$

Results

$$\begin{aligned}
\langle \mathbf{u}' \times \mathbf{b}' \rangle = &- (\beta + \zeta) \nabla \times \mathbf{B} + \alpha \mathbf{B} - (\nabla \zeta) \times \mathbf{B} + \gamma \nabla \times \mathbf{U} \\
&- \chi_{\overline{\rho}} \nabla \overline{\rho} \times \mathbf{B} - \chi_Q \nabla Q \times \mathbf{B} - \chi_D \frac{D\mathbf{U}}{Dt} \times \mathbf{B}
\end{aligned}$$

"magnetoclinicity"

Transport coefficients

$$\chi_{\overline{\rho}} = \frac{1}{3} (\gamma_s - 1)^2 \frac{Q}{\overline{\rho}} \int d\mathbf{k}\ k^2 \int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \int_{-\infty}^{\tau_2} d\tau_3 \times G_{u\mathrm{C}}(k; \tau, \tau_1) G_q(k; \tau, \tau_2) G_b(k; \tau, \tau_3) Q_{u\mathrm{C}}(k; \tau_2, \tau_3),$$

$$\chi_Q = \frac{1}{3} (\gamma_s - 1) \int d\mathbf{k}\ k^2 \int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \int_{-\infty}^{\tau_2} d\tau_3 \times G_{u\mathrm{C}}(k; \tau, \tau_1) G_\rho(k; \tau, \tau_2) G_b(k; \tau, \tau_3) Q_{u\mathrm{C}}(k; \tau_2, \tau_3),$$

$$\chi_D = \frac{1}{3} \int d\mathbf{k}\ k^2 \int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \int_{-\infty}^{\tau_2} d\tau_3 \times G_{u\mathrm{C}}(k; \tau, \tau_1) G_\rho(k; \tau, \tau_2) G_b(k; \tau, \tau_3) Q_{u\mathrm{C}}(k; \tau_2, \tau_3).$$

$$\langle \mathbf{u}' \times \mathbf{b}' \rangle = -(\beta + \zeta)\nabla \times \mathbf{B} + \alpha \mathbf{B} - (\nabla \zeta) \times \mathbf{B} + \gamma \nabla \times \mathbf{U} \\ - \chi_{\overline{\rho}} \nabla \overline{\rho} \times \mathbf{B} - \chi_Q \nabla Q \times \mathbf{B} - \chi_D \frac{D\mathbf{U}}{Dt} \times \mathbf{B}$$

where

$$\beta = \frac{1}{3} I_0 \{G_b, Q_{u\mathrm{S}} + Q_{u\mathrm{C}}\} + \frac{1}{3} I_0 \{G_{u\mathrm{S}} + G_{u\mathrm{C}}, Q_b\},$$

$$\zeta = \frac{1}{3} I_0 \{G_b, Q_{u\mathrm{S}}\} - \frac{1}{3} I_0 \{G_{u\mathrm{S}}, Q_b\},$$

$$\alpha = -\frac{1}{3} I_0 \{G_b, H_{uu}\} + \frac{1}{3} I_0 \{G_{u\mathrm{S}}, H_{bb}\},$$

$$\gamma = \frac{1}{3} I_0 \{G_{u\mathrm{S}} + G_{u\mathrm{C}} + G_b, Q_{w\mathrm{S}}\} + \frac{1}{3} I_0 \{G_{u\mathrm{S}} + G_b, Q_{w\mathrm{C}}\},$$

$$\chi_{\overline{\rho}} = \frac{1}{3} (\gamma_\mathrm{s} - 1)^2 \frac{Q}{\overline{\rho}} I_1 \left\{ G_{u\mathrm{C}}^{(1)}, G_q^{(2)}, G_b^{(3)}, Q_{u\mathrm{C}}^{(3)} \right\},$$

$$\chi_Q = \frac{1}{3} (\gamma_\mathrm{s} - 1) I_1 \left\{ G_{u\mathrm{C}}^{(1)}, G_\rho^{(2)}, G_b^{(3)}, Q_{u\mathrm{C}}^{(3)} \right\},$$

$$\chi_D = \frac{1}{3} I_1 \left\{ G_{u\mathrm{S}}^{(1)} + G_{u\mathrm{C}}^{(1)}, G_\rho^{(2)}, G_b^{(3)}, Q_{u\mathrm{C}}^{(3)} \right\}.$$

with

$$I_0 \{A, B\} = \int d\mathbf{k} \int_{-\infty}^{\tau} d\tau_1 A(k; \tau, \tau_1) B(k; \tau, \tau_1),$$

$$I_{2n} \left\{ A^{(1)}, B^{(2)}, C^{(3)}, D^{(3)} \right\} = \int d\mathbf{k}\ k^{2n} \int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau} d\tau_2 \int_{-\infty}^{\tau} d\tau_3 \\ \times A(k; \tau, \tau_1) B(k; \tau, \tau_2) C(k; \tau, \tau_3) D(k; \tau_2, \tau_3).$$

Turbulence model in terms of density variance

$$\int d\mathbf{k}\ k^2 Q_{u\mathrm{C}}(k;\tau_1,\tau_2) = \frac{1}{\tau_\rho^2}\frac{\langle \rho'^2\rangle}{\overline{\rho}^2} \quad \longleftarrow \quad ik^a {u'_{\mathrm{B}}}^a(\mathbf{k};\tau) = \frac{1}{\tau_\rho \overline{\rho}}\rho'_0(\mathbf{k};\tau)$$

$$\langle \mathbf{u}' \times \mathbf{b}' \rangle = -(\beta+\zeta)\nabla\times\mathbf{B} + \alpha\mathbf{B} - (\nabla\zeta)\times\mathbf{B} + \gamma\nabla\times\mathbf{U}$$
$$- \chi_{\overline{\rho}}\nabla\overline{\rho}\times\mathbf{B} - \chi_Q \nabla Q\times\mathbf{B} - \chi_D\frac{D\mathbf{U}}{Dt}\times\mathbf{B}$$

Transport coefficients

$\beta = \tau_b\langle \mathbf{u}'^2\rangle/2 + \tau_u\langle \mathbf{b}'^2\rangle/(2\mu_0\overline{\rho})$ MHD energy

$\zeta = \tau_b\langle \mathbf{u}'^2\rangle/2 - \tau_u\langle \mathbf{b}'^2\rangle/(2\mu_0\overline{\rho})$ Residual energy

$\alpha = \tau_b\langle -\mathbf{u}'\cdot\boldsymbol{\omega}'\rangle + \tau_u\langle \mathbf{b}'\cdot\mathbf{j}'\rangle/\overline{\rho}$ Residual helicity

$\gamma = (\tau_u+\tau_b)\langle \mathbf{u}'\cdot\mathbf{b}'\rangle$ Cross helicity

$$\chi_\rho = (\gamma_{\mathrm{s}}-1)^2\frac{\tau_u\tau_q\tau_b}{\tau_\rho^2}\frac{Q}{\overline{\rho}}\frac{\langle\rho'^2\rangle}{\overline{\rho}^2}$$

$$\chi_Q = (\gamma_{\mathrm{s}}-1)\frac{\tau_u\tau_b}{\tau_\rho}\frac{\langle\rho'^2\rangle}{\overline{\rho}^2}$$

$$\chi_D = \frac{\tau_u\tau_b}{\tau_\rho}\frac{\langle\rho'^2\rangle}{\overline{\rho}^2}$$

} Density variance

Density-variance effects

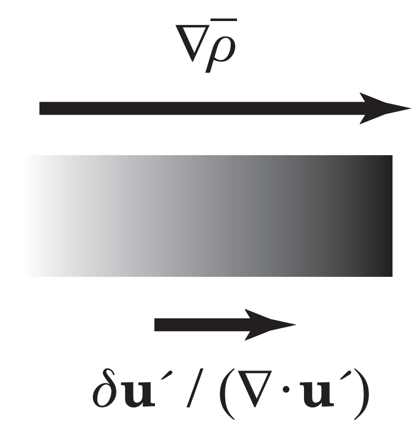
Density variance $$\left(\frac{\partial}{\partial t}+\mathbf{U}\cdot\nabla\right)\left\langle\rho'^2\right\rangle=-2\left\langle\rho'\mathbf{u}'\right\rangle\cdot\nabla\overline{\rho}-2\left\langle\rho'^2\right\rangle\nabla\cdot\mathbf{U}+\cdots$$

Magnetoclinicity: $$\left\langle\mathbf{u}'\times\mathbf{b}'\right\rangle/\mu_0=-\chi_\rho\nabla\overline{\rho}\times\mathbf{B}\qquad \chi_\rho\propto\left\langle\rho'^2\right\rangle$$

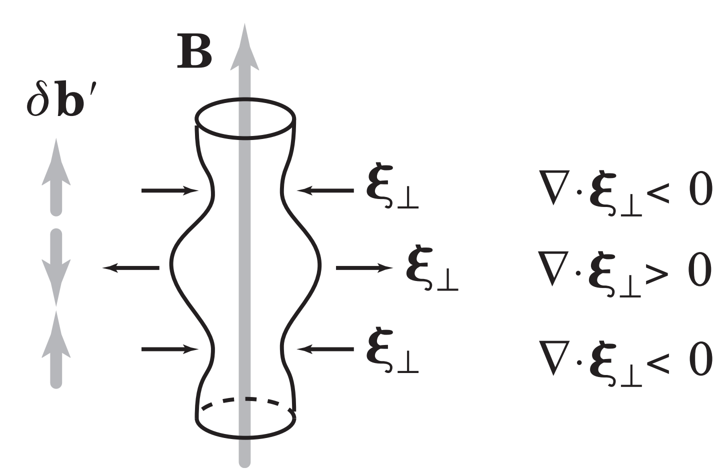
Simplest expressions for the density and internal-energy fluctuations

Turbulent dilatation $$\rho'=-\tau_\rho\overline{\rho}\nabla\cdot\mathbf{u}'\qquad q'=-(\gamma_{\rm s}-1)\tau_q Q\nabla\cdot\mathbf{u}'$$

$$\frac{\partial\mathbf{u}'}{\partial t}=\cdots-(\gamma_{\rm s}-1)\frac{q'}{\overline{\rho}}\nabla\overline{\rho}+\cdots$$
$$=\cdots+(\gamma_{\rm s}-1)^2\tau_q\frac{Q}{\overline{\rho}}(\nabla\cdot\mathbf{u}')\nabla\overline{\rho}+\cdots$$

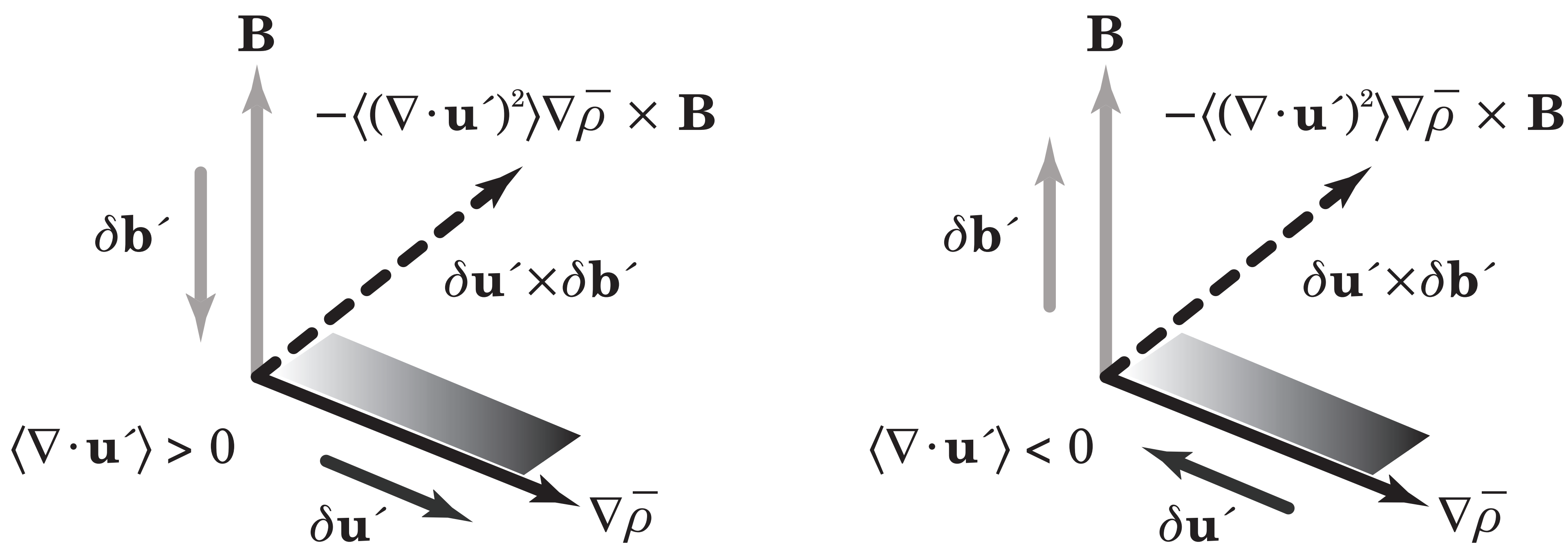


$$\frac{\partial\mathbf{b}'}{\partial t}=\cdots-(\nabla\cdot\mathbf{u}')\mathbf{B}+\cdots$$



$$\frac{\partial}{\partial t}\langle \mathbf{u}' \times \mathbf{b}' \rangle \simeq \cdots + (\gamma_{\rm s} - 1)\frac{1}{\overline{\rho}}\langle q' \nabla \cdot \mathbf{u}' \rangle \nabla \overline{\rho} \times \mathbf{B} + \cdots$$

$$= \cdots - (\gamma_{\rm s} - 1)^2 \tau_q \langle (\nabla \cdot \mathbf{u}')^2 \rangle \frac{Q}{\overline{\rho}} \nabla \overline{\rho} \times \mathbf{B} + \cdots$$



Irrespective of the sign of dilatation, the electromotive force is generated in the direction of **B** x **∇ρ̄**

$$\rho' = -\tau_\rho \overline{\rho} \nabla \cdot \mathbf{u}' \qquad \langle \rho'^2 \rangle = \tau_\rho^2 \overline{\rho}^2 \langle (\nabla \cdot \mathbf{u}')^2 \rangle$$

Energy generation due to the mean density variation

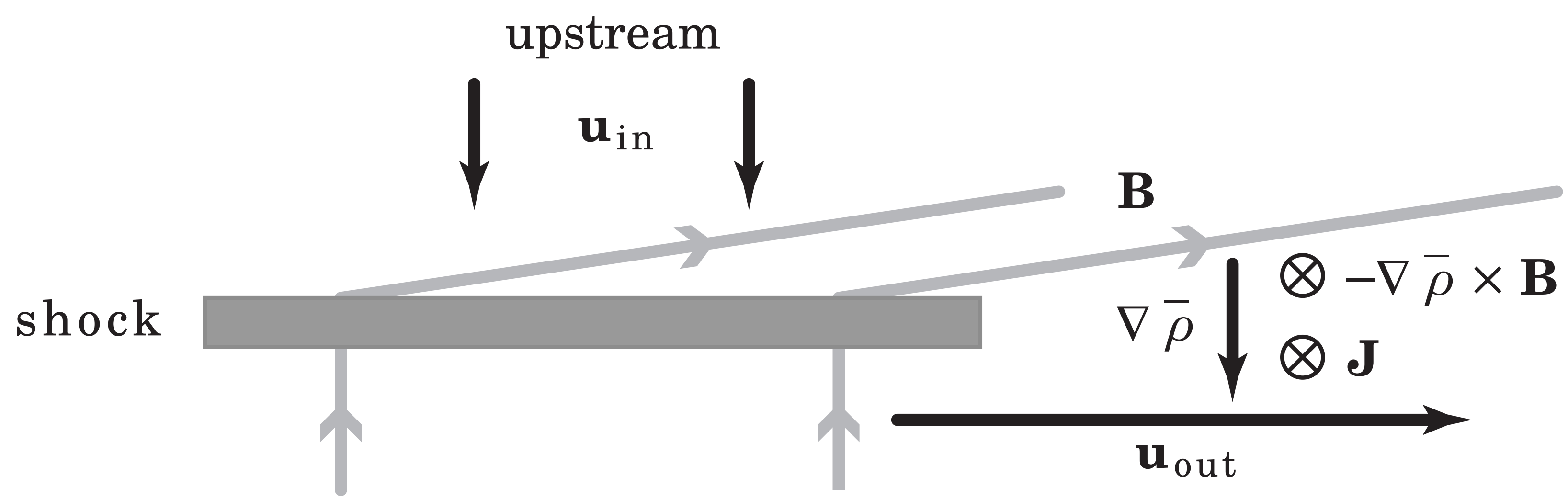
$$\frac{D}{Dt}\frac{1}{2}\left\langle \mathbf{u}'^2 \right\rangle = -\frac{1}{2\overline{\rho}}\left\langle \mathbf{u}' \times \mathbf{b}' \right\rangle \cdot \mathbf{J} - \left\langle u'^a u'^b \right\rangle \frac{\partial U^a}{\partial x^b} + \frac{1}{2\mu_0 \overline{\rho}} \left\langle u'^a b'^b \right\rangle \left(\frac{\partial B^b}{\partial x^a} + \frac{\partial B^a}{\partial x^b} \right)$$

$$- (\gamma_{\mathrm{s}} - 1) \frac{1}{\overline{\rho}} \left(\left\langle \rho' \mathbf{u}' \right\rangle \cdot \nabla Q + \left\langle q' \mathbf{u}' \right\rangle \cdot \nabla \overline{\rho} \right) - \frac{1}{\overline{\rho}} \left\langle \rho' \mathbf{u}' \right\rangle \cdot \frac{D\mathbf{U}}{Dt} - \varepsilon_u + T_u$$

where

$$\left\langle \mathbf{u}' \times \mathbf{b}' \right\rangle / \mu_0 = +\chi_\rho \mathbf{B} \times \nabla \overline{\rho} + \cdots$$

Note that $\chi_\rho \propto K_\rho = \left\langle \rho'^2 \right\rangle$



In the slow shocks in magnetic reconnection, the turbulent electromotive force due to the mean density variation is expected to enhance the turbulence generation in the foreshock (upstream) region.
Turbulence intensity should be enhanced across the shock.

