Freely falling bodies in Standing Wave Spacetime*

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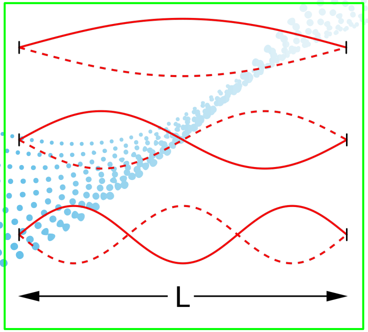
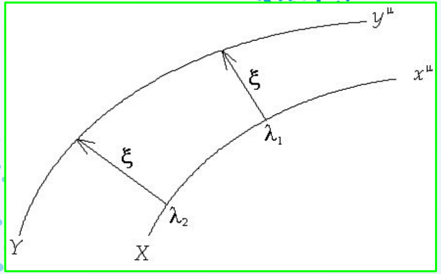
*Based on '*Freely falling bodies in standing wave spacetime*' Sebastian Szybka, Syed Naqvi

Outline:

- ➔ Gravitational Waves as Exact Solutions
- ➔ How to define Standing Gravitational Waves in General Relativity?
- ➔ How does Standing Gravitational Waves affect geodesics?



Standing GWs ???



Grav. Waves as Exact Solutions to E.F.E

Suppose you want to study :

- ❖ Gravitational field outside a spherical mass → **Schwarzschild**
- ❖ A spatially homogeneous and isotropic universe → **FLRW**
- ❖ Solutions model radiation moving at the speed of light → **Plane GW solution**

A special class of vacuum pp-wave spacetime

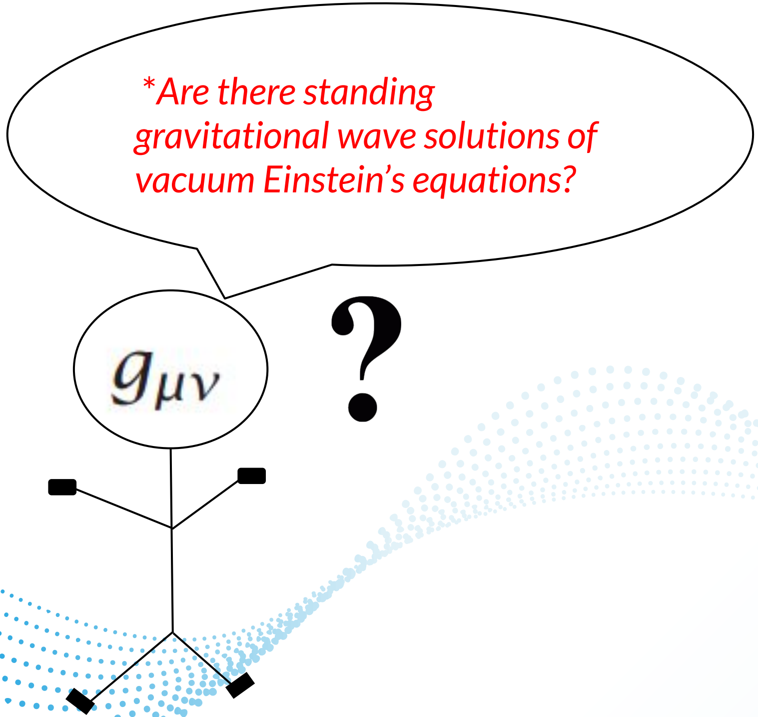
There exist many exact solutions which are interpreted as exact GWs.

★ What type of different GW solutions exist ?

*plane-fronted waves with parallel propagation

Exact Solutions of Einstein's Field Equations- Dietrich Kramer, Hans Stephani, and Malcolm MacCallum

Motivation



- #Bondi - Studied GWs of unsymmetric body rotating about z-axis

- **Issue -** If nonlinearities are taken into account, the lack of superposition principle complicates studies.
- *Stefani - Look for exact solutions with some particular metric functions
- **Issue -** Method is not covariant

#H. Bondi. Gravitational waves in general relativity XVI. Standing waves.
*H. Stephani. Some remarks on standing gravitational waves.

