Wave-optics limit of the stochastic gravitational wave background

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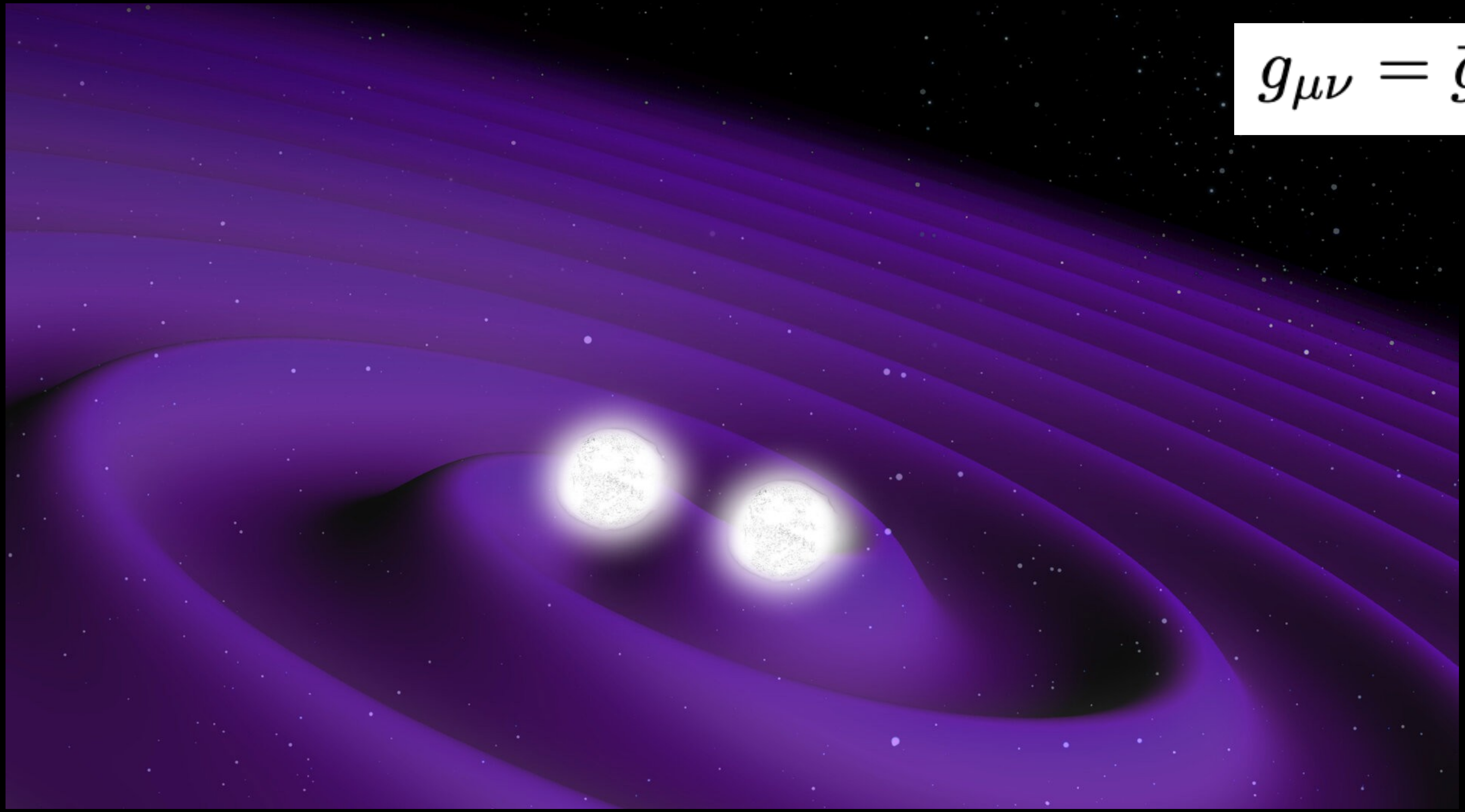
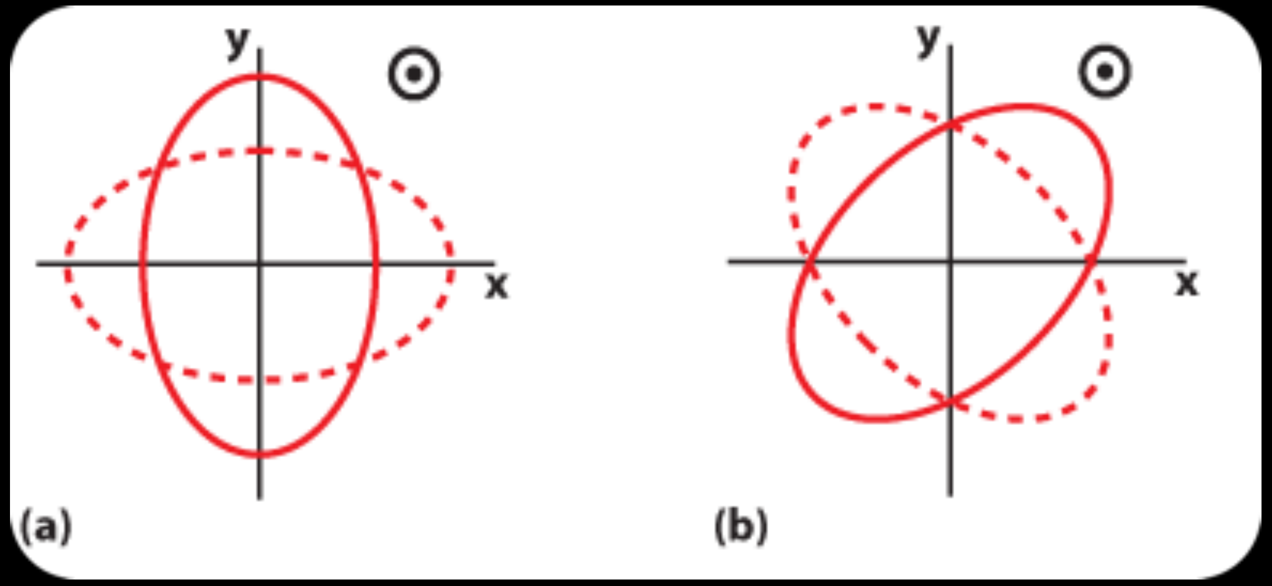
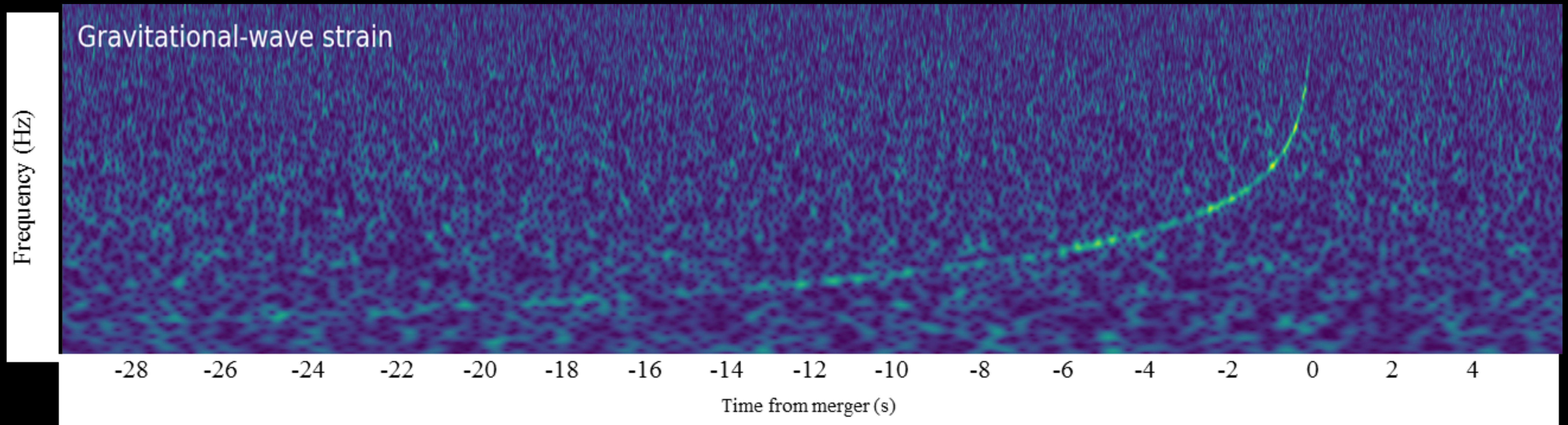
Gravitational waves: a historical detection



$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \alpha\, h_{\mu\nu}$$

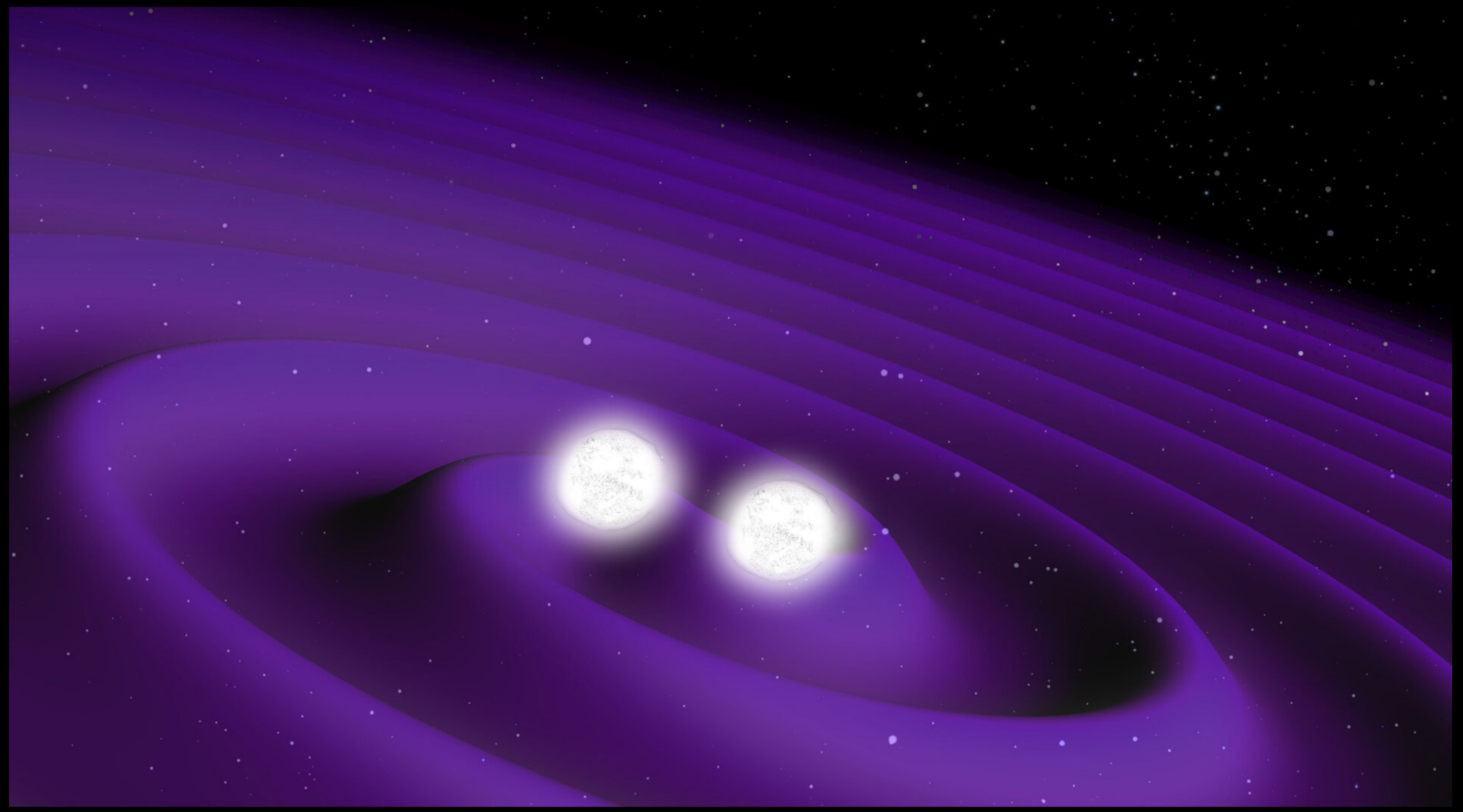


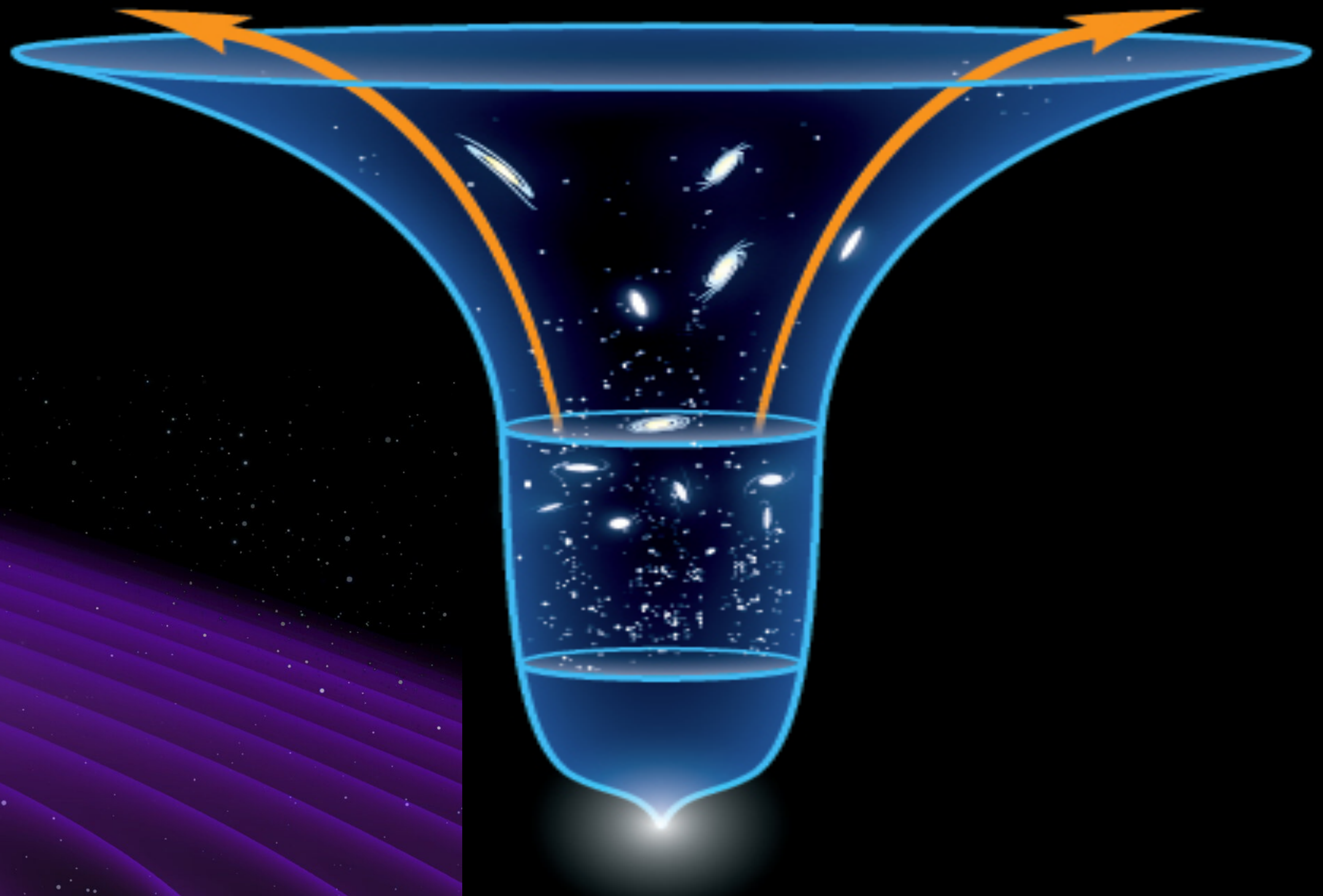


Omega Scan of LIGO's First Neutron Star Merger Detection

My research Interest: GW cosmology

What is the source of the late time cosmic acceleration?