Turbulent mass flux

$$\begin{aligned}\langle \rho' \mathbf{u}' \rangle &= \langle \rho'_0 \mathbf{u}'_0 \rangle + \langle \rho'_0 \mathbf{u}'_1 \rangle + \langle \rho'_1 \mathbf{u}'_0 \rangle + \cdots \\ &= \langle \rho'_{\mathrm{B}} \mathbf{u}'_{\mathrm{B}} \rangle + \langle \rho'_{\mathrm{B}} \mathbf{u}'_{01} \rangle + \langle \rho'_{\mathrm{B}} \mathbf{u}'_{10} \rangle + \cdots \\ &\quad + \langle \rho'_{01} \mathbf{u}'_{\mathrm{B}} \rangle + \langle \rho'_{01} \mathbf{u}'_{01} \rangle + \cdots + \langle \rho'_{10} \mathbf{u}'_{\mathrm{B}} \rangle + \cdots\end{aligned}$$

$$\langle \rho' \mathbf{u}' \rangle = -\kappa_{\overline{\rho}} \nabla \overline{\rho} - \kappa_Q \nabla Q - \kappa_D \frac{D\mathbf{U}}{DT} - \kappa_B \mathbf{B}$$

with

$$\kappa_{\overline{\rho}} = \frac{1}{3} I_0 \left\{ G_\rho, 2Q_{u\mathrm{S}} + Q_{u\mathrm{C}} \right\}$$ Gradient diffusion

$$\kappa_Q = \frac{1}{3} (\gamma_{\mathrm{s}} - 1) \frac{1}{\overline{\rho}} I_0 \left\{ 2G_{u\mathrm{S}} + G_{u\mathrm{C}}, Q_\rho \right\}$$ Cross diffusion

$$\kappa_D = \frac{1}{3} \frac{1}{\overline{\rho}} I_0 \left\{ 2G_{u\mathrm{S}} + G_{u\mathrm{C}}, Q_\rho \right\}$$ Non-equilibrium

$$\kappa_B = \frac{1}{3\mu_0} I_1 \left\{ G_\rho^{(1)}, G_{u\mathrm{C}}^{(2)}, Q_{w\mathrm{C}}^{(2)} \right\}$$ Along magnetic field

where

$$I_0 \left\{ A, B \right\} = \int d\mathbf{k} \int_{-\infty}^{\tau} d\tau_1 A(k;\tau,\tau_1) B(k;\tau,\tau_1)$$

$$I_{2n} \left\{ A^{(1)}, B^{(2)}, C^{(3)}, D^{(3)} \right\} = \int d\mathbf{k}\ k^{2n} \int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau} d\tau_2 \int_{-\infty}^{\tau} d\tau_3 \times A(k;\tau,\tau_1) B(k;\tau,\tau_2) C(k;\tau,\tau_3) D(k;\tau_2,\tau_3)$$

Turbulent internal-energy flux

$$\langle q'\mathbf{u}'\rangle = \langle q_0'\mathbf{u}_0'\rangle + \langle q_0'\mathbf{u}_1'\rangle + \langle q_1'\mathbf{u}_0'\rangle + \cdots$$
$$= \langle q_{\rm B}'\mathbf{u}_{\rm B}'\rangle + \langle q_{\rm B}'\mathbf{u}_{01}'\rangle + \langle q_{\rm B}'\mathbf{u}_{10}'\rangle + \cdots$$
$$+ \langle q_{01}'\mathbf{u}_{\rm B}'\rangle + \langle q_{01}'\mathbf{u}_{01}'\rangle + \cdots + \langle q_{10}'\mathbf{u}_{\rm B}'\rangle + \cdots$$

$$\langle q'\mathbf{u}'\rangle = -\eta_Q \nabla Q - \eta_{\overline{\rho}} \nabla \overline{\rho} - \eta_B \mathbf{B}$$

with $$\eta_Q = \frac{1}{3} I_0 \left\{G_q, 2Q_{u{\rm S}} + Q_{u{\rm C}}\right\}$$ Gradient diffusion

$$\eta_{\overline{\rho}} = \frac{1}{3}(\gamma_{\rm s} - 1)\frac{1}{\overline{\rho}} I_0 \left\{2G_{u{\rm S}} + G_{u{\rm C}}, Q_q\right\}$$ Cross diffusion

$$\eta_B = \frac{1}{3\mu_0 \overline{\rho}}(\gamma_{\rm s} - 1) Q I_1 \left\{G_q^{(1)}, G_{u{\rm C}}^{(2)}, Q_{w{\rm C}}^{(2)}\right\}$$ Along magnetic field

Comparison with MHD waves

Transverse Alfvén and Magnetoacoustic waves

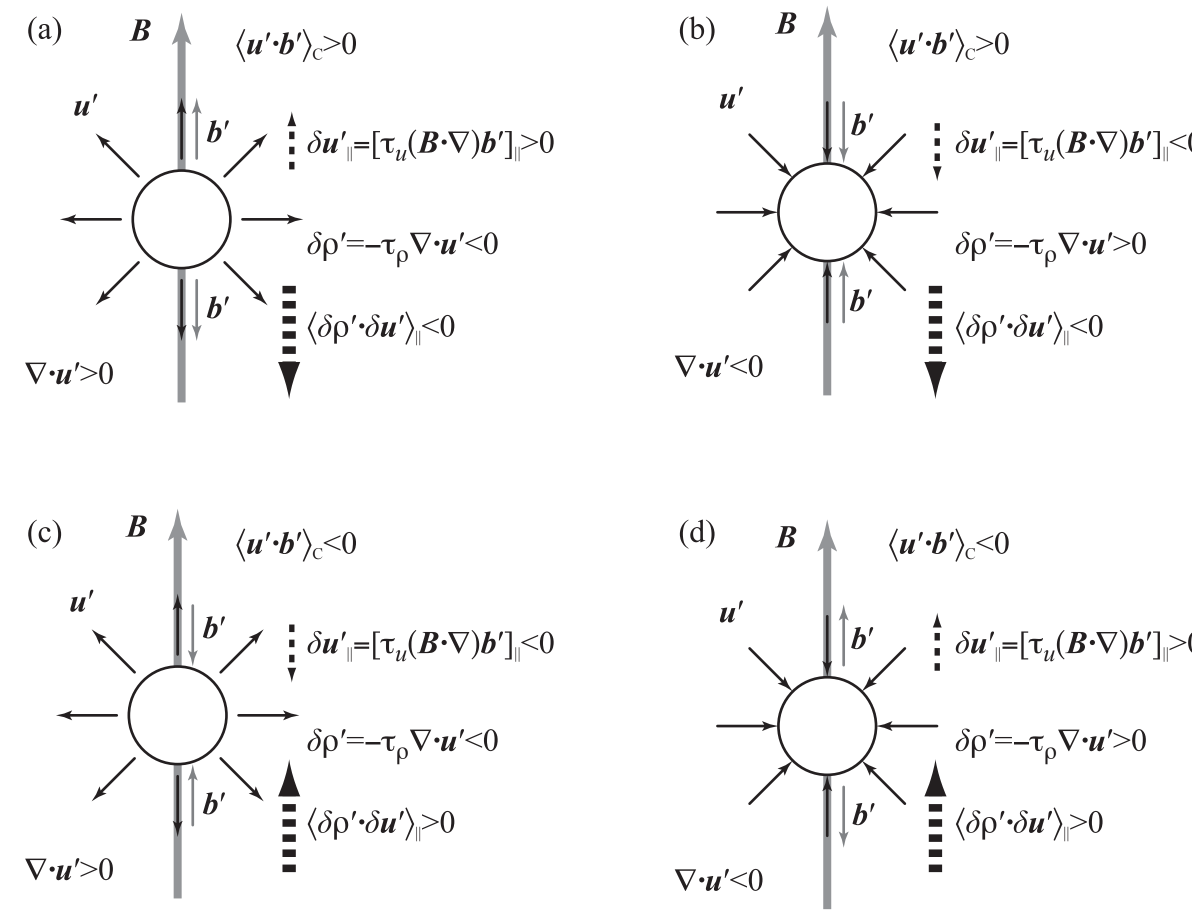
Material (and energy) flux along the mean magnetic field

$$\langle \rho' \mathbf{u}' \rangle_B = -\kappa_B \mathbf{B}$$

$$\kappa_B = \frac{1}{3\mu_0} \int d\mathbf{k}\ k^2 \int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau} d\tau_2\ G_\rho(k;\tau,\tau_1) G_{uc}(k;\tau,\tau_2) Q_{wC}(k;\tau_1,\tau_2)$$

$$\equiv \frac{1}{3\mu_0} I_1 \left\{ G_\rho^{(1)}, G_{u\mathrm{C}}^{(2)}, Q_{w\mathrm{C}}^{(2)} \right\}$$

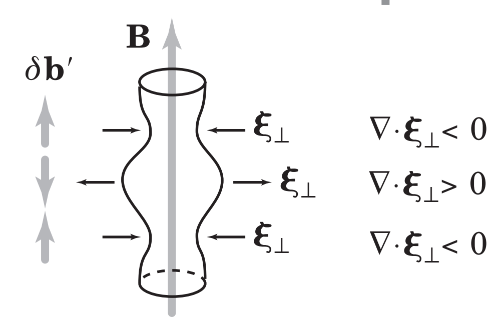
$$= C_{\kappa B} \frac{1}{\mu_0 \overline{\rho}} \tau_\rho \tau_{uc} \overline{\rho} \left\langle \mathbf{u}' \cdot \mathbf{b}' \right\rangle_{\mathrm{C}}$$

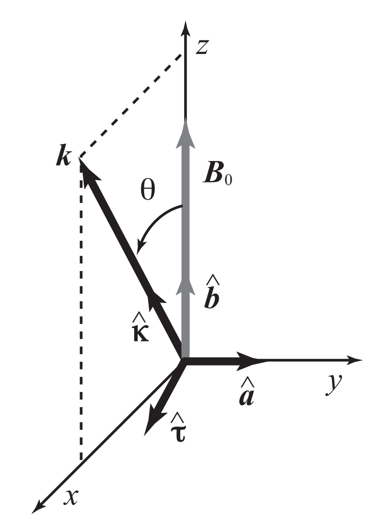


$$\delta\rho' \simeq -\tau_\rho \overline{\rho} \nabla \cdot \mathbf{u}',$$

$$\delta\mathbf{u}' \simeq \tau_u \frac{1}{\mu_0 \overline{\rho}} (\mathbf{B} \cdot \nabla) \mathbf{b}'$$

Magnetoacoustic wave

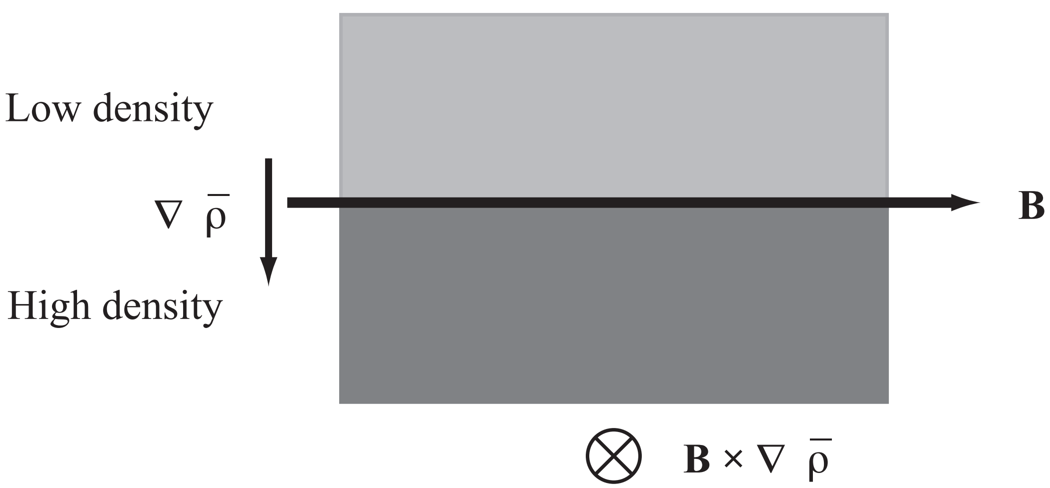


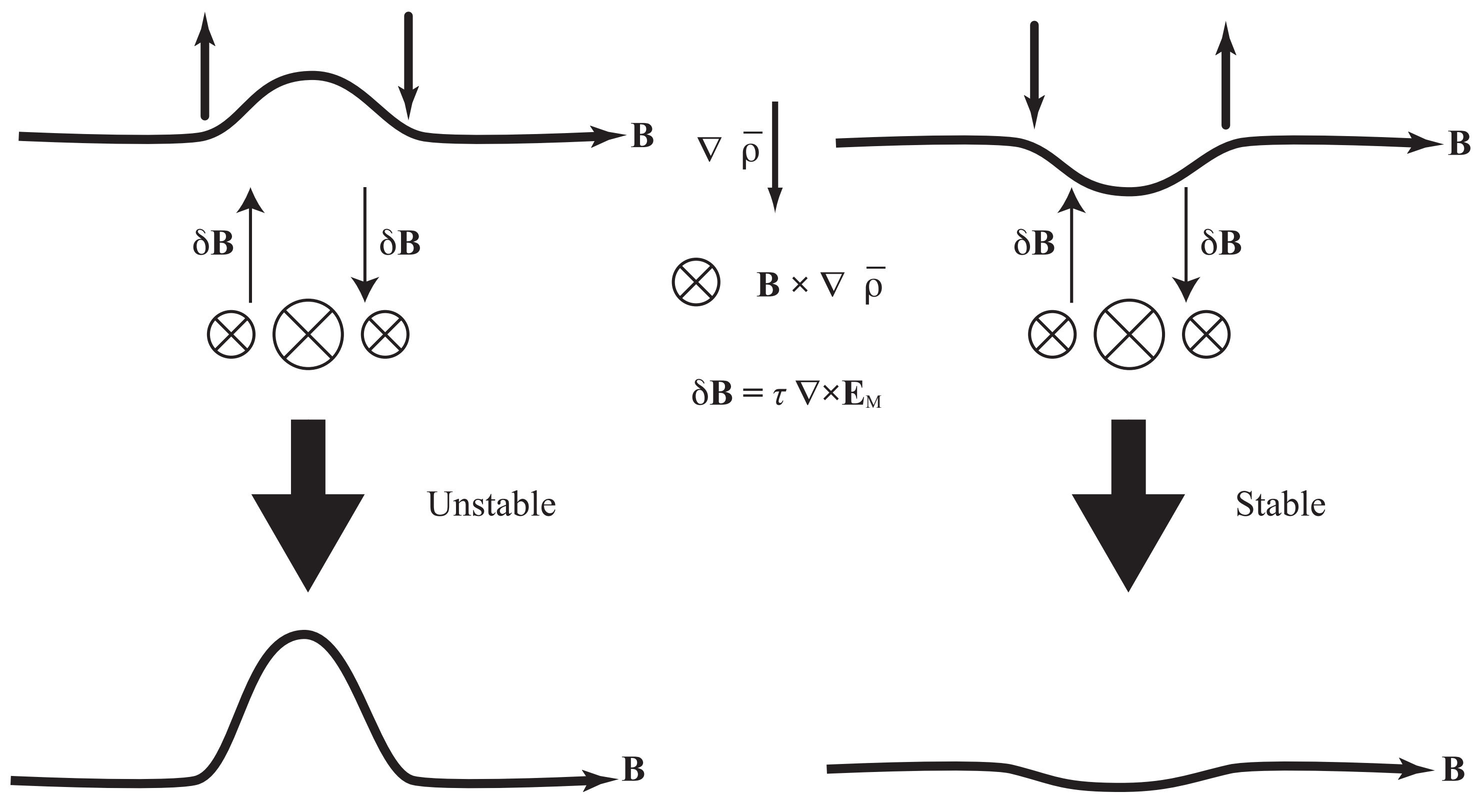


$$\delta\mathbf{u} \cdot \delta\mathbf{b} \propto \left[\cos\theta(\omega^2 - c_{\mathrm{S}}^2 k^2)\hat{\tau} + \sin\theta \omega^2 \hat{\kappa} \right] \cdot \hat{\tau} = \cos\theta(\omega^2 - c_{\mathrm{S}}^2 k^2)$$

“Magnetoclinicity” instability

$$\langle \mathbf{u}' \times \mathbf{b}' \rangle / \mu_0 = +\chi_\rho \mathbf{B} \times \nabla \overline{\rho} + \cdots$$





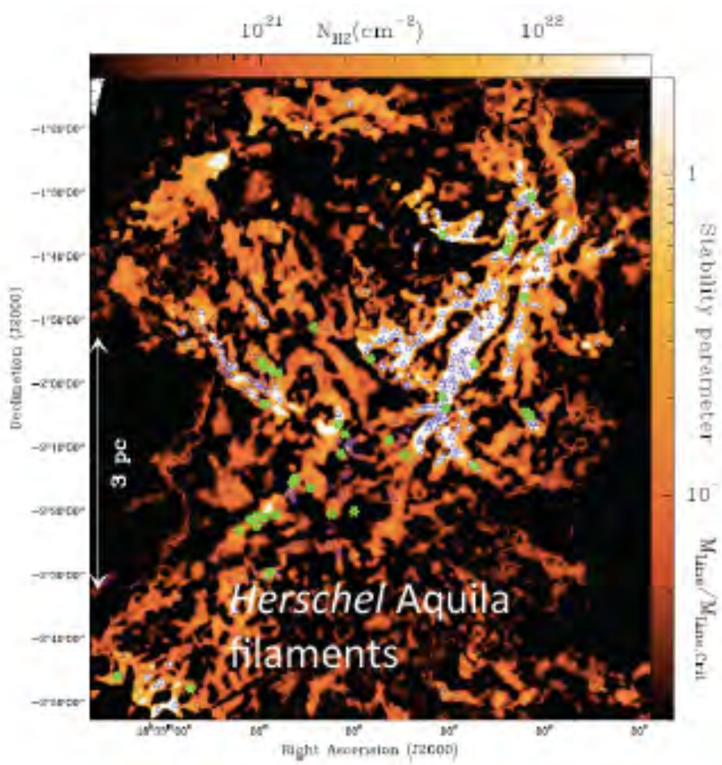
$$\frac{\partial \overline{\rho}}{\partial t} + \nabla \cdot (\overline{\rho} \mathbf{U}) = -\nabla \cdot \langle \rho' \mathbf{u}' \rangle$$

