Does inflation always start with a Bang?

Jerome Quintin
University of Waterloo
and
Perimeter Institute for Theoretical Physics

Copernicus Webinar, July 18, 2023

Inflation

$$ds^2 = -\,dt^2 + a(t)^2 d\vec{x}^2, \qquad \dot{a} > 0, \qquad \ddot{a} > 0$$

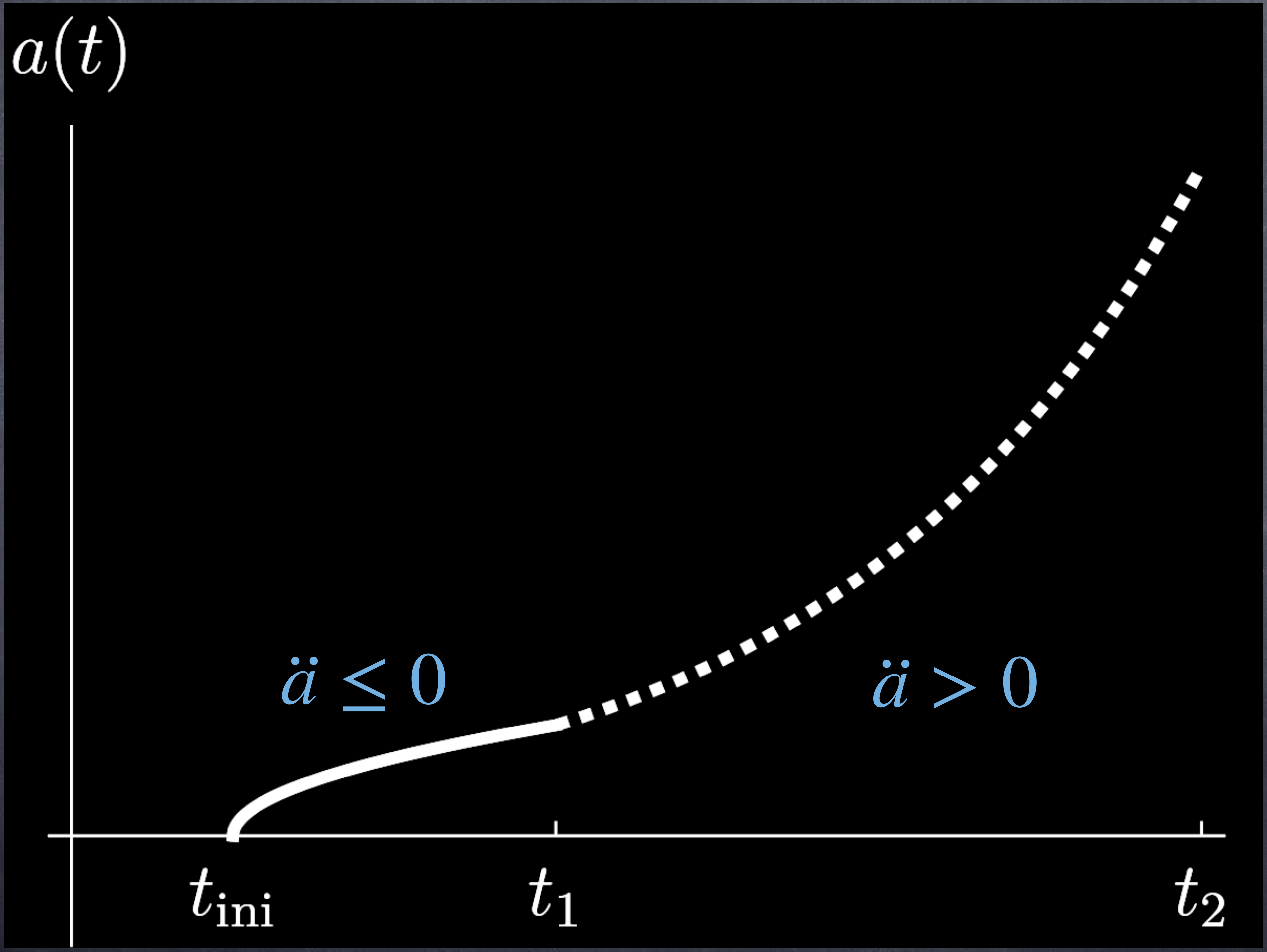
What about the beginning or before?

at the level of the classical geometry
(Lorentzian metric)

mainly based on arXiv:2305.01676
with Ghazal Geshnizjani and Eric Ling,

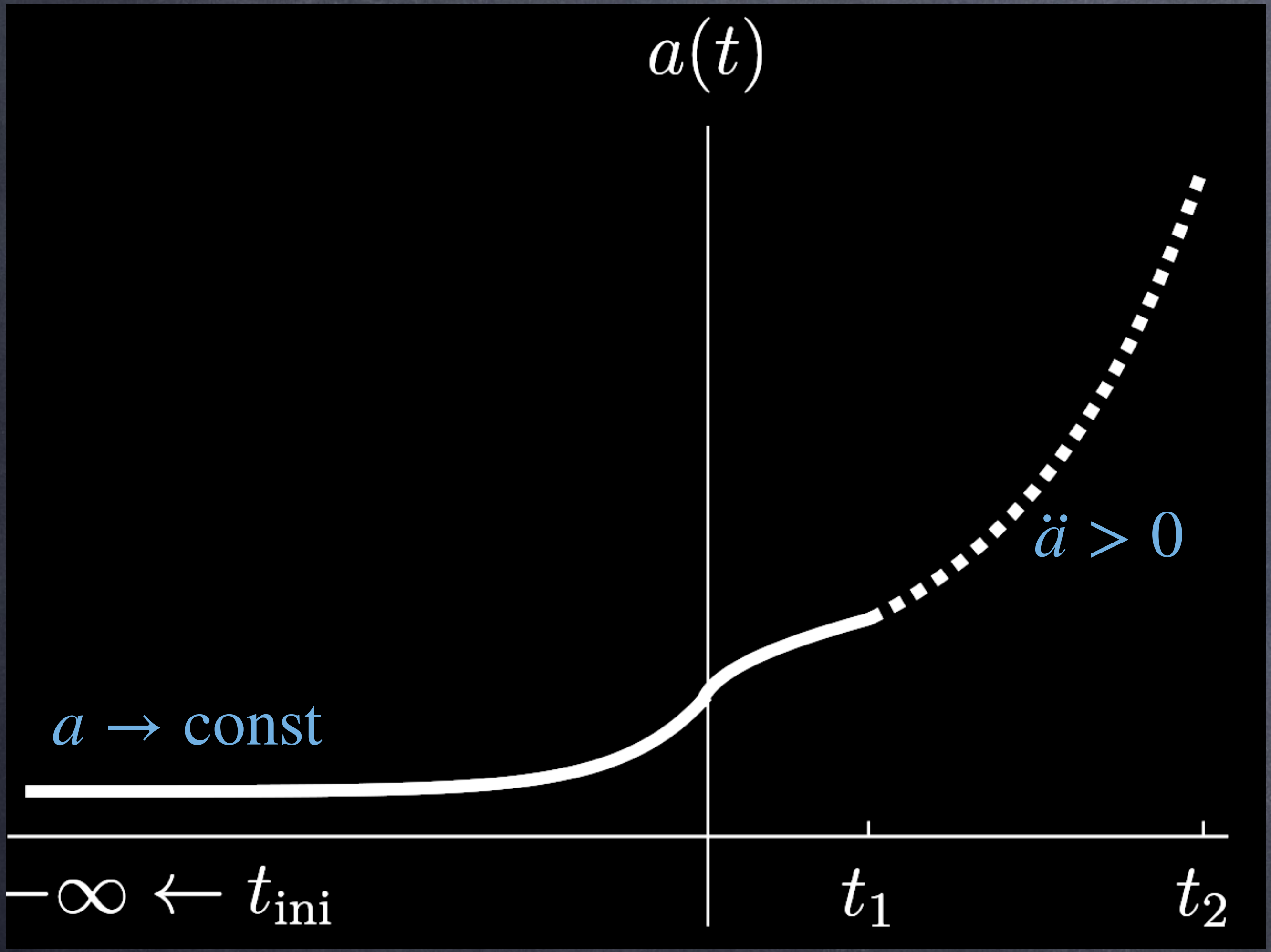
but also 1803.07085 with Daisuke Yoshida

