Quantum Gravity $\to$ GR

Classical Physics : $F = ma$

Must come from $i\partial_t \Psi = H\Psi$

Ehrenfest's Theorem

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$$

$$\partial_t \langle \hat{x} \rangle = \langle i [\hat{H}, \hat{x}] \rangle = \left\langle \frac{\hat{p}}{m} \right\rangle$$

$$\partial_t \langle \hat{p} \rangle = \langle i [\hat{H}, \hat{p}] \rangle = -\left\langle \frac{\partial V(\hat{x})}{\partial x} \right\rangle$$

$$\Rightarrow \hat{\phi}, \hat{\pi} \qquad [\hat{\phi}_x, \hat{\pi}_y] = i\delta^3(\bar{x}-\bar{y})$$

$$\partial_t \langle \hat{\phi} \rangle = \left\langle \frac{\partial H}{\partial \pi} \right\rangle$$

$$\partial_t \langle \hat{\pi} \rangle = -\left\langle \frac{\partial H}{\partial \phi} \right\rangle$$

$$\int D\varphi \, e^{iS[\varphi]} \to \int D\varphi \, e^{iS[\varphi+\delta\varphi]}$$

$$\varphi \to \varphi + \delta\varphi$$

$$\simeq \int D\varphi \left(1 + i\frac{\delta S}{\delta\varphi}\delta\varphi\right) e^{iS[\varphi]}$$

$$\longrightarrow \left\langle \frac{\delta S}{\delta \varphi} \right\rangle = 0$$

$$\left\langle \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \varphi_\mu} \right\rangle = 0$$

Ehrenfest's Theorem in Gauge Theories

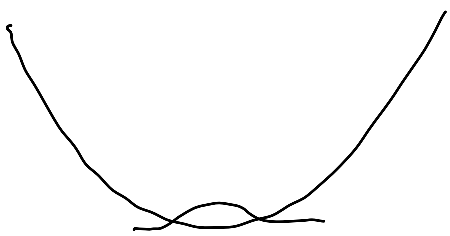
(thy. w/ redundancies)

$$H_{cl} \longrightarrow \hat{H}$$

- involves gauge fixing
- lose e.o.m.

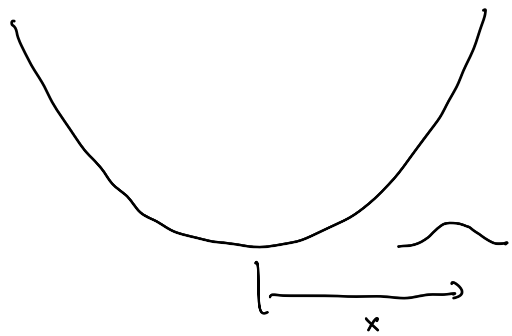
$$\int D\lambda \int DA \; e^{iS[A] + iS_{g.f}[\lambda, A]}$$

Coherent States



$e^{-i\hat{H}t}|0\rangle$

$= e^{-iE_0 t}|0\rangle$



Not eigenstate of $\hat{H}$

$$\langle E \rangle \sim \langle \tfrac{1}{2} m \omega^2 x^2 \rangle$$

$$\sqrt{\langle x^2 \rangle} \sim \sqrt{n}$$

Outline

* Minisuperspace
* Einstein's Eq.s

(* Maxwell's Eq.s)

Minisuperspace

$$ds^2 = -N^2(t)\,dt^2 + a^2(t)\,d\vec{x}^2$$

N ~ lapse

$$S = \int d^4x \left[\sqrt{-g}\,M_{pl}^2 R + \sqrt{-g}\,\underbrace{\mathcal{L}_{matter}}_{\phi(t)}\right] + S_{GHY}$$

$$= \int dt \left[-3M_{pl}^2 \frac{a\dot{a}^2}{N} + \frac{a^3\dot{\phi}^2}{2N} - N a^3 V(\phi)\right]$$

Classically

1st Friedmann $\frac{\delta S}{\delta N} = a^3 \left[3M_{pl}^2 \frac{\dot{a}^2}{N^2 a^2} - \frac{\dot{\phi}^2}{2N^2} - V(\phi) \right) = 0$

2nd Friedmann $\frac{\delta S}{\delta a} = 3Na^2 \left[2M_{pl}^2 \frac{\ddot{a}}{N^2 a} + M_{pl}^2 \frac{\dot{a}^2}{N^2 a^2} - 2M_{pl}^2 \frac{\dot{a}\dot{N}}{N^3 a} - \frac{\dot{\phi}^2}{2N^2} + V(\phi) \right] = 0$