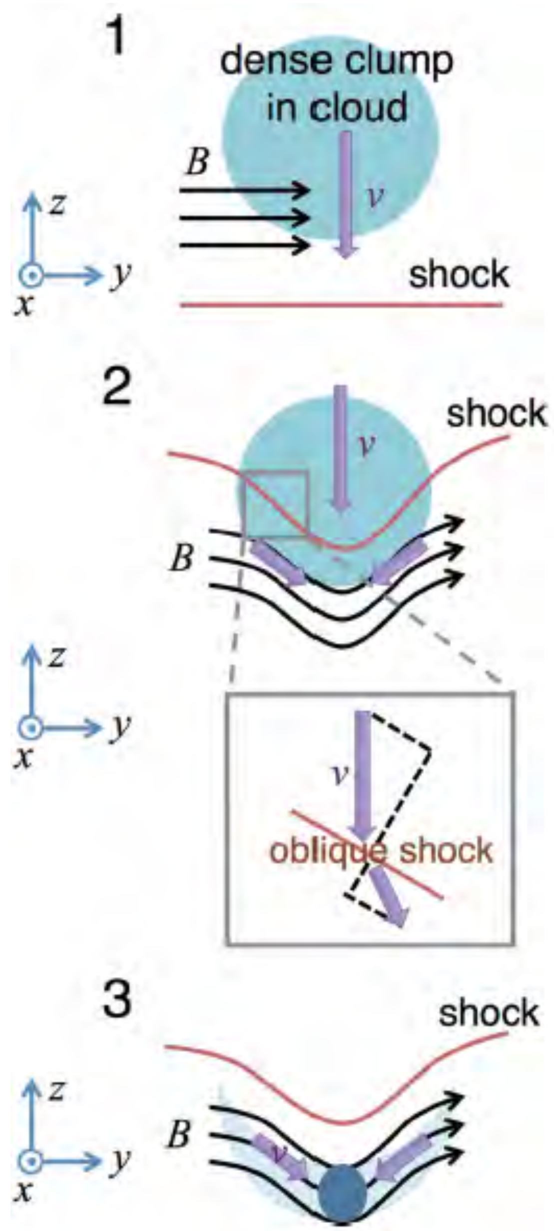
$$\langle \rho' \mathbf{u}' \rangle = -\kappa_{\overline{\rho}} \nabla \overline{\rho} - \kappa_Q \nabla Q - \kappa_D \frac{D\mathbf{U}}{DT} - \kappa_B \mathbf{B}$$

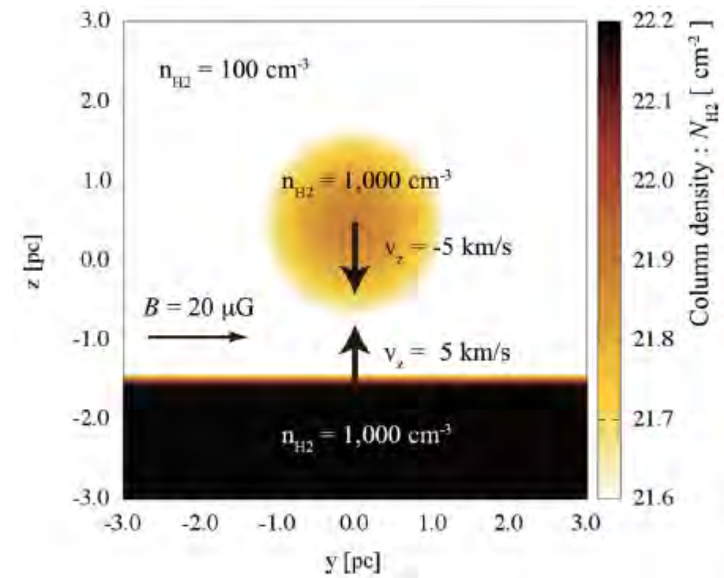
$$\frac{D}{Dt} \langle \mathbf{u}' \cdot \mathbf{b}' \rangle = -\frac{1}{2} \left\langle u'^a u'^b - \frac{1}{\mu_0 \overline{\rho}} b'^a b'^b \right\rangle \left(\frac{\partial B^b}{\partial x^a} + \frac{\partial B^a}{\partial x^b} \right) - \langle \mathbf{u}' \times \mathbf{b}' \rangle \cdot \mathbf{\Omega}$$
$$- (\gamma_s - 1) \frac{1}{\overline{\rho}} \langle \rho' \mathbf{b}' \rangle \cdot \nabla Q - (\gamma_s - 1) \frac{1}{\overline{\rho}} \langle q' \mathbf{b}' \rangle \cdot \nabla \overline{\rho} - \frac{1}{\overline{\rho}} \langle \rho' \mathbf{b}' \rangle \cdot \frac{D\mathbf{U}}{Dt}$$
$$- \langle \mathbf{u}' \cdot \mathbf{b}' \rangle \nabla \cdot \mathbf{U} + \mathbf{B} \cdot \nabla \left\langle \frac{1}{2} \mathbf{u}'^2 \right\rangle - \varepsilon_W + T_W + \text{R.T.},$$

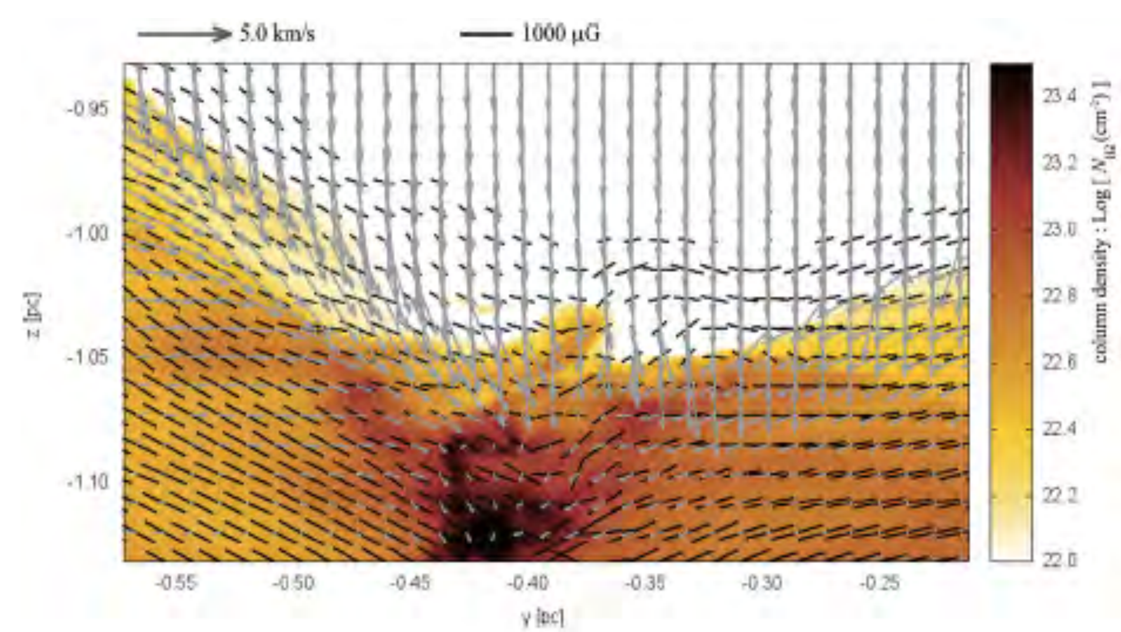
Molecular cloud filament formation through shock

Inoue, Hennebelle, Fukui, et al. PASJ, **70**, S53-1-11 (2018)









Self-consistent model (closure)

$$\langle \mathbf{u}' \times \mathbf{b}' \rangle = -(\beta + \zeta)\nabla \times \mathbf{B} + \alpha \mathbf{B} - (\nabla \zeta) \times \mathbf{B} + \gamma \nabla \times \mathbf{U}$$

$$- \chi_{\overline{\rho}} \nabla \overline{\rho} \times \mathbf{B} - \chi_Q \nabla Q \times \mathbf{B} - \chi_D \frac{D\mathbf{U}}{Dt} \times \mathbf{B}$$

$$\langle \rho' \mathbf{u}' \rangle = -\kappa_{\overline{\rho}} \nabla \overline{\rho} - \kappa_Q \nabla Q - \kappa_D \frac{D\mathbf{U}}{DT} - \kappa_B \mathbf{B}$$

Turbulent energy

$$\left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \right) \langle \mathbf{u}'^2 + \mathbf{b}'^2 \rangle / 2 = -\langle \mathbf{u}'\mathbf{u}' - \mathbf{b}'\mathbf{b}' \rangle \cdot \nabla \mathbf{U} - \langle \mathbf{u}' \times \mathbf{b}' \rangle \cdot \nabla \times \mathbf{B} - \varepsilon_K + \cdots$$

Turbulent cross helicity

$$\left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \right) \langle \mathbf{u}' \cdot \mathbf{b}' \rangle = -\langle \mathbf{u}'\mathbf{u}' - \mathbf{b}'\mathbf{b}' \rangle \cdot \nabla \mathbf{B} - \langle \mathbf{u}' \times \mathbf{b}' \rangle \cdot \nabla \times \mathbf{U} - \varepsilon_W + \cdots$$

Density variance

$$\left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \right) \left\langle \rho'^2 \right\rangle = -2 \left\langle \rho' \mathbf{u}' \right\rangle \cdot \nabla \overline{\rho} - 2 \left\langle \rho'^2 \right\rangle \nabla \cdot \mathbf{U} + \cdots$$

Turbulent residual helicity

$$\left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \right) \langle \mathbf{b}' \cdot \mathbf{j}' - \mathbf{u}' \cdot \boldsymbol{\omega} \rangle = -\langle \mathbf{u}'\mathbf{u}' - \mathbf{b}'\mathbf{b}' \rangle \cdot \nabla \boldsymbol{\Omega} - \frac{1}{\tau\beta} \langle \mathbf{u}' \times \mathbf{b}' \rangle \cdot \mathbf{B} - \varepsilon_H + \cdots$$

$$\mathbf{j}' = \nabla \times \mathbf{b}', \quad \boldsymbol{\omega}' = \nabla \times \mathbf{u}', \quad \boldsymbol{\Omega} = \nabla \times \mathbf{U}$$

Summary

- **Theory for strongly nonlinear and inhomogeneous compressible MHD turbulence**
- **Transports in strongly compressible MHD turbulence**
- **Dynamo in the presence of strong compressibility: magnetoclinicity (density variance) effect**
- **Cross diffusion (density variance) and transport along the mean magnetic field (compressional cross helicity)**

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Dynamo coupled
with large-scale flows

Connotations of Mean-Field Dynamo

- Kinematic

  Velocity prescribed

  No back-reaction through **J** x **B** nor the Reynolds (Maxwell) stress

- α and Ω effects

  $$\mathbf{E}_{\mathrm{M}} \equiv \langle \mathbf{u}' \times \mathbf{b}' \rangle = \alpha \mathbf{B} - \beta \nabla \times \mathbf{B}$$

- Transport coefficients as parameters

  α and β are adjustable parameters

- Azimuthal average

  Axisymmetry or homogeneity in the azimuthal direction

- Incompressible

Modelling in dynamos

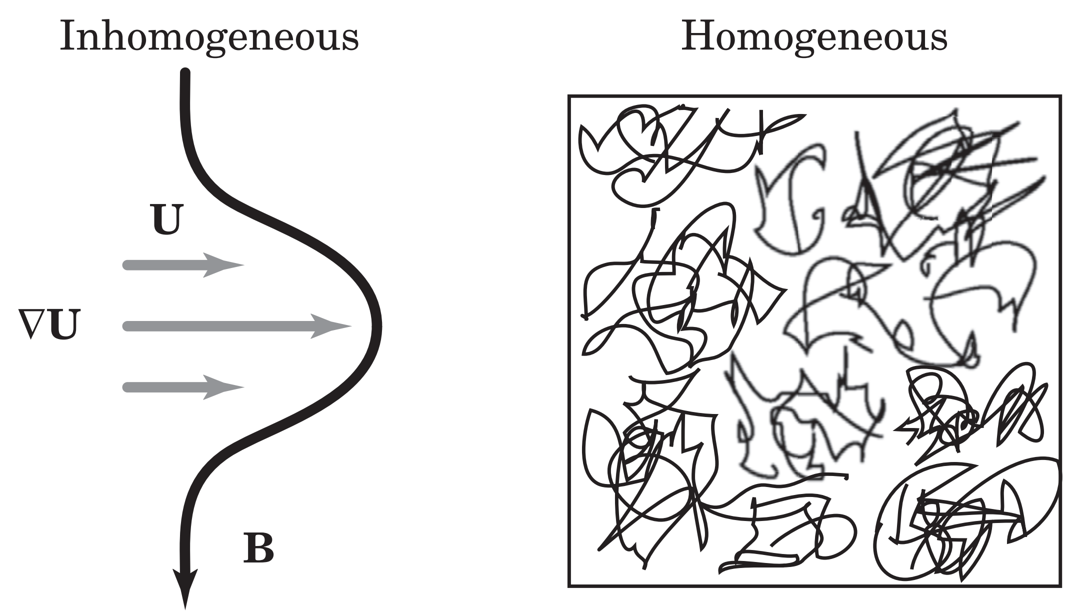
$$\langle \mathbf{u}' \times \mathbf{b}' \rangle^\alpha = \alpha^{\alpha a} B^a + \beta^{\alpha a b} \frac{\partial B^a}{\partial x^b} + \cdots$$

Mean field

$$\mathbf{b} = \mathbf{B} + \mathbf{b}', \quad \mathbf{B} = \langle \mathbf{b} \rangle$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \nabla \times \langle \mathbf{u}' \times \mathbf{b}' \rangle + \eta \nabla^2 \mathbf{B}$$

$(\mathbf{B} \cdot \nabla)\mathbf{U}$ ⟶ differential rotation, "Ω effect"



Turbulence

$\mathbf{U} = \mathbf{U}_0(\text{constant}) \ \text{ or } \ \mathbf{0}$

$$\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{u}' = (\mathbf{B} \cdot \nabla)\mathbf{b}' + (\mathbf{b}' \cdot \nabla)\mathbf{B} - (\mathbf{u}' \cdot \nabla)\mathbf{U} + \cdots$$

$$\frac{\partial \mathbf{b}'}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{b}' = (\mathbf{B} \cdot \nabla)\mathbf{u}' - (\mathbf{u}' \cdot \nabla)\mathbf{B} + (\mathbf{b}' \cdot \nabla)\mathbf{U} + \cdots$$

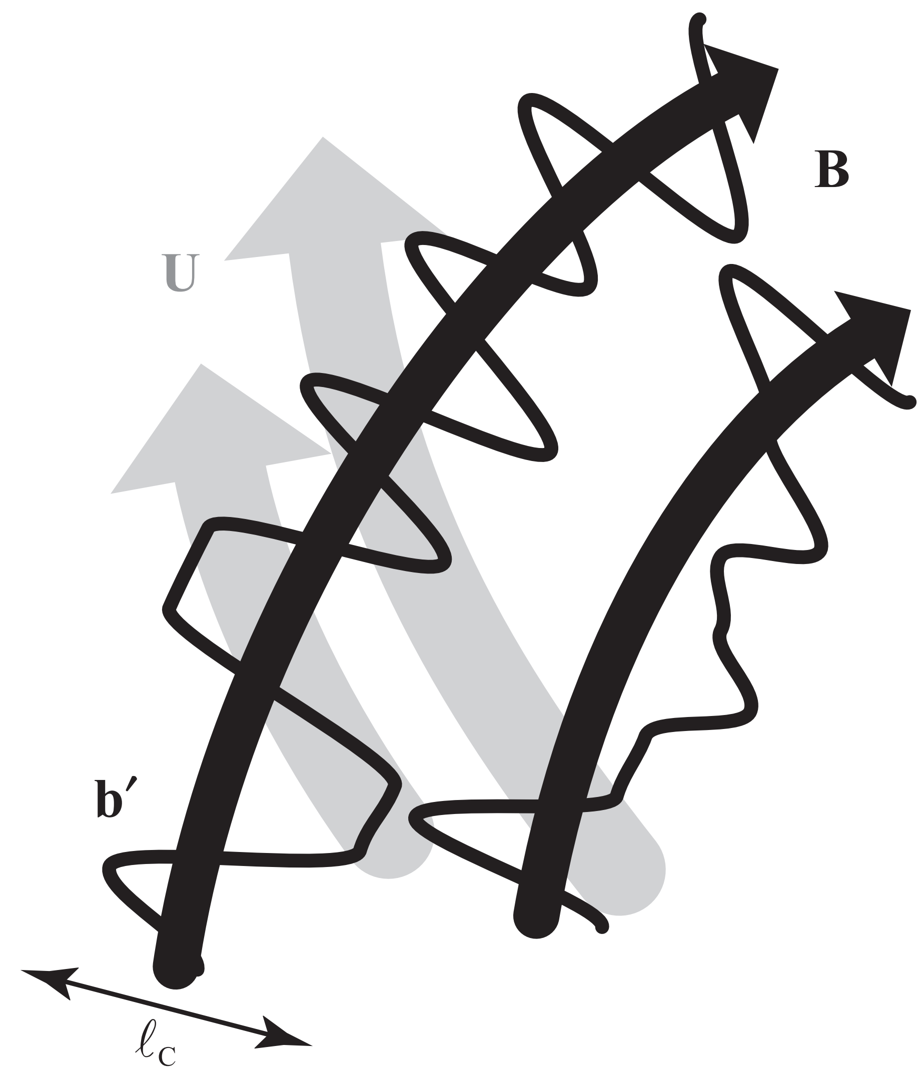
⟶ $\langle \mathbf{u}' \times \mathbf{b}' \rangle^\alpha = \alpha^{\alpha a} B^a + \beta^{\alpha a b} \dfrac{\partial B^a}{\partial x^b} + \cdots$ "Ansatz"

$$\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{u}' = (\mathbf{B}\cdot\nabla)\mathbf{b}' + (\mathbf{b}'\cdot\nabla)\mathbf{B} - (\mathbf{u}'\cdot\nabla)\mathbf{U} + \cdots$$

$$\frac{\partial \mathbf{b}'}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{b}' = (\mathbf{B}\cdot\nabla)\mathbf{u}' - (\mathbf{u}'\cdot\nabla)\mathbf{B} + (\mathbf{b}'\cdot\nabla)\mathbf{U} + \cdots$$

$$\left\langle \frac{\partial \mathbf{u}'}{\partial t} \times \mathbf{b}' \right\rangle + \left\langle \mathbf{u}' \times \frac{\partial \mathbf{b}'}{\partial t} \right\rangle = \cdots$$

$$\tau\langle \mathbf{u}' \times [(\mathbf{b}'\cdot\nabla)\mathbf{U}] + [(\mathbf{u}'\cdot\nabla)\mathbf{U}] \times \mathbf{b}'\rangle^{\alpha}$$
$$= \epsilon^{\alpha a b}\tau\langle u'^{a} b'^{c}\rangle \frac{\partial U^{b}}{\partial x^{c}} - \epsilon^{\alpha b a}\tau\langle b'^{a} u'^{c}\rangle \frac{\partial U^{b}}{\partial x^{c}}$$
$$= \tau\left(\langle u'^{a} b'^{c}\rangle + \langle u'^{c} b'^{a}\rangle\right)\epsilon^{\alpha a b}\frac{\partial U^{b}}{\partial x^{c}}$$