Standing Grav. Wave Solution*

- **analyze and admit arbitrary wavelength gravitational waves in expanding universe**

*#Metric

$$\hat{g} = e^f(-dt^2 + dz^2) + t(e^p dx^2 + e^{-p} dy^2)$$

where $0 \leq z < 2\pi$, $t > 0$ (t is a cosmic time function), $0 \leq x, y < 2\pi$, $f = f(t, z)$ and $p = p(t, z)$. The particular solution we are interested in is given by

After solving Einstein Field Equations

Parameters

$$p = -\ln t + 2\beta\sqrt{\lambda}J_0(\frac{t}{\lambda})\sin(\frac{z}{\lambda});,$$

$$f = \frac{\beta^2}{\lambda}t^2\left[J_0^2(\frac{t}{\lambda}) + J_1^2(\frac{t}{\lambda}) - 2\frac{\lambda}{t}J_0(\frac{t}{\lambda})J_1(\frac{t}{\lambda})\sin^2(\frac{z}{\lambda})\right] - 2\beta\sqrt{\lambda}J_0(\frac{t}{\lambda})\sin(\frac{z}{\lambda}),$$

*Standing waves in general relativity, Sebastian J. Szybka and Adam Cieślik(2019).
*#Backreaction for Einstein-Rosen waves coupled to a massless scalar field, Sebastian J. Szybka,Michał J. Wyrębowski(2016).

Standing Grav. Wave Solution*

Metric

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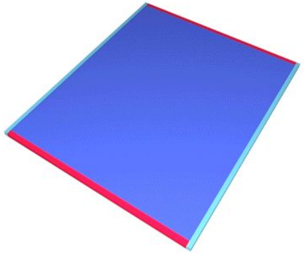
This solution is T^3 Gowdy Model(polarised 3-torus)*

2-torus

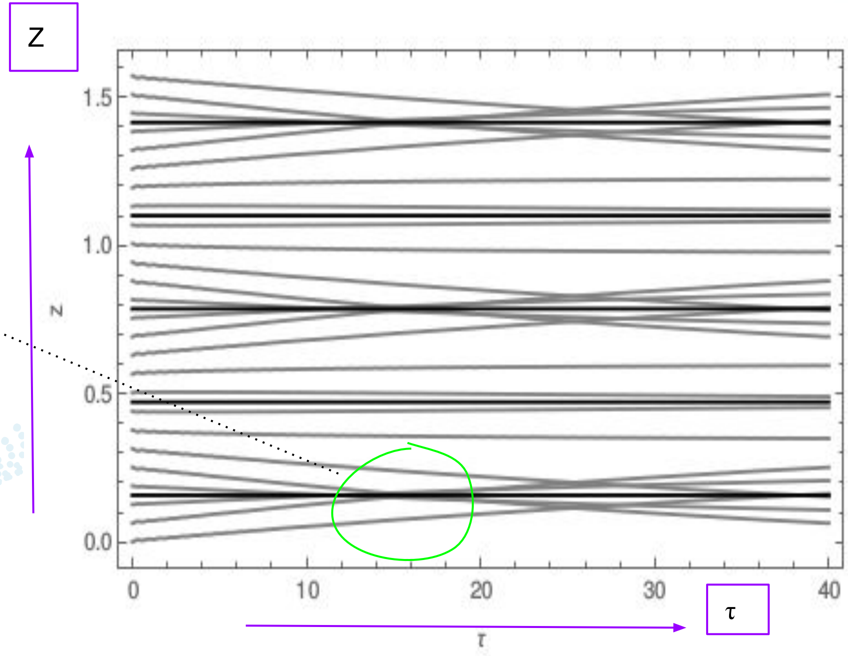


- Can be interpreted as a `non-linear' superposition of incoming gravitational waves and their reflections from the symmetry axis
- Amplitude of wave decreases as the universe expands

*Standing waves in general relativity Sebastian J. Szybka and Adam Cieślik.
*Robert H. Gowdy, "Of gravitational waves and spherical chickens" in: Einstein Online Band 03 (2007), 03-1008

Trajectories of Test Particles- Geodesic Eqn.