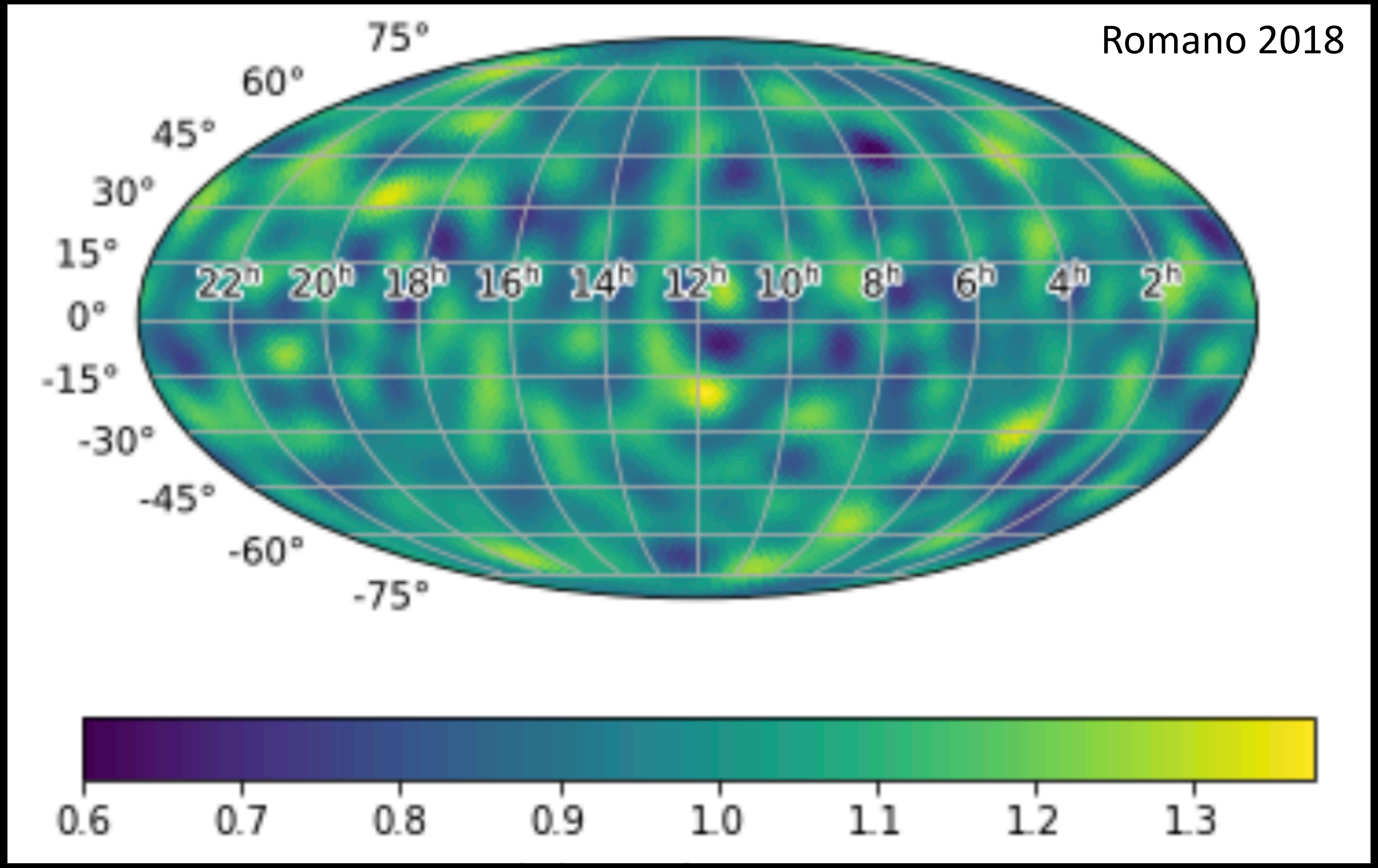




Is DE = Λ?

1. AG et al. *JCAP*, 11 (2020) 040
2. AG et al. *Phys.Rev.D*, 103 (2021) 8, 083506.
3. AG et al. *JCAP*, 06 (2021) 050
4. AG et al. arXiv: 2110.14689
5. AG et al. arXiv: 2210.06398

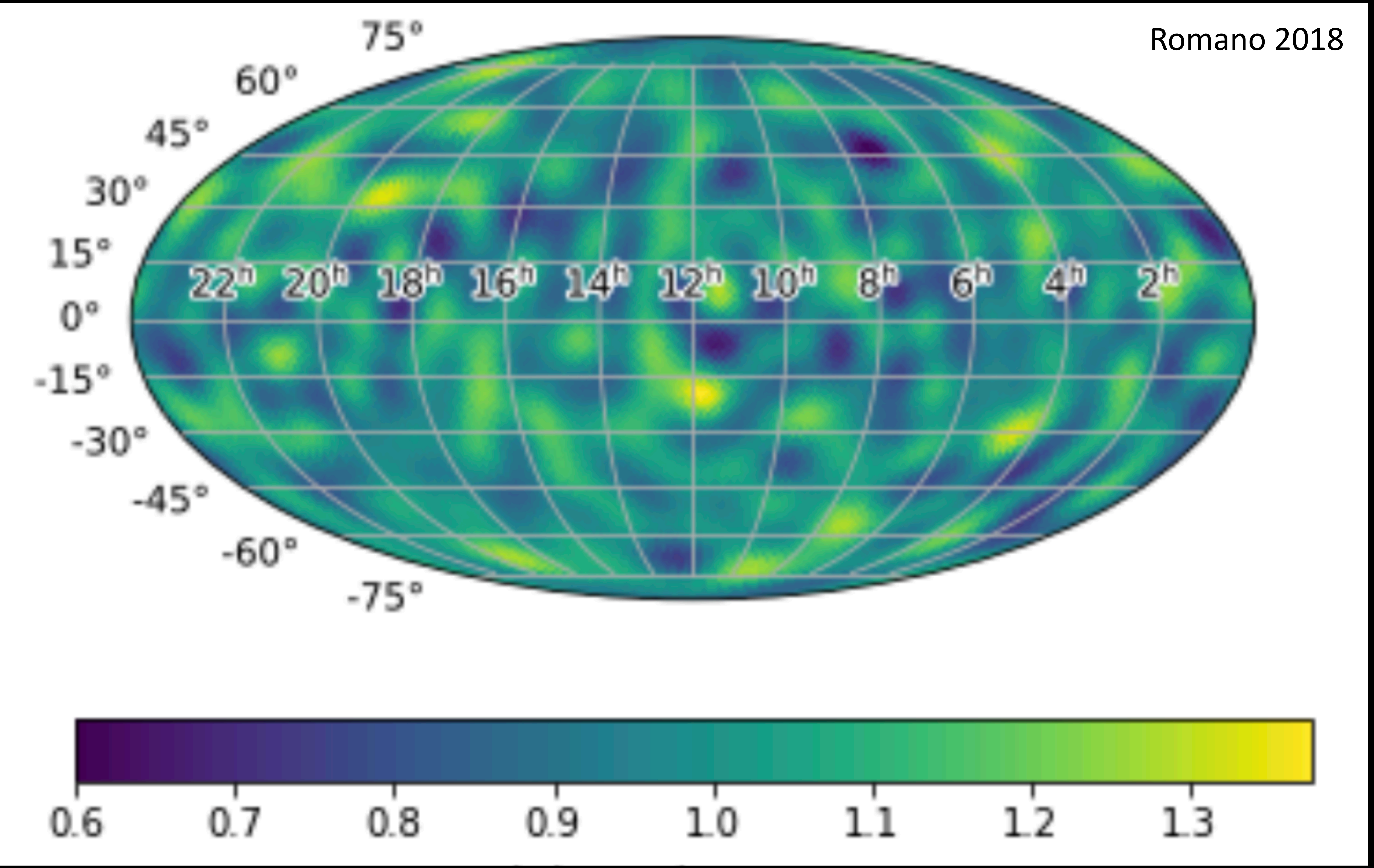
Which cosmological information is encoded in the SGWB?



1. AG arXiv: 2210.05718

Today!

Talk's plan:

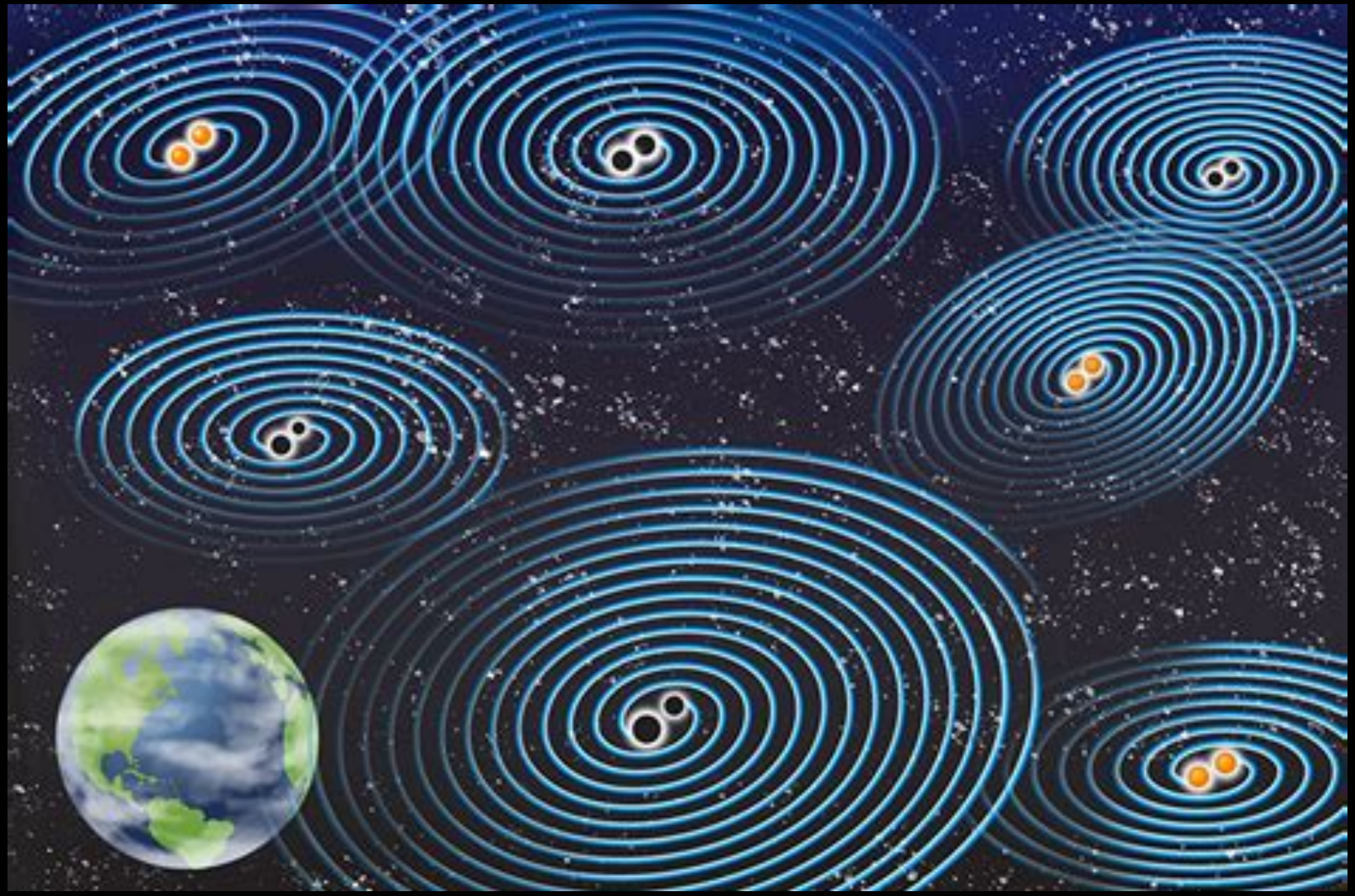


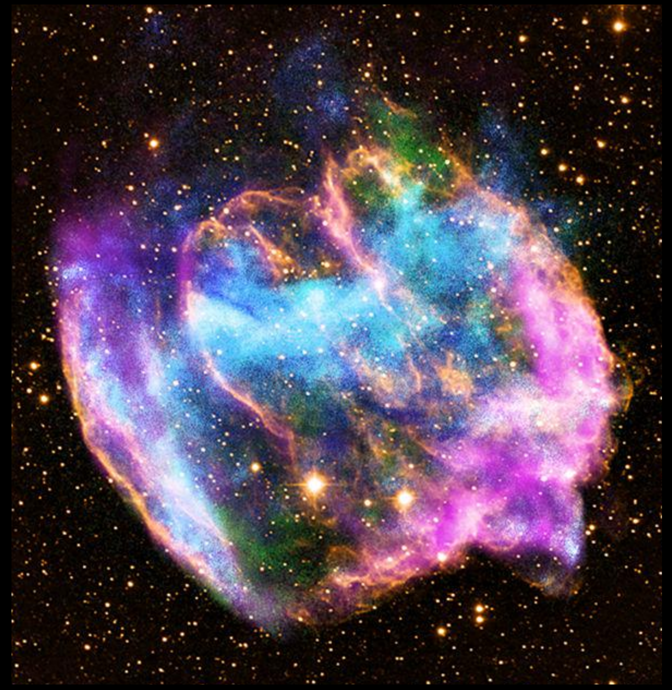
- Introduce SGWB
- State of the art and limitations
- New formalism in AG, 2210.05718