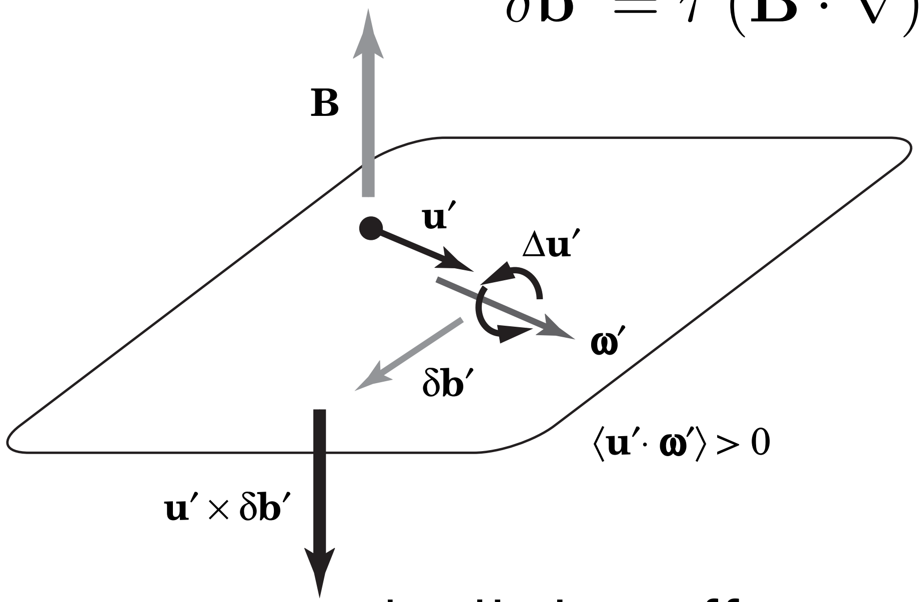
→ $$\langle \mathbf{u}' \times \mathbf{b}'\rangle = \cdots + \tau\langle \mathbf{u}'\cdot\mathbf{b}'\rangle \nabla\times\mathbf{U} + \cdots$$

cross helicity

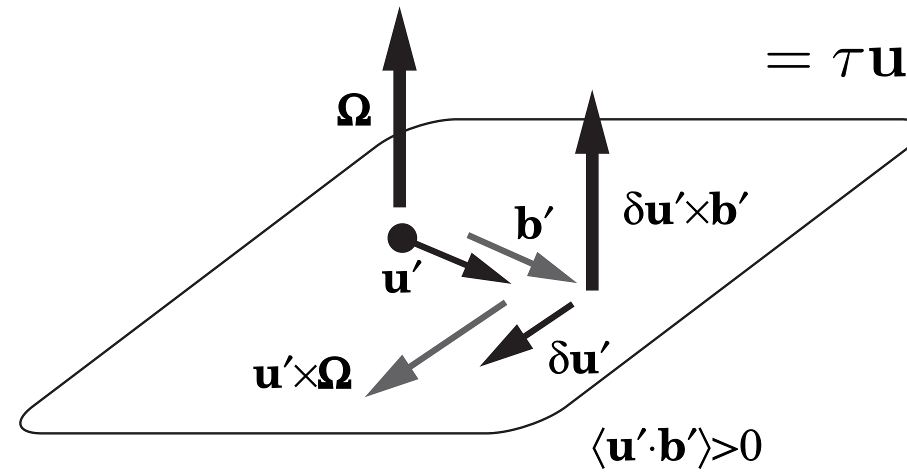
α and cross-helicity effects

(Yokoi, GAFD **107**, 114, 2013)

$$\delta \mathbf{b}' = \tau \left(\mathbf{B} \cdot \nabla \right) \mathbf{u}' + \cdots$$



$$\delta \mathbf{u}' = \tau \left(\mathbf{u}' \cdot \nabla \right) \mathbf{U} + \cdots$$
$$= \tau \mathbf{u}' \times \mathbf{\Omega} + \cdots$$

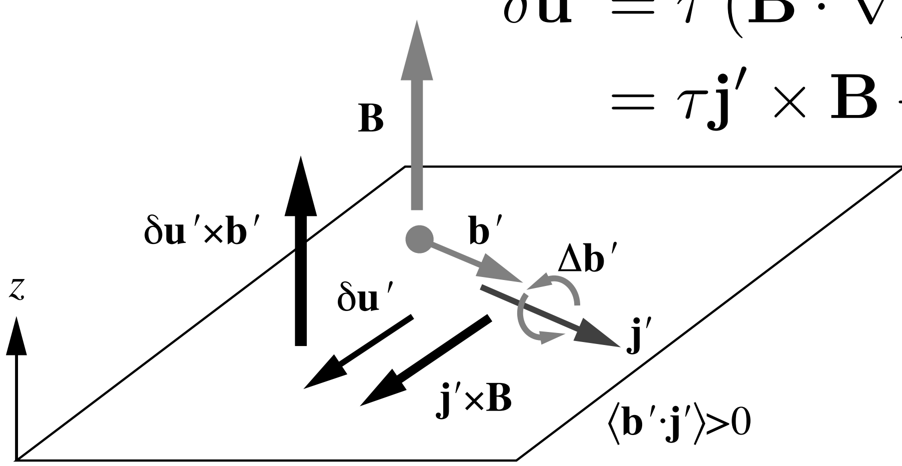


mean vorticity
$\mathbf{\Omega} = \nabla \times \mathbf{U}$

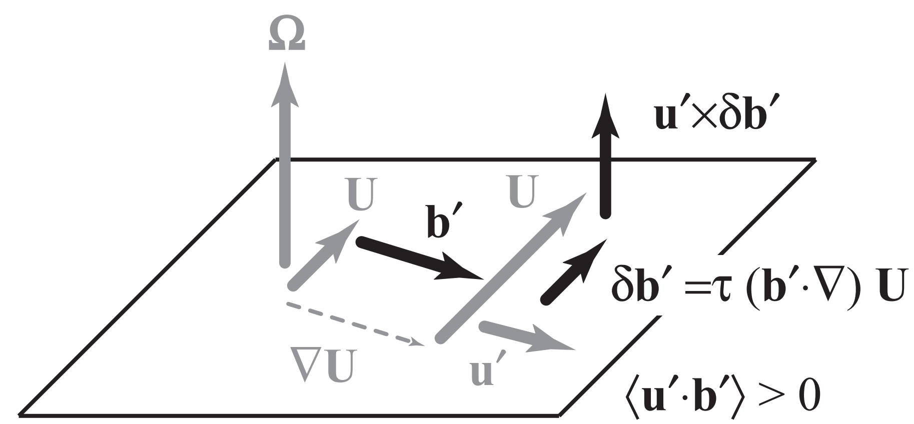
helicity effect

cross-helicity effect

$$\delta \mathbf{u}' = \tau \left(\mathbf{B} \cdot \nabla \right) \mathbf{b}' + \cdots$$
$$= \tau \mathbf{j}' \times \mathbf{B} + \cdots$$



$$\delta \mathbf{b}' = \tau \left(\mathbf{b}' \cdot \nabla \right) \mathbf{U} + \cdots$$



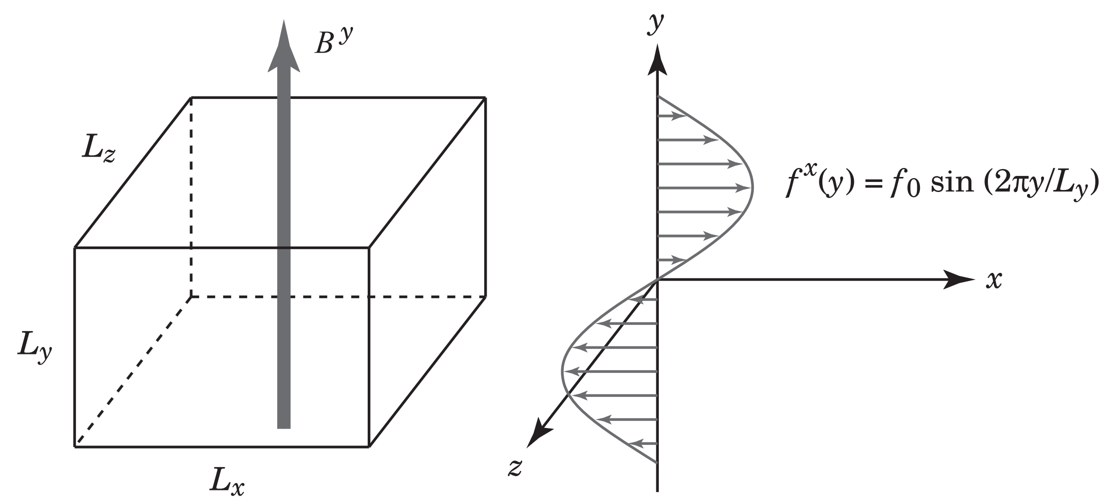
α dynamo

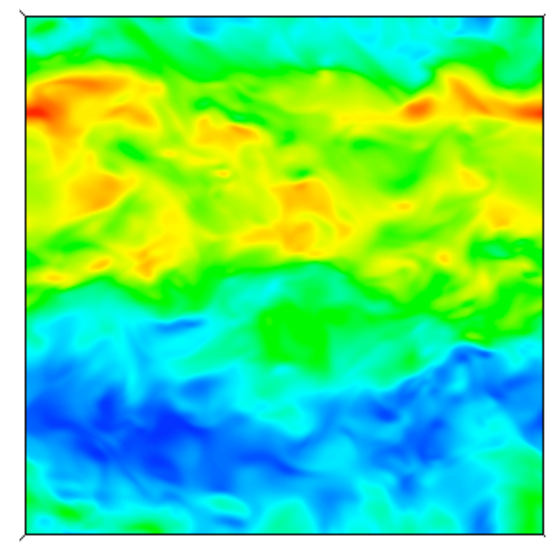
$$\langle \mathbf{u}' \times \mathbf{b}' \rangle = \alpha \mathbf{B} - \beta \nabla \times \mathbf{B} + \gamma \nabla \times \mathbf{U}$$

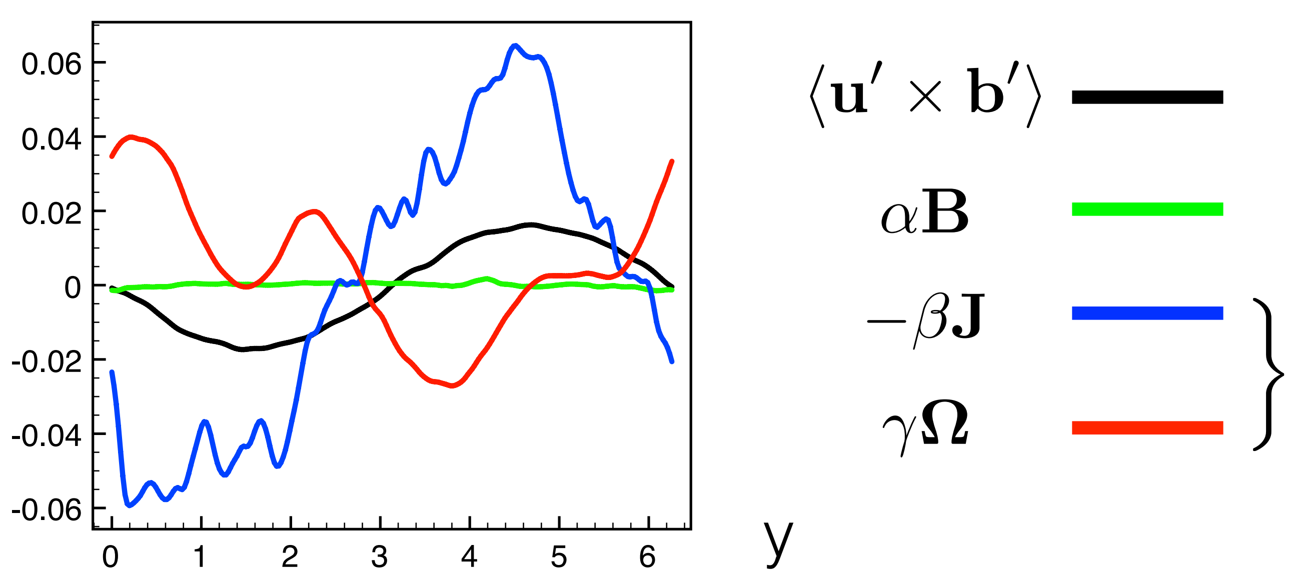
cross-helicity dynamo

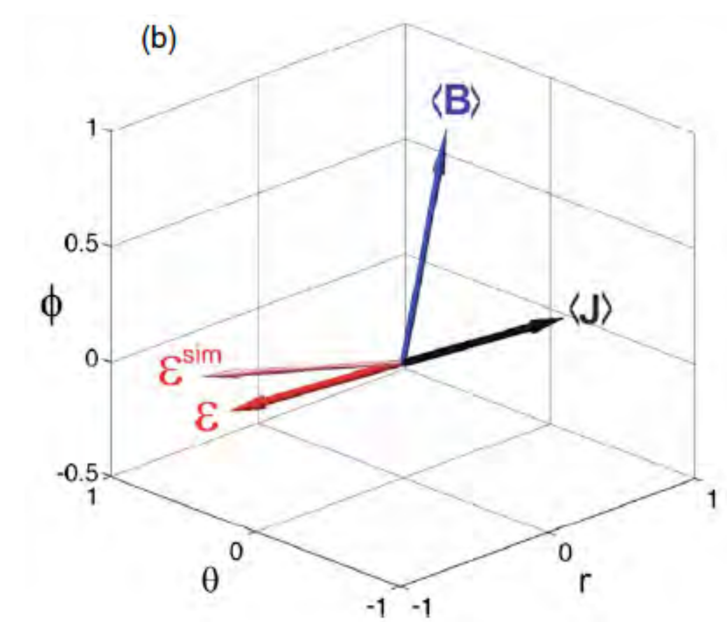
Validation of expressions

DNS of electromotive force in Kolmogorov flow (Yokoi & Balarac, 2011)









(Rahbarnia, et al. ApJ, 2012)

Toy model for solar-activity cycle

Yokoi, Schmitt, Pipin, et al., Astrophys. J. **824**, 67 (2016)

$$\begin{cases} \dfrac{\partial A}{\partial t} = \beta \dfrac{\partial^2 A}{\partial x^2} + \alpha B \\[2ex] \dfrac{\partial B}{\partial t} = \beta \dfrac{\partial^2 B}{\partial x^2} - \dfrac{\partial}{\partial x}\left(\gamma \dfrac{\partial U}{\partial x} \right) \\[2ex] \dfrac{\partial \gamma}{dt} = \beta \dfrac{\partial^2 \gamma}{\partial x^2} - \alpha\tau \dfrac{\partial U}{\partial x} \dfrac{\partial A}{\partial x} \end{cases}$$

$$\longleftarrow \quad \frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\cancel{\mathbf{U} \times \mathbf{B}_1} + \alpha \mathbf{B}_0 - \beta \mathbf{J}_1)$$

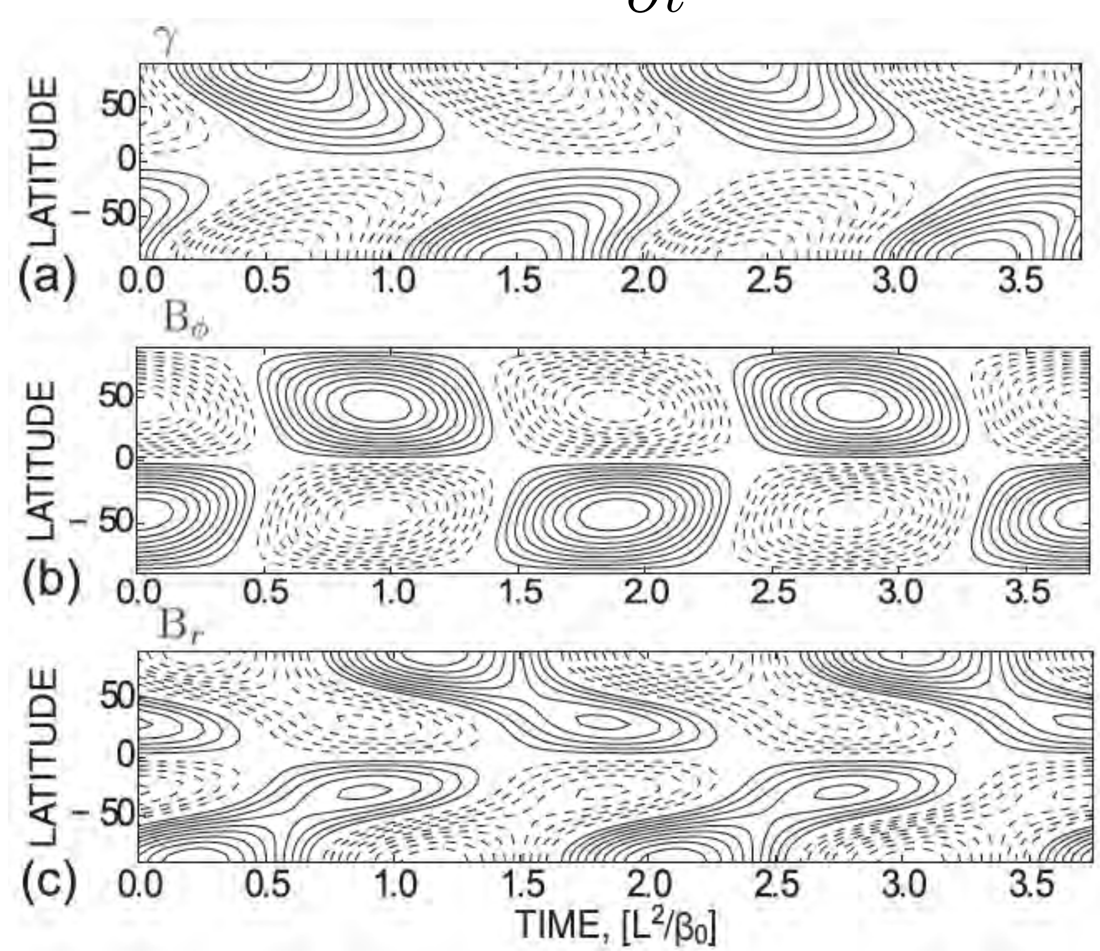
$$\longleftarrow \quad \frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times (\cancel{\mathbf{U} \times \mathbf{B}_0} - \beta \mathbf{J}_0 + \gamma \mathbf{\Omega})$$

$$\longleftarrow \quad \frac{\partial W}{\partial t} = -\alpha \mathbf{B}_1 \cdot \mathbf{\Omega} + \cdots$$