Possible pre-inflationary phase

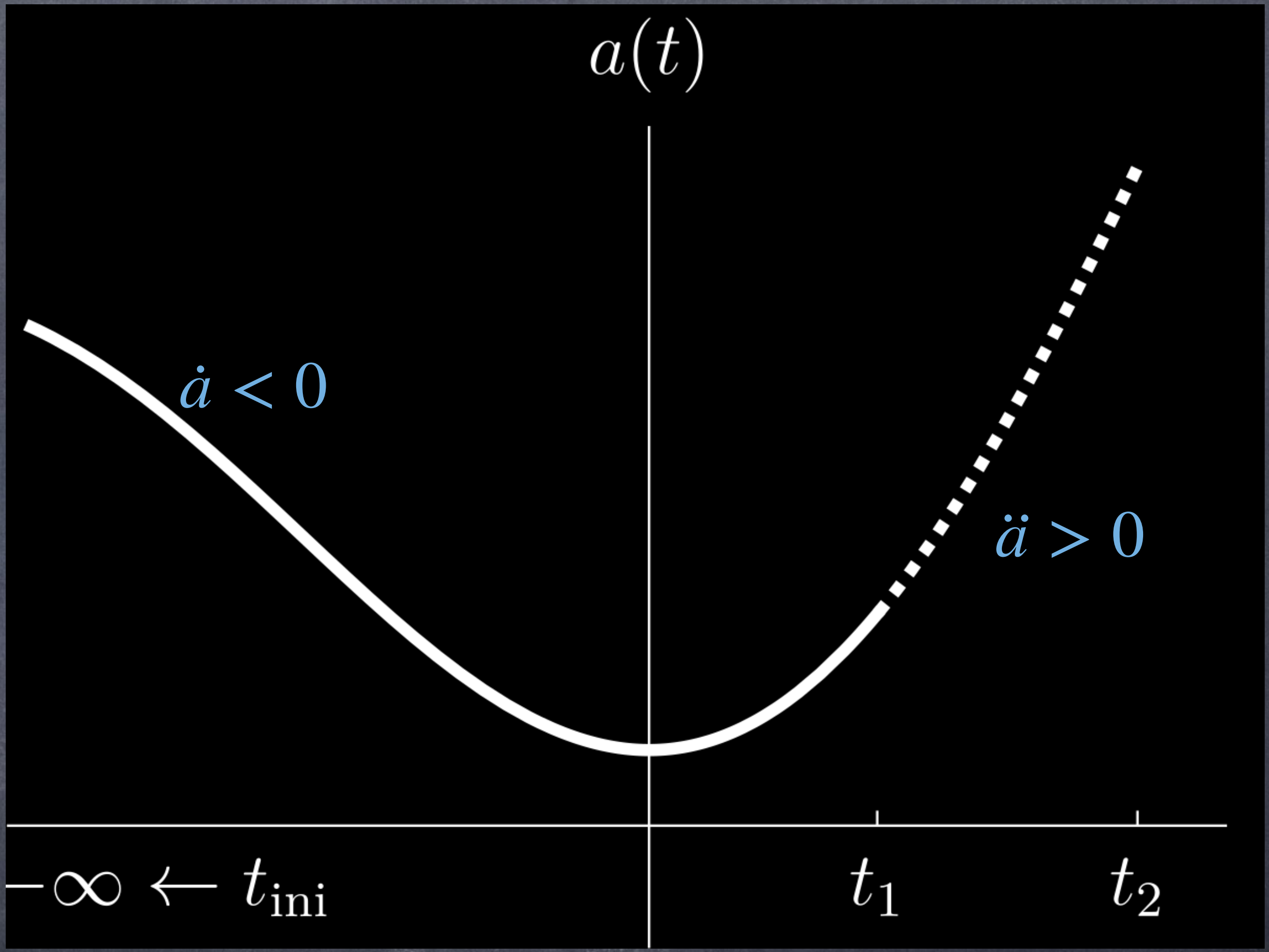


Scalar curvature singularity at $t_{\rm ini}$ (e.g., $|R_{\mu\nu}R^{\mu\nu}| \to \infty$)