Particles attracted at antinodes



Wave is 'z' direction

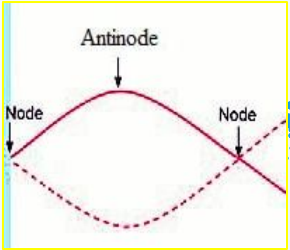
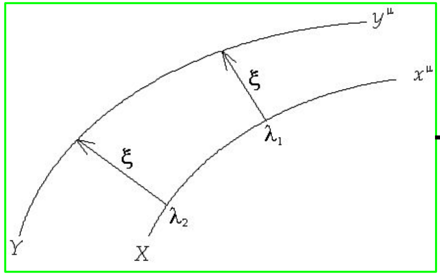


Figure : Geodesic equation for 'z' vs proper time τ, Here λ=1/10, β=0.2

Geo. Eqn has stationary soln. at antinodes **&** Study the motion of of freely-falling bodies at anitnodes

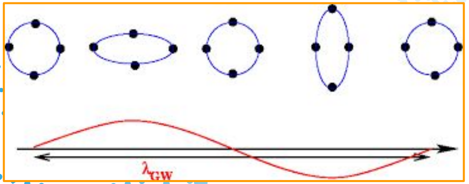
Freely-falling particles - Geodesic Deviation Eqn

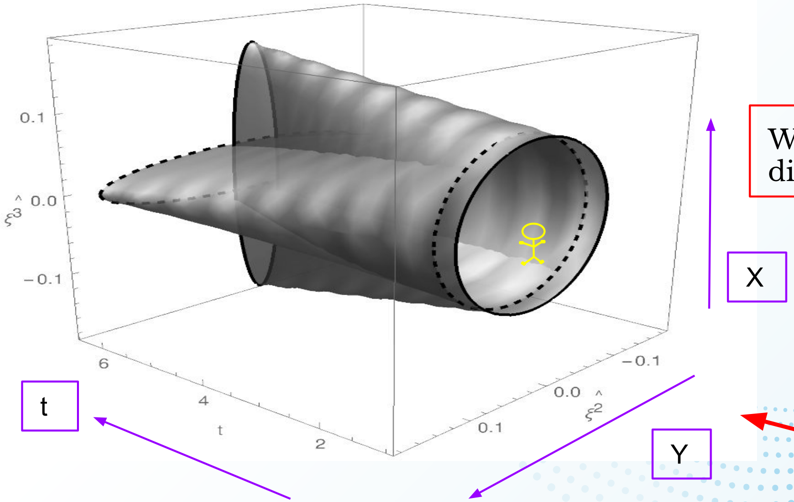


Freely-falling frame

$$\frac{d^2\xi^{\hat{\alpha}}}{d\tau^2} = -R^{\hat{\alpha}}{}_{\hat{0}\hat{\beta}\hat{0}}\,\xi^{\hat{\beta}}$$

General behaviour of GW(linearized):





Wave in 'Z' direction

Hint of Grav. Wave Memory??

Figure : Tissot plot showing how a ring of test particles will behave for waves for slightly different initial conditions. Here λ=1/10, β=0.2

Conclusion

❖ **Motivation** → Musical instruments have standing waves → What are Standing Gravitational waves? → Behaviors of test particles in such a spacetime

❖ Analyzed the Geodesic Equation ⇒ **Particles are attracted to antinodes**

❖ Analyzed the Geodesic Deviation Eqn ⇒ permanent deformation of ring of test particles(still need to workout details,since many waves are there)

❖ Future Work : electromagnetic standing waves coupled to gravity???

★ Some Trivia - Grishchuk & Sazhin(1975)* : - toroidal electromagnetic resonator with alternating current

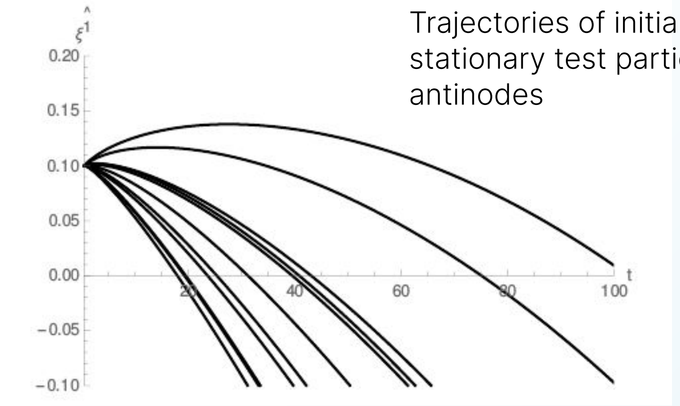
Interference of radiated GWs ⇒ Standing GW

*Excitation and detection of standing gravitational waves L. P. Grishchuk and M. V. Sazhin

Back-Up

X,Y,Z components of Geodesic Deviation Eqn.