Cross helicity

Toroidal

Poloidal

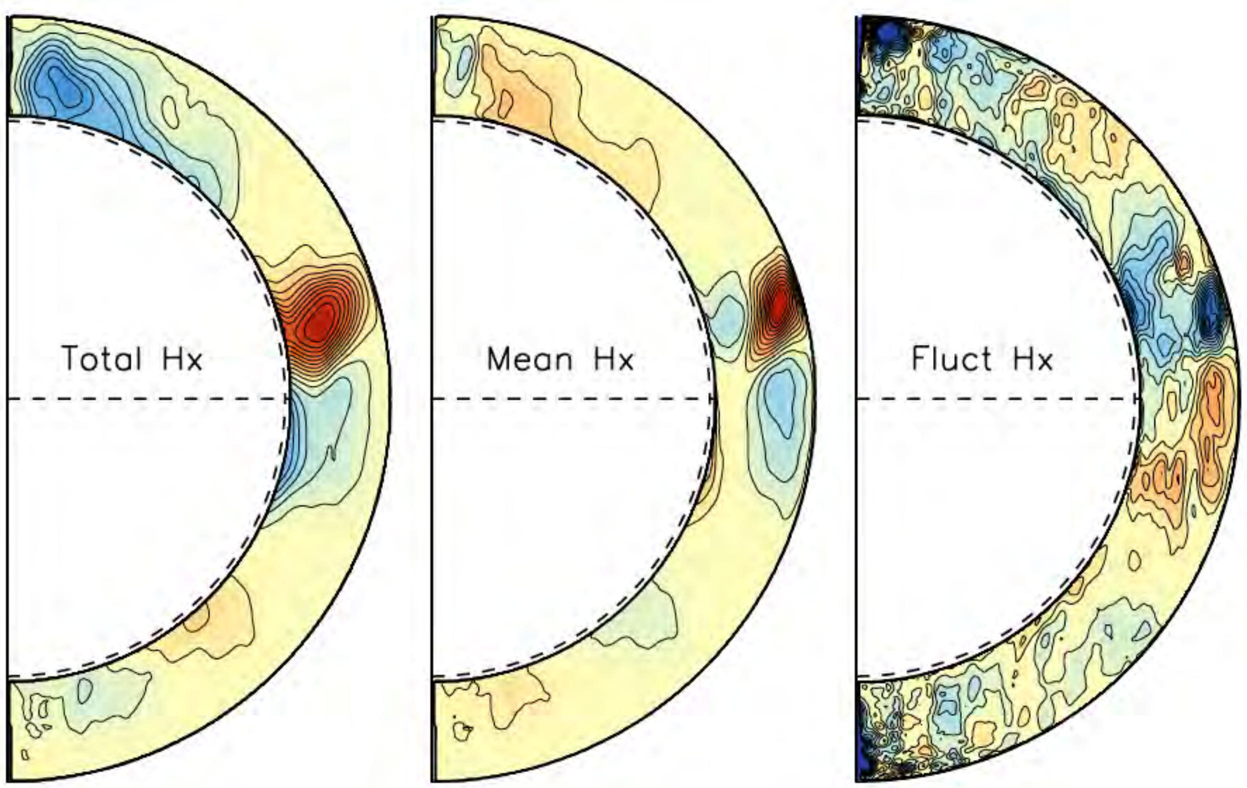


Relative importance of cross-helicity to differential-rotation effects

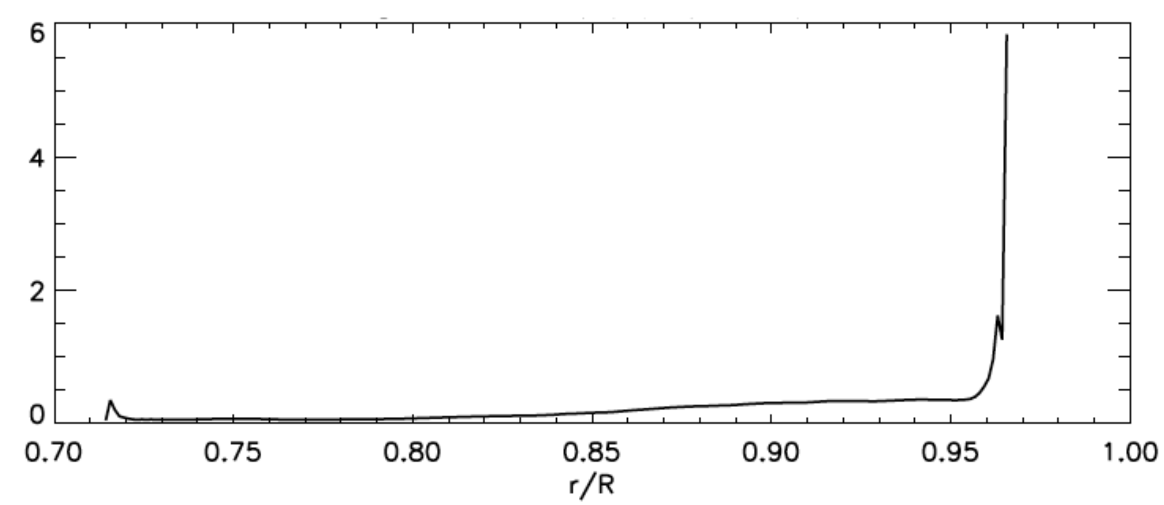
$$\frac{\text{(cross-helicity effect)}}{\text{(differential-rotation effect)}} = \frac{|\nabla \times (\gamma \nabla \times \mathbf{U})|}{|\nabla \times (\mathbf{U} \times \mathbf{B})|}$$

$$\sim \frac{\langle \mathbf{u}' \cdot \mathbf{b}' \rangle}{D \left(\dfrac{\partial U}{\partial r} \right) B^r} \frac{\tau_{\rm turb}}{\tau_{\rm mean}} \sim \frac{\langle \mathbf{u}' \cdot \mathbf{b}' \rangle}{\delta U B^r} Ro^{-1} = \frac{\langle \mathbf{u}' \cdot \mathbf{b}' \rangle}{\delta U B^r} \frac{K/\varepsilon}{D/\delta U}$$

Spatial distribution of cross helicity



Relative magnitude of the cross-helicity to the differential rotation terms

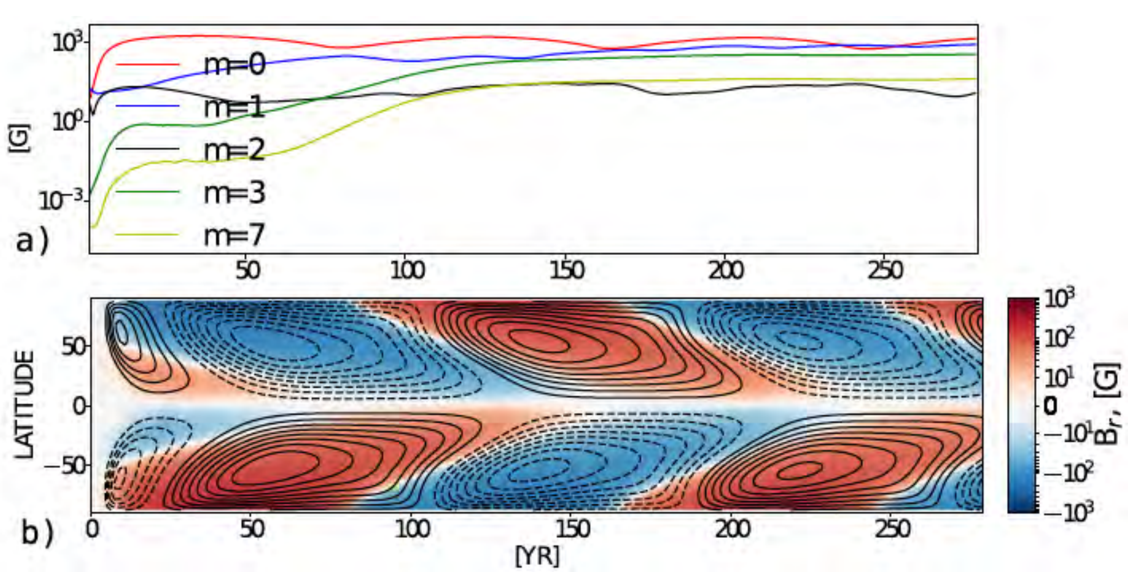


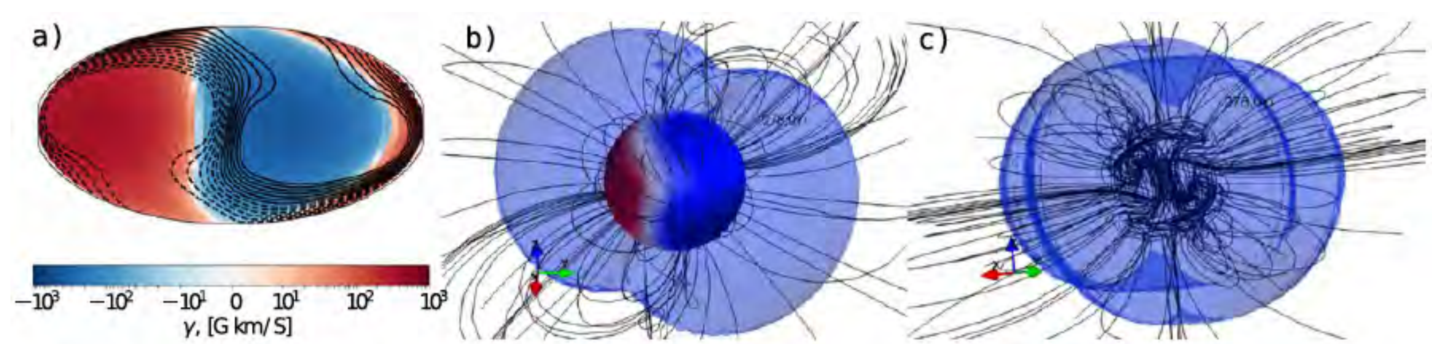
Provided by Mark Miesch (2016)

Cross-helicity dynamo for fully convective stars (cool stars)

Pipin & Yokoi, Astrophys. J. **859**, 18 (2018)

For a particular case of the fast rotating stars with solid body rotation regime, we show a possibility to sustain the strong dipolar B-field via $\alpha^2\gamma^2$ dynamo.





Solution for magnetic-field variation

Mean induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B} + \underline{\mathbf{E}_{\rm M}}) + \eta \nabla^2 \mathbf{B} \qquad \mathbf{E}_{\rm M} = -\beta \mathbf{J} + \alpha \mathbf{B} + \gamma \boldsymbol{\Omega}$$

Turbulence

$$\mathbf{B} = \mathbf{B}_0 + \delta \mathbf{B}, \quad \mathbf{J} = \mathbf{J}_0 + \delta \mathbf{J}$$

Reference $$\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}_0) + \nabla \times (\alpha \mathbf{B}_0 - \beta \nabla \times \mathbf{B}_0)$$

Modulation $$\frac{\partial \delta \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \delta \mathbf{B}) - \nabla \times \left[\beta \nabla \times \left(\delta \mathbf{B} - \frac{\gamma}{\beta} \mathbf{U} \right) \right]$$

→ $$\delta \mathbf{B} = \frac{\gamma}{\beta} \mathbf{U} = C_{\rm W} \frac{W}{K} \mathbf{U} \qquad \frac{|W|}{K} = \frac{|\langle \mathbf{u}' \cdot \mathbf{b}' \rangle|}{\langle \mathbf{u}'^2 + \mathbf{b}'^2 \rangle / 2} \leq 1$$

c.f. $$\nabla \times \left(\frac{\gamma}{\beta} \mathbf{U} \right) = \frac{\gamma}{\beta} \nabla \times \mathbf{U} + \nabla \left(\frac{\gamma}{\beta} \right) \times \mathbf{U}$$

Solution for momentum variation

Mean momentum equation

$$\frac{\partial \mathbf{U}}{\partial t} = \mathbf{U} \times \boldsymbol{\Omega} + \mathbf{J} \times \mathbf{B} \underline{- \nabla \cdot \boldsymbol{\mathcal{R}}} + \mathbf{F} - \nabla \left(P + \frac{1}{2}\mathbf{U}^2 + \left\langle \frac{1}{2}\mathbf{b}'^2 \right\rangle \right)$$

Turbulence

$$\mathbf{J} = \sigma \left(\mathbf{E} + \mathbf{U} \times \mathbf{B} + \underline{\mathbf{E}_{\rm M}} \right)$$

Turbulence

$$\begin{cases} \mathcal{R}^{\alpha\beta} = \frac{2}{3} K_{\rm R} \delta^{\alpha\beta} - \nu_{\rm K} \mathcal{S}^{\alpha\beta} + \nu_{\rm M} \mathcal{M}^{\alpha\beta} \\ \mathbf{E}_{\rm M} = -\beta \mathbf{J} + \alpha \mathbf{B} + \gamma \boldsymbol{\Omega} \end{cases}$$

Mean Lorentz force $\quad \mathbf{J} \times \mathbf{B} = \frac{1}{\beta} \left(\mathbf{U} \times \mathbf{B} \right) \times \mathbf{B} + \frac{\gamma}{\beta} \boldsymbol{\Omega} \times \mathbf{B} - \frac{1}{\beta} \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \varphi \right) \times \mathbf{B}$

$$\mathbf{U} = \mathbf{U}_0 + \delta \mathbf{U}, \quad \boldsymbol{\Omega} = \boldsymbol{\Omega}_0 + \delta \boldsymbol{\Omega}$$

Reference $\quad \frac{\partial \boldsymbol{\Omega}_0}{\partial t} = \nabla \times \left[\mathbf{U}_0 \times \boldsymbol{\Omega}_0 + \nu_{\rm K} \nabla^2 \mathbf{U_0} + \mathbf{F} - \frac{1}{\beta} \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \varphi \right) \times \mathbf{B} \right]$

Modulation $\quad \frac{\partial \delta \boldsymbol{\Omega}}{\partial t} = \nabla \times \left[\left(\delta \mathbf{U} - \frac{\gamma}{\beta} \mathbf{B} \right) \times \boldsymbol{\Omega}_0 + \nu_{\rm K} \nabla^2 \left(\delta \mathbf{U} - \frac{\gamma}{\beta} \mathbf{B} \right) \right]$

$\longrightarrow \quad \delta \mathbf{U} = \frac{\gamma}{\beta} \mathbf{B} = C_\gamma \frac{W}{K} \mathbf{B}$

$$\frac{|W|}{K} = \frac{|\langle \mathbf{u}' \cdot \mathbf{b}' \rangle|}{\langle \mathbf{u}'^2 + \mathbf{b}'^2 \rangle / 2} \leq 1$$

Global flow generation

http://www.inflowimages.com/

Lake Michigan (Hess et al. 1988)

on the Mars, imaged by rover
(Greeley et al. 2007)

Vortical structure at the solar surface
(Wedemeyer-Böhm et al. 2012)

Vortex generation

Vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) + \underbrace{\frac{\nabla \rho \times \nabla p}{\rho^2}}_{\text{baroclinicity}} + \nu \nabla^2 \boldsymbol{\omega}$$

cf., Biermann battery $-\frac{\nabla n_e \times \nabla p_e}{n_e^2 e}$

Mean vorticity

$$\boldsymbol{\Omega} = \nabla \times \mathbf{U}$$

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} = \nabla \times (\mathbf{U} \times \boldsymbol{\Omega}) + \nabla \times \underbrace{\langle \mathbf{u}' \times \boldsymbol{\omega}' \rangle}_{\mathbf{V}_{\rm M}} + \nu \nabla^2 \boldsymbol{\Omega}$$

vortexmotive force

cf., Mean magnetic field $\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \nabla \times \langle \mathbf{u}' \times \mathbf{b}' \rangle + \eta \nabla^2 \mathbf{B}$

electromotive force

Reynolds stress $\mathcal{R}^{ij} = \langle u'^i u'^j \rangle$ $\qquad V_{\rm M}^i = -\frac{\partial \mathcal{R}^{ij}}{\partial x^j} + \frac{\partial K}{\partial x^i}$

Theoretical formulation

Basic field: homogeneous isotropic but non-mirrosymmetric

$$\frac{\langle u'_{0\alpha}(\mathbf{k};\tau) u'_{0\beta}(\mathbf{k};\tau) \rangle}{\delta(\mathbf{k}+\mathbf{k}')} = D_{\alpha\beta}(\mathbf{k}) Q_0(k;\tau,\tau') + \frac{i}{2}\frac{k_a}{k^2}\epsilon_{\alpha\beta a} H_0(k;\tau,\tau')$$