More pre-inflationary possibilities



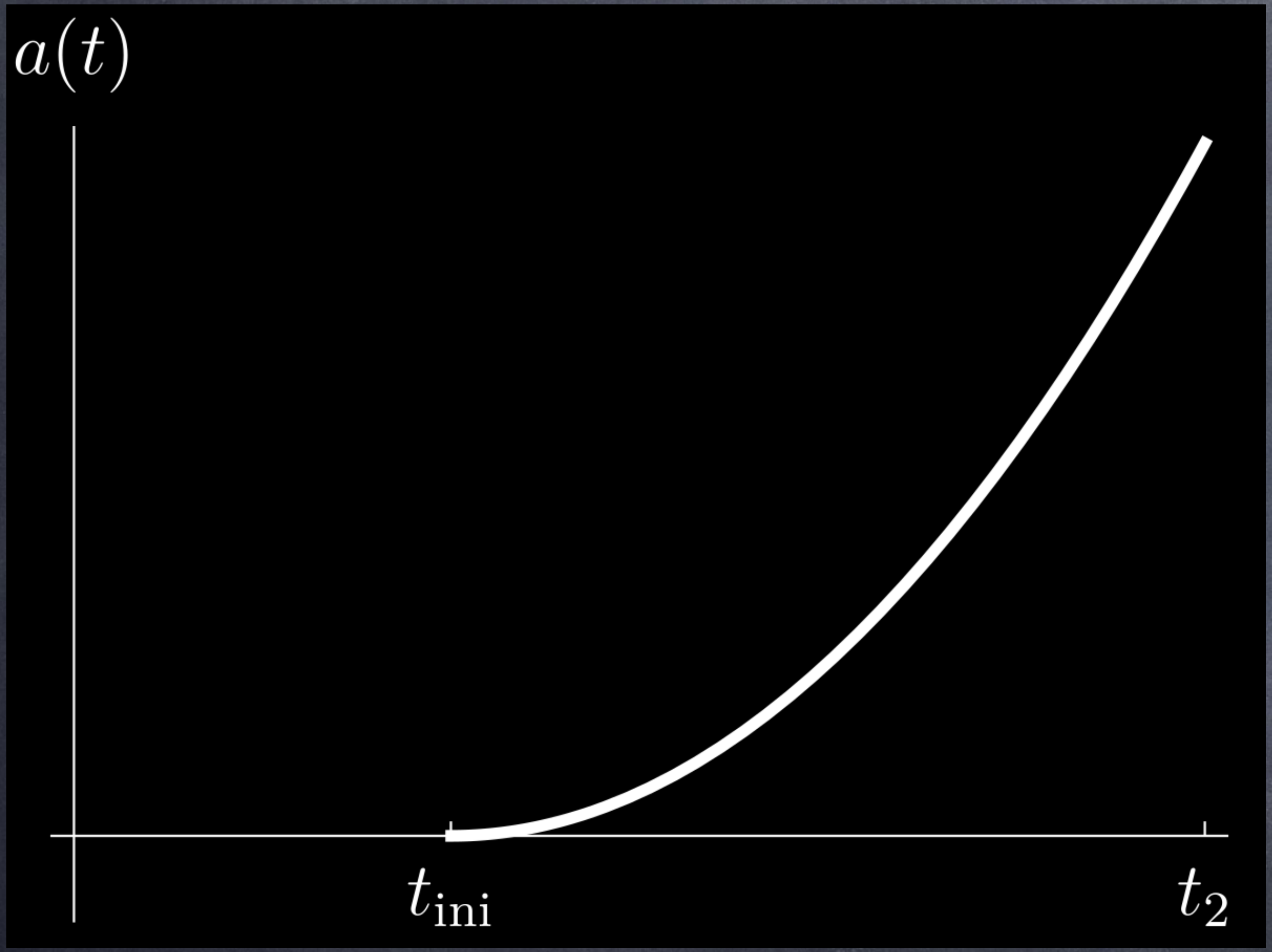
Loitering

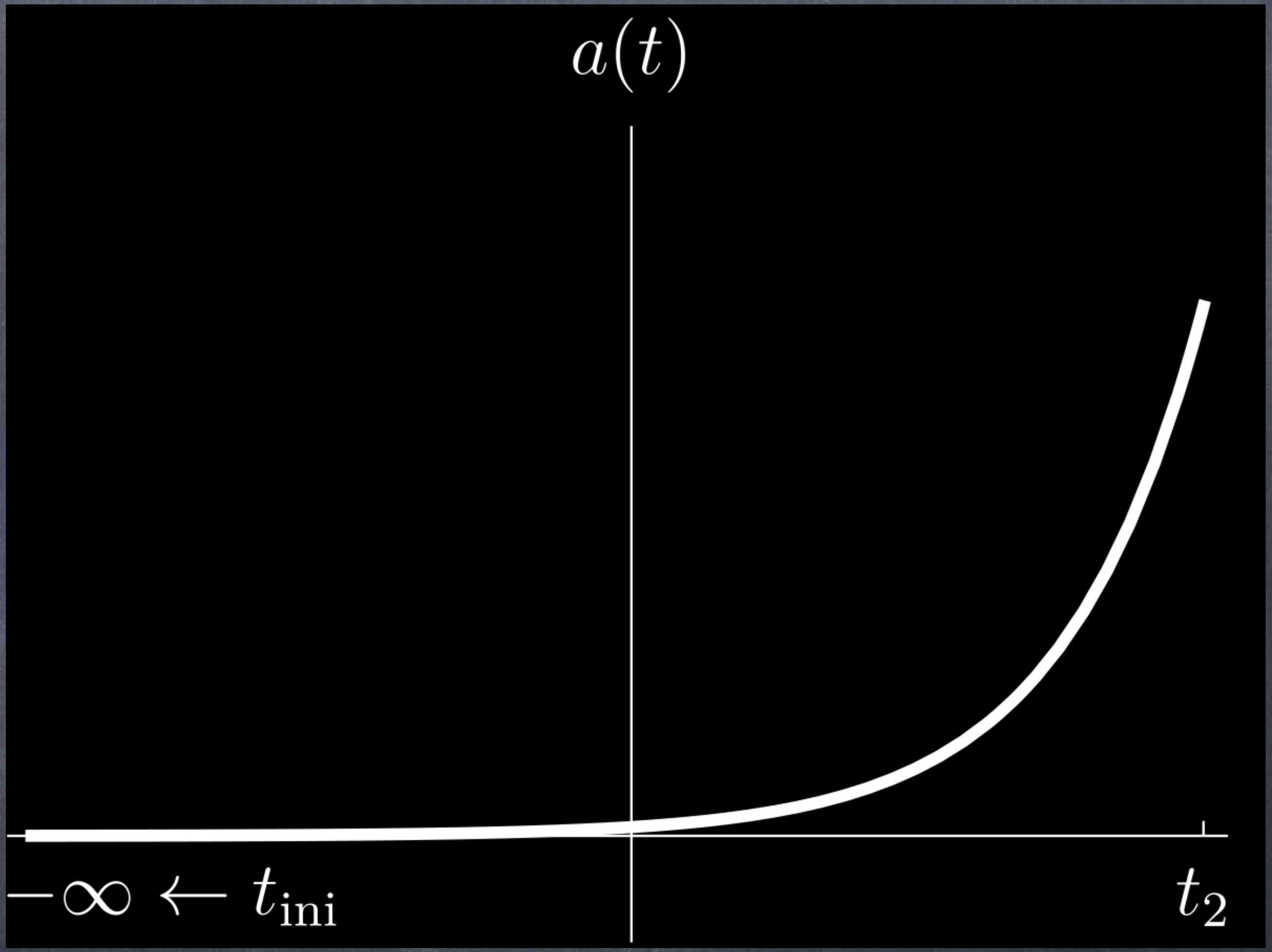


Bounce

No singularity! (But often requires new physics)