Deviation in ‘z’



Trajectories of initially stationary test particles near antinodes

$$\frac{d^2\xi^{\hat{1}}}{dt^2} - \frac{1}{2} f_{,t}\frac{d\xi^{\hat{1}}}{dt} = \frac{1}{2}(f_{,tt} - f_{,zz})\xi^{\hat{1}} = -\frac{\beta}{\lambda t} J_1\left(\frac{t}{\lambda}\right)\left[\sqrt{\lambda} + t\beta J_1\left(\frac{t}{\lambda}\right)\right]\xi^{\hat{1}}$$

Deviation in ‘x’

$$\frac{d^2\xi^{\hat{2}}}{dt^2} - \frac{1}{2} f_{,t}\frac{d\xi^{\hat{2}}}{dt} = \frac{1}{4t}\left[f_{,t} + (p_{,t}(2 - t f_{,t}) + 2t\, p_{,tt})\right]\xi^{\hat{2}}$$
$$= -\frac{\beta}{\lambda^{3/2}}\left[J_0\left(\frac{t}{\lambda}\right) - \frac{\lambda^2}{4\beta^2 t} J_1^{-1}\left(\frac{t}{\lambda}\right) f_{,t}^2\right]\xi^{\hat{2}},$$

Deviation in ‘y’

$$\frac{d^2\xi^{\hat{3}}}{dt^2} - \frac{1}{2} f_{,t}\frac{d\xi^{\hat{3}}}{dt} = \frac{1}{4t}\left[f_{,t} - (p_{,t}(2 - t f_{,t}) + 2t\, p_{,tt})\right]\xi^{\hat{3}}$$
$$= \frac{\beta}{\lambda^{3/2}}\left[J_0\left(\frac{t}{\lambda}\right) - \frac{\lambda}{2} J_1\left(\frac{t}{\lambda}\right) f_{,t}\right]\xi^{\hat{3}},$$

Back-Up

From pp-wave to plane wave

A *pp-wave spacetime* is any Lorentzian manifold whose metric tensor can be described, with respect to Brinkmann coordinates, in the form

$$ds^2 = H(u,x,y)\,du^2 + 2\,du\,dv + dx^2 + dy^2$$

The most important class of particularly symmetric pp-waves are the plane wave spacetimes, which were first studied by Baldwin and Jeffery.

$$H(u,x,y) = a(u)\,(x^2 - y^2) + 2\,b(u)\,xy + c(u)\,(x^2 + y^2)$$

a,b describe the wave profiles of the two linearly independent polarization modes while c describes the wave profile of any nongravitational radiation

In general relativity, a gravitational plane wave is a special class of a vacuum pp-wave spacetime, and may be defined in terms of Brinkmann coordinates by

$$ds^2 = [a(u)(x^2 - y^2) + 2b(u)xy]du^2 + 2dudv + dx^2 + dy^2$$

Back-Up

X,Y,Z,t Geodesic Equation :

$$\ddot{t} + \frac{1}{2}\left\{f_{,t}\dot{t}^2 + 2f_{,z}\dot{t}\dot{z} + f_{,t}\dot{z}^2 + e^{-f}\left[e^p\dot{x}^2(1+p_{,t}t) + e^{-p}\dot{y}^2(1-p_{,t}t)\right]\right\} = 0$$

$$\ddot{x} + \dot{x}(\dot{t}/t + p_{,t}\dot{t} + p_{,z}\dot{z}) = 0\ ,$$

$$\ddot{y} + \dot{y}(\dot{t}/t - p_{,t}\dot{t} - p_{,z}\dot{z}) = 0\ ,$$

$$\ddot{z} + \frac{1}{2}\left[f_{,z}\dot{z}^2 + 2f_{,t}\dot{t}\dot{z} + f_{,z}\dot{t}^2 + tp_{,z}e^{-f}(-e^p\dot{x}^2 + e^{-p}\dot{y}^2)\right] = 0\ ,$$