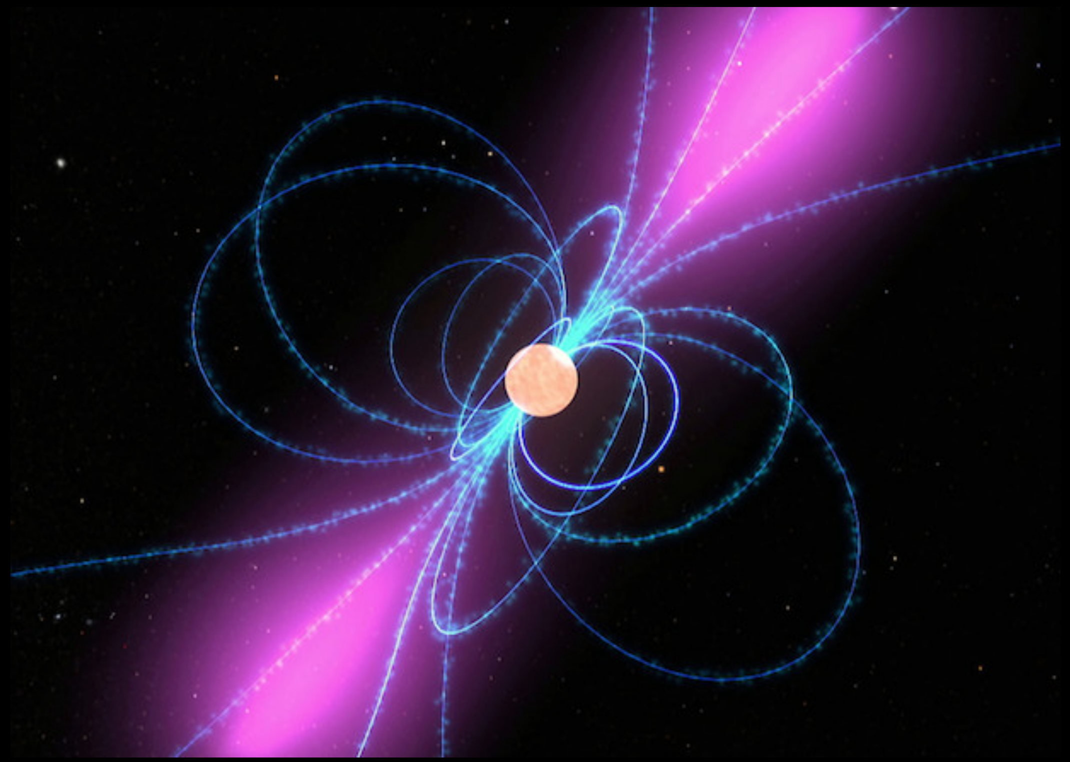
Take home message:
This is a new and very flexible investigation tool

SGWB sources

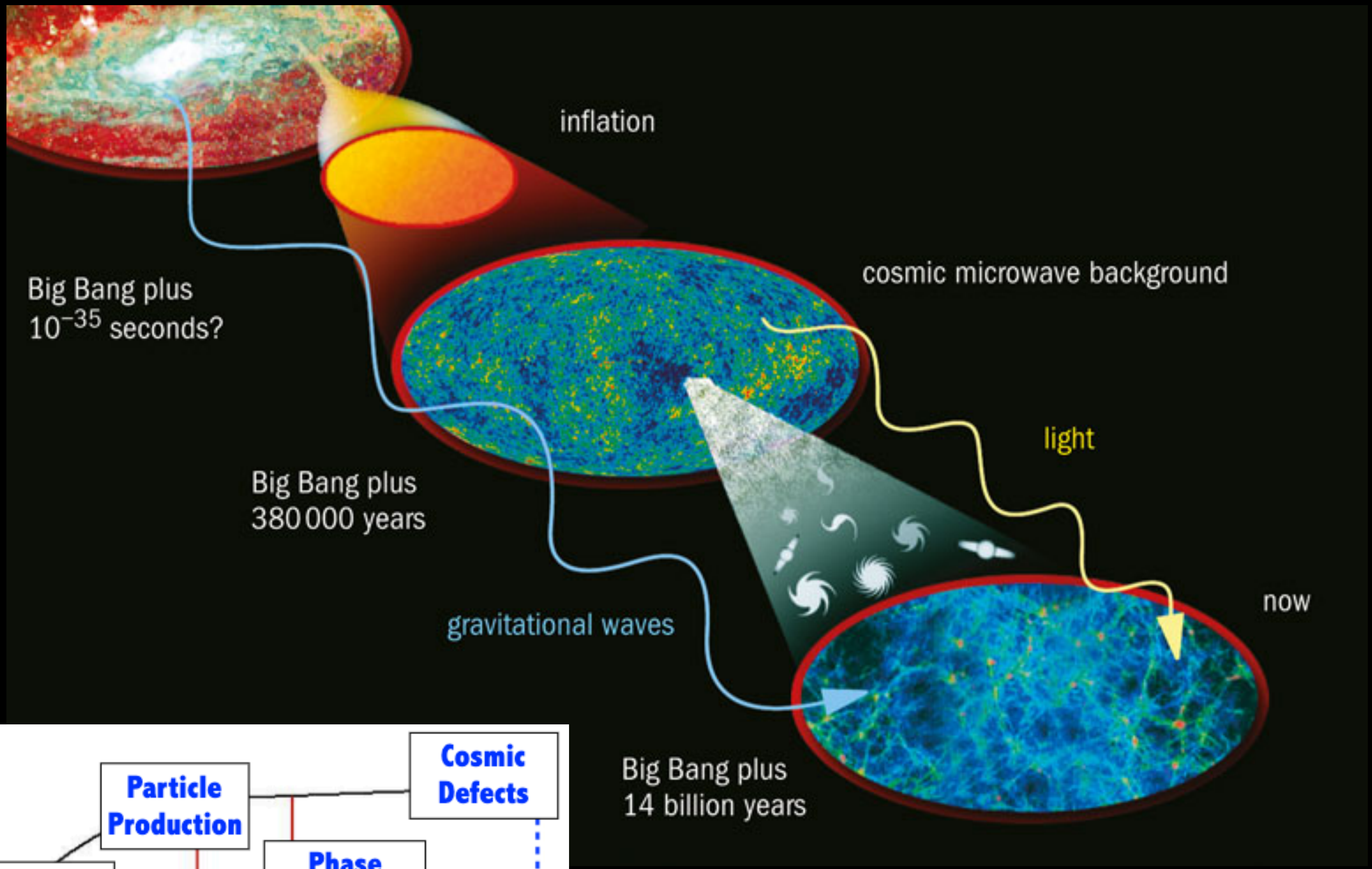
Astrophysical SGWB:
Incoherent superposition of many unresolved GW from astrophysical sources

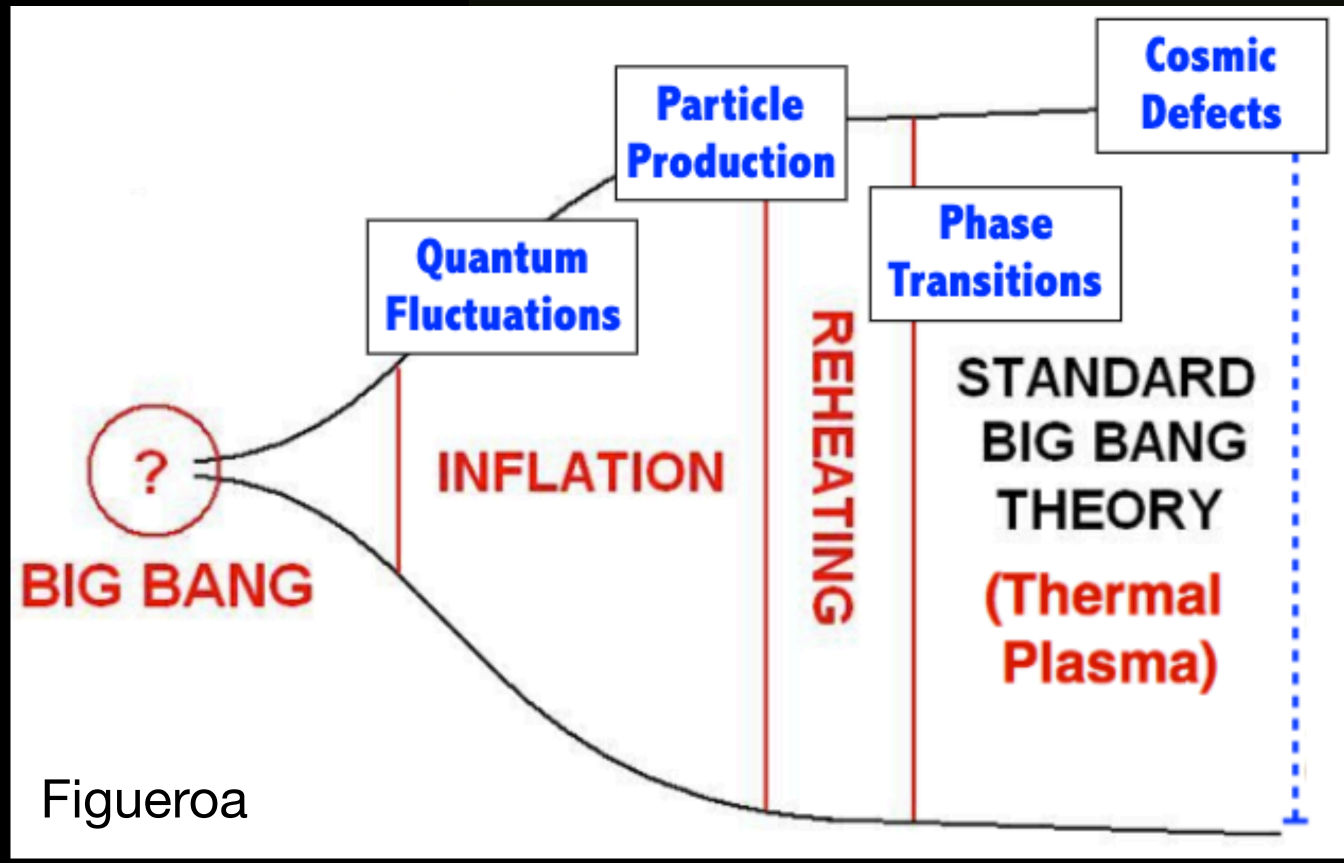






Cosmological SGWB:
Intrinsically stochastic processes



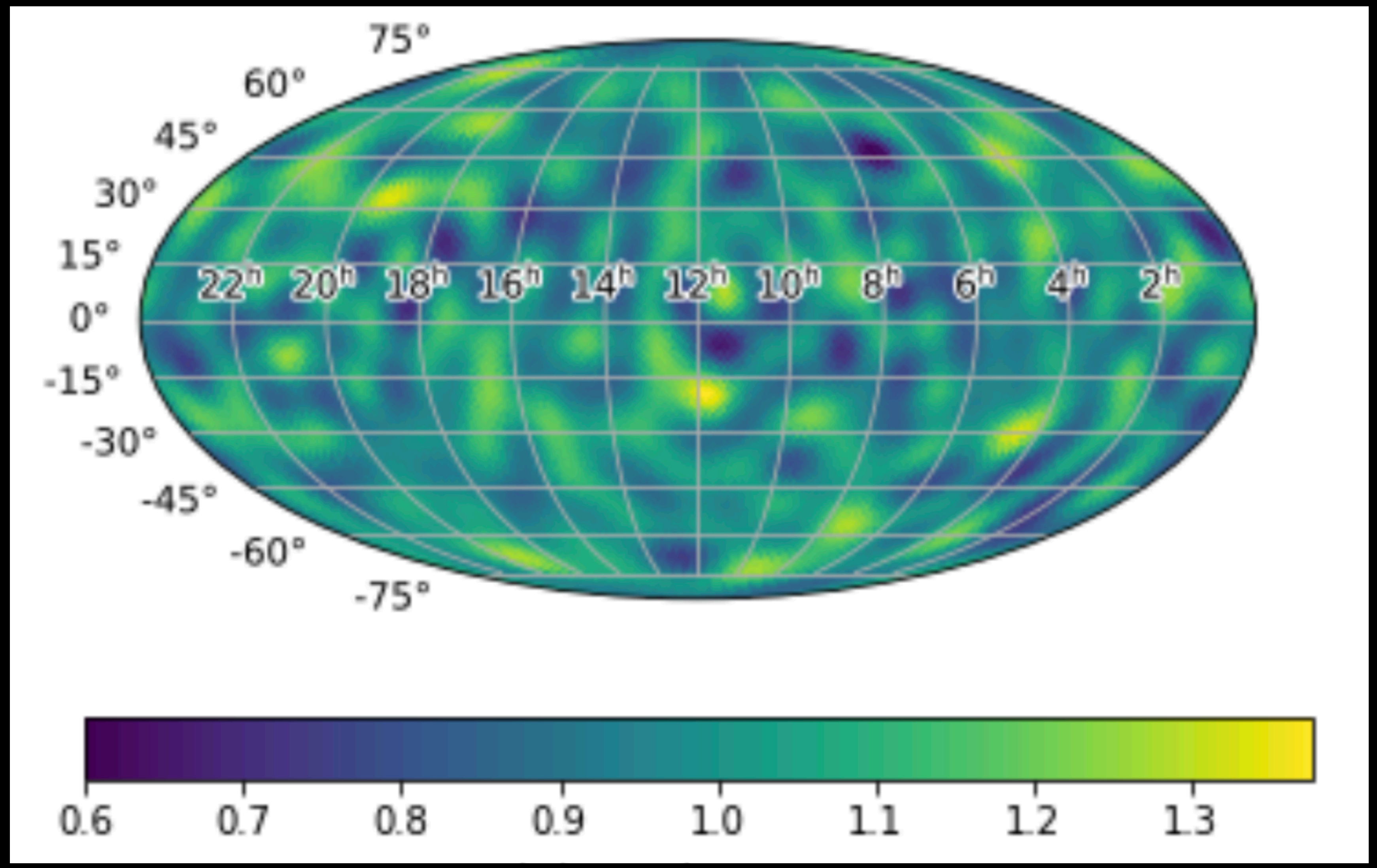


Statistical description of SGWB

Promote GW amplitude to random variable and study n-point function

$$\left\langle H_\lambda(\mathbf{x},\tau_1)\, H_\sigma(\mathbf{y},\tau_2)\right\rangle = \int d^3k\, d^3p\, e^{i(\mathbf{y}\cdot\mathbf{p}-\mathbf{x}\cdot\mathbf{k})} \left\langle H^*_\lambda(\mathbf{k},\tau_1)\, H_\sigma(\mathbf{p},\tau_2)\right\rangle$$

$\lambda, \sigma = L, R$ helicity eigenstates



Romano 2018

Wave equation on FRW

$$(H^{(0)}_{\lambda,\mathbf{k}})'' + 2\mathcal{H}(H^{(0)}_{\lambda,\mathbf{k}})' + k^2 H^{(0)}_{\lambda,\mathbf{k}} = 0$$

Damped waves

Transfer function

$$H^{(0)}_\lambda(\tau,\mathbf{k}) \equiv H_\lambda(\tau_s,\mathbf{k})\,\mathcal{T}^H(\tau,k)$$