Calculation of the Reynolds stress

$$\left\langle {u'}^\alpha {u'}^\beta \right\rangle = \left\langle {u'_{\mathrm{B}}}^\alpha {u'_{\mathrm{B}}}^\beta \right\rangle + \left\langle {u'_{\mathrm{B}}}^\alpha {u'_{01}}^\beta \right\rangle + \left\langle {u'_{01}}^\alpha {u'_{\mathrm{B}}}^\beta \right\rangle + \cdots$$
$$+ \left\langle {u'_{\mathrm{B}}}^\alpha {u'_{10}}^\beta \right\rangle + \left\langle {u'_{10}}^\alpha {u'_{\mathrm{B}}}^\beta \right\rangle + \cdots$$

$$\left\langle {u'}^\alpha {u'}^\beta \right\rangle_{\mathrm{D}} = -\nu_{\mathrm{T}} \mathcal{S}^{\alpha\beta} + \left[\Gamma^\alpha \left(\Omega^\beta + 2\omega_{\mathrm{F}}^\beta \right) + \Gamma^\beta \left(\Omega^\alpha + 2\omega_{\mathrm{F}}^\alpha \right) \right]_{\mathrm{D}}$$

where $\mathcal{S}^{\alpha\beta} = \frac{\partial U^\alpha}{\partial x^\beta} + \frac{\partial U^\beta}{\partial x^\alpha} - \frac{2}{3}\nabla\cdot\mathbf{U}\delta^{\alpha\beta}$

mixing length

$\nu_{\mathrm{T}} \sim \tau u^2 \sim u\ell$

Eddy viscosity $\nu_{\mathrm{T}} = \frac{7}{15}\int \mathrm{d}\mathbf{k} \int_{-\infty}^{t} d\tau_1 \; G(k;\tau,\tau_1) Q(k;\tau,\tau_1)$

Helicity-related coefficient $\mathbf{\Gamma} = \frac{1}{30}\int k^{-2}\mathrm{d}\mathbf{k} \int_{-\infty}^{t} d\tau_1 \; G(k;\tau,\tau_1) \nabla H(k;\tau,\tau_1)$

helicity inhomogeneity is essential

Eddy viscosity + Helicity model

Reynolds stress

Yokoi & Yoshizawa (1993) Phys. Fluids A**5**, 464

$$\mathcal{R}_{\alpha\beta} \equiv \langle u'_\alpha u'_\beta \rangle$$

$$= \frac{2}{3} K \delta_{\alpha\beta} - \nu_\mathrm{T} \left(\frac{\partial U_\alpha}{\partial x_\beta} + \frac{\partial U_\beta}{\partial x_\alpha} \right) + \eta \left[\Omega_\alpha \frac{\partial H}{\partial x_\beta} + \Omega_\beta \frac{\partial H}{\partial x_\alpha} - \frac{2}{3} \delta_{\alpha\beta} \left(\boldsymbol{\Omega} \cdot \nabla \right) H \right]$$

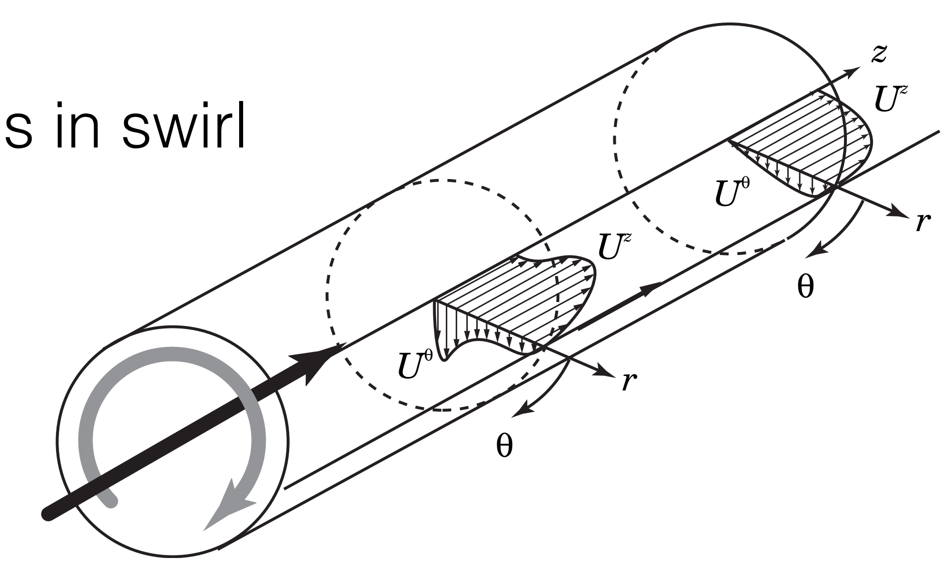
$$\nu_\mathrm{T} = C_\nu \tau K, \quad \tau = K/\epsilon, \quad \eta = C_H \tau (K^3/\epsilon^2)$$

Turbulence quantities

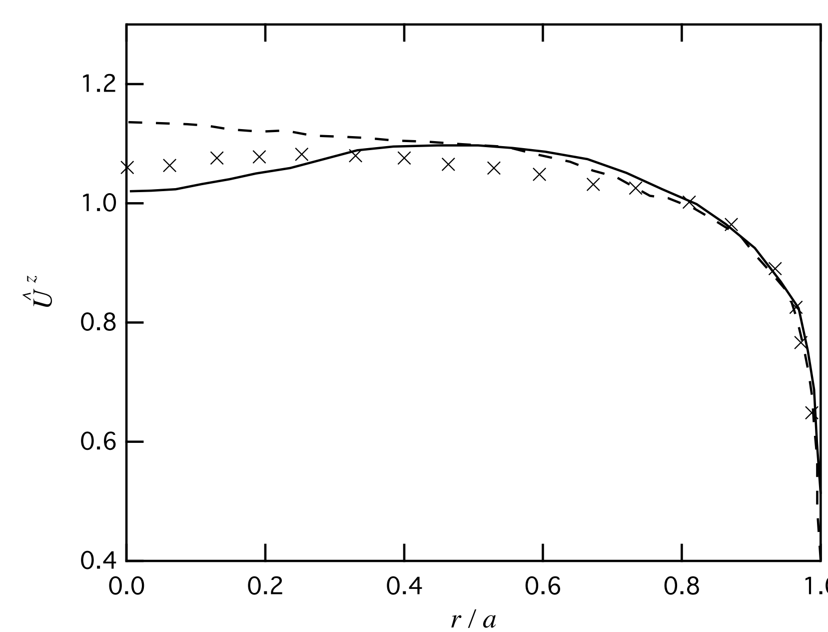
$$K \equiv \frac{1}{2} \langle \mathbf{u}'^2 \rangle, \quad \epsilon \equiv \nu \left\langle \frac{\partial u'_b}{\partial x_a} \frac{\partial u'_b}{\partial x_a} \right\rangle,$$

$$H \equiv \langle \mathbf{u}' \cdot \boldsymbol{\omega}' \rangle, \quad \epsilon_H \equiv 2\nu \left\langle \frac{\partial u'_b}{\partial x_a} \frac{\partial \omega'_b}{\partial x_a} \right\rangle$$

Velocity profiles in swirl

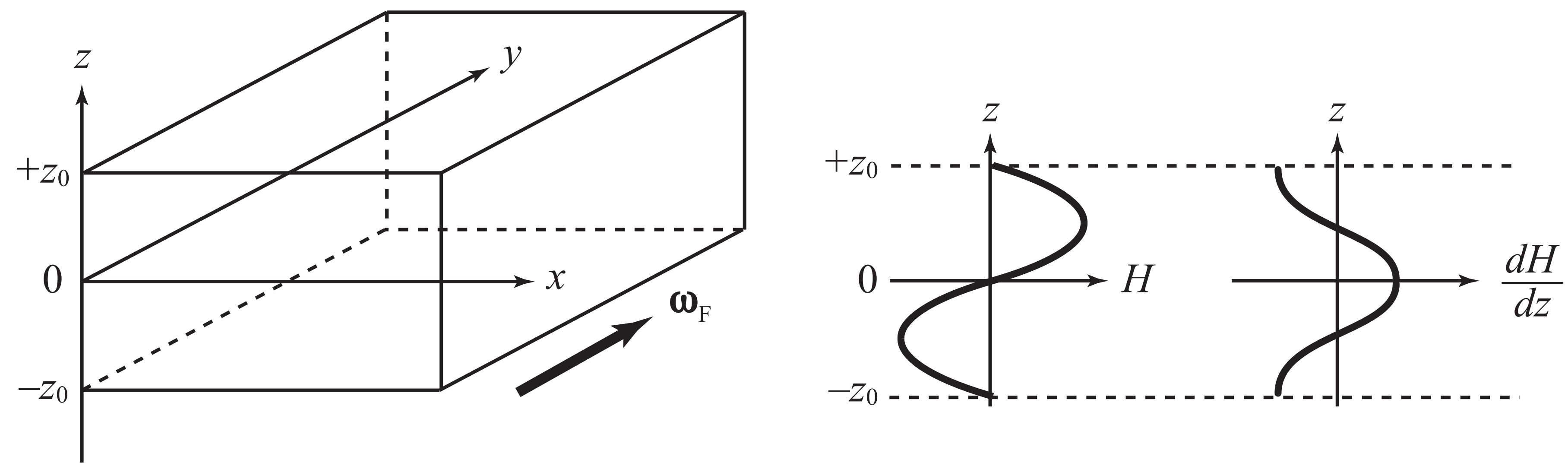


Helicity turbulence model



Yokoi & Brandenburg, Phys. Rev. E. (2016)

DNS set-up



Set-up of the turbulence and rotation $\boldsymbol{\omega}_F$ (left), the schematic spatial profile of the turbulent helicity H (= $\langle \mathbf{u}' \cdot \boldsymbol{\omega}' \rangle$) (center) and its derivative dH/dz (right).

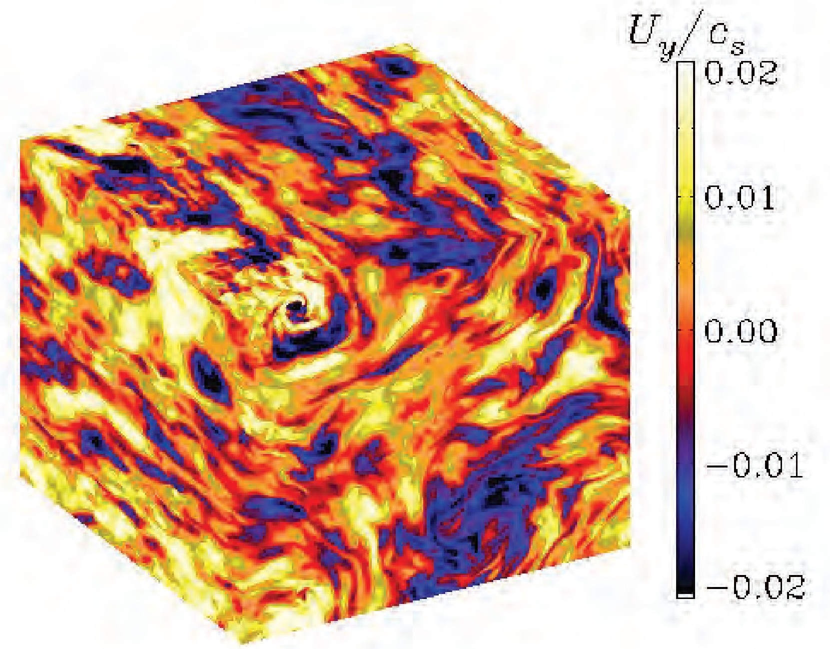
Rotation $\boldsymbol{\omega}_{\mathrm{F}} = (\omega_{\mathrm{F}}^x, \omega_{\mathrm{F}}^y, \omega_{\mathrm{F}}^z) = (0, \omega_{\mathrm{F}}, 0)$

Inhomogeneous turbulent helicity $H(z) = H_0 \sin(\pi z / z_0)$

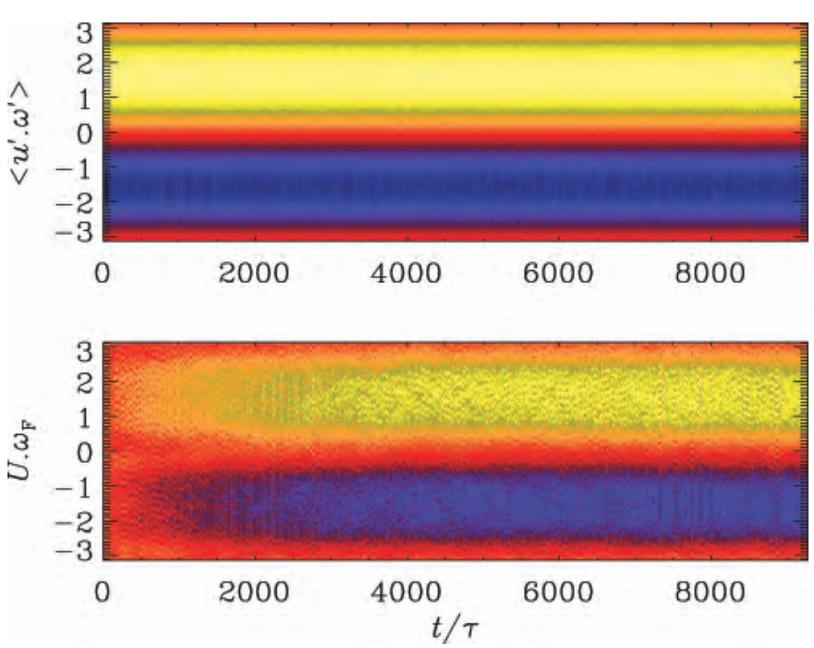
Run	k_f/k_1	Re	Co	$\eta/(\nu_T \tau^2)$
A	15	60	0.74	0.22
B1	5	150	2.6	0.27
B2	5	460	1.7	0.27
B3	5	980	1.6	0.51
C1	30	18	0.63	0.50
C2	30	80	0.55	0.03
C3	30	100	0.46	0.08

Summary of DNS results

Global flow generation



Axial flow component U^y on the periphery of the domain



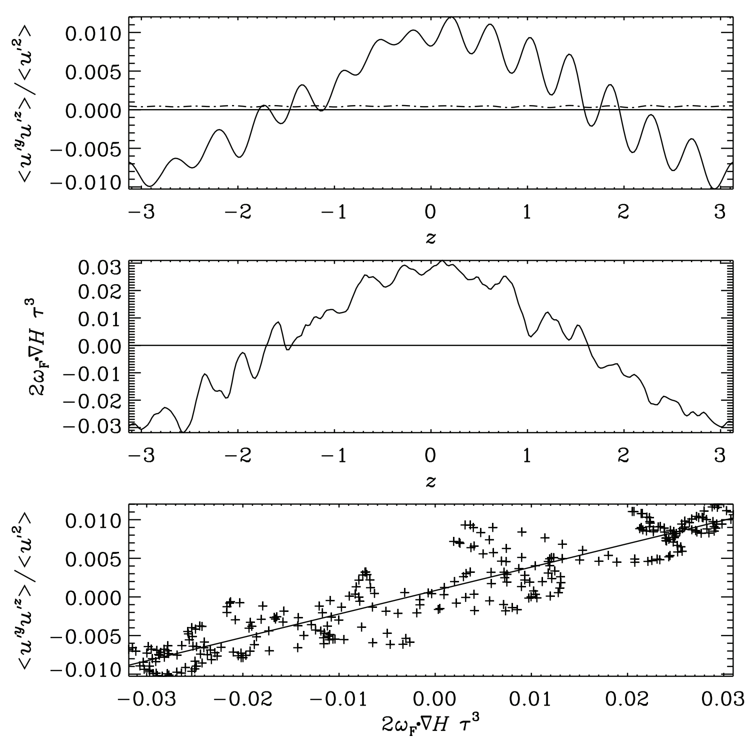
Turbulent helicity $\langle \mathbf{u}' \cdot \boldsymbol{\omega}' \rangle$ (top) and mean-flow helicity $\mathbf{U} \cdot 2\boldsymbol{\omega}_F$ (bottom)

Reynolds stress

$$\left\langle u'^{\alpha} u'^{\beta} \right\rangle_{\rm D} = -\nu_{\rm T} \mathcal{S}^{\alpha\beta} + \left[\Gamma^{\alpha} \left(\Omega^{\beta} + 2\omega_{\rm F}^{\beta} \right) + \Gamma^{\beta} \left(\Omega^{\alpha} + 2\omega_{\rm F}^{\alpha} \right) \right]_{\rm D}$$

Early stage

$$\langle u'^{y} u'^{z} \rangle = \eta 2 \omega_{\rm F}^{y} \frac{\partial H}{\partial z}$$

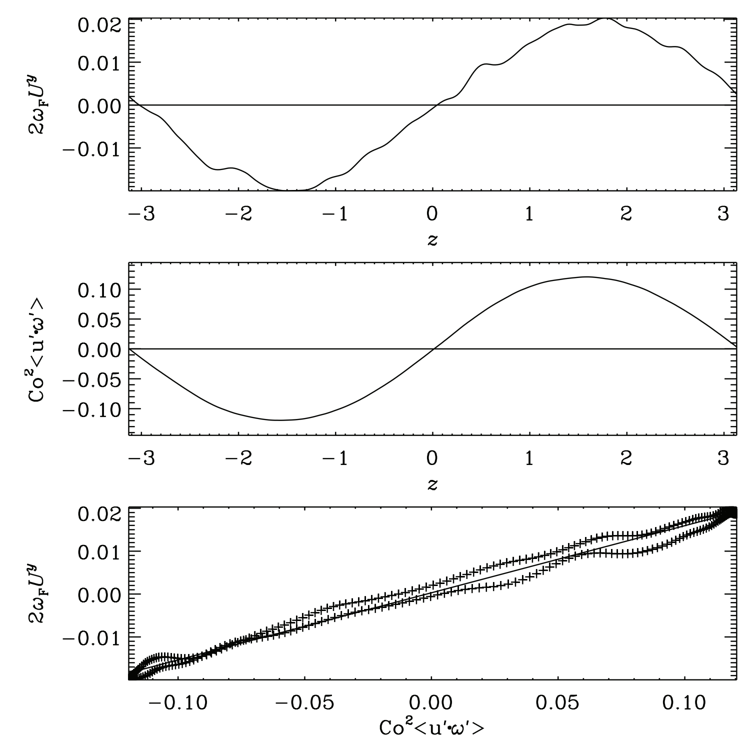


Reynolds stress $\langle u'^{y} u'^{z} \rangle$ (top), helicity-effect term $(\nabla H)^{z}\, 2\omega_{\rm F}{}^{y}$ (middle), and their correlation (bottom).