"Always" inflating

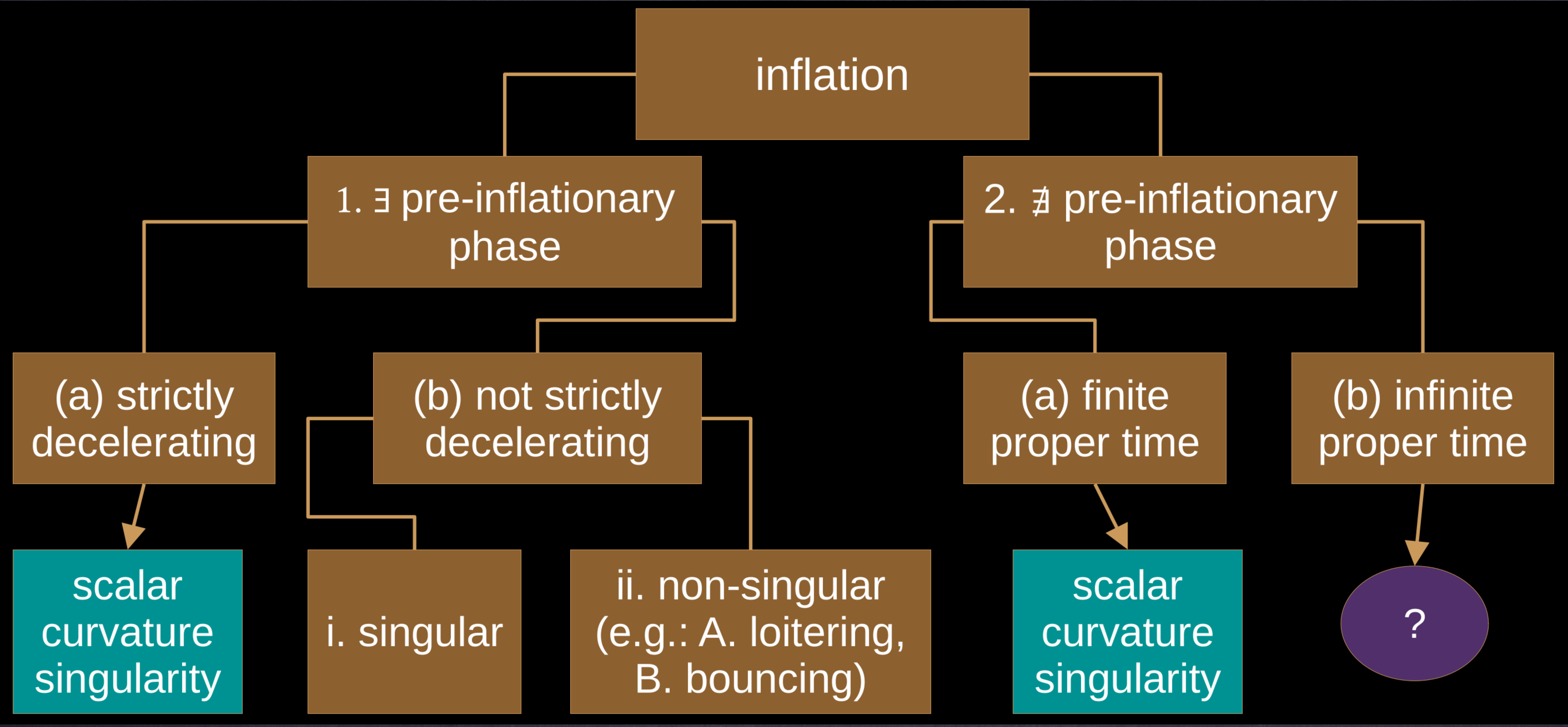




Scalar curvature singularity at t_{ini} (e.g., $|R_{\mu\nu}R^{\mu\nu}| \to \infty$)

Unclear what happens as $t_{\text{ini}} \to -\infty$

In flat FLRW



inflation
1. ∃ pre-inflationary phase
2. ∄ pre-inflationary phase
(a) strictly decelerating
(b) not strictly decelerating
(a) finite proper time
(b) infinite proper time
scalar curvature singularity
i. singular
ii. non-singular (e.g.: A. loitering, B. bouncing)
scalar curvature singularity
?

Borde-Guth-Vilenkin (2003)

Consider an affine parameter λ of a null geodesic in FLRW

$$\mathrm{d}\lambda = a(t)\,\mathrm{d}t$$

Given the Hubble parameter $H = \dfrac{\mathrm{d}\ln a}{\mathrm{d}t}$ we find

$$\int_{\lambda_i}^{\lambda_f} \mathrm{d}\lambda\, H = a_f - a_i < a_f$$

Thus defining an average Hubble parameter

$$H_{\rm av} \equiv \frac{1}{\lambda_f - \lambda_i} \int_{\lambda_i}^{\lambda_f} d\lambda \, H < \frac{a_f}{\lambda_f - \lambda_i}$$

we find

$$H_{\rm av} > 0 \implies \lambda_f - \lambda_i < \infty$$

Geodesic incompleteness!

However, is this coordinate independent?
Does it always imply some kind of singularity?

de Sitter (dS) $\Lambda = \frac{3}{8\pi G_{\mathrm{N}}}$

- Flat dS

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + e^{2t}\,\mathrm{d}\vec{x}^2$$

- Closed dS

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \cosh^2(t)\,\mathrm{d}\Omega_{(3)}^2$$

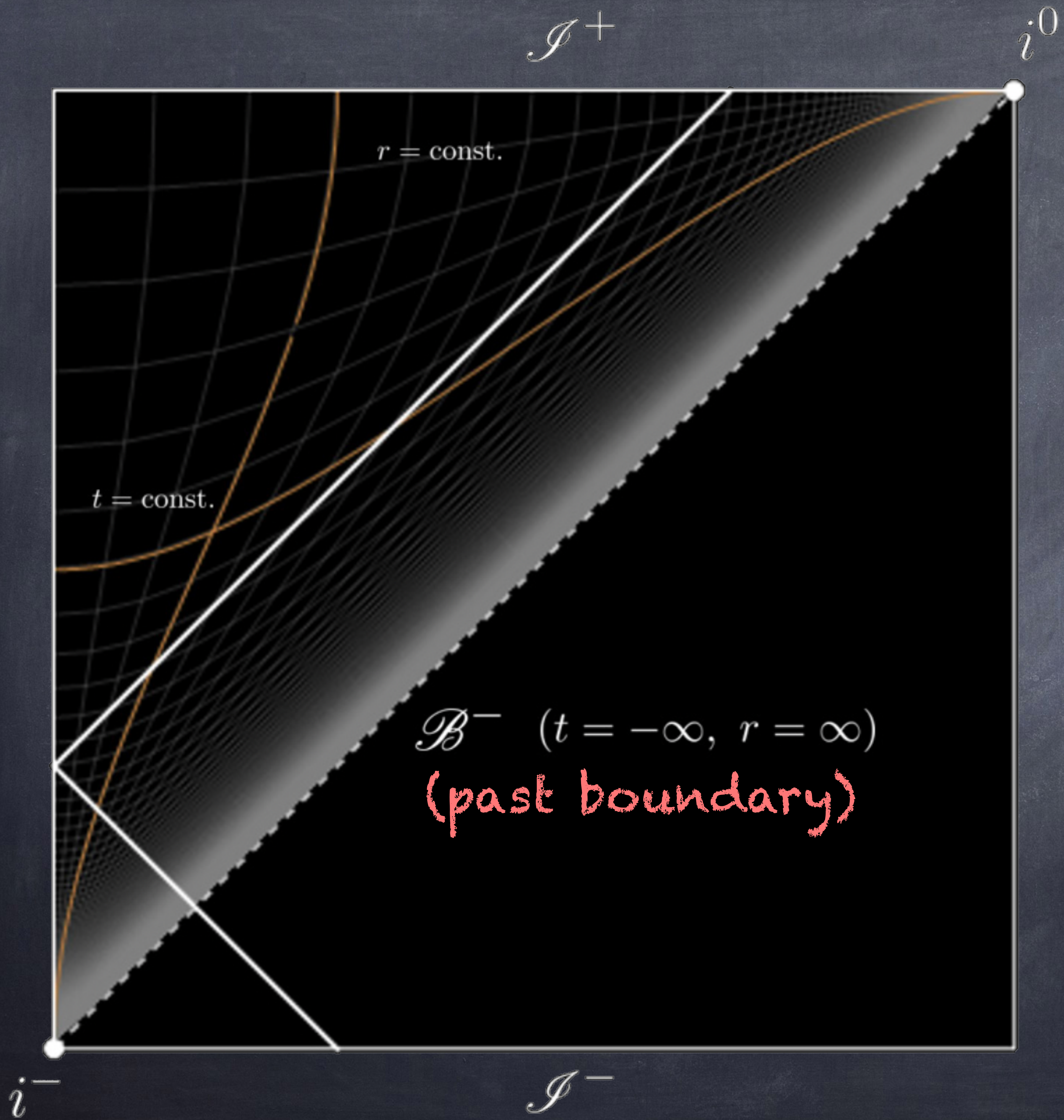
- Open dS

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \sinh^2(t)\,(\mathrm{d}\psi^2 + \sinh^2\psi\,\mathrm{d}\Omega_{(2)}^2)$$