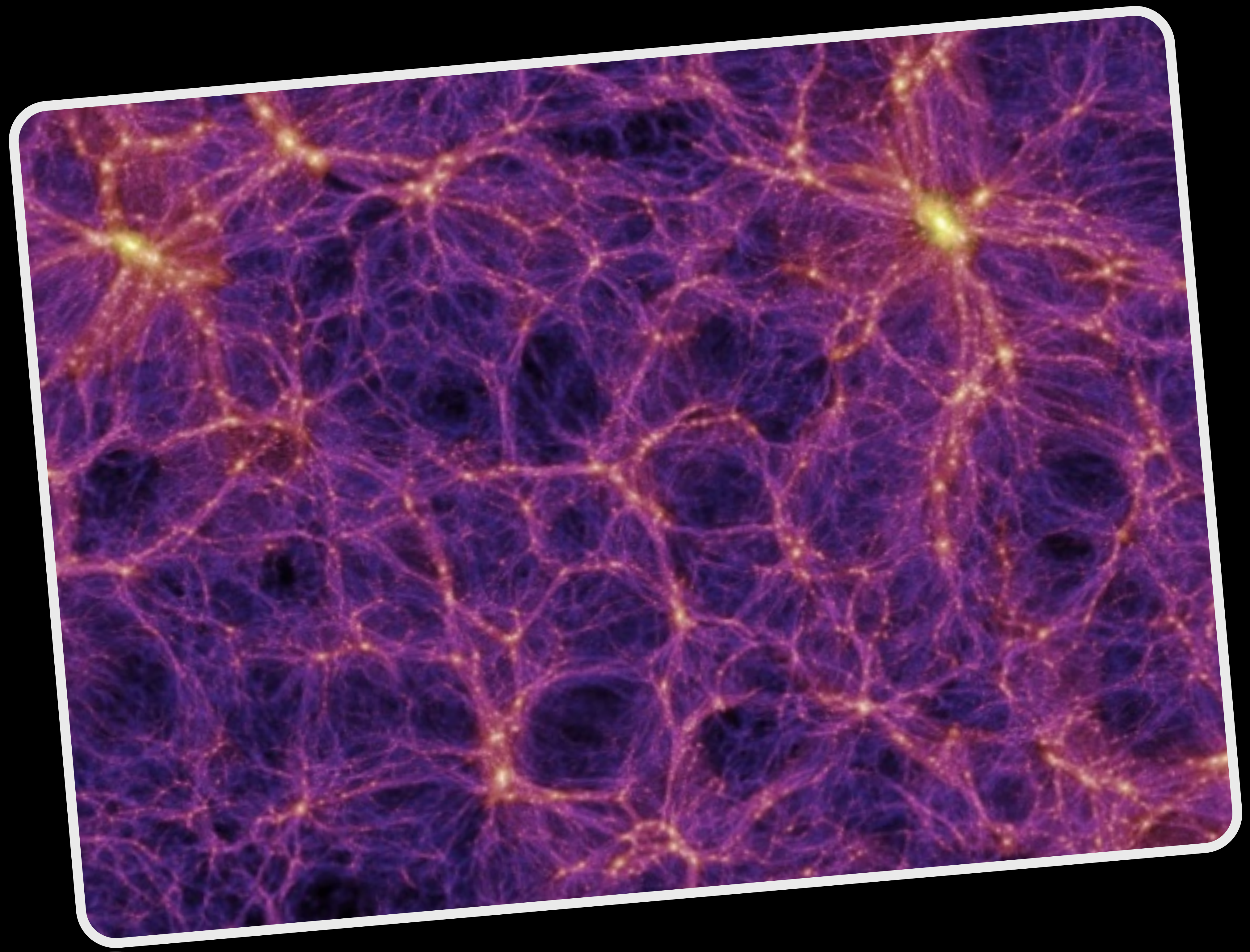
Sources statistics

$$\left\langle H^*_\lambda(\mathbf{k},\tau_s^1) H_\lambda(\mathbf{p},\tau_s^2)\right\rangle$$

on FRW: SGWB statistics is source statistics

Propagation effects due to structures

Universe has Dark Matter structures

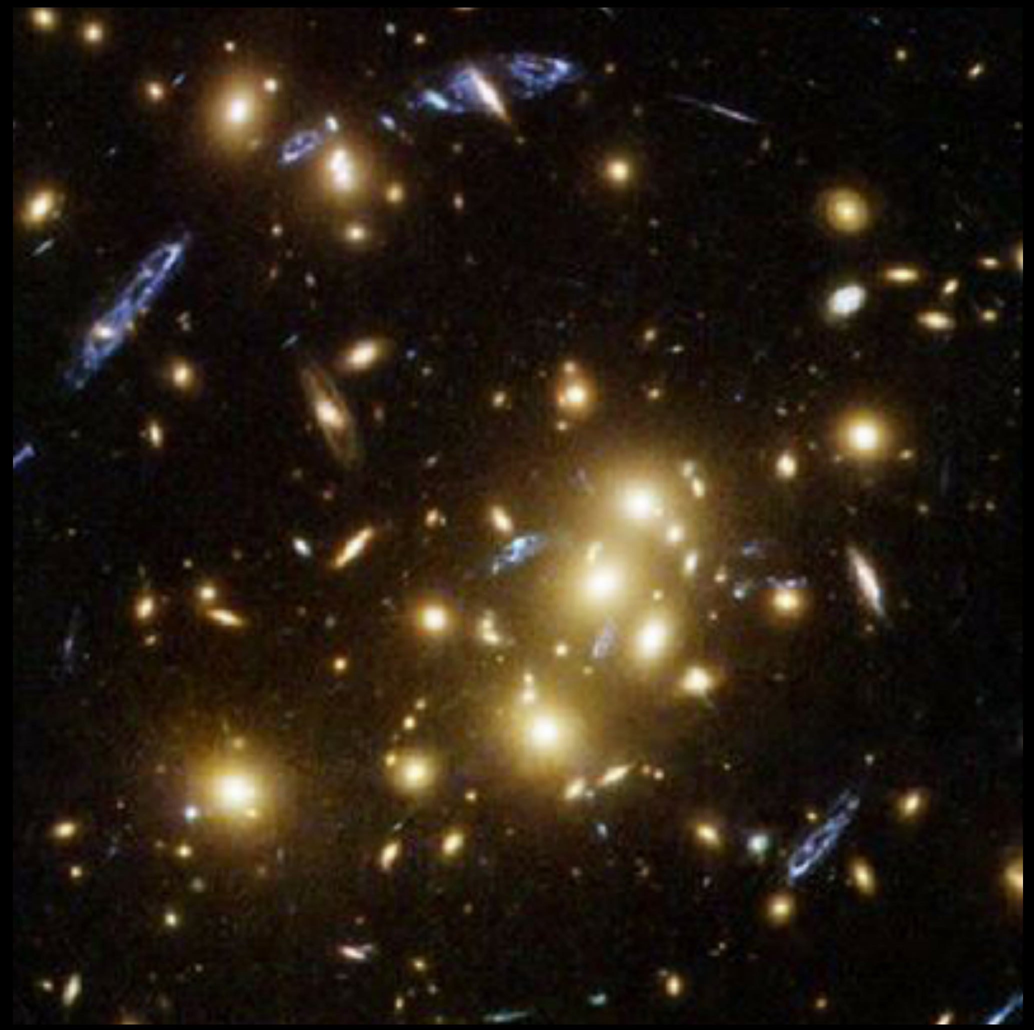


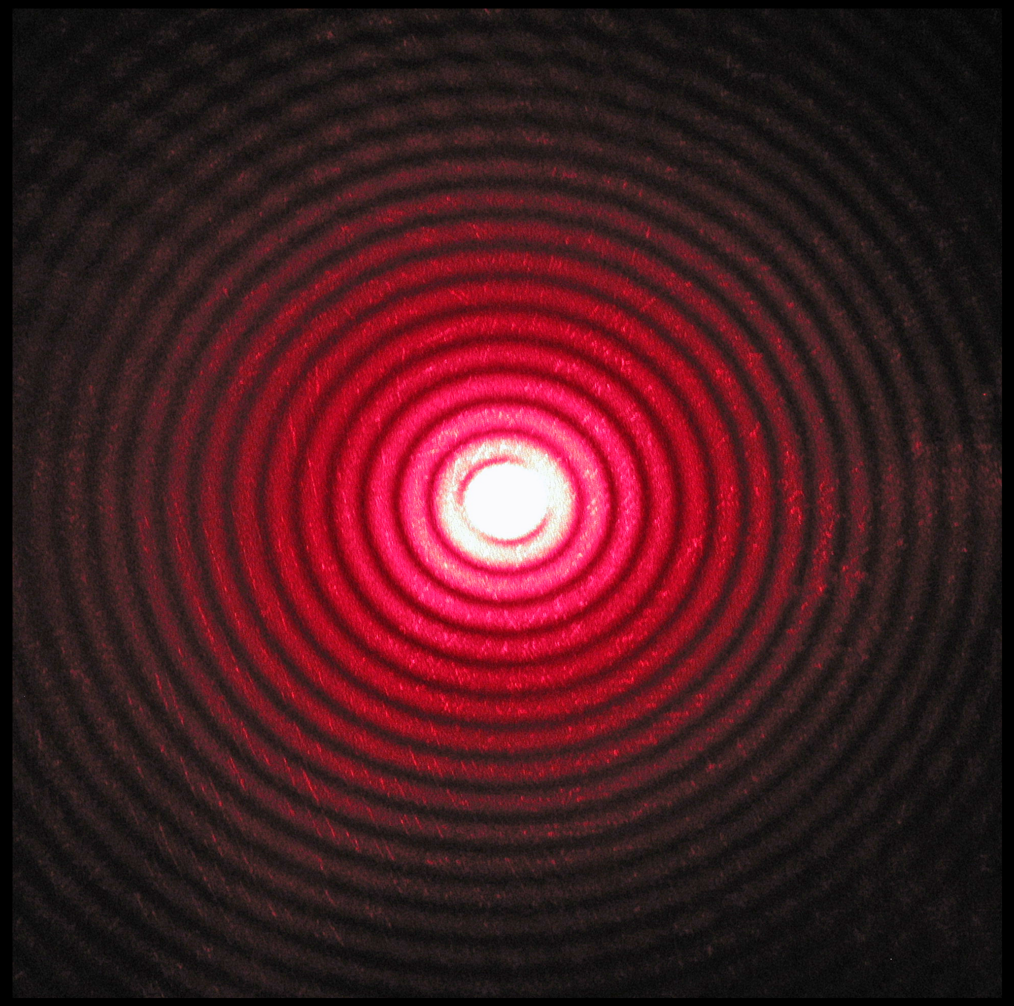
Spacetime:

$$\bar{g}_{\mu\nu}dx^{\mu}dx^{\nu} = a^2(\tau)\left[-(1+2\epsilon\phi(x))d\eta^2 + (1-2\epsilon\phi(x))d\mathbf{x}^2\right]$$

GW travels through potential wells. Possible effects?

(Can't have pictures of GW, but they work the same as photons)





SGWB: contains information about evolution of the Universe and cosmic web.

Standard treatment: Boltzmann equation (BE)

SGWB described by distribution function in phase-space, $f(x^\mu, p_\mu)$, satisfying a "continuity equation"

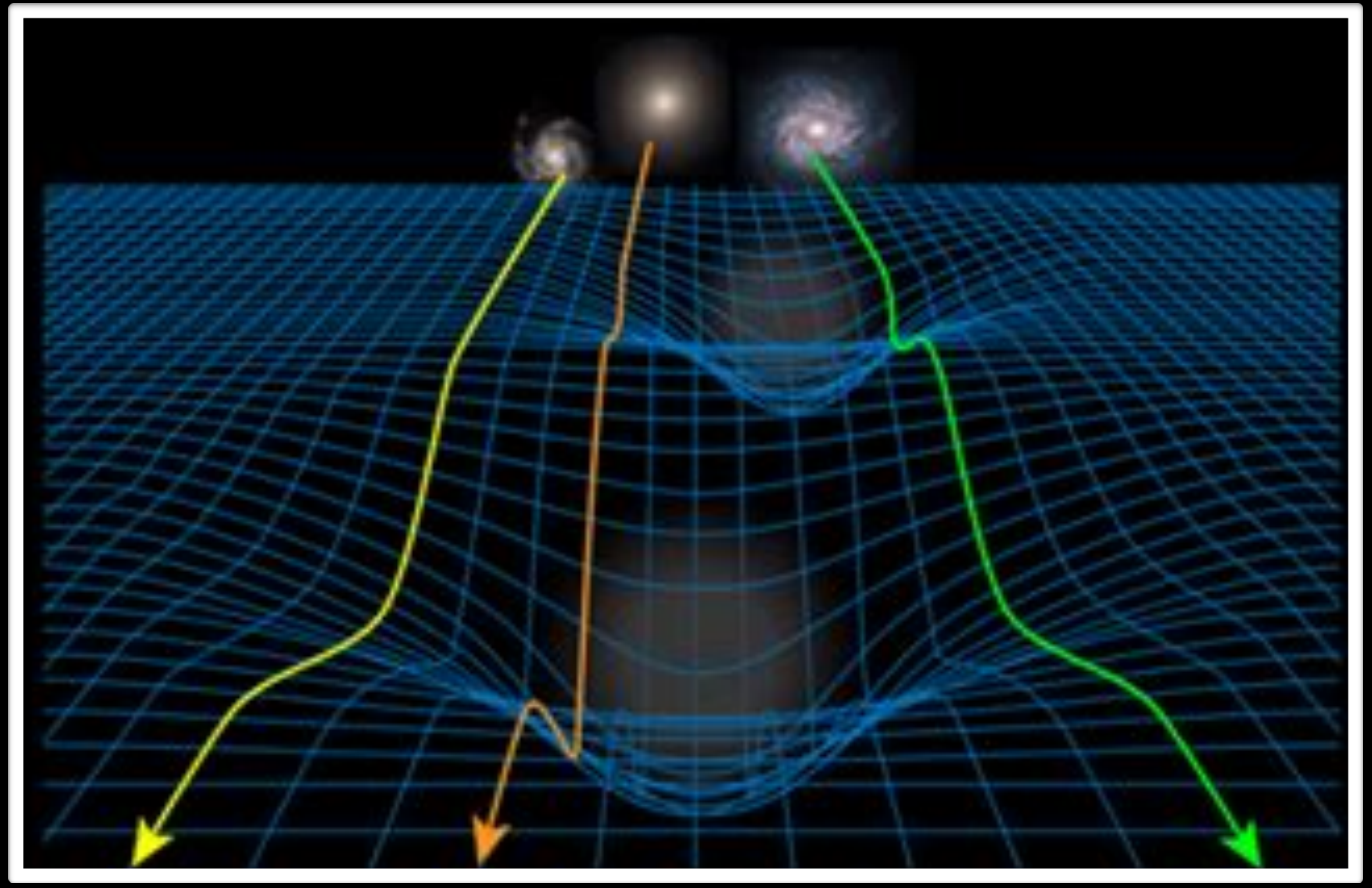
arXiv: 1609.08168, 2201.08782

Number density of "gravitons"

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = \mathcal{J}[f(\lambda)]$$

Derivative of *graviton* distribution function along its *trajectory*

Emissivity: accounts for sources



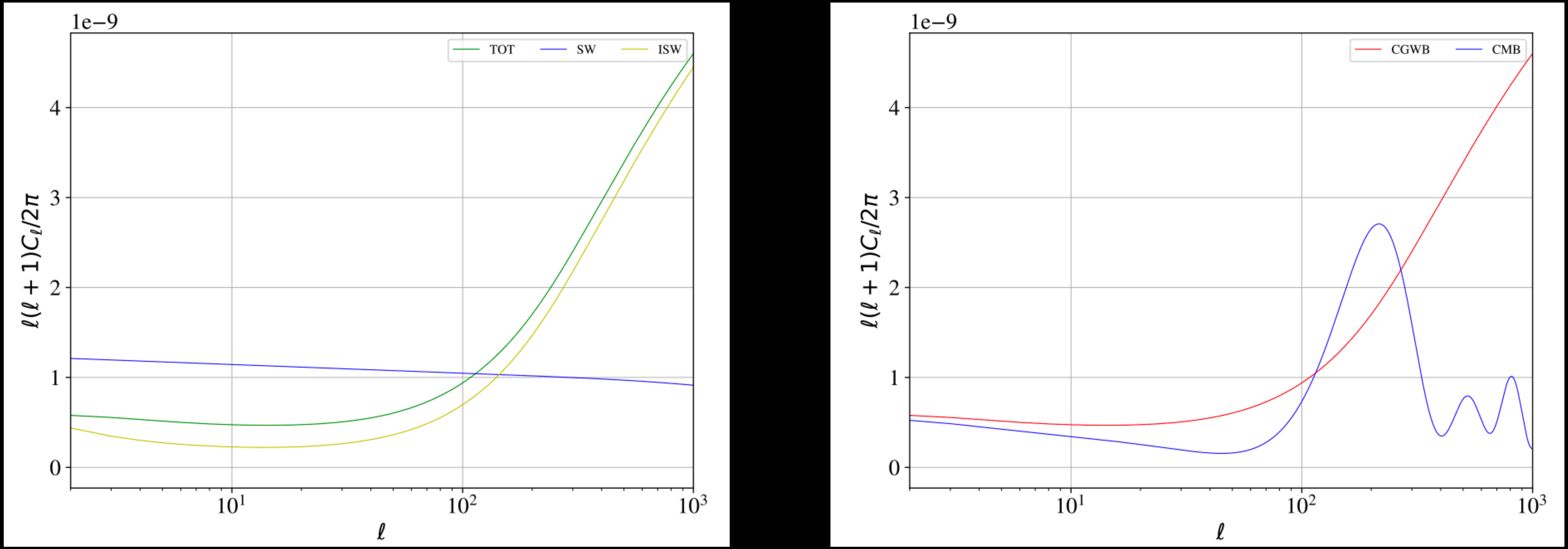
Trajectory: use geodesics equation with perturbed metric

$$\bar{g}_{\mu\nu} dx^\mu dx^\nu = a^2(\tau)\left[-(1+2\epsilon\phi(x))d\eta^2 + (1-2\epsilon\phi(x))d\mathbf{x}^2\right]$$