Back-Up

Gowdy Spacetime

- Choice of R=t : T^3 model
- Choice of R=sin(t)sin(θ) : $S^1 \times S^2$ model
- Choice of R=t : S^3 model

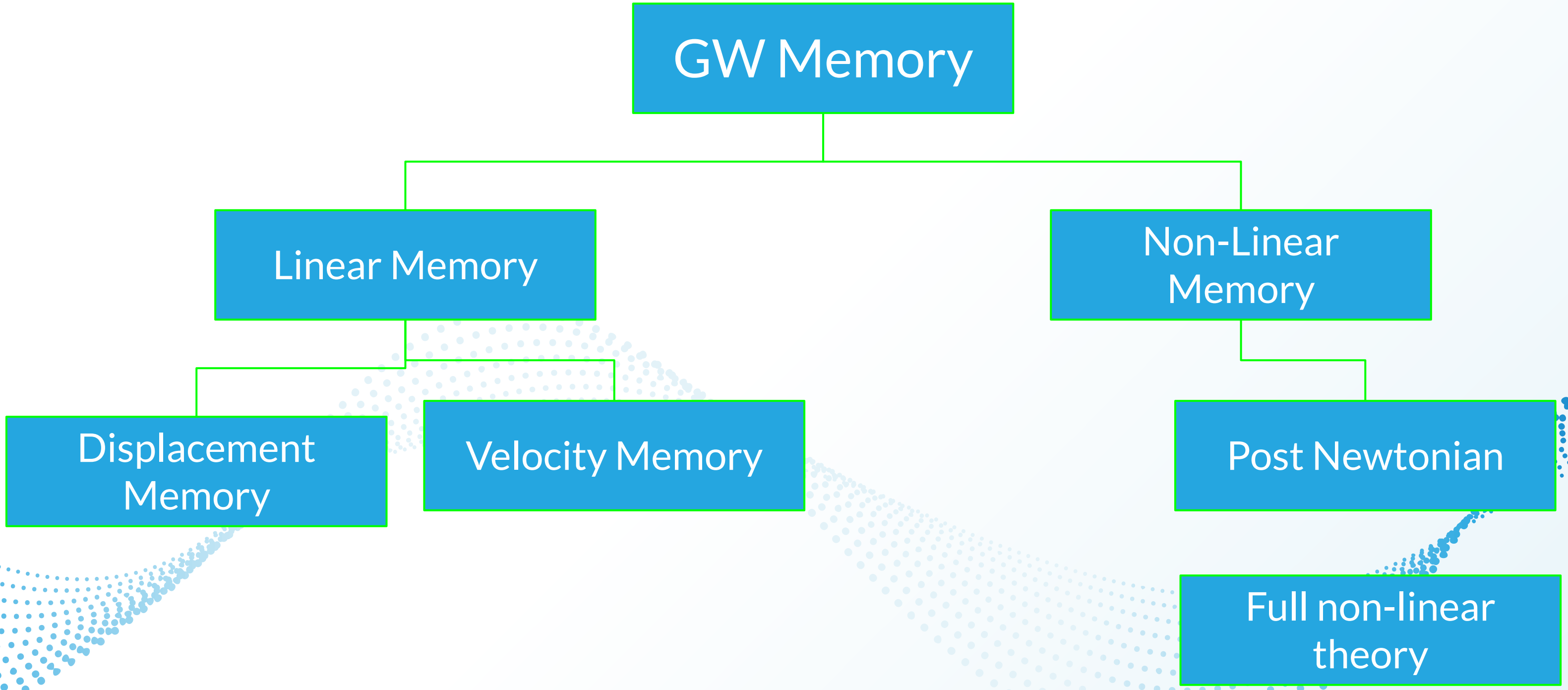
$$ds^2 = L^2\left\{e^{2a}\left(d\theta^2 - dt^2\right) + R\left[e^P\left(d\sigma + Q\,d\delta\right)^2 + e^{-P}d\delta^2\right]\right\}$$