- Conformal dS

$$\mathrm{d}s^2 = \frac{1}{\cos^2 T}(-\mathrm{d}T^2 + \mathrm{d}\Omega_{(3)}^2)$$

- ...



Flat dS

↓ extendible

Global dS

Close enough to open dS:

Milne-like $\Longrightarrow C^0$ extendible

(Eric Ling's talk last week)

Close enough to flat dS:

quasi-dS $\Longrightarrow ?$

$$ds^2 = a(\eta)^2(-d\eta^2 + dr^2 + r^2 d\Omega_{(2)}^2)\,, \quad a = e^t = -1/\eta$$

"Eddington-Finkelstein coordinates"

$$\lambda = \int dt\, a(t) = \int d\eta\, a(\eta)^2\,, \quad v = \eta + r$$

$$\downarrow$$

$$ds^2 = -2\, d\lambda\, dv + a(\lambda)^2 dv^2 + a(\lambda)^2 (v - \eta(\lambda))^2 d\Omega_{(2)}^2$$

dS: $ds^2 = -2\, d\lambda\, dv + \lambda^2 dv^2 + (1 + \lambda v)^2 d\Omega_{(2)}^2$

$a = \lambda > 0 \longrightarrow$ extendible to $\lambda \in \mathbb{R}$

$$\mathrm{d}s^2 = -2\,\mathrm{d}\lambda\,\mathrm{d}v + a(\lambda)^2\mathrm{d}v^2 + a(\lambda)^2(v-\eta(\lambda))^2\mathrm{d}\Omega^2_{(2)}$$

Generally if

$$t \to -\infty\,,\ \eta \to -\infty\,,\ a \to 0^+\,,\ \lambda \to 0^+$$

C^k extendibility of the metric requires

$$a^2,\ a^2\eta,\ a^2\eta^2 \in C^k$$

If $a(t)/e^{H_\Lambda t} \to 1$ or $H(t) \to H_\Lambda$ as $t \to -\infty$ for some $H_\Lambda > 0$,

then $\exists\ C^0$ extension

If $\dot{H}/a^2$ converges to a finite limit as $t \to -\infty$,

then $\exists\ C^2$ extension

If $\dot{H}/a^2$ is smooth in a as $a \to 0^+$,

then $\exists\ C^\infty$ extension

Toy example:

$$a(t) = e^t + \sin^2(e^{-3t})e^{2t} \implies \lim_{t\to-\infty} \frac{a(t)}{e^t} = 1 \implies \exists\, C^0 \text{ extension}$$

But $H = \dfrac{\dot{a}}{a}$ does not have a limit as $t \to -\infty$!

⟶ curvature singularity

take-home #1: coordinate singularities and curvature singularities are not mutually exclusive

$C^2 \longrightarrow$ geodesics, curvature

$H \to H_\Lambda, \dot{H} \to 0 \implies$ asymptotically dS

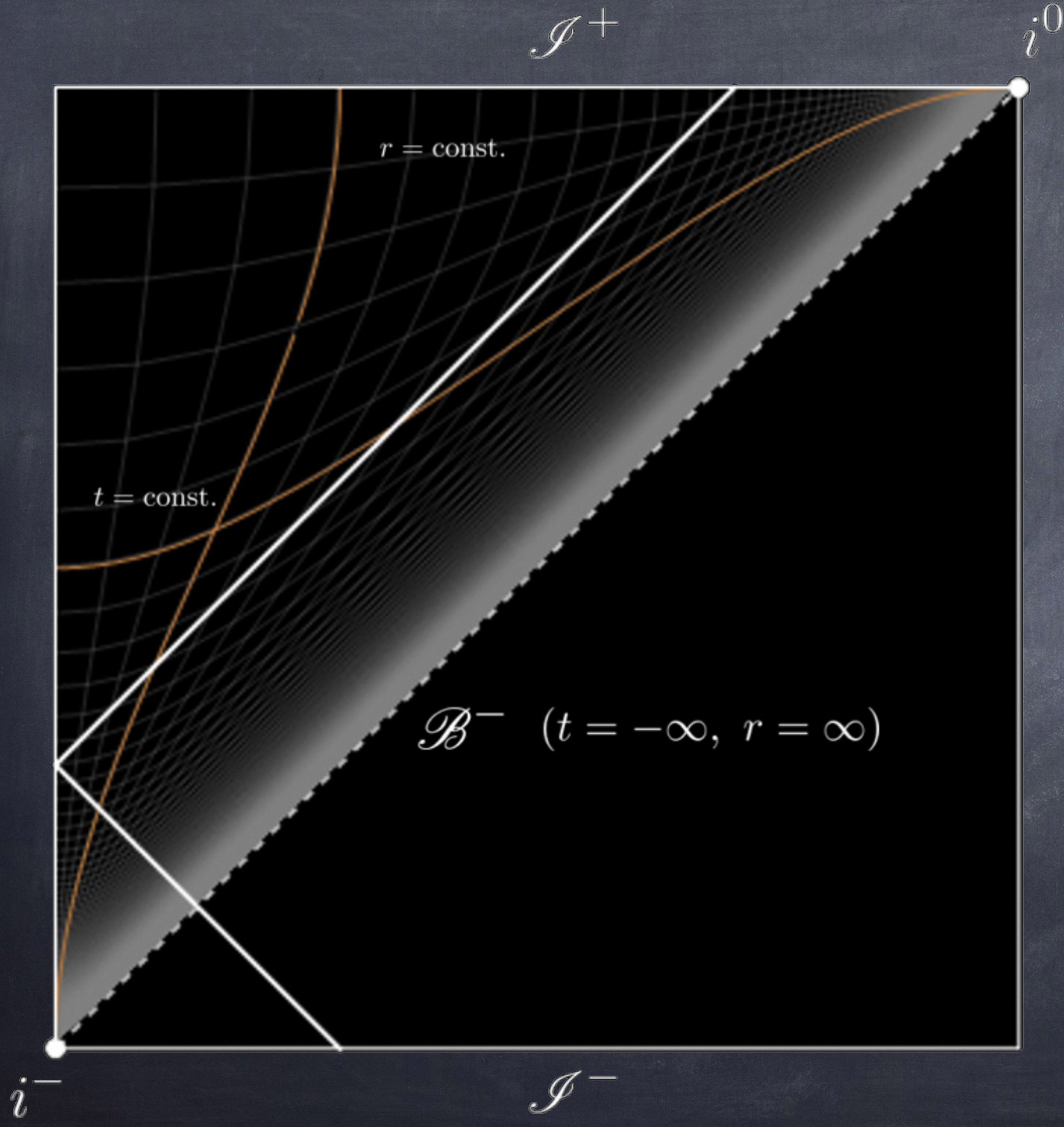
no scalar curvature singularity

not enough!

need $\dot{H}/a^2$ to converge

$a(t) = e^{H_\Lambda t} + O(e^{3H_\Lambda t})$ as $t \to -\infty$ does it

$a(t) = e^{H_\Lambda t} + e^{3H_\Lambda t} \implies \exists\, C^\infty$ extension



take-home #2:
want to approach dS "very fast" to get C^2 extension beyond past boundary

Why $\dot{H}/a^2$?

