Effective Cuscuton Theory

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The cuscuton

$$S=\int \mathrm{d}^4x\sqrt{-g}\left(\frac{M_P^2}{2}R^{(4)}+\mu^2\sqrt{X}-V(\phi)\right),\quad X=-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,\quad X>0$$

- Does not propagate scalar dof [Afshordi et al (2007)]

- Applications to modified gravity: extensions for cuscuton like theories in the unitary gauge
  [Gao et al (2011); Hiramatsu & Kobayashi (2022); K. Aoki, A. De Felice, C. Lin, S. Mukohyama (2019)]

- Applications for VSL theories – the cuscuton appears in the UV limit of anti-DBI
  [D. Bessada, W. H. Kinney, D. Stojkovic and J. Wang (2010)
  Afshordi & Magueijo (2016),
  MM, Moschou, Afshordi & Magueijo, (2021)]

$$\mathcal{L}_{aDBI}\sim\frac{1}{B(\phi)}\sqrt{1+2B(\phi)X}\Big|_{X\gg1}\to\sim\sqrt{X}$$

[Mukhanov, Vikman, (2006)]

- In flat spacetimes the cuscuton possesses a scalarless symmetry [Tasinato (2020)]

- The cuscuton as the low-energy limit to Horava-Liftzing gravity [Afshordi (2009)]

The cuscuton

Afshordi, Chung, Geshnizjani, (2007)

Action:

$$S=\int \mathrm{d}^4x\sqrt{-g}\left(\frac{M_P^2}{2}R+\mu^2\sqrt{X}-V(\phi)\right),\quad X=-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,\quad X>0$$

EOM in FLRW:

$$H^2=-\frac{1}{3}V(\phi)$$
$$\dot{H}=-\frac{3H^2}{2}+\frac{1}{2}\mu^2\sqrt{\dot{\phi}^2}-\frac{1}{2}V(\phi)$$
$$3\mu^2H=-V'(\phi)$$

Pure constraint system

$$\frac{M_P^2}{3\mu^4}V'^2(\phi)-V(\phi)=\rho_m$$

- Non-dynamical auxiliary field, provides constraint equations which modify the dynamics of the fields it couples to.
- If we fix ρ_m this consequently fixes ϕ.

Superluminal field

Afshordi, Chung, Geshnizjani (2007)
Afshordi (2009)

$$c_s = \infty$$

- Infinite sound speed – but does not propagate information outside the light cone (it has no internal dynamics – no phase space). $(\mathrm{d}\Pi \wedge \mathrm{d}\phi = 0)$

Conditions:

- In the frame the cuscuton is uniform which lead to elliptical equation.
- Geometric perspective: CMC surfaces that do not intersect in the bulk.

Enlarged set of symmetries - protected from radiative corrections

Geometric picture of the cuscuton

Afshordi, Chung, Geshnizjani 2007

Scalar field equ:

$$\boxed{\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\frac{g^{\mu\nu}\partial_\nu\phi}{\sqrt{X}}\right]+V'(\phi)=0}$$

Unit normal vectors for constant ϕ hypersurfaces:

$$\boxed{n^\mu \equiv \frac{\partial^\mu\phi}{\sqrt{X}}}$$

From the extrinsic curvature $K=\nabla_\mu n^\mu$:

$$\boxed{K(\phi)=-\frac{V'(\phi)}{\mu^2}}$$

- The mean curvature on constant ϕ hypersurfaces is only a function of ϕ and hence constant (CMC)

Geometric picture of the cuscuton

Afshordi, Chung, Geshnizjani 2007

Think of soap bubbles and films!

CMC: in Euclidean space can be seen as a surface where the exterior pressure and surface tension forces balance

$$S_\phi = \int \mathrm{d}^4x\sqrt{-g}\left[\mu^2|n^\mu\partial_\mu\phi| - V(\phi)\right]$$
$$= \mu^2\int_\phi \mathrm{d}\phi\,\Sigma(\phi) - \int \mathrm{d}^4x\sqrt{-g}V(\phi)$$

Pressure difference across the surface

- Solve to find the surface of the bubble

$$-\mu^2\Delta\phi K_{dis} = \Delta V$$

surface tension

Mean extrinsic curvature of the constant-ϕ surface

- Theory that generalized the surface and volume terms

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Extending the cuscuton theory

Will the theory be cuscuton-like if we add higher-order curvature terms?

i.e., at certain energy scales gravity may need to be supplemented by higher-order operators.