Developed stage

$$\langle u'^{y} u'^{z} \rangle = -\nu_{\rm T} \frac{\partial U^{y}}{\partial z} + \eta 2 \omega_{\rm F}^{y} \frac{\partial H}{\partial z}$$

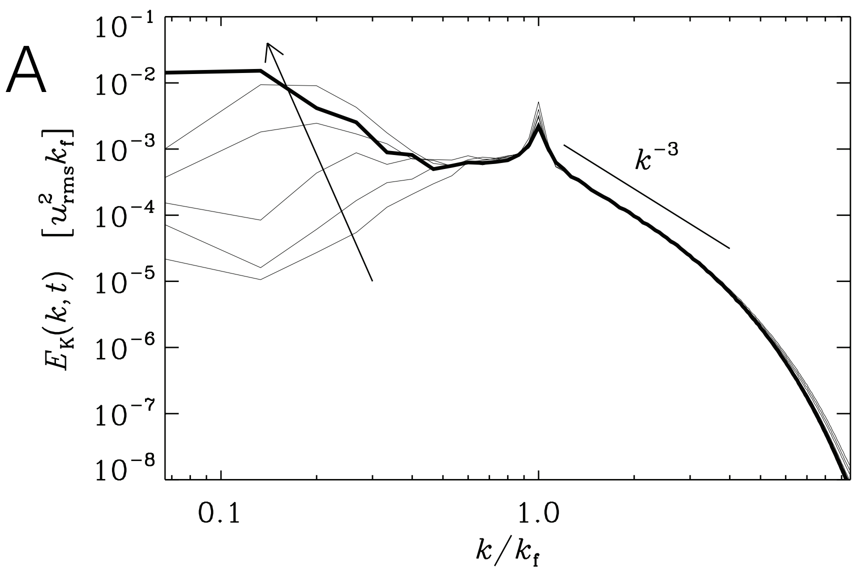
$$U^{y} = (\eta / \nu_{\rm T})\, 2\omega_{\rm F}^{y} H$$



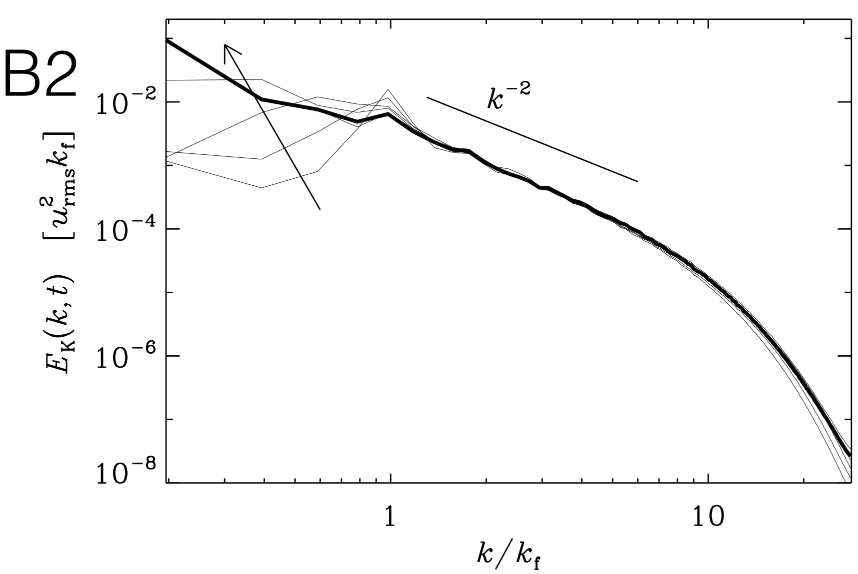
Mean axial velocity U^{y} (top), turbulent helicity multiplied by rotation $2\omega_{\rm F} H$ (middle), and their correlation (bottom).

Spectra

Run A



Run B2



Physical origin

Reynolds stress $\mathcal{R}^{ij} \equiv \langle u'^i u'^j \rangle$

Vortexmotive force $\mathbf{V}_\mathrm{M} \equiv \langle \mathbf{u}' \times \boldsymbol{\omega}' \rangle$

$$V_\mathrm{M}^i = -\frac{\partial \mathcal{R}^{ij}}{\partial x^j} + \frac{\partial K}{\partial x^i}$$

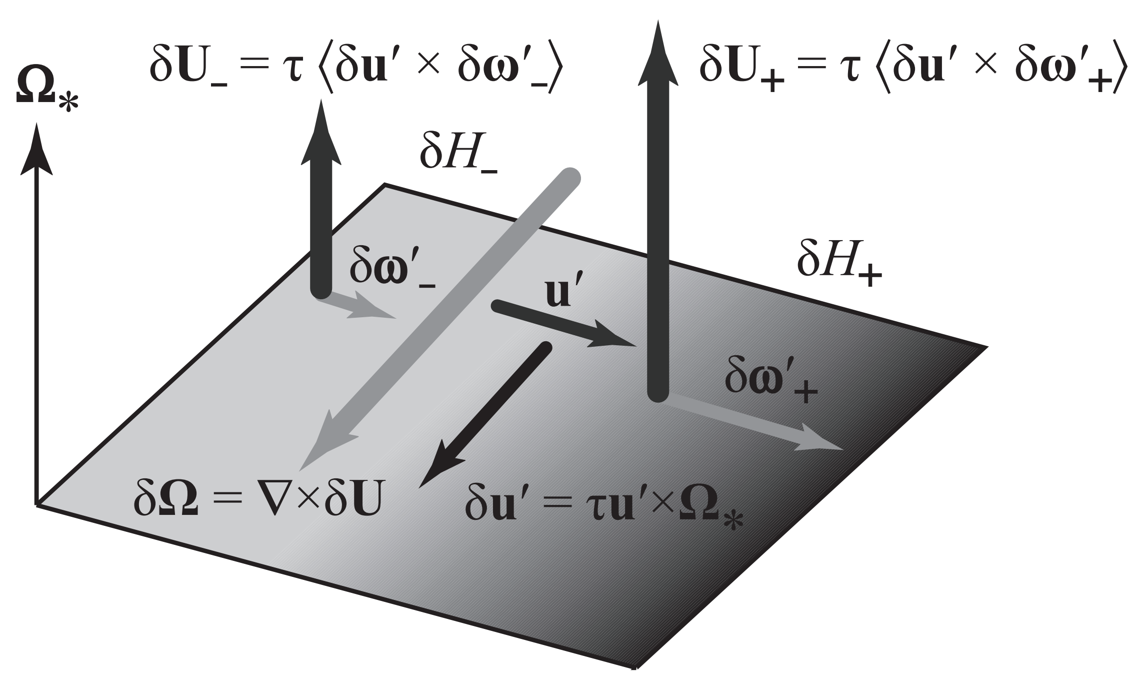
$$\frac{\partial \boldsymbol{\Omega}}{\partial t} = \nabla \times (\mathbf{U} \times \boldsymbol{\Omega}) + \nabla \times \mathbf{V}_\mathrm{M} + \nu \nabla^2 \boldsymbol{\Omega}$$

$$\mathbf{V}_\mathrm{M} = -D_\Gamma 2\boldsymbol{\omega}_\mathrm{F} - \nu_\mathrm{T} \nabla \times \boldsymbol{\Omega}$$

$$D_\Gamma = \nabla \cdot \boldsymbol{\Gamma} \propto \nabla^2 H$$

→ $$\delta \mathbf{U} \sim -(\nabla^2 H) \boldsymbol{\Omega}_*$$

$$\nabla^2 H \simeq -\frac{\delta H}{\ell^2} = -\frac{\langle \mathbf{u}' \cdot \delta \boldsymbol{\omega}' \rangle}{\ell^2}$$



Reynolds stress evolution

(Inagaki, Yokoi & Hamba, submitted to Phys. Rev. Fluids)

Local helical forcing

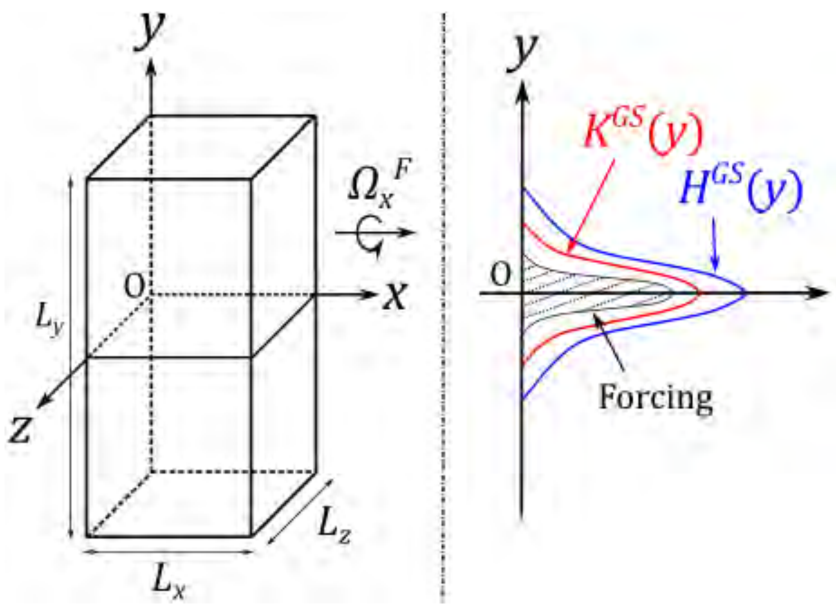
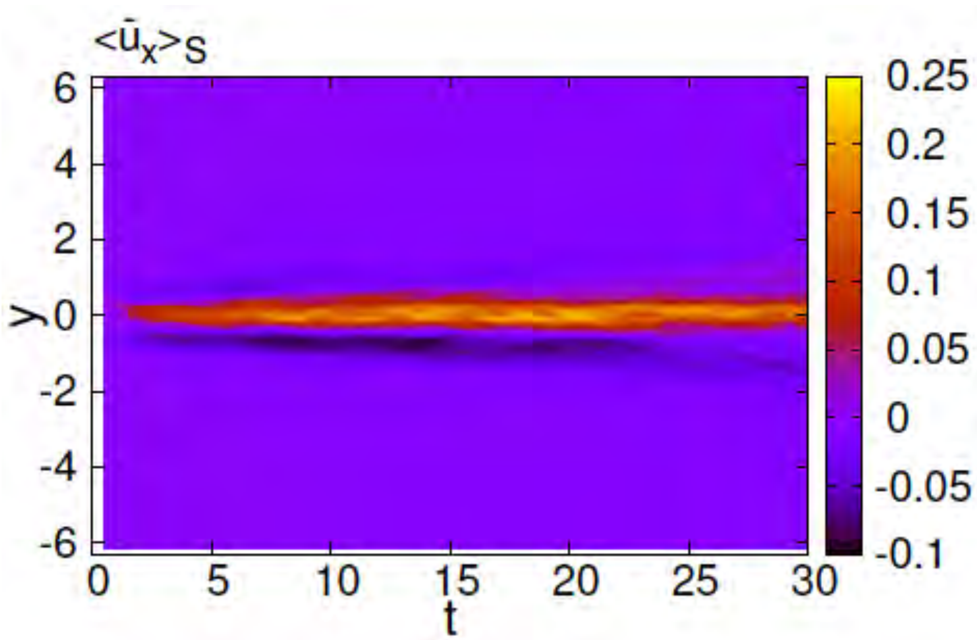
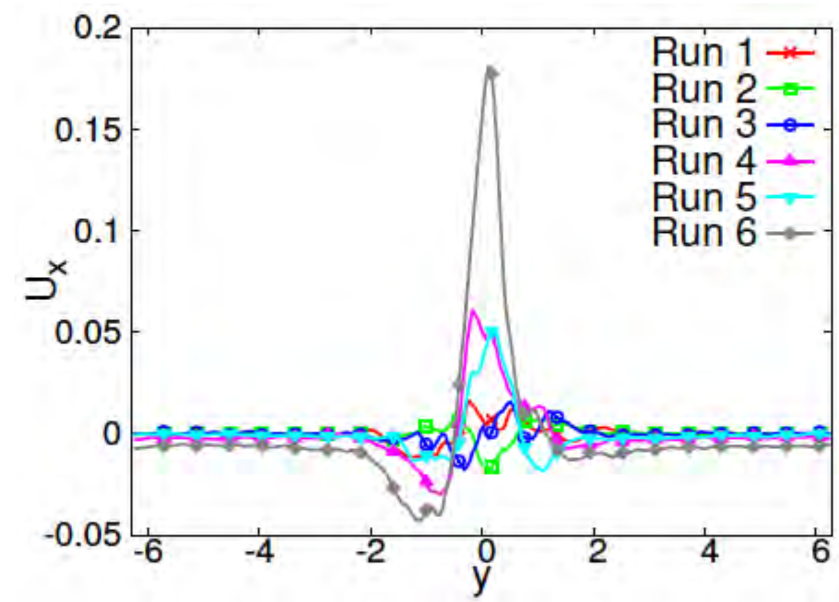


TABLE I. Calculation parameters.

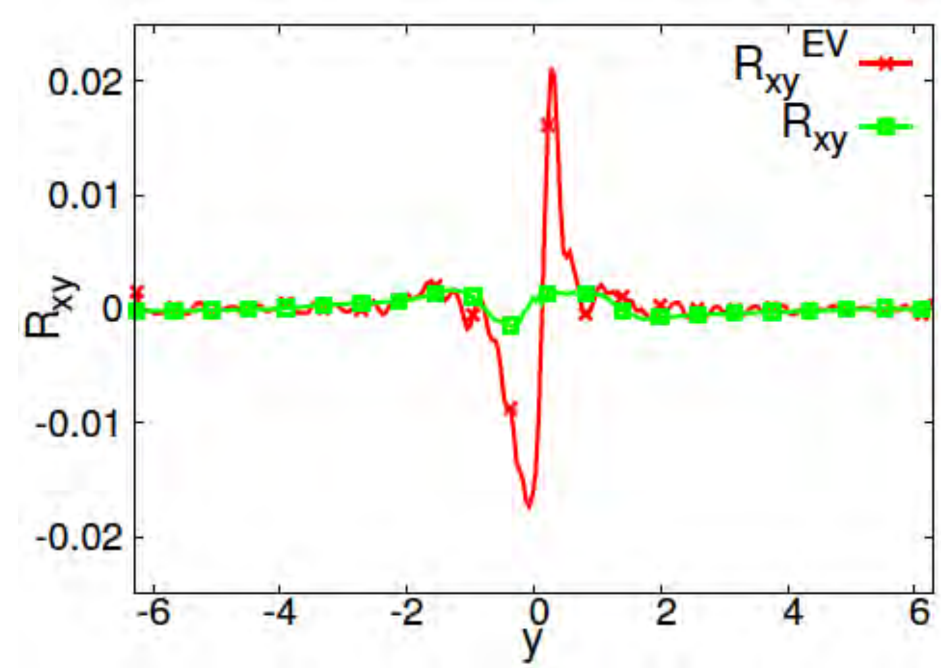
Run	α	Ω_x^F	L_0^{GS}	Ro_0^{GS}
1	0	0	0.506	∞
2	0.5	0	0.547	∞
3	0	5	0.542	0.185
4	0.2	5	0.550	0.182
5	0.5	2	0.544	0.459
6	0.5	5	0.602	0.166





Reynolds-stress budget

$$R_{xy} = \nu_{\mathrm{T}} \frac{\partial U_x}{\partial y} + N_{xy}$$



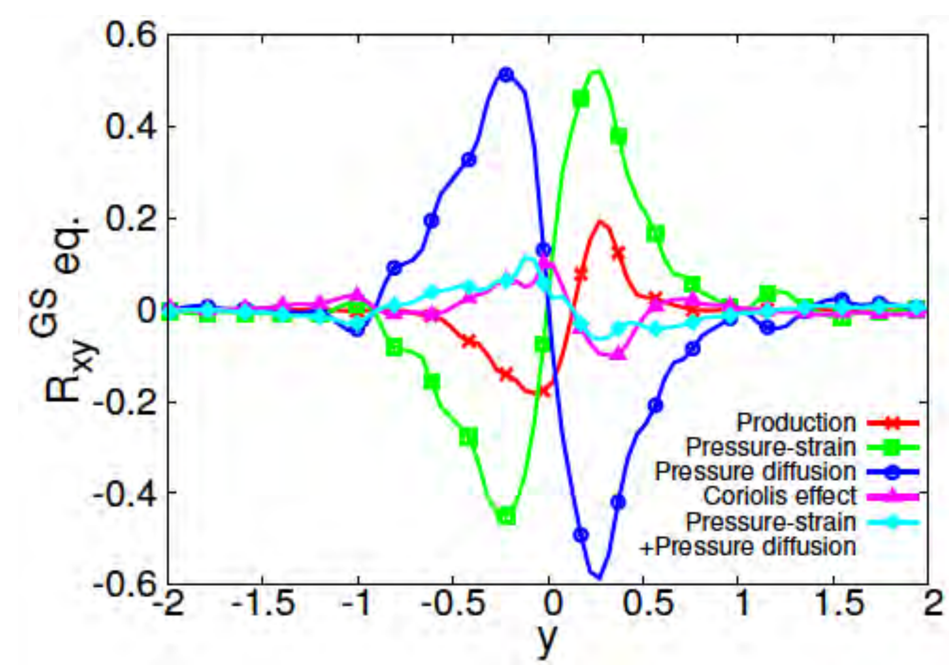
$$\frac{\partial R_{xy}^{GS}}{\partial t} \simeq P_{xy}^{GS} + \Phi_{xy}^{GS} + \Pi_{xy}^{GS} + C_{xy}^{GS} \simeq 0$$

Production $$P_{xy}^{GS} = -\frac{2}{3} K^{GS} \frac{\partial U_x}{\partial y} - B_{yy}^{GS} \frac{\partial U_x}{\partial y} - B_{xz}^{GS} \frac{\partial U_z}{\partial y}$$

Press. strain $$\Phi_{xy}^{GS} = 2 \left\langle \overline{p}' \overline{s}'_{xy} \right\rangle$$