* There are other approaches, but they rely on GO approximation using similar assumptions

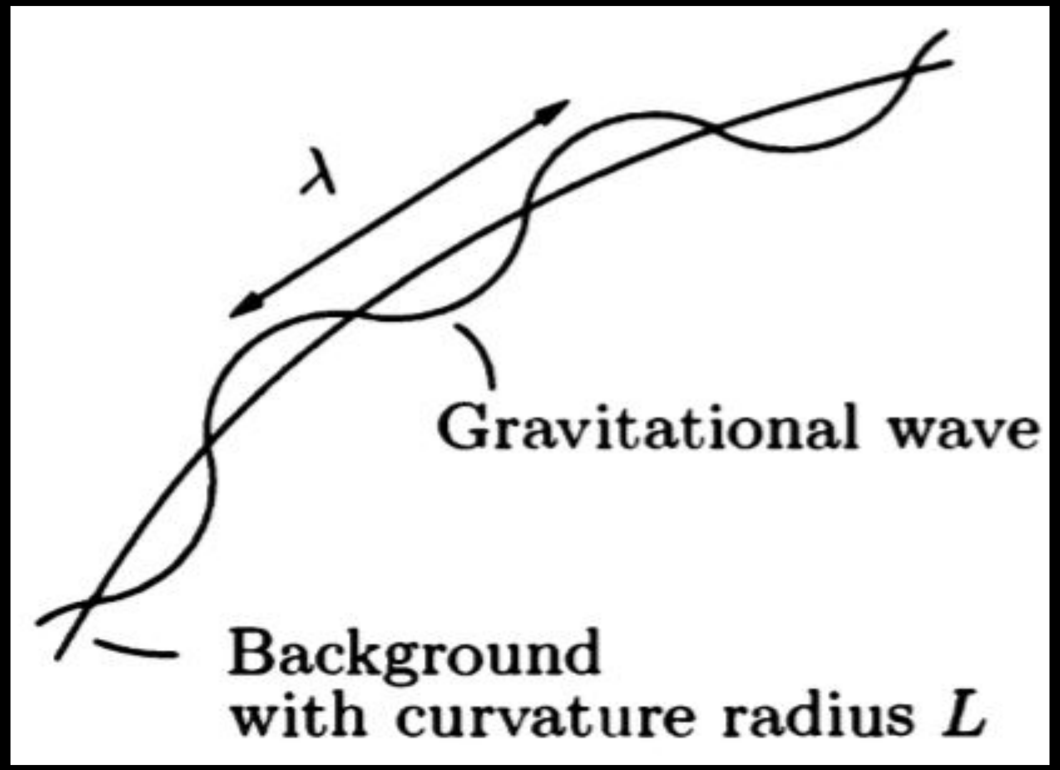
Solution of Boltzmann equation

Graviton loses/gains energy falling in and climbing out of potential wells
(ISW and SW)



arXiv: 2201.08782

Limitation of BE: phase-space and WKB



Isaacson '68

$$h_{\mu\nu} = \mathscr{A}_{\mu\nu}\, e^{i\theta/\omega}$$
$$\omega = \lambda/L$$