What makes a theory cuscuton-like?

How to build an EFT for a non-propagating degree of freedom?

- ~~Fundamental symmetries.~~
- ~~Available dof's.~~

Extending the geometric picture

- Replace 2d surfaces or soap films with 3d spatial hypersurfaces of constant ϕ

- Replace Euclidean space with 4d Lorentzian spacetime

S-branes

(spacelike branes)

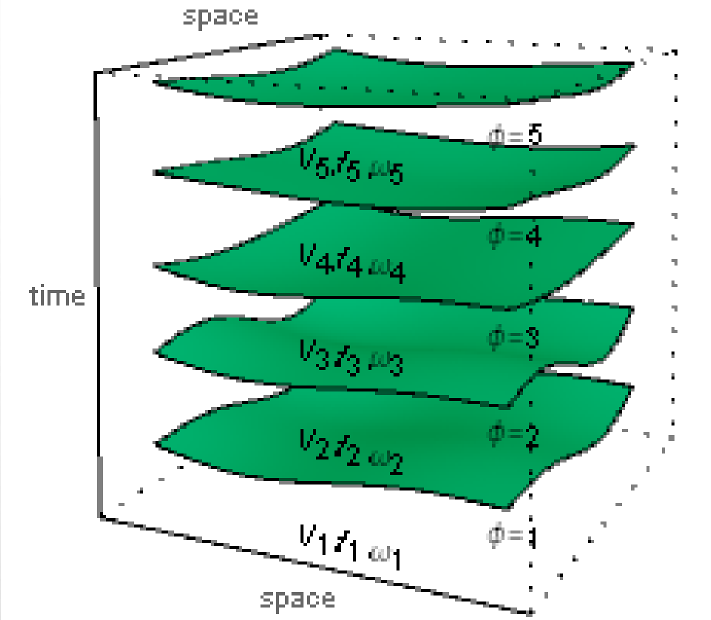
Topological defects localised on a spatial hypersurface representing an instant in time.

- corrections to the cuscuton kinetic term live on the boundaries or S-brane

- corrections to the potential live in the 4-dimensional bulk

Extending the geometric picture

- Consider a stack of spatial (3+0d) branes living in a 3+1d bulk. Surfaces do not talk to each other.
- Assume cuscuton is a discrete field (labelling transitions by $\phi = 1, \ldots, n$. It becomes continues in the limit of many such transitions.
- Discontinuous jumps in spacetime (deformed branes interface between different phases/vacua).
- We expect, bulk terms (E.H., Lovelock, e.t.c.) will have associated boundary terms.
- Cuscuton action: sum and take the continuous limit of the discrete transitions.



Sum over geometric invariants of the boundaries and bulks, ordered in powers of curvature

Approx. continuous limit

- Approximate the sum as an integral over geometric invariants
- Unitary gauge (homogeneous scalar field ϕ=ϕ(t), gradient points in the direction of time).

Discrete action:

$$\boxed{S_{\rm disc} = \sum_\phi \mathcal{V}(\phi, \phi+\Delta\phi) \times V(\phi) + \sum_\phi \mathcal{S}(\phi) \times c_1(\phi)\Delta\phi}$$

- $\mathcal{V}(\phi, \phi+\Delta\phi)$ is the volume enclosed between ϕ and $\phi+\Delta\phi$ surfaces
- $\mathcal{S}(\phi)$ is the area of the ϕ-hypersurface.