The usual flat FLRW basis is parallel along comoving timelike observers:

tetrad basis: $E^0 = \mathrm{d}t,\ E^1 = a\mathrm{d}r,\ E^2 = ar\mathrm{d}\theta,\ E^3 = ar\sin\theta\mathrm{d}\phi$

$$\implies g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{ab}E^a E^b$$

$$u^\mu u_\mu = -1 \implies u^\mu \nabla_\mu E^a = 0$$

Curvature in this basis is what we expect:

$$R_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -2\dot{H}(E^0)^2 + (3H^2 + \dot{H})\eta_{ab}E^a E^b$$

no scalar curvature singularity if $H \to H_\Lambda,\ \dot{H} \to 0$

Why $\dot{H}/a^2$?

If we instead go to a basis that is parallel along **null** directions:

tetrad basis: $\tilde{E}^0 = \frac{a}{\sqrt{2}}(E^0 - E^1),\ \tilde{E}^1 = \frac{1}{\sqrt{2}a}(E^0 + E^1),\ \tilde{E}^2 = E^2,\ \tilde{E}^3 = E^3$

Nomura & Yoshida [2105.05642]

$$k^\mu k_\mu = 0 \implies k^\mu \nabla_\mu \tilde{E}^a = 0$$

Then components of the curvature are different:

$$R_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\frac{\dot{H}}{a^2}(\tilde{E}^0)^2 - a^2\dot{H}(\tilde{E}^1)^2 - 2(3H^2 + 2\dot{H})\tilde{E}^0\tilde{E}^1 + (3H^2 + \dot{H})((\tilde{E}^2)^2 + (\tilde{E}^3)^2)$$

If $\dot{H}/a^2 \to \pm\infty$, then there is a null parallelly propagated (p.p.) curvature singularity

Recall the $\{\lambda, v, \theta, \phi\}$ coordinates:

$$ds^2 = -2\,d\lambda\,dv + a(\lambda)^2 dv^2 + a(\lambda)^2 (v - \eta(\lambda))^2 d\Omega^2_{(2)}$$

$$\downarrow$$

$$R_{\mu\nu} dx^\mu dx^\nu = -2\frac{\dot{H}}{a^2} d\lambda^2 + (3H^2 + \dot{H})(-2\,d\lambda\,dv + a(\lambda)^2 dv^2 + a(\lambda)^2 (v - \eta(\lambda))^2 d\Omega^2_{(2)})$$

$a(t) = e^t + \sin^2(e^{-3t})e^{2t}$	$a(t) = e^t + e^{2t}$	$a(t) = e^t + e^{3t}$
$\lim_{t\to-\infty} H \nexists$	$H \to 1,\ \dot{H} \to 0$ $\dot{H}/a^2 \to \infty$	$\dot{H}/a^2 \to 4$ $\dot{H}/a^2$ analytic in a
coordinate singularity C^0 extendible	coordinate singularity C^0 extendible	coordinate singularity C^∞ extendible
C^1 inextendible? geodesically incomplete?	C^1 inextendible? geodesically incomplete?	geodesically complete
scalar curvature singularity	no scalar curvature singularity	no scalar curvature singularity
null p.p. curvature singularity	null p.p. curvature singularity	no null p.p. curvature singularity

take-home #3: one has to be careful about what one means by a "singularity"

(coordinate singularity,
spacetime inextendibility,
geodesic incompleteness,
scalar curvature singularity,
p.p. curvature singularity)

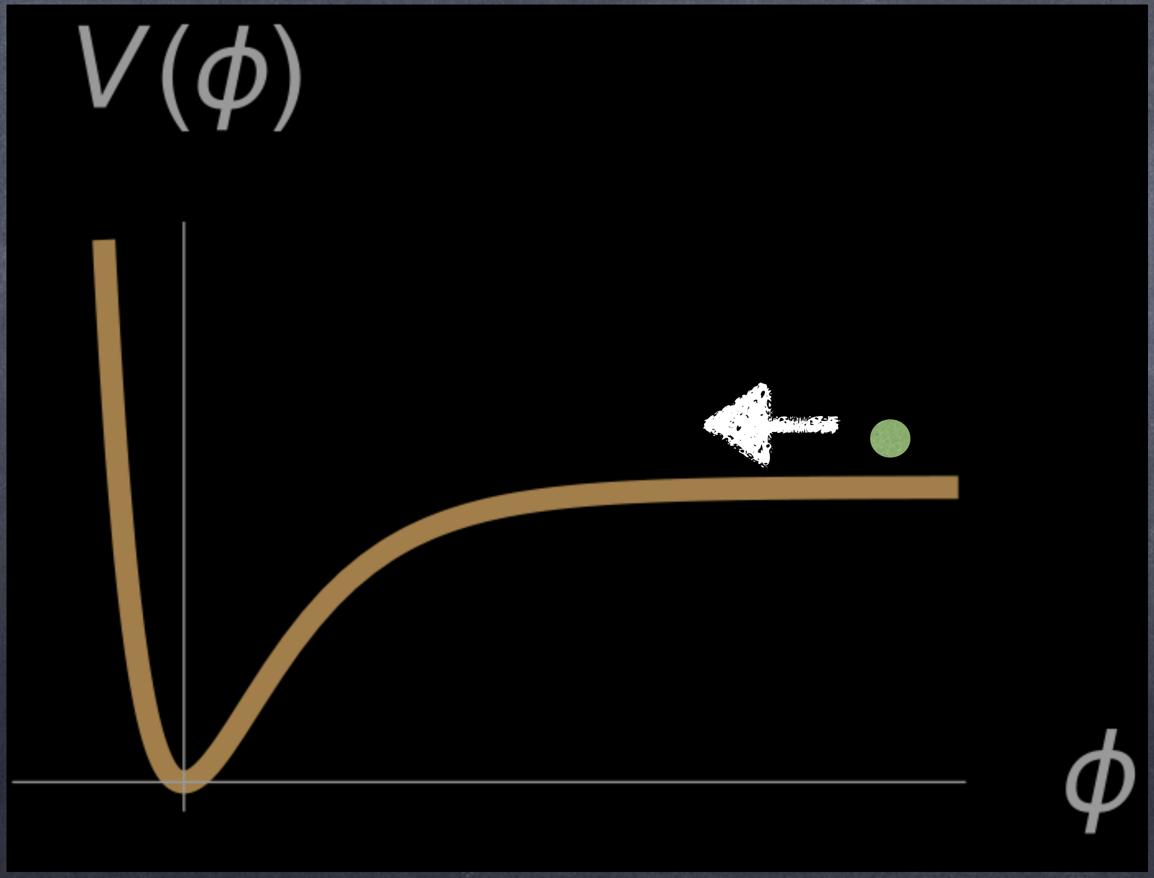
Pure geometrical statements so far, independent of any field equations (we have not solved any)