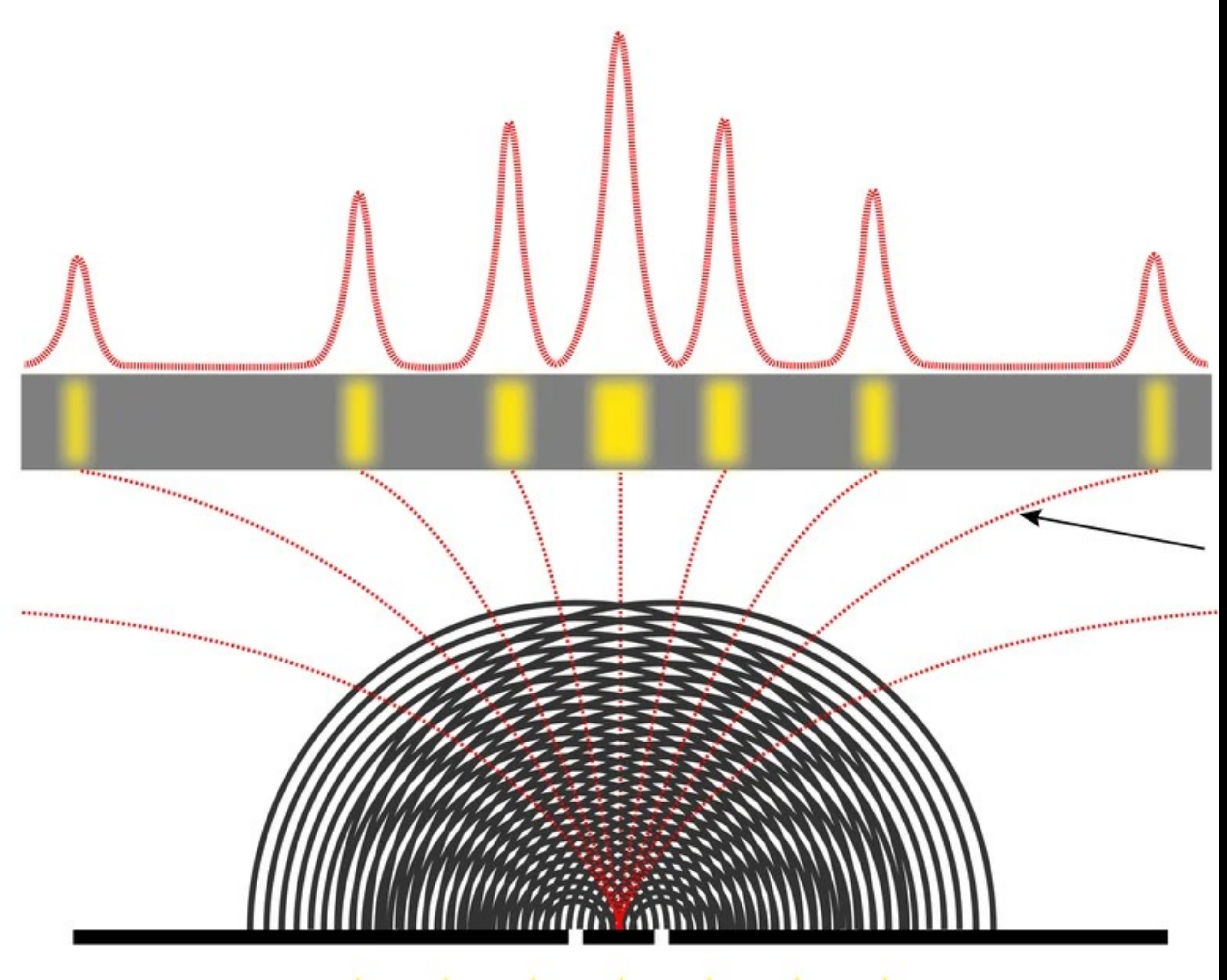
Einstein

Continuity eq. with
$$k_\mu = \partial_\mu \theta$$

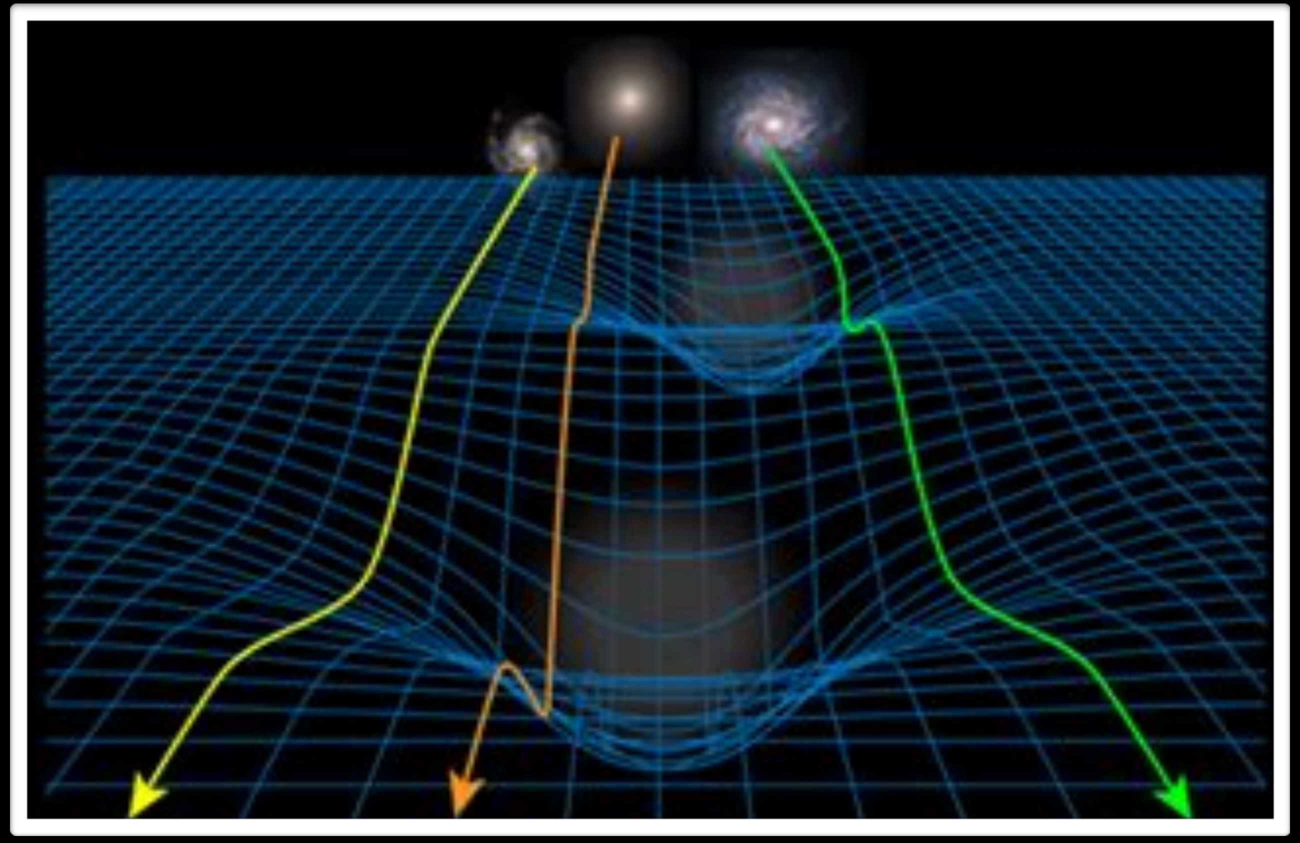
$$\psi = \sqrt{\rho}\, e^{iS/\hbar}$$

Schrodinger

Continuity eq. with
$$k_i = \partial_i S$$



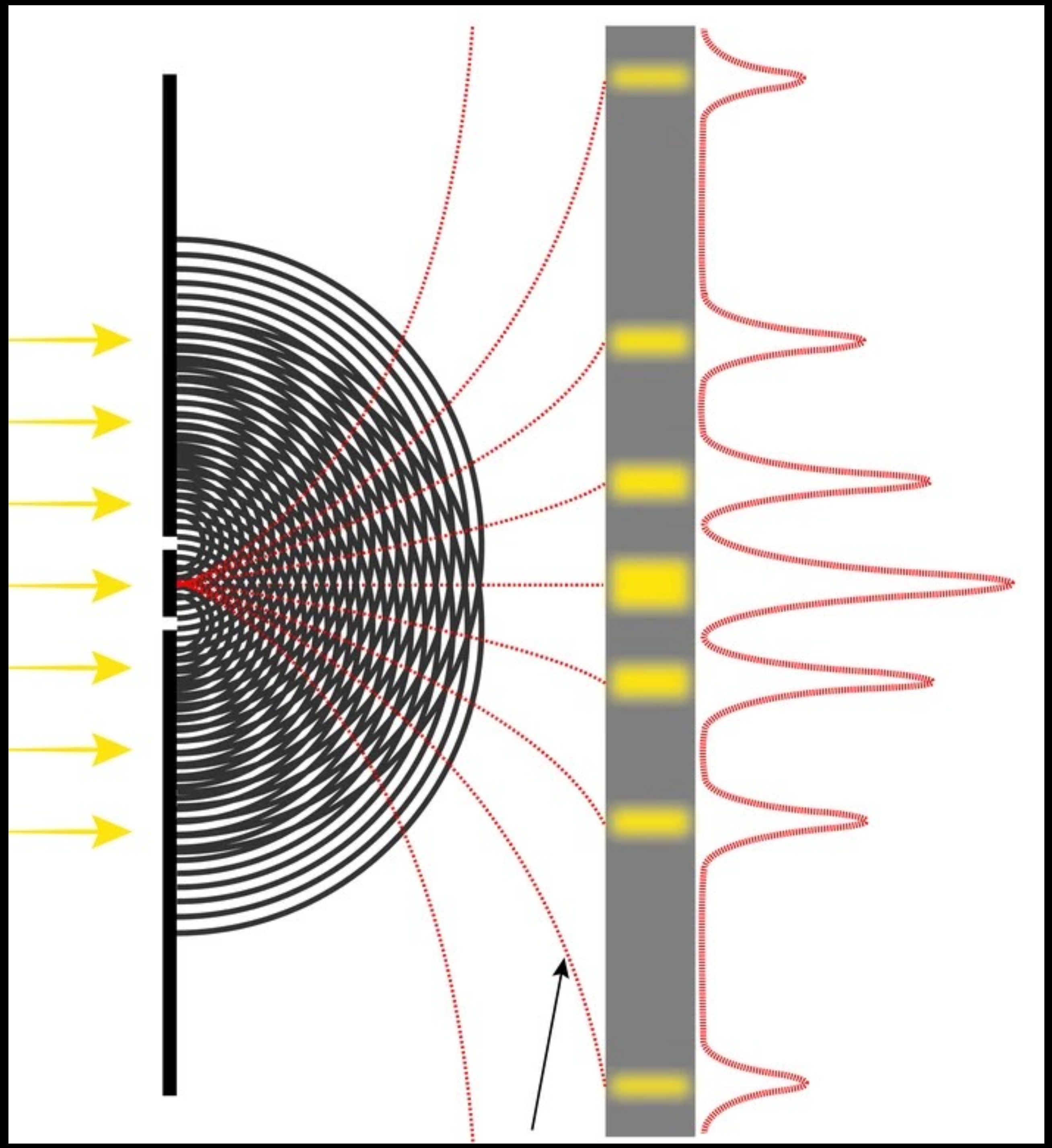
WKB



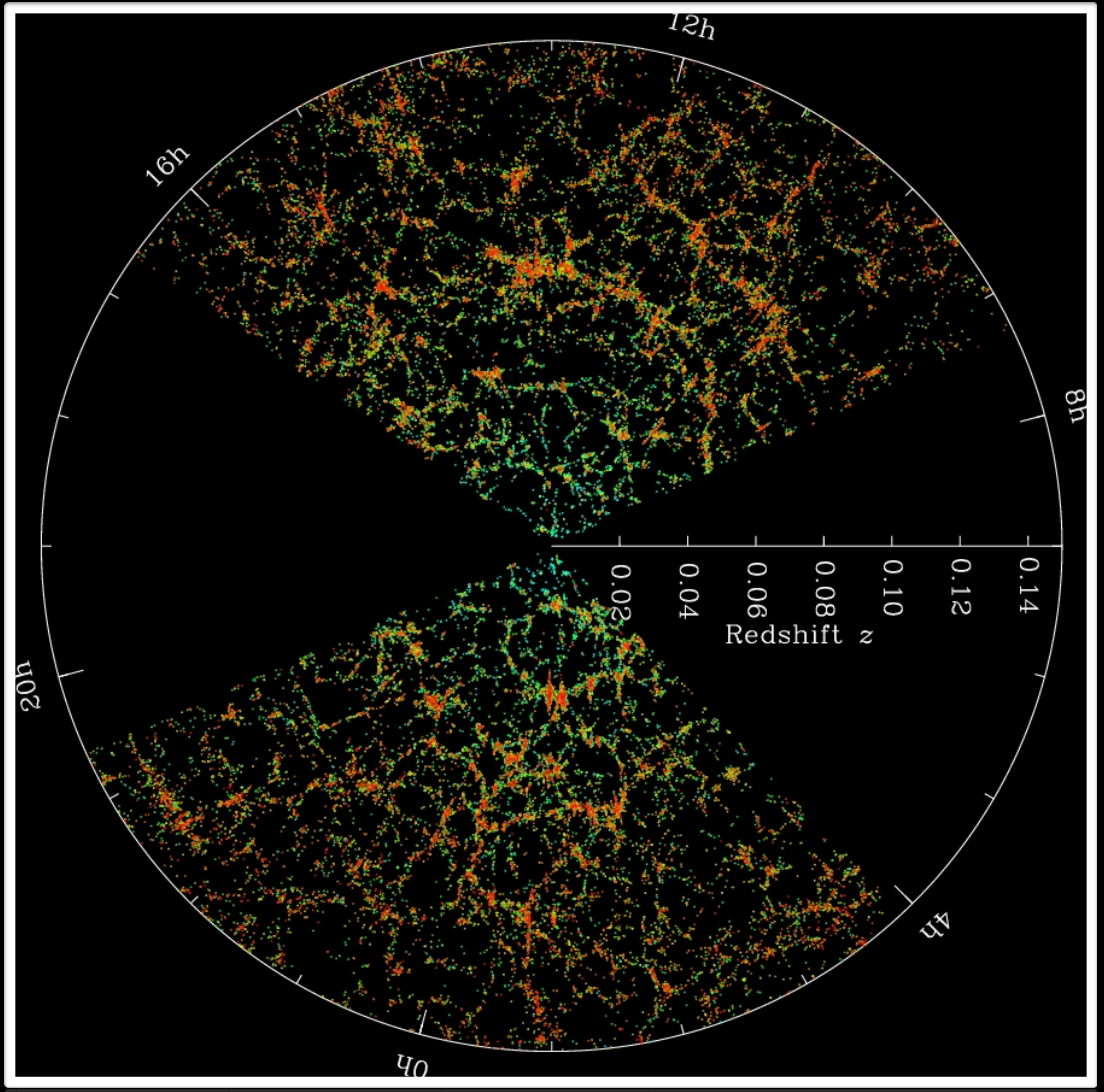
BE is WKB approximation. It cannot account for:

1) Interference and diffraction
2) Polarization effects
3) Washes away scale dependencies

Goal: wave-optics limit of the SGWB



When $\lambda / L \gtrsim 1$ ***diffraction*** and ***interference*** become important

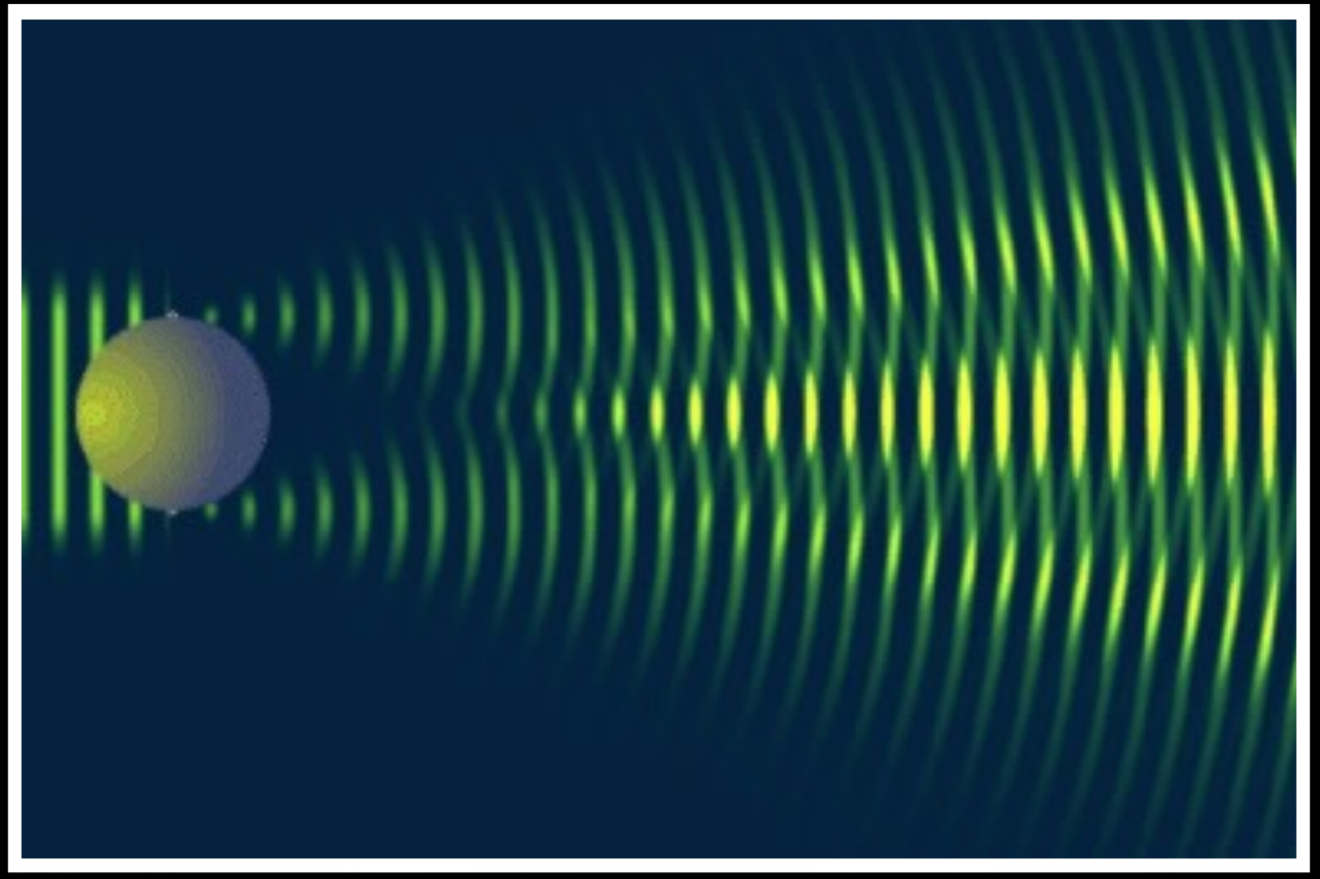


In LISA band:

resolved events with
$M \in (10^5 - 10^{6.5}) M_\odot$

wave-optics effects in
$(0.1 - 1.6)\ \%$

arXiv: 2204.05434



Since SGWB contains all wavelengths, some of them will be in the wave-optics limit

Strategy and results

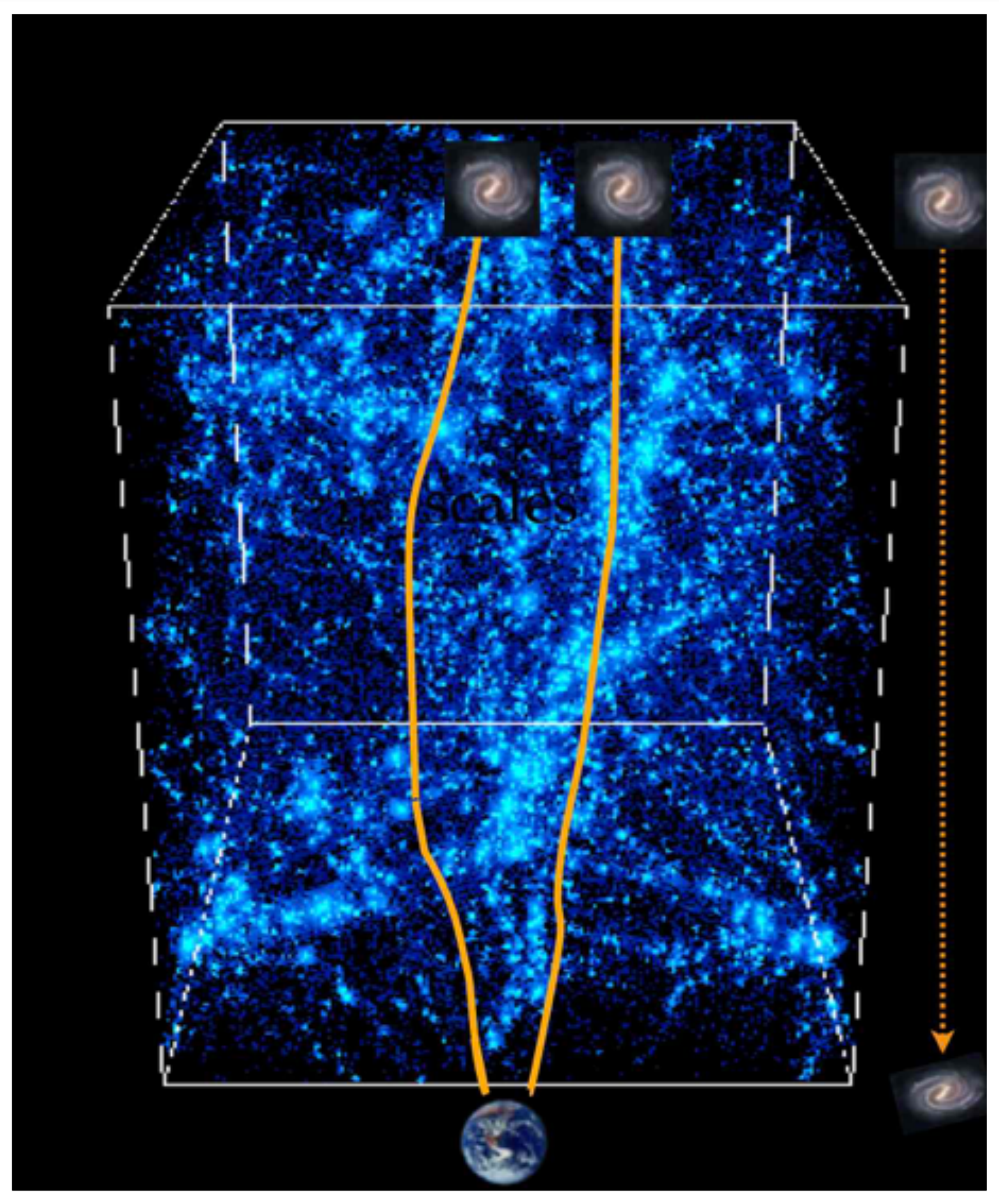
Avoid BE by solving directly Einstein's equations

- Linearize Einstein equations and Classical matter approximation
- Add cosmic structures
- Iterative scheme and solutions
- Computation of two point function

And obtain

- New polarization modes
- Wave-optics (WO) effects

Classical Matter approximation



$$\bar{g}_{\mu\nu} dx^\mu dx^\nu = a^2(\tau) \left[-(1 + 2\epsilon\phi(x)) d\eta^2 + (1 - 2\epsilon\phi(x)) d\mathbf{x}^2 \right]$$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \alpha\, h_{\mu\nu}$$

GW

Linearize in α
Einstein equations:

$$\delta_\alpha \left[G^\mu{}_\nu - \frac{1}{2} T^\mu{}_\nu \right]$$

External Matter:

$$\delta_\alpha T^\mu{}_\nu = 0$$

GWs propagating in a "frozen" perturbed Universe.

Understanding Classical matter approx.