Take the continuous limit:

$$\sum_\phi \mathcal{V}(\phi, \phi+\Delta\phi) \times V(\phi)\Big|_{\Delta\phi\to 0} = \int \mathrm{d}^4x\sqrt{-g}\, V(\phi), \quad \sum_\phi \mathcal{S}(\phi) \times c_1(\phi)\Delta\phi\Big|_{\Delta\phi\to 0} = \int \mathrm{d}^3x\, \mathrm{d}\phi\, \sqrt{\gamma}\, c_1(\phi),$$

$$\int \mathrm{d}^3x\, \mathrm{d}\phi\, \sqrt{\gamma}\, c_1(\phi) = \int \mathrm{d}^4x\, \sqrt{-g}\, c_1(\phi)\frac{1}{N} = \int \mathrm{d}^4x\, \sqrt{-g}\, c_1(\phi)\sqrt{X}$$

Effective Cuscuton Theory (ECT)

No surface term

terms that live in the bulks

dimensions $\phi = \tilde{\phi}/\Lambda^3$

Action:

$$S_{CET} = \Lambda^4 \int \mathrm{d}^4x\sqrt{-g}\Big[V(\phi) + \frac{f(\phi)}{2\Lambda^2}R + \frac{\omega(\phi)}{2\Lambda^4}R_{GB}$$
$$+ \frac{c_1(\phi)}{\Lambda}\sqrt{X} - \varepsilon\frac{c_2(\phi)}{\Lambda^2}\sqrt{X}K + \frac{c_3(\phi)\sqrt{X}}{\Lambda^3}\mathcal{R} + \frac{c_4(\phi)}{\Lambda^4}\sqrt{X}\mathcal{K}_{GB}\Big]$$

terms that live on the boundaries

Brane thickness $\longrightarrow 0$

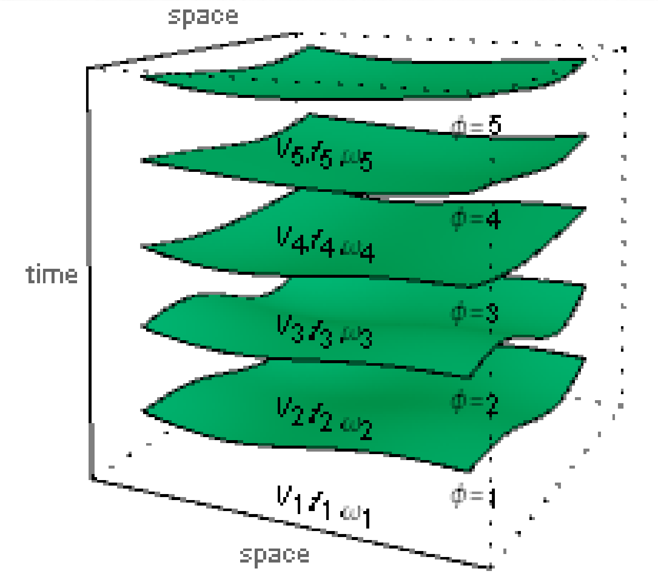
$$\mathcal{K}_{\rm GB} = -2\left(J - 2\varepsilon G^{ij}K_{ij}\right), \quad J_{ij} = \frac{1}{3}\left(2KK_{ic}K^{c}{}_{j} + K_{cd}K^{cd}K_{ij} - 2K_{ic}K^{cd}K_{dj} - K^2K_{ij}\right)$$

[Davis 2003]

In the continuous limit, the couplings are taken to be slow-varying functions of the scalar field ϕ.

Note: we can recover the original cuscuton action: $f = M_{Pl}^2, \quad c_1 = -\mu^2$ and set rest to zero.

Effective Cuscuton Theory (ECT)



Action:

$$S_{CET} = \Lambda^4 \int \mathrm{d}^4x\sqrt{-g}\Big[V(\phi) + \frac{f(\phi)}{2\Lambda^2} R + \frac{\omega(\phi)}{2\Lambda^4} R_{GB} + \frac{c_1(\phi)}{\Lambda}\sqrt{X} - \varepsilon\frac{c_2(\phi)}{\Lambda^2}\sqrt{X}K + \frac{c_3(\phi)\sqrt{X}}{\Lambda^3}\mathcal{R} + \frac{c_4(\phi)}{\Lambda^4}\sqrt{X}\mathcal{K}_{GB}\Big]$$

- Each bulk contains distinct values for the cosmological constant, gravitational constant and couplings to the Lovelock terms.
- Each brane contains distinct values for the coefficients of the gravitational surface terms.
- Whenever there is a discontinuity/jump in the bulk couplings, there will be a corresponding surface term associated with it.