When the function Q is zero, this system reduces to a linear wave equation for P. That case is usually referred to as a **Polarized Gowdy spacetime**. Solutions for the polarized case can be constructed at will from well-known functions.

$$\hat{g} = e^f(-dt^2 + dz^2) + t(e^p dx^2 + e^{-p} dy^2)$$

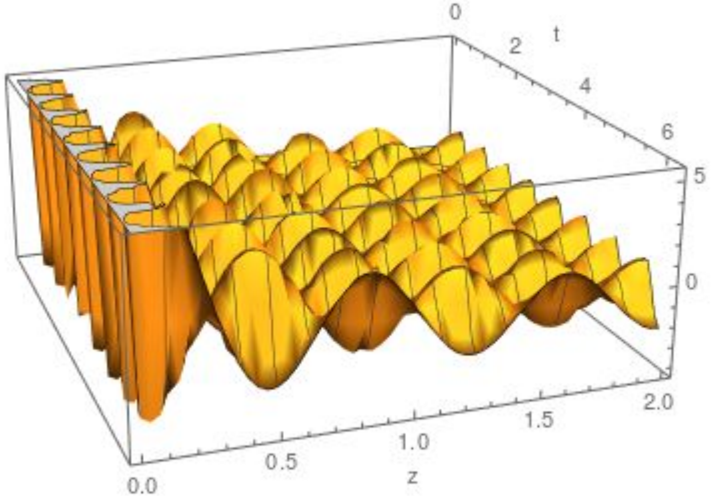
Back-Up

Types of Gravitational Wave Memory

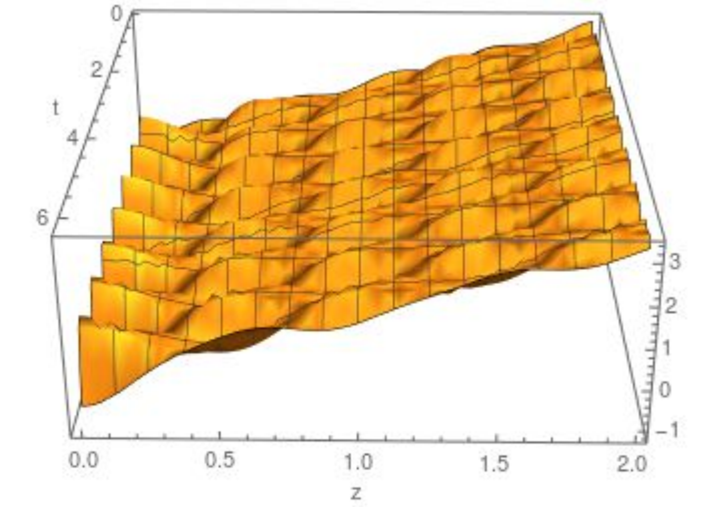


Back-Up

Plot for ‘p’ for some β and λ



Plot for ‘f’ for some β and λ



- ❖ Model is expanding hence amplitude of wave decreases in time
- ❖ Initial conditions are crucial for Tissot diagrams

Back-Up

2 Standing waves

In the search for a reliable criterion, we propose to define standing waves using Burnett's [11] formulation of the Isaacson high frequency limit [12] in the form generalized to nonvacuum spacetimes by Green and Wald [13].

Definition 2.1. Let (M, g) be a spacetime satisfying vacuum Einstein's equations. We say that (M, g) contains standing gravitational wave if

(i) it belongs to a one-parameter family of spacetimes $(M, g(\lambda))$ satisfying the Green-Wald assumptions [13] [we denote the background spacetime with $(M, g^{(0)})$],

(ii) the Ricci tensor of the background metric $g^{(0)}$ is of a Serge type $[(11)1, 1]$ (in Plebański notation $[2S_1{-}S_2{-}T]_{(111)}$) with the degenerate eigenvalue equal to zero and remaining eigenvalues $1, -1$.

S. R. Green and R. M. Wald. New framework for analyzing the effects of small scale inhomogeneities in cosmology. Physical Review D, 83:084020, 2011.