ϕ E.O.M. $\frac{\delta S}{\delta \phi} = -\frac{a^3}{N} \left(\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \underline{\frac{\dot{N}\dot{\phi}}{N}} + N^2 \frac{\partial V}{\partial \phi} \right) = 0$

$N = 1$

Build $\hat{H}$ from H_{cl}

find Conjugates

$\rightarrow \frac{\delta L}{\delta \dot{N}} = 0$

$\frac{\delta L}{\delta \dot{a}} \equiv \pi_a$

$\frac{\delta L}{\delta \dot{\phi}} \equiv \pi_\phi$

$H_{cl} = \pi_a \dot{a} + \pi_\phi \dot{\phi} - L \Big|_{\substack{\dot{a} = \cdots \pi_a \\ \dot{\phi} = \cdots \pi_\phi}}$

$= -\frac{N}{12 M_{pl}^2} \left\langle \widehat{\frac{\pi_a^2}{a}} \right\rangle + \frac{N}{2} \left\langle \widehat{\frac{1}{a^3}} \right\rangle \tilde{\pi}_\phi^2 + N a^3 V(\phi)$

$\frac{\pi_{cl}^2}{a_{cl}}$

$= N \tilde{H}$

Now Quantize! N?

$$i \frac{\partial}{\partial t} \Psi = N \hat{\tilde{H}} \Psi \rightarrow \underline{\frac{i \partial}{N \partial t}} \Psi = \hat{\tilde{H}} \Psi$$

$N = 1!$

$N \partial t \rightarrow \partial \tau$

$$\left.\begin{aligned} \partial_t \langle \hat{a} \rangle &= \langle i [\hat{H}, \hat{a}] \rangle \sim \left\langle \frac{\partial H}{\partial \pi_a} \right\rangle \\ \partial_t \langle \hat{\pi}_a \rangle &= \langle i [\hat{H}, \hat{\pi}_a] \rangle \sim -\left\langle \frac{\partial H}{\partial a} \right\rangle \end{aligned}\right] \rightarrow$$

2nd Friedmann Eq.

$$\frac{\delta N \tilde{H}}{\delta N} = 0 \Rightarrow \underline{\tilde{H} = 0}$$

$\hookrightarrow$ 1st Friedmann Equation!

$\cancel{\hat{\tilde{H}} = 0}$ not consistent with commutators

e.g. $[\hat{H}, \hat{a}]$

$$\hat{\tilde{H}} | \Psi \rangle_{phys} = 0$$

$$i \partial_t | \Psi \rangle = \hat{\tilde{H}} | \Psi \rangle = 0$$

Choose

$$\underline{\langle \hat{\tilde{H}} \rangle = 0}$$

$$\underline{\partial_t \langle \hat{H} \rangle = \langle i [\hat{H}, \hat{H}] \rangle = 0}$$

$$\hookrightarrow \int Da\, DN\, D\varphi \; e^{iS[N,a,\varphi]}$$

$$N \to N + \underline{\delta N}$$

$$\int Da\, DN\, D\varphi \; e^{iS[\underline{N+\delta N}, a, \phi]}$$

$$dt' = N\, dt$$

$$S[N+\delta N, a, \phi] = S[N, a, \phi]$$

$|x\rangle$ state of fields

$$\langle x|\hat{H}|x\rangle = \underline{C} = \underline{3M_{pl}^2 a \dot{a}^2 - a^3 \rho}$$

$$\left[\begin{array}{l} \underline{\left(\frac{\dot{a}}{a}\right)^2} = \frac{8\pi}{3} G_N \rho + \frac{\mathbb{H}_0}{a^3} \\ \frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = -4\pi G_N P \end{array}\right.$$

x

$$\langle x|\hat{H}|x\rangle \simeq H_{cl}[a_{cl}, \pi_{cl}, \dots]$$

More General Relativity

$$S = \int d^4x \left(\sqrt{-g}\, M_{pl}^2 R + \sqrt{-g}\, \underbrace{\mathcal{L}_{matter}}_{\frac{1}{2} g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi + \cdots} \right) + S_{GHY}$$

$$\frac{\delta S}{\delta g_{\mu\nu}} = \sqrt{-g}\, M_{pl}^2\, G^{\mu\nu} - \sqrt{-g}\, T^{\mu\nu}$$