Effective Cuscuton Theory (ECT)

Not all couplings are independent!

The effective couplings corresponding to the E.H. and G.B are complately determined by the geometry.

In the continuous limit…

$$c_2(\phi) = \lim_{\Delta\phi\to 0} \frac{f(\phi+\Delta\phi) - f(\phi)}{\Delta\phi} = \frac{\partial f(\phi)}{\partial\phi},$$

$$c_4(\phi) = \lim_{\Delta\phi\to 0} \frac{\omega(\phi+\Delta\phi) - \omega(\phi)}{\Delta\phi} = \frac{\partial \omega(\phi)}{\partial\phi}$$

… the surface couplings represent the rate of change of the bulk curvature couplings as we transition from one vacuum to another.

The boundary is bookkeeping that something physical is changing.

(Ensures the ECT propagates only two tensorial dof's)

ADM decomposition

Ingredients:

$$g^{\alpha\beta} = \gamma^{\alpha\beta} - n^\alpha n^\beta,$$
$$K_{ab} \equiv \nabla_b n_a,$$
$$a_a = -n^b \nabla_b n^a = -D_a \ln N = -\frac{D_a N}{N},$$
$$\mathcal{L}_{\mathbf{n}} \gamma_{ab} = 2K_{ab}$$

Extrinsic curvature & trace:

$$K_{ij} = \frac{1}{2N}(\partial_t \gamma_{ij} - D_i N_j - D_j N_i)$$
$$K = \gamma^{ij} K_{ij} = \frac{1}{2N}\left(\gamma^{ij}\partial_t \gamma_{ij} - 2D_i N^i\right)$$

Projections (ADM)

Projection of 4-D Riemann tensor:

$$ {}^{(4)}R_{ijkl} = K_{ik}K_{jl} - K_{il}K_{jk} + {}^{(3)}R_{ijkl}, $$

$$ {}^{(4)}R_{ijk\mathbf{n}} \equiv n^{\mu}\, {}^{(4)}R_{ijk\mu} = D_i K_{jk} - D_j K_{ik}, $$

$$ {}^{(4)}R_{i\mathbf{n}k\mathbf{n}} \equiv n^{\mu}n^{\nu}\, {}^{(4)}R_{i\mu j\nu} = -\frac{1}{N}(\dot{K}_{ij} - \mathcal{L}_{\mathbf{N}}K_{ij}) + K_{ik}K_{j}{}^{k} + \frac{1}{N}D_iD_jN $$

[Deruelle, Sasaki Sendouda, Yamauchi 2010]

Projection of 4-D Ricci tensor & scalar:

$$ {}^{(4)}R_{ij} = KK_{ij} - 2K_{i}{}^{a}K_{ja} + {}^{(3)}R_{ij} - \frac{D_jD_iN}{N} + \left(\dot{K}_{ij} - \mathcal{L}_{\mathbf{N}}K_{ij}\right) $$

$$ {}^{(4)}R_{\mathbf{n}j} = D^{a}K_{ja} - D_jK, $$

$$ {}^{(4)}R_{\mathbf{nn}} = -K_{ij}K^{ij} + \frac{D_iD^iN}{N} - \left(\dot{K} - \mathcal{L}_{\mathbf{N}}K\right) $$

$$ {}^{(4)}R = K_{ij}K^{ij} + K^2 + {}^{(3)}R - \frac{D^aD_aN}{N} + 2\left(\dot{K} - \mathcal{L}_{\mathbf{N}}K\right) $$