Switch off cosmic structures:
$\epsilon = 0$

Choose Poisson gauge, γ_{ij} is transverse, traceless (TT)

$$(\bar{g}_{\mu\nu} + \alpha h_{\mu\nu})dx^\mu dx^\nu = a^2(\eta)\left\{-(1+2\alpha H_{00})d\eta^2 + 2\alpha H_{0i}\, d\eta dx^i + [(1-2\alpha H)\delta_{ij} + \alpha\gamma_{ij}]dx^i dx^j\right\}$$

Perturbed Einstein equations:

$$\nabla^2 H = 4\pi G a^2(\cancel{\delta\rho +} 3\mathcal{H} H_f)\,, \quad \text{with} \quad -\boldsymbol{\nabla} H_f = [(\bar\rho + \bar P)(\cancel{v +} \boldsymbol{H}_{0i})]_{\parallel}$$

$$\nabla^2 H_{0i} = 16\pi G a^2[(\bar\rho + \bar P)(\cancel{v +} \boldsymbol{H}_{0i})]_{\perp}\,,$$

$$(\partial_\tau^2 + 2\mathcal{H}\partial_\tau - \Delta)\gamma_{ij} = 8\pi G a^2\,(\cancel{\Sigma_{ij}})_T$$

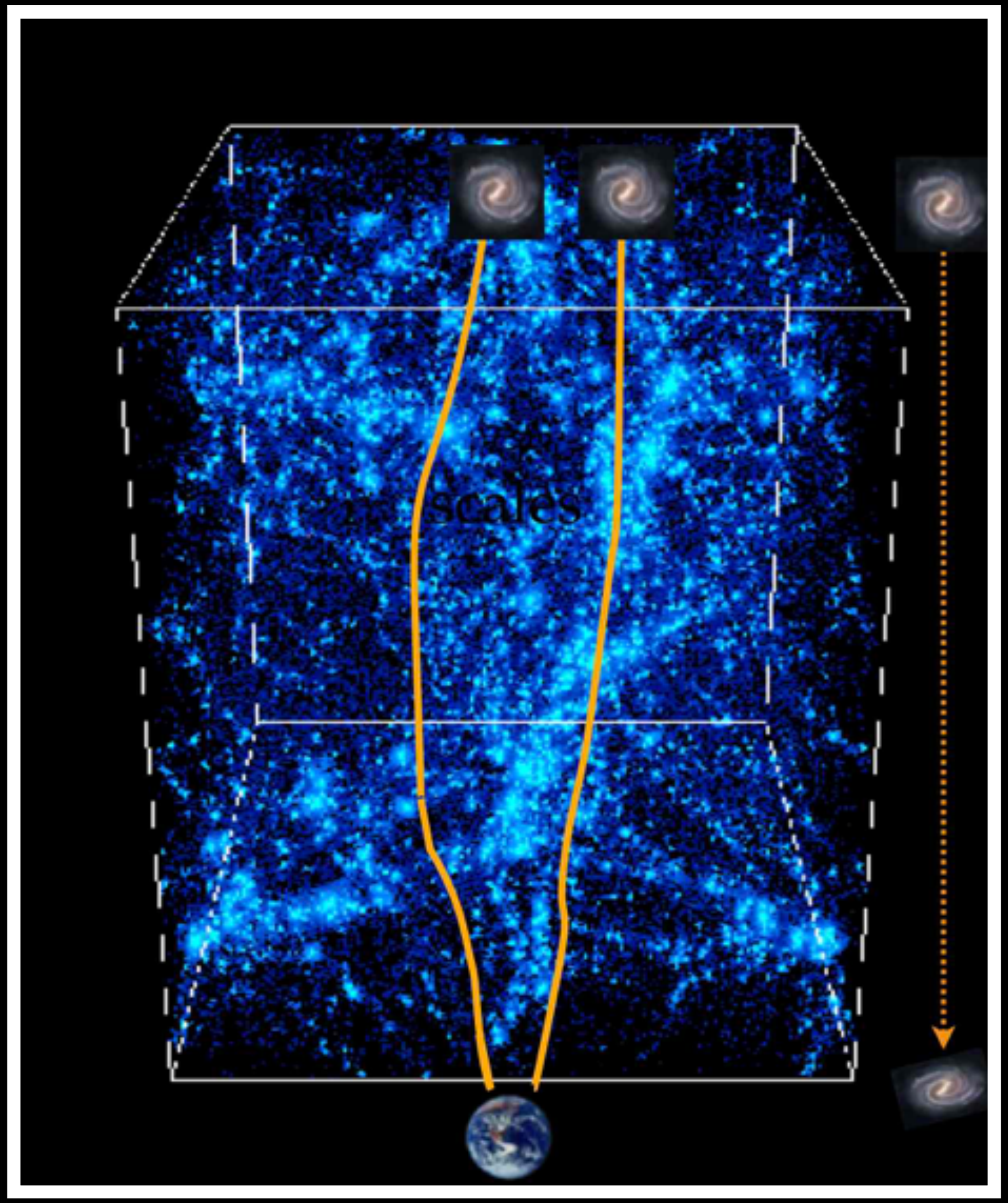
$$\delta\rho\,, v^i\,, (\Sigma_{ij})_T \in \delta_\alpha T_{\mu\nu}$$

Bertschinger 95'

Sourceless Poisson eqs. → Solution: $H = H_{0i} = 0$ → Only γ_{ij} survives

GR gauge theory: for every symmetry there is a "Gauss's law".
$H, H_{0i} \neq 0$ to ensure $\delta_\alpha[\nabla_\mu T^\mu_\nu] = 0$.

GWs through cosmic structures



Double expansion: ϵ and α

$$\mathrm{d}s^2 = a^2(\eta)\Big\{ \left[-\mathrm{d}\eta^2 + \mathrm{d}\mathbf{x}^2\right] - 2\epsilon\phi \left[\mathrm{d}\eta^2 + \mathrm{d}\mathbf{x}^2\right] + \alpha\, H_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \Big\}$$

FRW — Cosmic structures — GW

Plug in $\delta_\alpha G_{\mu\nu} = 0$ and keep: $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha\epsilon)$ orders

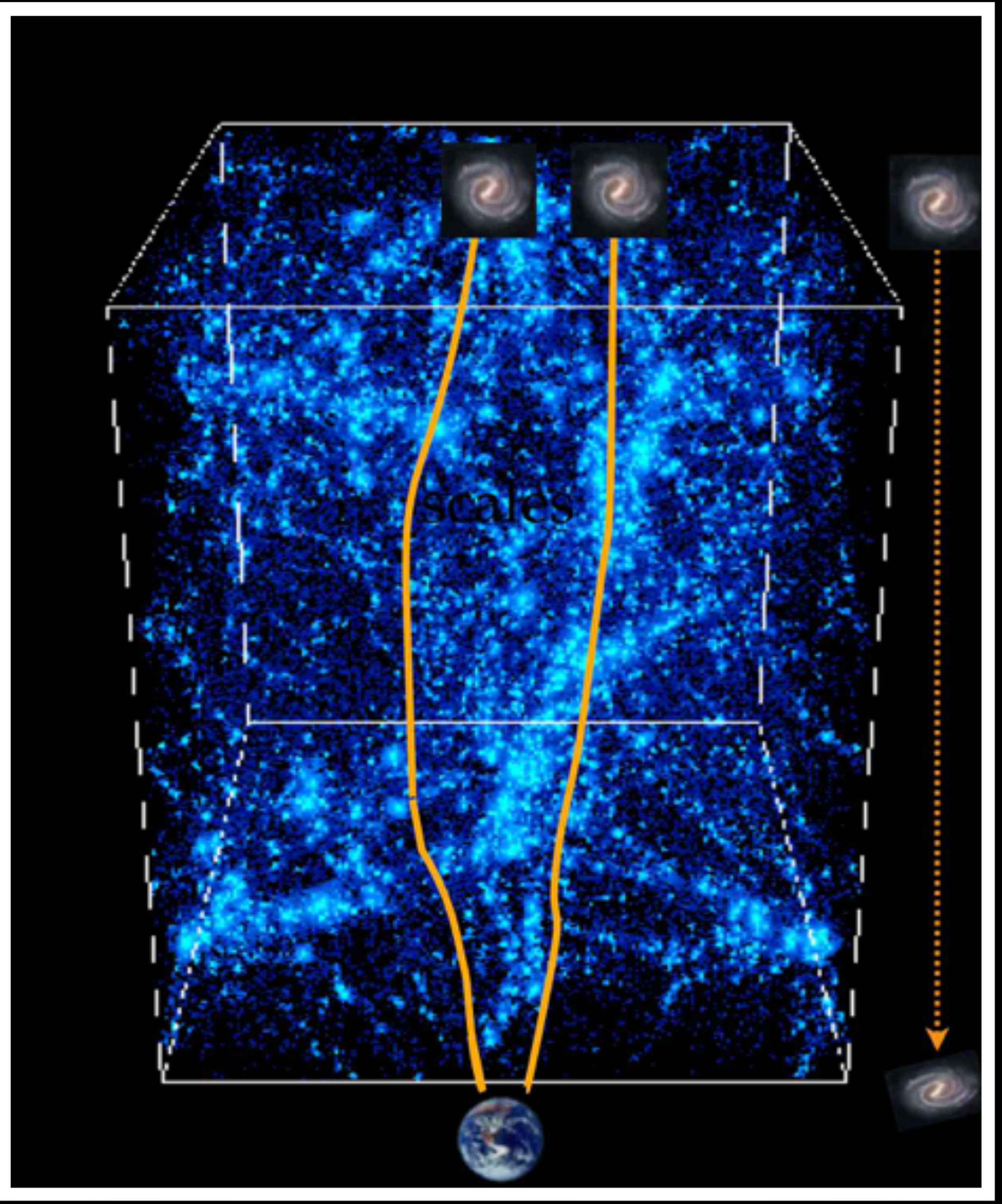
$$[\mathcal{O}_0\, H]_{\mu\nu} + \epsilon\, [\mathcal{O}_1\, H]_{\mu\nu} = 0$$

$\sim \Box H_{\mu\nu}, \dots$ $\qquad$ $\sim \phi\, H_{\mu\nu},\ \partial^k\phi\, \partial_k H_{\mu\nu}, \dots$

+ Do same for gauge choice: $\bar{\nabla}_\mu h^\mu{}_\nu = 0$

Very long equation (~100 terms), need to find scheme to solve them

Iterative solution



Expand GW in ϵ:

$$H_{\mu\nu} = H^{(0)}_{\mu\nu} + \epsilon H^{(1)}_{\mu\nu}$$

$$[\mathcal{O}_0\, H]_{\mu\nu} + \epsilon\,[\mathcal{O}_1\, H]_{\mu\nu} = 0$$

same order in ϵ

ϵ^0 : $\left[\mathcal{O}_0\, H^{(0)}\right]_{\mu\nu} = 0$ → Free GW

ϵ^1 : $\left[\mathcal{O}_0\, H^{(1)}\right]_{\mu\nu} = -\left[\mathcal{O}_1\, H^{(0)}\right]_{\mu\nu}$ → GW with Source

$$\sim \phi\, H^{(0)}_{\mu\nu},\ \partial^k\phi\, \partial_k H^{(0)}_{\mu\nu}, \ldots$$

New <u>source</u> for $H^{(1)}_{\mu\nu}$: interaction between $H^{(0)}_{\mu\nu}$ and ϕ.

Free solution

0^{th} Einstein eqs.

$$\left[\mathcal{O}_0\, H^{(0)}\right]_{\mu\nu} = 0$$

00,0i components

$$\Box H^{(0)} + \ldots = 0$$
$$\Box H^{(0)}_{0i} + \ldots = 0$$

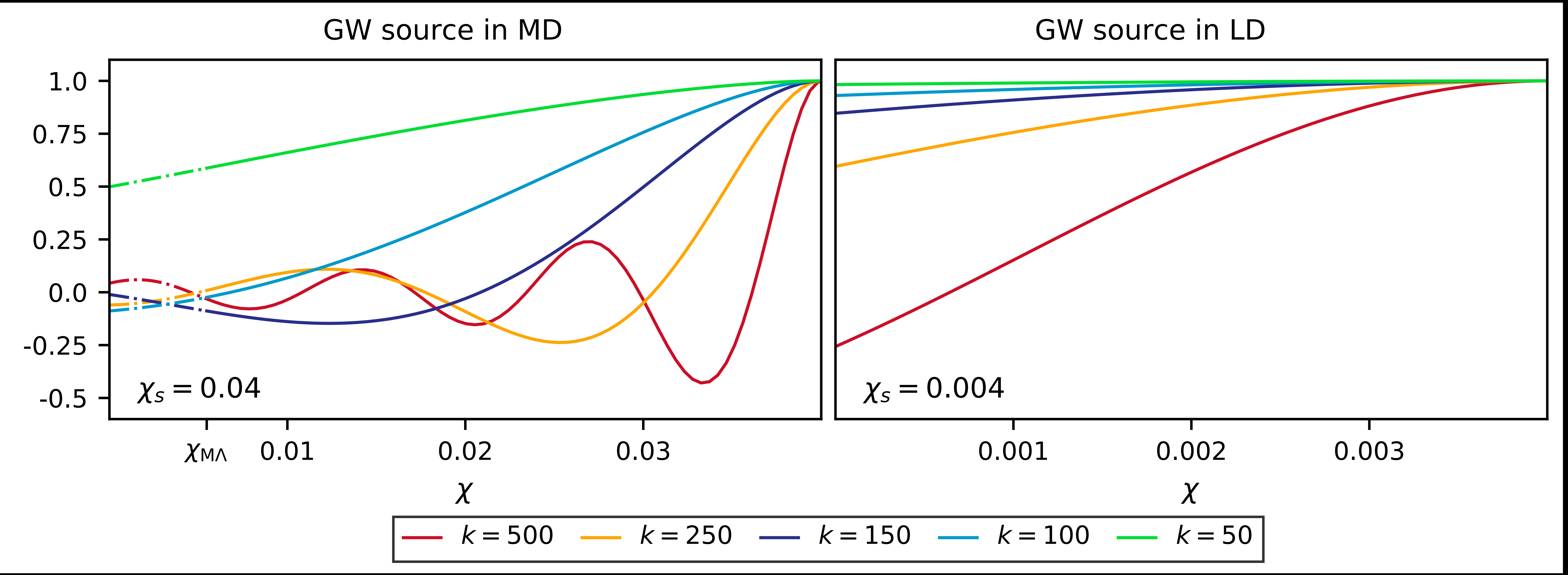
No scalar / vector sources in GR

$$H^{(0)} = H^{(0)}_{0i} = 0$$

In covariant gauge: “constraint equations” aren’t first order PDE.

Only TT components $\neq 0$

$$(H^{(0)}_{\lambda,\mathbf{k}})'' + 2\mathcal{H}(H^{(0)}_{\lambda,\mathbf{k}})' + k^2 H^{(0)}_{\lambda,\mathbf{k}} = 0$$



First order solution: new polarization modes

1^{st} Einstein eqs.

$$\left[\mathcal{O}_0\, H^{(1)}\right]_{\mu\nu} = -\left[\mathcal{O}_1\, H^{(0)}\right]_{\mu\nu}$$

$$H^{(1)}_{\mu\nu} = \left[\begin{array}{c|c} 0 & H^{(1)}_{0i} \\ \hline H^{(1)}_{i0} & \gamma^{(1)}_{ij} + E^{(1)}_{ij} + \frac{1}{3}\delta_{ij} H^{(1)} \end{array}\right]$$

00,0i components

$$\Box H^{(1)} + \ldots = -4\, H^{(0)}_{ij}\, \partial^i \partial^j \phi$$

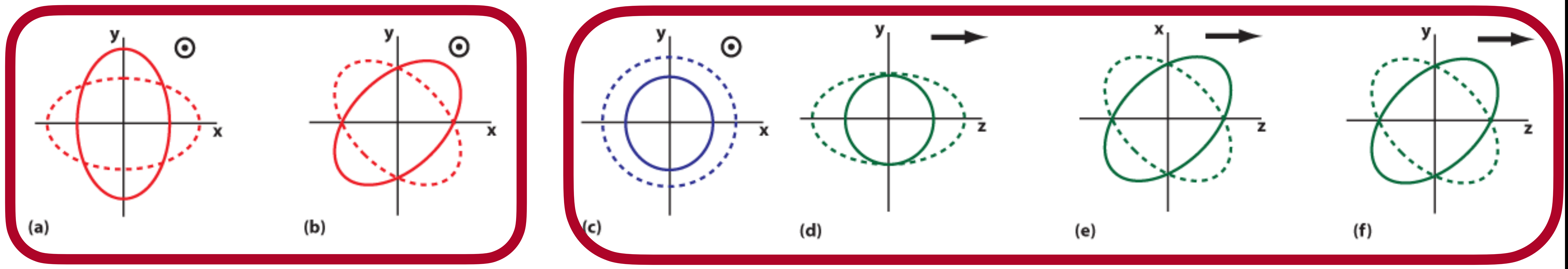
$$\Box H^{(1)}_{0i} + \ldots = -\frac{2}{a} \partial^k \phi\, (a H^{(0)}_{ki})'$$

$$\partial^j E^{(1)}_{ij} = (H^{(1)}_{0i})' + 4\mathcal{H} H^{(1)}_{0i} + \frac{\partial_i H^{(1)}}{6}$$

$$H^{(1)}, H^{(1)}_{0i}, E_{ij} \neq 0$$

- New polarization
- Not new d.o.f

Compute perturbed Riemann tensor



AG et al. 2110.14689: scalar wave in scalar-tensor theories is screened!