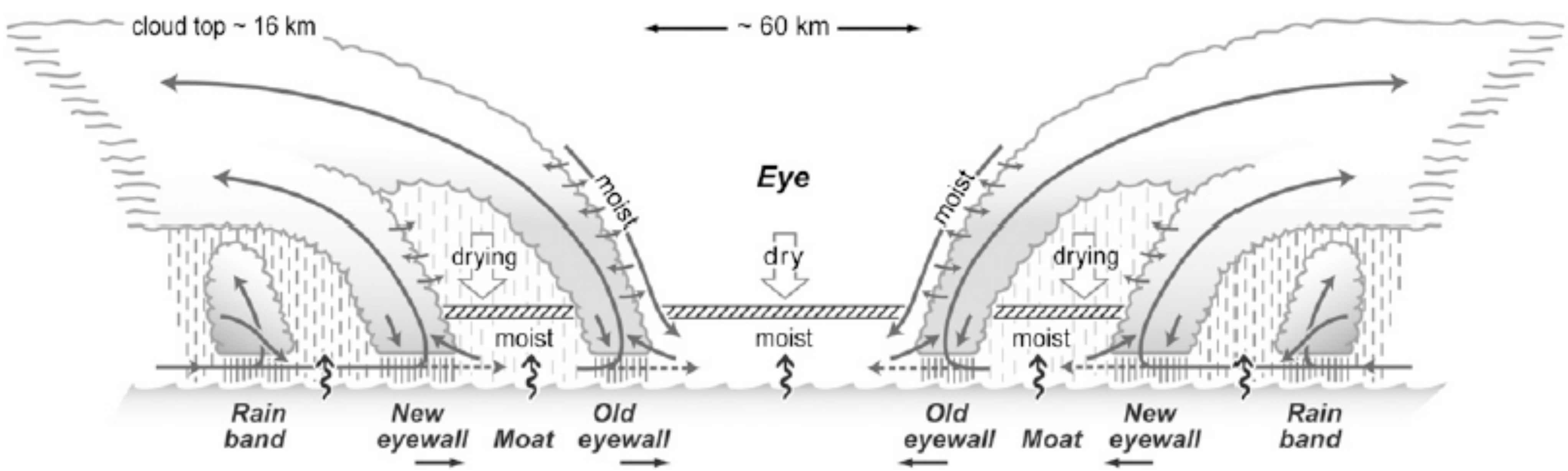
Press. diff. $$\Pi_{xy}^{GS} = -\frac{\partial}{\partial y} \left\langle \overline{p}' \overline{u}'_x \right\rangle$$

Coriolis $$C_{xy}^{GS} = 2 R_{xz}^{GS} \Omega_x^F$$

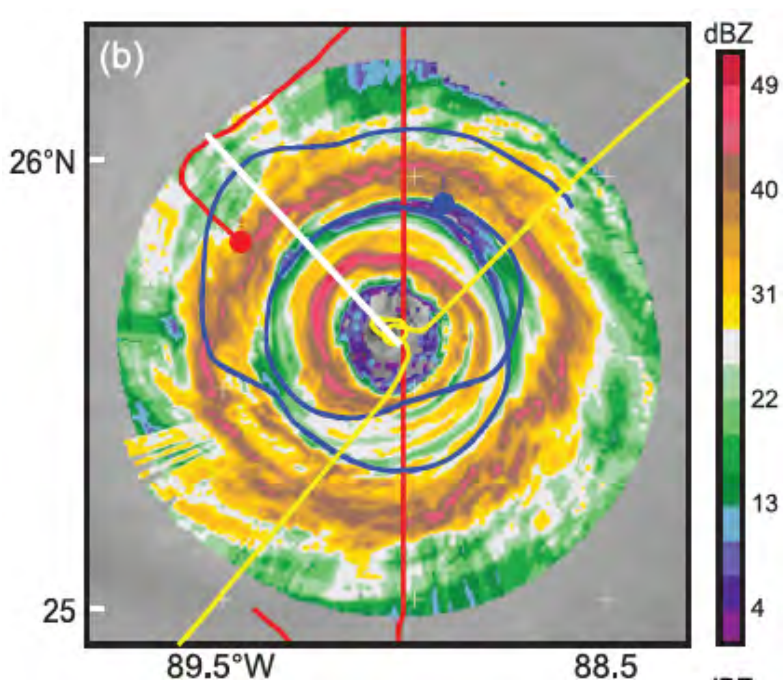


Flow generation in tropical cyclone

Kosuke ITO, Geophys. Fluid Dyn. Seminar
at Shikotsuko, 25-27 Aug. 2017

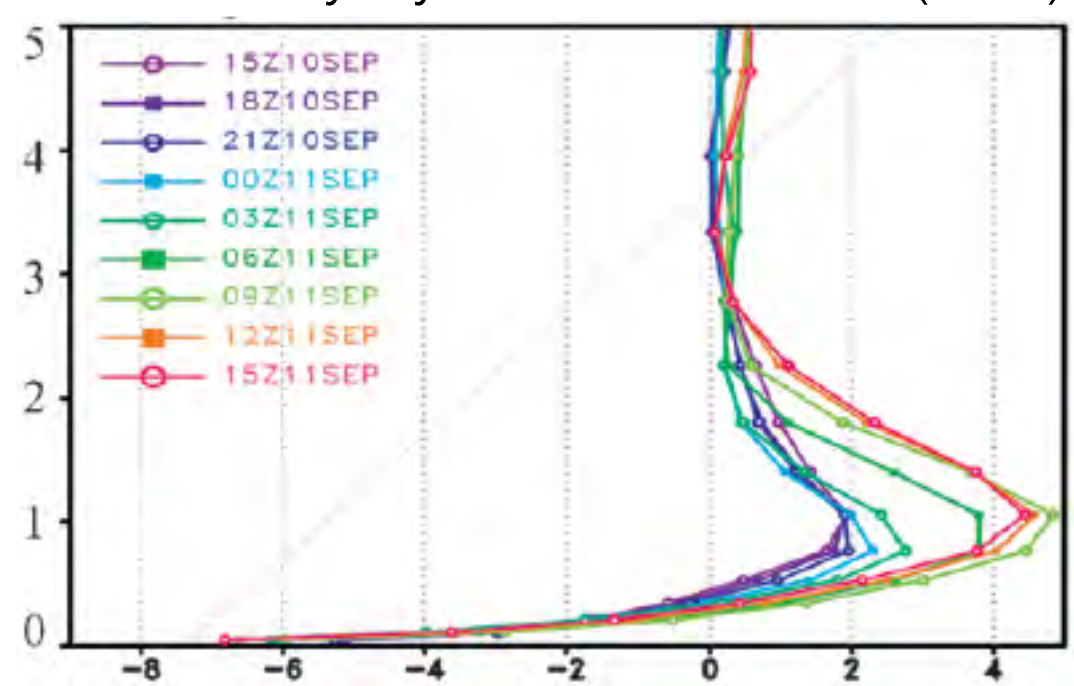


Flow acceleration



Houze, 2009

Secondary Eyewall Formation (SEF)



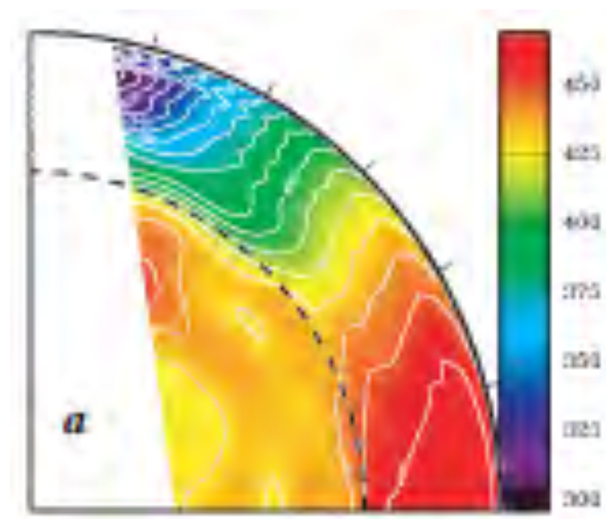
Huang et al., 2012

Angular-momentum transport in the solar convection zone

Angular momentum around the rotation axis

$$L = \Gamma r^2 \omega_{\rm F} + \Gamma r U^\phi \qquad \Gamma = \sin\theta$$

$$\frac{\partial}{\partial t}\rho L + \nabla \cdot (\rho \mathbf{F}_L) = 0$$



Miesch (2005) Liv. Rev. Sol. Phys. 2005-1

Vector flux of angular momentum $\mathbf{F}_L$

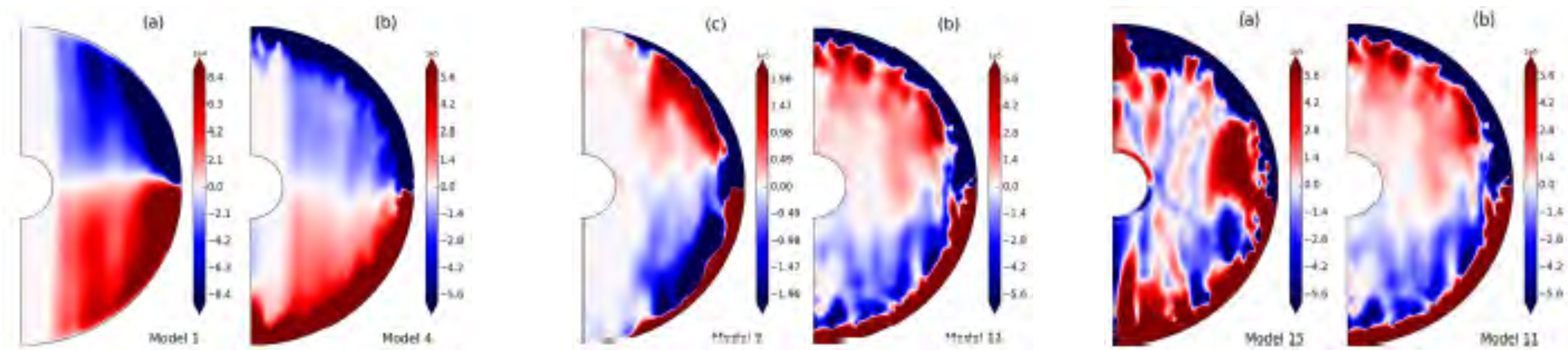
$$F_L^r = L U^r + r\Gamma \mathcal{R}^{r\phi}$$

$$F_L^\theta = L U^\theta + r\Gamma \mathcal{R}^{\theta\phi}$$

Helicity effect $$\mathcal{R}_H^{r\phi} = +\frac{\partial H}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r} r U^\theta - \frac{1}{r}\frac{\partial U^r}{\partial \theta}\right)$$

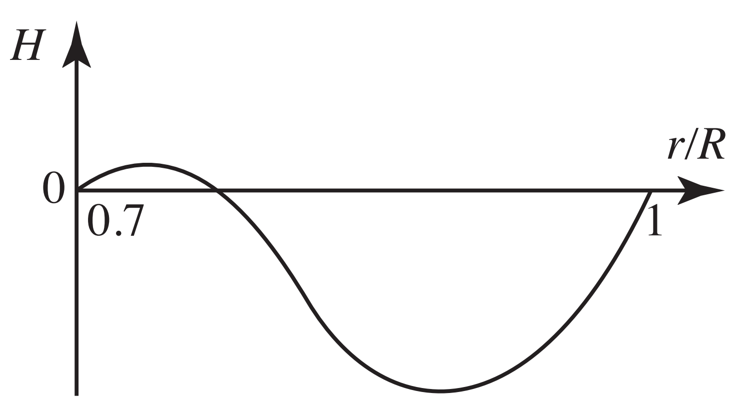
$$\mathcal{R}_H^{\theta\phi} = +\frac{1}{r}\frac{\partial H}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial r} r U^\theta - \frac{1}{r}\frac{\partial U^r}{\partial \theta}\right)$$

Helicity effect in the stellar convection zone



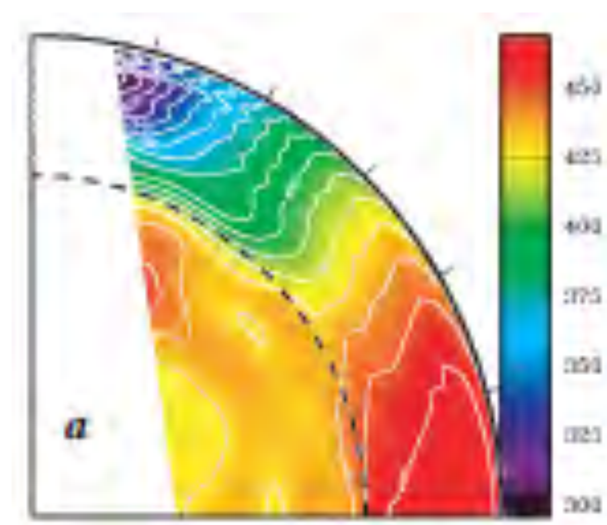
Duarte, et al, (2016) MNRAS **456**, 1708

Schematic helicity distribution



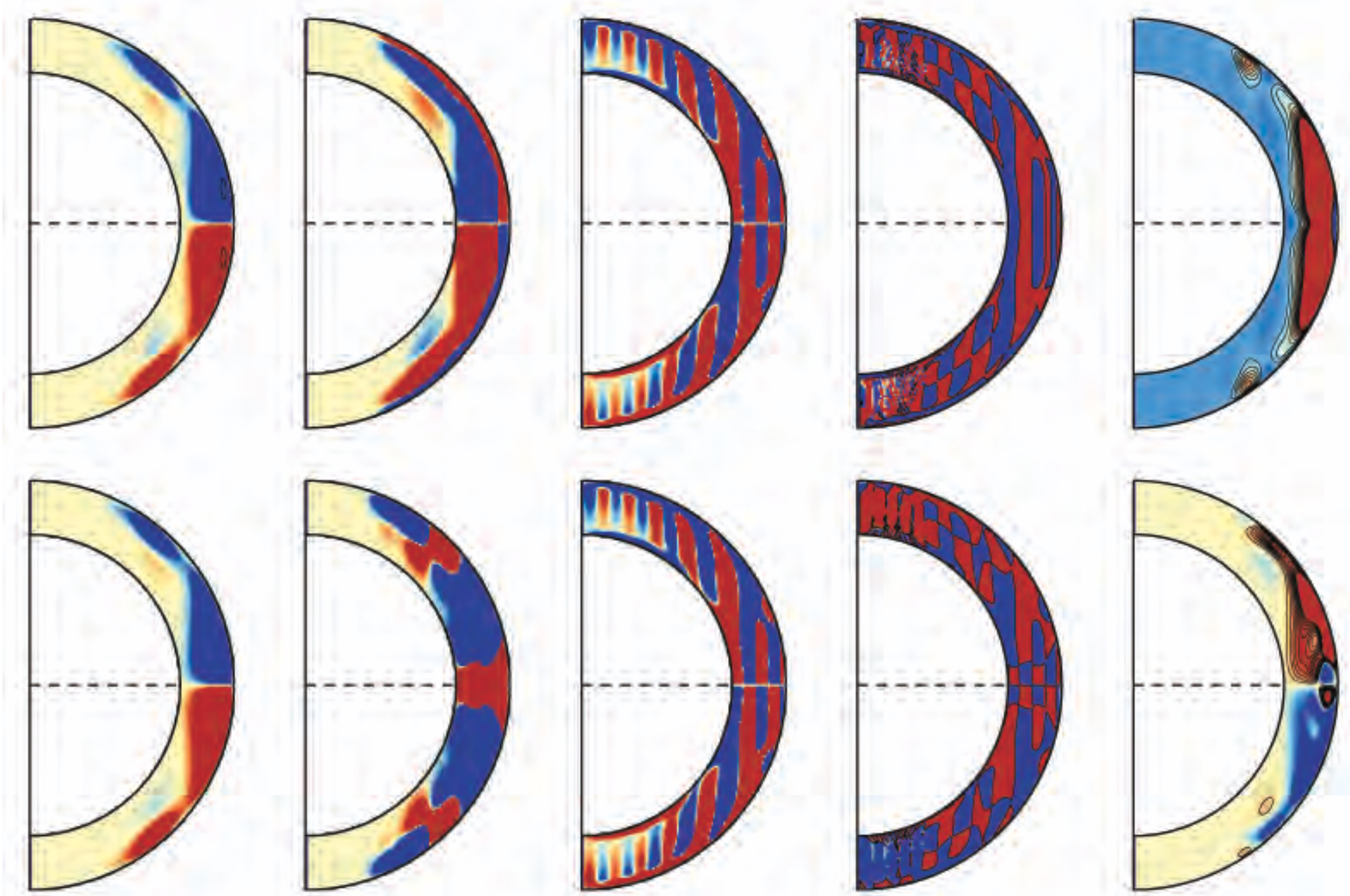
$$\nabla^2 H > 0$$

$$\delta \mathbf{U} \sim -(\nabla^2 H)\mathbf{\Omega}_*$$



Helicity effect in the Reynolds stress

Helicity	Helicity Gradient	Azimuthal Vorticity	Helicity effect	Reynolds stress
$\mathbf{u}\cdot\boldsymbol{\omega}$	$\frac{\partial H}{\partial r}$	$\overline{\Omega}^\phi$	$\frac{\partial H}{\partial r}\overline{\Omega}^\phi$	$\overline{u'^r u'^\phi}$



$\mathbf{u}\cdot\boldsymbol{\omega} - \overline{\mathbf{u}}\cdot\overline{\boldsymbol{\omega}}$ $(\equiv H)$	$\frac{1}{r}\frac{\partial H}{\partial \theta}$	$\overline{\Omega}^\phi$	$\frac{1}{r}\frac{\partial H}{\partial \theta}\overline{\Omega}^\phi$	$\overline{u'^\theta u'^\phi}$

(provided by Mark Miesch)

$C_\eta \tau \ell^2 |(\nabla^2 H)\Omega_*|$

Solar parameters

$$v \sim 200 \text{ m s}^{-1} = 2\times 10^4 \text{cm s}^{-1}$$
$$\ell \sim 200 \text{ Mm} = 2\times 10^{10}\text{cm}$$
$$\tau \sim \ell/v \sim 10^6 \text{ s}$$

$r\phi$ component

$$\left|\overline{u'^r u'^\phi}\right| \sim 1.2\times 10^9$$
$$\left|\frac{\partial H}{\partial r}\overline{\Omega}^\phi\right| \sim 9.4\times 10^{-15}$$
$$\tau\ell^2\left|\frac{\partial H}{\partial r}\overline{\Omega}^\phi\right| \sim 10^{12} \longrightarrow 10^9$$

with $C_\eta = O(10^{-3})$

$\theta\phi$ component

$$\left|\overline{u'^\theta u'^\phi}\right| \sim 5.6\times 10^8$$
$$\left|\frac{1}{r}\frac{\partial H}{\partial \theta}\overline{\Omega}^\phi\right| \sim 2.6\times 10^{-15}$$
$$\tau\ell^2\left|\frac{1}{r}\frac{\partial H}{\partial \theta}\overline{\Omega}^\phi\right| \sim 10^{11} \longrightarrow 10^8$$

Magnitude same as the Reynolds stress

Summary

- Theory for strongly nonlinear and inhomogeneous turbulence
- Electromotive force in strongly compressible MHD turbulence
- Dynamo with large-scale flow
- Global flow generation