Homogeneous scalar field

Cuscuton in ADM

$$\boxed{\sqrt{X} = \sqrt{-g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi} \to \sqrt{(\mathcal{L}_{\mathbf{n}}\phi)^2 - \cancel{\gamma^{ij}D_i\phi D_j\phi}}}$$

unitary gauge
$\delta\phi = 0$

$$\mathcal{L}_{\mathbf{n}} = \frac{1}{N}(\partial_t - \mathcal{L}_{\mathbf{N}})$$

Cuscuton simplifies

$$\boxed{\sqrt{X} = \mathcal{L}_{\mathbf{n}}\phi}$$

normalisation: $\varepsilon \equiv n_\alpha n^\alpha = -1$

n^α point in the direction of increasing ϕ

Scalar-Tensor theory

Action:

$$S_{ECT} = \Lambda^4 \int \mathrm{d}^4x\, N\sqrt{\gamma}\Big[V(\phi) + c_1(\phi)\frac{\sqrt{X}}{\Lambda} + \frac{f(\phi)}{2\Lambda^2}\Big(K_{ab}K^{ab} + K^2 + \mathcal{R} - \frac{2D^aD_aN}{N} + 2\mathcal{L}_{\mathbf{n}}K\Big) - \varepsilon c_2(\phi)\frac{\sqrt{X}K}{\Lambda^2}\Big]$$

IBP:

$$S_{ECT} = \Lambda^4 \int \mathrm{d}^4x\, N\sqrt{\gamma}\Big[V(\phi) + c_1(\phi)\frac{\sqrt{X}}{\Lambda} + \frac{f(\phi)}{2\Lambda^2}\Big(K^2 - K_{ab}K^{ab} + \mathcal{R}\Big) - \cancel{\frac{1}{\Lambda^2}f'(\phi)\sqrt{X}K} - \cancel{\varepsilon c_2(\phi)\frac{\sqrt{X}K}{\Lambda^2}}\Big]$$

use: $\varepsilon = -1$ and $c_2(\phi) = f'(\phi)$

$$\Rightarrow \quad \boxed{S_{ECT} = \Lambda^4 \int \mathrm{d}^4x\, N\sqrt{\gamma}\Big[V(\phi) + c_1(\phi)\frac{\sqrt{X}}{\Lambda} + \frac{f(\phi)}{2\Lambda^2}\Big(K^2 - K_{ab}K^{ab} + \mathcal{R}\Big)\Big]}$$

Subset of MMG-II [Lin, Mukohyama (2017)]with $L = NF(K_{ij}, R_{ij}, \gamma^{ij}, t) + G(K_{ij}, R_{ij}, \gamma^{ij}, t)$ with no Einstein frame.

Some intuition from the geometric picture

The contributions from the surface terms, induce cancellations such that we get a constraint equation.

EOMs:

$$0 = V(\phi) - \frac{f(\phi)}{2}\left(K^2 - K_{cd}K^{cd} - \mathcal{R}\right),$$

$$0 = V'(\phi) - c_1(\phi)K\frac{\mathcal{L}_{\mathbf{n}}\phi}{\sqrt{X}} + \frac{f'(\phi)}{2}\Big(K^2 - K_{ab}K^{ab} + \mathcal{R}\Big),$$

Generalization of the CMC condition

$$\boxed{K = \frac{1}{c_1(\phi)}\left[\frac{3}{2}(V'(\phi) + f'(\phi)\mathcal{R})\right]}$$

$\longrightarrow$

Background level

$$\boxed{V'(\phi) - c_1(\phi)\sqrt{\frac{3\,V'(\phi)}{4\,f'(\phi)}} = 0}$$

Need to add matter ρ_m to the mix.