↓

Let's introduce some physics now

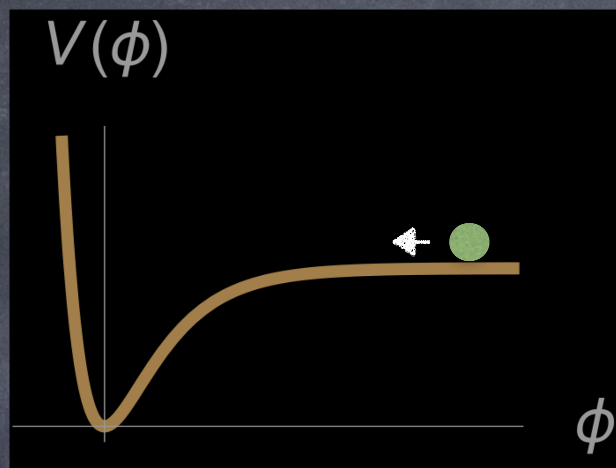
Slow-roll inflation

$$3H^2 \simeq V(\varphi)\,, \quad 3H\dot{\varphi} + V'(\varphi) \simeq 0$$



Starobinsky

$$V(\varphi) = \frac{3}{4}m^2(1 - e^{-\sqrt{2/3}\varphi})^2$$



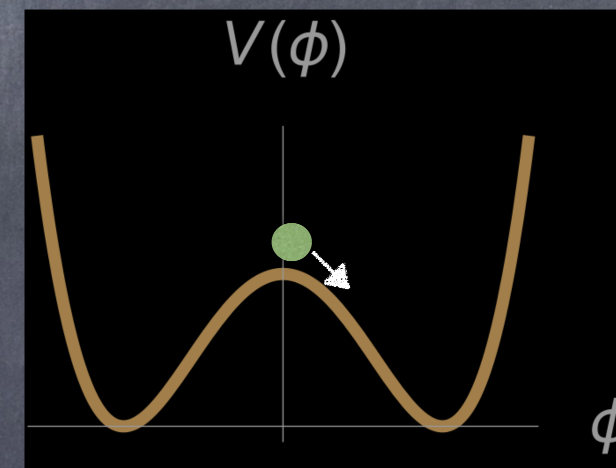
$$a(t) \simeq a_e e^{\frac{m}{2}t}\left(1 - \frac{2}{3} m e^{-\sqrt{\frac{2}{3}}\varphi_e t}\right)$$

$$H \to \frac{m}{2}, \dot{H} \to 0$$

$$\dot{H}/a^2 \to -\infty$$

Small-field

$$V(\varphi) = V_0\left(1 - \left(\frac{\varphi}{2m}\right)^2\right)$$



$$a(t) \simeq a_e e^{\sqrt{\frac{V_0}{3}}t - \frac{\varphi_e^2}{8}\exp(\frac{2}{m^2}\sqrt{\frac{V_0}{3}}t)}$$

$$H \to \sqrt{\frac{V_0}{3}}, \dot{H} \to 0$$

$\dot{H}/a^2$ smooth in a

Starobinsky inflation	Small-field inflation
$H \to \text{const}, \dot{H} \to 0$	$\dot{H}/a^2 \to \text{const}$
$\dot{H}/a^2 \to \infty$	$\dot{H}/a^2$ smooth in a
coordinate singularity	coordinate singularity
C^0 extendible	C^∞ extendible
C^1 inextendible?	
geodesically incomplete?	geodesically complete
no scalar curvature singularity	no scalar curvature singularity
null p.p. curvature singularity	no null p.p. curvature singularity

However, both require extreme fine-tuning initially!

$\dot{\varphi} = 0$ at $t = -\infty$

When $V'(\varphi) \simeq 0$, expect $\ddot{\varphi} + 3H\dot{\varphi} \simeq 0 \implies \dot{\varphi}^2 \sim a^{-6}$

$\downarrow$

Looking backwards in time, we do not dynamically expect to approach dS

$$H \to H_\Lambda,\ \dot{H} \to 0 \iff p \to -\rho$$

$p \to -\rho \implies C^0$ extendibility

but likely not enough to get C^2, etc.

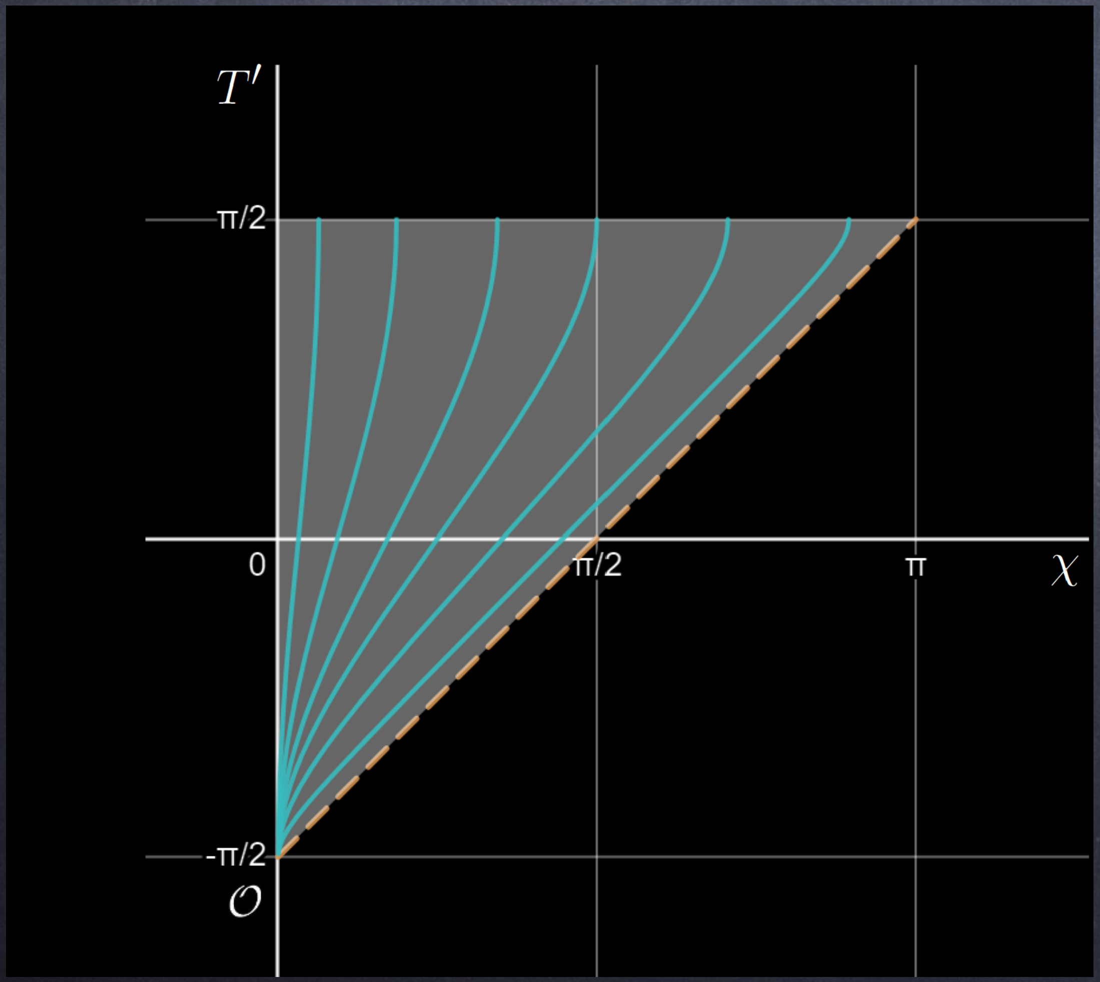
Consider adding a subdominant matter component to a c.c. with EoS w and demand C^2:

$$\rho = \Lambda + \rho_m, \quad p = -\Lambda + w\rho_m$$

$$\frac{\dot{H}}{a^2} \propto a^{-5-3w} < \infty \text{ as } a \to 0^+ \iff w \le -\frac{5}{3}$$

$p \to -\rho \implies C^0$ extendibility

converse also true!
(in a specific context, though without the FLRW symmetry assumptions)



- If we solve the Einstein equations with a perfect fluid,
- if we have a continuous conformal extension with conformal factor Ω in which the integral curves of the fluid's vector field have past endpoint at the origin (as in dS),
- if ρ, p, Ω^2Ric extend continuously through the origin,
- if we have strong causality near the origin, then

$p = -\rho$ at the origin

$p = -\rho$ at the origin is 'special'

either some principle selects this

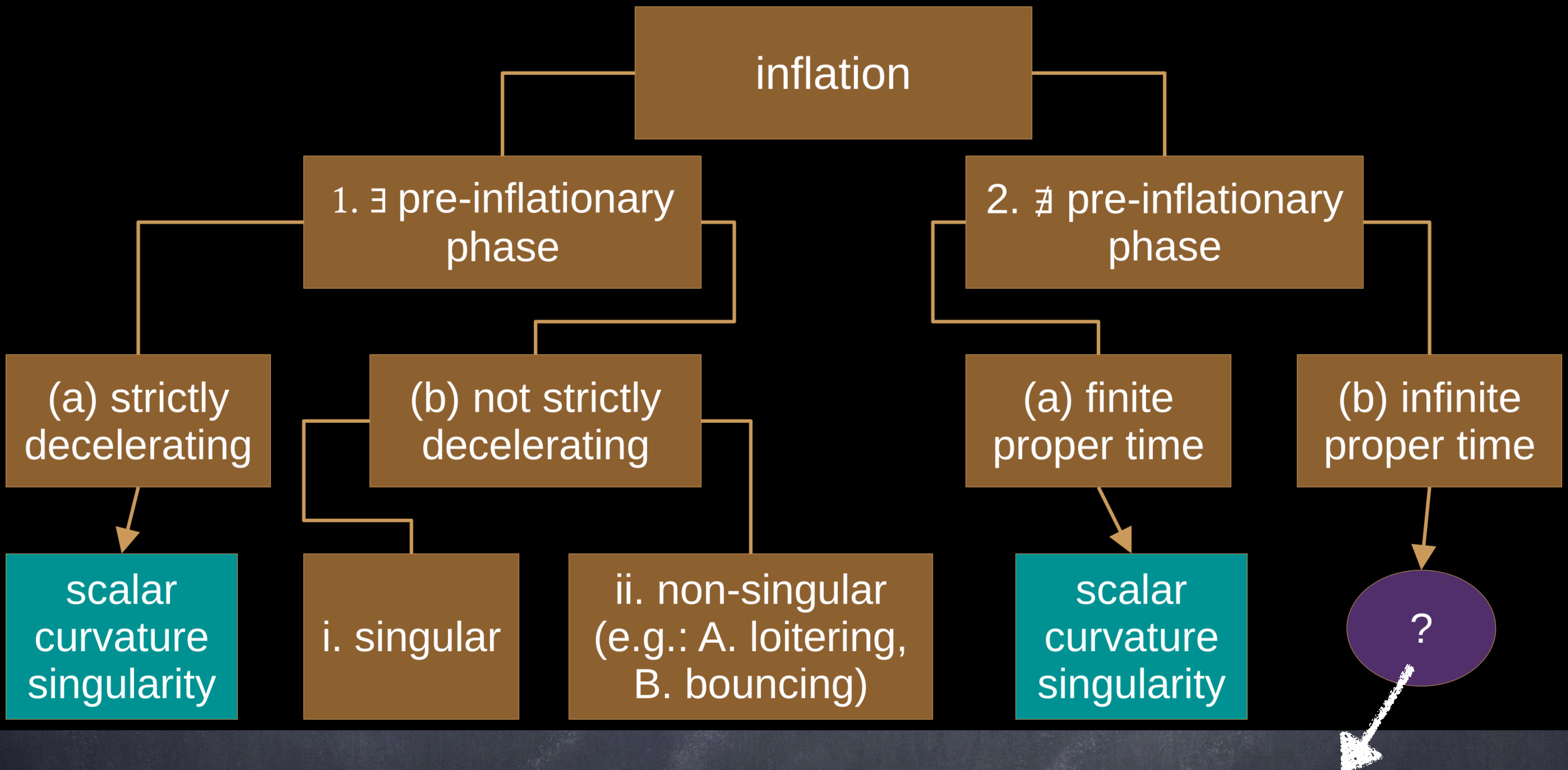
either $p \nrightarrow -\rho \implies$

- we do not have a continuous conformal extension in which the integral curves of the fluid's vector field have past endpoint at the origin (as in dS)
- or ρ, p, $\Omega^2 Ric$ do not extend continuously through the origin
- Or another assumption fails (Einstein equations or strong causality)

$a \nearrow$ $\quad$ $a \searrow$

$$3H^2 = \Lambda - \frac{k}{a^2} + \frac{\rho_{m0}}{a^3} + \frac{\rho_{r0}}{a^4} + \frac{\sigma_0^2}{a^6}$$

anisotropic approach to singularity



take-home #4: scalar curvature singularity unless some principle selects just the right past asymptotic conditions

Summary

- Geometrical examples where past boundary (i.e. where null geodesics appear to end) is more like a "conical singularity"
- Geometrical cases that are fully non-singular, but need to approach dS "fast enough"
- Need to be precise about type of singularity
- Physically unlikely, classically, that inflation happening for infinite proper time in the past is non-singular

Outlook

- Quantum effects?

- Does the last theorem hold beyond GR?

- Classical geometries that are extendible, do they bounce as global dS?

Thank you for your attention!

Questions?