$$= 0 \Rightarrow \text{Einstein's Eq.s}$$

Build Quantum Theory

$$\left.\begin{array}{l} \dfrac{\delta \mathcal{L}}{\delta \dot{g}_{00}} = 0 \\[2ex] \dfrac{\delta \mathcal{L}}{\delta \dot{g}_{0i}} = 0 \end{array}\right| \Rightarrow \begin{array}{l} g_{00} = -1 \\ g_{0i} = 0 \\ g_{ij} \equiv \gamma_{ij} \end{array}$$

$$\frac{\delta \mathcal{L}}{\delta \dot{g}_{ij}} \equiv \pi^{ij}$$

$$\mathcal{H} = \frac{1}{\sqrt{\gamma}} \left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl} \right) \pi^{ij}\pi^{kl}$$

$$- \sqrt{\gamma}\, {}^{(3)}R + \text{matter terms}$$

Quantum operator

$$[\hat{\gamma}_{ij}(x), \hat{\pi}^{kl}(y)] = i(\delta\delta + \delta\delta)\,\delta^3(\vec{x}-\vec{y})$$

build coherent states $|\gamma_{cl}\,\pi_{cl}\rangle$

$$\left.\begin{aligned} \partial_t\langle\gamma_{ij}\rangle &\sim \left\langle \frac{\delta \hat{H}}{\delta \pi^{ij}} \right\rangle \\ \partial_t\langle\pi^{ij}\rangle &\sim -\left\langle \frac{\delta \hat{H}}{\delta \hat{\gamma}_{ij}} \right\rangle \end{aligned}\right]$$ only spatial E.O.M.

$$\Rightarrow \quad \sqrt{\gamma}\, G^{ij} = 8\pi G_N \sqrt{\gamma}\, T^{ij}$$

$$\not\propto$$

$$G^{00} = 8\pi G_N T^{00} + \frac{\mathbb{H}(x)}{\sqrt{\gamma}}$$

$$G^{0i} = 8\pi G_N T^{0i} + \frac{\mathbb{P}^i(x)}{\sqrt{\gamma}}$$

$$G^{ij} = 8\pi G_N T^{ij}$$

$$\nabla_\mu G^{\mu\nu} = 0$$

$$\nabla_\mu T^{\mu\nu} = 0$$

$$T^{\mu\nu}_{Aux} = \frac{1}{\sqrt{\gamma}} \left(\begin{array}{c|c} \mathbb{H} & \mathbb{P}^i \\ \hline \mathbb{P}^j & 0 \end{array} \right)$$

$$\to \quad \nabla_\mu T^{\mu\mu}_{Aux} = 0$$

$$\to \quad \partial_0 \mathbb{H} + \partial_i \mathbb{P}^i = 0$$

$$\partial_0 (\gamma_{ij} \mathbb{P}^j) = 0$$

In Pert. Thy. (linear expasion around Homogeneous b.g.)

$$ds^2 = -dt^2 + a^2(t)\,(\delta_{ij} + \underline{h_{ij}})\,d\vec{x}^2$$

$$h_{ij} = \delta_{ij} h + D_{ij}\eta + \text{vect.} + \text{tensor}$$

$$\hookrightarrow \quad \frac{\partial_i\partial_j - \frac{1}{3}\partial^2\delta_{ij}}{\partial^2}$$

$$\mathbb{P}^i = 0 \quad , \quad \mathbb{H} \equiv \mathbb{H}_0 \quad \text{(homogeneous)}$$

$$T_{AUX}^{00} = \frac{\mathbb{H}_0}{\sqrt{\gamma}} \simeq \frac{\mathbb{H}_0}{a^3}\left(1 - \tfrac{1}{2} h\right)$$

$$T_{AUX}{}^{0}{}_{0} = -\frac{\mathbb{H}_0}{a^3}\left(1 - \tfrac{1}{2} h\right) \equiv -\bar{\rho}_S (1 + \delta)$$

$$\dot{\delta} = -\tfrac{1}{2}\dot{h}$$

$$\delta \sim -h$$

Conclusion

Quantizing "GR" $\rightsquigarrow$ physical states
classical-like
states
(not $\hat{H}|\psi\rangle = 0$)

Can Add Non-Dynamical Source terms
(to G_{00}, G_{0i}) and get consistent
time evolution w/o
additional dynamical
d.o.f.

EM

$$\hat{G} \equiv \nabla \cdot \hat{\vec{E}} - e\hat{J}^0 \qquad A_0 = 0$$

$$[\hat{G}, \hat{H}] = 0$$

$$\hat{G}\,|\Psi\rangle_{phys} = 0$$

$$\hat{G}\,|\Psi\rangle_{phys} = G(\vec{x})\,|\Psi\rangle_{phys}$$

$\vec{\nabla} \cdot \vec{E}_{back}$