With GB

The action has the form:

$$S_{ECT} = \Lambda^4 \int \mathrm{d}^4x\, N\sqrt{\gamma} F(K_{ij}, R_{ij}, D_i, \gamma^{ij}, \mathcal{L}_{\mathbf{n}}, t)$$

which is of the form [Z.-B. Yao, M. Oliosi, X. Gao and S. Mukohyama, (2023)] apart of the Lie derivative…

(a Hamiltonian analysis may not be possible due to the mixing of time and spatial partial derivatives of the metric)

but with a suitable substitution…

$$K = \frac{1}{c_1(\phi)}\left[\frac{3}{2}V'(\phi) + f'(\phi)\mathcal{R} + \omega'(\phi)(B+C)\right]$$

$$B = 12 K_a{}^c K^{ab} K_b{}^d K_{cd} + \frac{8}{3} K K_a{}^c K^{ab} K_{cd} + K_{ab}K^{ab}K_{cd}K^{cd} - 2K^2 K_{cd}K^{cd} + \frac{1}{3}K^4,$$

$$C = \mathcal{R}_{GB} + 8\, D_b D_a G^{ab} + 16 \frac{D_c D_a N}{N}\left(K_a{}^c K^{ab} - K K^{bc}\right) + 8 \frac{D_c D^c N}{N}\left(K^2 - K_{ab}K^{ab}\right),$$

Discussion

- Will the EFT propagate scalar dof?
- Is ECT ghost free?
- Perturbation theory?
- Phenomenology
- Relationship with VSL theories?
- Can ECT be extended further?

It's TURTLES all the way down!

Scalar-Tensor (covariant form)

Lagrangian:

$$\mathcal{L}_{ECT} = V(\phi) + c_1(\phi)\sqrt{X} + \frac{f(\phi)}{2}R - \varepsilon^2\alpha_3(\phi)g^{\rho\sigma}\sqrt{X}\nabla_\rho\left(\frac{\nabla_\sigma\phi}{\sqrt{X}}\right)$$

Variation wrt metric:

$$\begin{aligned}0 = &-g_{\mu\nu}V(\phi) - c_1(\phi)\left(g_{\mu\nu}\sqrt{X} + \frac{\nabla_\mu\phi\nabla_\nu\phi}{\sqrt{X}}\right) + f(\phi)\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + f'(\phi)\left(-\nabla_\nu\nabla_\mu\phi + g_{\mu\nu}\nabla_\rho\nabla^\rho\phi\right)\\ &+ f''(\phi)\left(-\nabla_\mu\nabla_\nu\phi + g_{\mu\nu}\nabla_\rho\nabla^\rho\phi\right) - \varepsilon^2\alpha_3'(\phi)\left(g_{\mu\nu}\nabla_\alpha\phi\nabla^\alpha\phi - \nabla_\mu\phi\nabla_\nu\phi\right)\\ &+ \varepsilon^2\alpha_3(\phi)\frac{1}{X}\left(-g_{\mu\nu}\nabla^\alpha\phi\nabla_\beta\nabla_\alpha\phi\nabla^\beta\phi - \nabla_\alpha\nabla^\alpha\phi\nabla_\mu\phi\nabla_\nu\phi + \nabla^\alpha\phi\nabla_\mu\nabla_\alpha\phi\nabla_\nu\phi + \nabla^\alpha\phi\nabla_\mu\phi\nabla_\nu\nabla_\alpha\phi\right)\end{aligned}$$

Variation wrt scalar field:

$$\begin{aligned}0 = &V'(\phi) + c_1(\phi)g^{\mu\nu}\nabla_\mu\left(\frac{\nabla_\mu\phi}{\sqrt{X}}\right) + \frac{1}{2}f'(\phi)R + \varepsilon^2\alpha_3(\phi)\frac{1}{X}\left(\nabla_\alpha\nabla^\alpha\phi\nabla_\beta\nabla^\beta\phi - R_{\alpha\beta}\nabla^\alpha\phi\nabla^\beta\phi - \nabla_\beta\nabla_\alpha\phi\nabla^\beta\nabla^\alpha\phi\right)\\ &+ \varepsilon^2 2\alpha_3(\phi)\frac{1}{X^2}\left(\nabla^\alpha\phi\nabla_\beta\nabla_\alpha\phi\nabla^\beta\phi\nabla_\rho\nabla^\rho\phi - \nabla^\alpha\phi\nabla^\beta\phi\nabla_\rho\nabla_\beta\phi\nabla^\rho\nabla_\alpha\phi\right)\end{aligned}$$