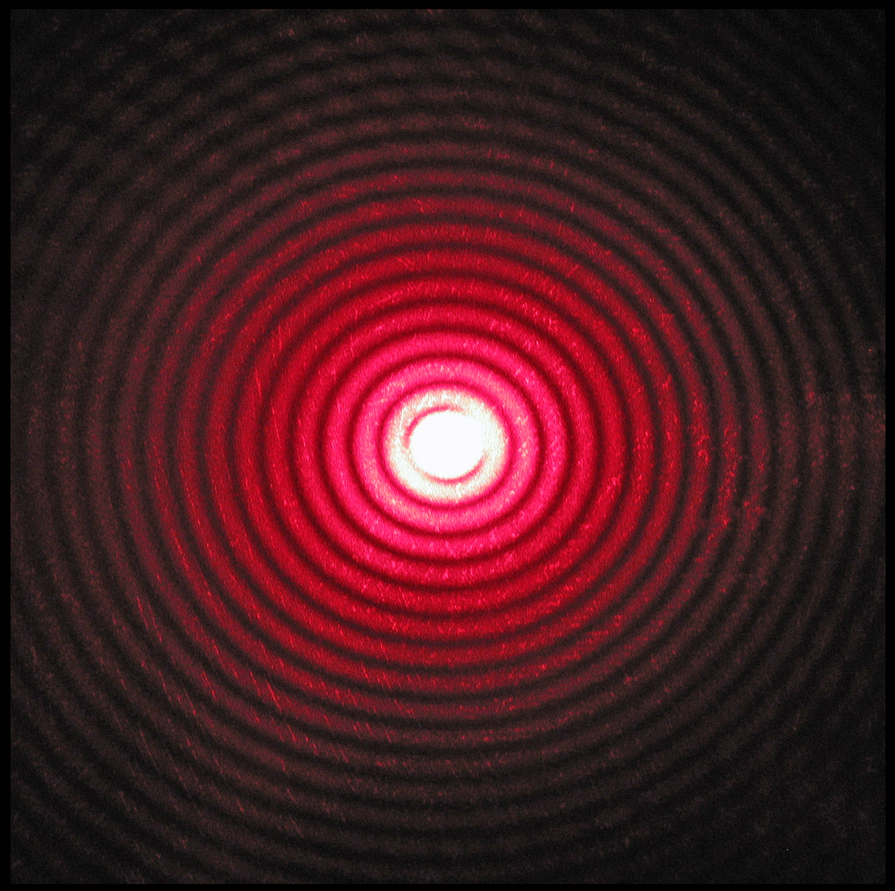
Components of $H^{(0)}_{ij}$ along $\partial_i \phi$ source scalar & vector modes in $H^{(1)}_{ij}$

First order solution: transverse-traceless part

1^{st} Einstein eqs.

$$\left[\mathcal{O}_0\, H^{(1)}\right]_{\mu\nu} = -\left[\mathcal{O}_1\, H^{(0)}\right]_{\mu\nu}$$

$$H^{(1)}_{\mu\nu} = \left[\begin{array}{c|c} 0 & H^{(1)}_{0i} \\ \hline H^{(1)}_{i0} & \gamma^{(1)}_{ij} + E^{(1)}_{ij} + \frac{1}{3}\delta_{ij} H^{(1)} \end{array}\right]$$



Extract TT part of Einstein eqs. Use Green's function to solve.

$$(\gamma^{(1)}_{\lambda,\mathbf{k}})'' + 2\mathcal{H}(\gamma^{(1)}_{\lambda,\mathbf{k}})' + k^2\gamma^{(1)}_{\lambda,\mathbf{k}} = \mathcal{S}_{\lambda,\mathbf{k}}$$

$$\mathcal{S}_{\lambda,\mathbf{k}} = -\frac{1}{2}\sum_{\sigma=L,R}\int d^3p\, H^{(0)}_{\sigma,\mathbf{p}}\left[\phi''_{\mathbf{k}-\mathbf{p}} + 2\mathcal{H}\phi'_{\mathbf{k}-\mathbf{p}} + \phi_{\mathbf{k}-\mathbf{p}}(p^2+k^2)\right][\mathcal{R}(\hat{p},\hat{k})]^{\sigma}{}_{\lambda}$$

Sum over different modes: interference

$$H_\lambda(\tau_s,\mathbf{k})\,\mathcal{T}^H(\tau,k)$$

$$\mathcal{F}(\mathbf{p},\mathbf{k}-\mathbf{p},\tau)\phi^{in}_{\mathbf{k}-\mathbf{p}}$$

Rotation matrix: Diagonal only if $\hat{p} \parallel \hat{k}$

Eq. of TT modes in coordinate basis

$$\tilde{\gamma}^{(1)}_{ij} = \gamma^{(1)}_{ij} + 2H^{(0)}_{ij}\phi$$

$$\Box_\eta\tilde{\gamma}^{(1)}_{ij} - 2\mathcal{H}(\tilde{\gamma}^{(1)}_{ij})' = -4\phi\,\Delta H^{(0)}_{ij}$$

Statistical description of SGWB

Promote amplitude to random variable, compute 2-point function

2x2 matrix

$$\left\langle \gamma^{(1)*}_{\lambda}(\mathbf{k}_1,\tau_1)\,\gamma^{(1)}_{\sigma}(\mathbf{k}_2,\tau_2)\right\rangle = \sum_{\tau_1^s,\tau_2^s}\int_{\tau_1^s}^{\tau_1} d\tau_1' \int_{\tau_2^s}^{\tau_2} d\tau_2'\, g^*_{k_1}(\tau_1,\tau_1')\, g_{k_2}(\tau_2,\tau_2')\left\langle \mathcal{S}^*_{\lambda}(\mathbf{k}_1,\tau_1')\, S_{\sigma}(\mathbf{k}_2,\tau_2')\right\rangle$$

Green's function

$$\left\langle \mathcal{S}^*_{\lambda}(\mathbf{k}_1,\tau_1')\, S_{\sigma}(\mathbf{k}_2,\tau_2')\right\rangle = \sum_{\lambda'\sigma'}\int d^3p_1 d^3p_2\, \mathcal{T}^H_{p_1}(\tau_1')\,\mathcal{T}^H_{p_2}(\tau_2')\,\mathcal{F}^*(\mathbf{p}_1,\mathbf{k}_1-\mathbf{p}_1,\tau_1')\,\mathcal{F}(\mathbf{p}_2,\mathbf{k}_2-\mathbf{p}_2,\tau_2')\times$$
$$\times\left[\mathcal{R}^*(\hat{p}_1,\hat{k}_1)\right]^{\lambda'}{}_{\lambda}\left[\mathcal{R}(\hat{p}_2,\hat{k}_2)\right]^{\sigma'}{}_{\sigma}\left\langle \phi^{\tau_{in}*}_{\mathbf{k}_1-\mathbf{p}_1}\,\phi^{\tau_{in}}_{\mathbf{k}_2-\mathbf{p}_2}\, H^{(0)*}_{\lambda'}(\mathbf{p}_1,\tau_1^s)\, H^{(0)}_{\sigma'}(\mathbf{p}_2,\tau_2^s)\right\rangle$$

Free GW: $\mathcal{T}^H_{p_1}(\tau_1')\,\mathcal{T}^H_{p_2}(\tau_2')$

Growth of cosmic structures: $\mathcal{F}^*(\mathbf{p}_1,\mathbf{k}_1-\mathbf{p}_1,\tau_1')\,\mathcal{F}(\mathbf{p}_2,\mathbf{k}_2-\mathbf{p}_2,\tau_2')$

Statistics of GW sources and cosmic structures: $\left\langle \phi^{\tau_{in}*}_{\mathbf{k}_1-\mathbf{p}_1}\,\phi^{\tau_{in}}_{\mathbf{k}_2-\mathbf{p}_2}\, H^{(0)*}_{\lambda'}(\mathbf{p}_1,\tau_1^s)\, H^{(0)}_{\sigma'}(\mathbf{p}_2,\tau_2^s)\right\rangle$

Assume gaussian random fields and use Wick's theorem:
sum of 3 products of two 2-point functions

Cosmological SGWB

Assume: unpolarized 0^{th} order SGWB, statistical homogeneity and isotropy and uncorrelated GW and ϕ:

$$\left\langle H^{(0)*}_{\lambda,\mathbf{p}_1}(\tau_1^s)\, H^{(0)}_{\sigma,\mathbf{p}_2}(\tau_2^s)\right\rangle = \delta_{\lambda\sigma}\,\delta^3(\mathbf{p}_1-\mathbf{p}_2)\,\frac{\mathrm{I}^{(0)}(p_1,\tau_1^s,\tau_2^s)}{2}$$

$$\left\langle \phi^{\tau_{in}*}_{\mathbf{k}}\,\phi^{\tau_{in}}_{\mathbf{p}}\right\rangle = \delta^3(\mathbf{k}-\mathbf{p})\,\frac{2\pi^2}{k^3}\,P^{\phi}_{in}(k)$$

Scale dependent features?

$$\left\langle \phi^{\tau_{in}}_{\mathbf{k}_2-\mathbf{p}_2}\,H^{(0)}_{\lambda,\mathbf{p}_2}(\tau_2^s)\right\rangle = 0$$

WO effects are frequency dependent: multi-band analysis!

$$\left\langle S^*_\lambda(\mathbf{k},\tau_1')\,S_\sigma(\mathbf{k},\tau_2')\right\rangle = \pi^2\int d^3p\,\left\{\begin{array}{c}\text{Deterministic evolution:}\\ \text{Transfer functions \&}\\ \text{Rotation matrix}\end{array}\right\}_{\lambda\sigma}\times\frac{P^{\phi}_{in}(|\mathbf{k}_1-\mathbf{p}|)}{|\mathbf{k}_1-\mathbf{p}|^3}\times\frac{\mathrm{I}^{(0)}(p,\tau_1^s,\tau_2^s)}{2}$$

Stokes parameters

Decompose 2-point function on Pauli matrices basis

$$\langle (\gamma^1_\lambda)^* \gamma^1_\sigma \rangle = \frac{1}{2}\begin{bmatrix} I+V & Q-iU \\ Q+iU & I-V \end{bmatrix}$$

$$I = |\gamma_R|^2 + |\gamma_L|^2$$

- Intensity of SGWB

$$V = |\gamma_R|^2 - |\gamma_L|^2$$

- Circular polarization due to different amounts of R/L modes

$$Q = 2\mathrm{Re}(\gamma_R^* \gamma_L)$$
$$U = 2\mathrm{Im}(\gamma_R^* \gamma_L)$$

- Linear polarization due to phase difference between R/L modes

Polarization in GO is transported trivially: must include WO effects!

Cosmological SGWB: Stokes parameters

$$I^{(1)}(\mathbf{k}) = \int d^3p \, \frac{P_{in}^{\phi}(|\mathbf{k}-\mathbf{p}|)}{|\mathbf{k}-\mathbf{p}|^3} \times \mathrm{I}^{(0)}(p) \times \left(4Y_0^0(\theta,\varphi) + \frac{8\sqrt{5}}{7}Y_0^2(\theta,\varphi) + \frac{2}{21}Y_0^4(\theta,\varphi)\right) \times \left\{\begin{array}{c}\text{Deterministic evolution:}\\ \text{Transfer functions, Green's}\\ \text{function}\end{array}\right\}$$

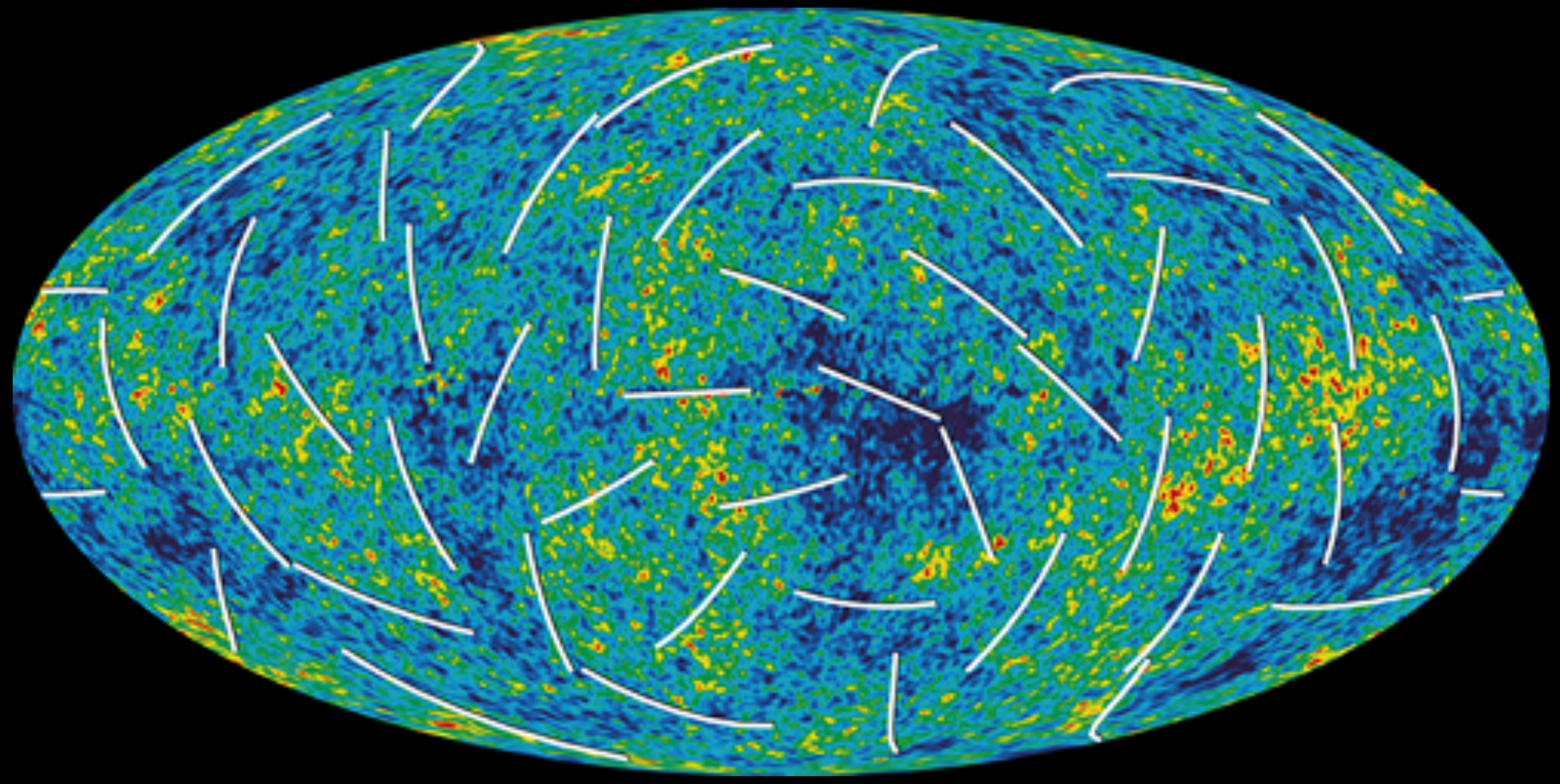
$$V^{(1)}(\mathbf{k}) = 0$$

$$\sim e^{\pm 4i\phi}$$

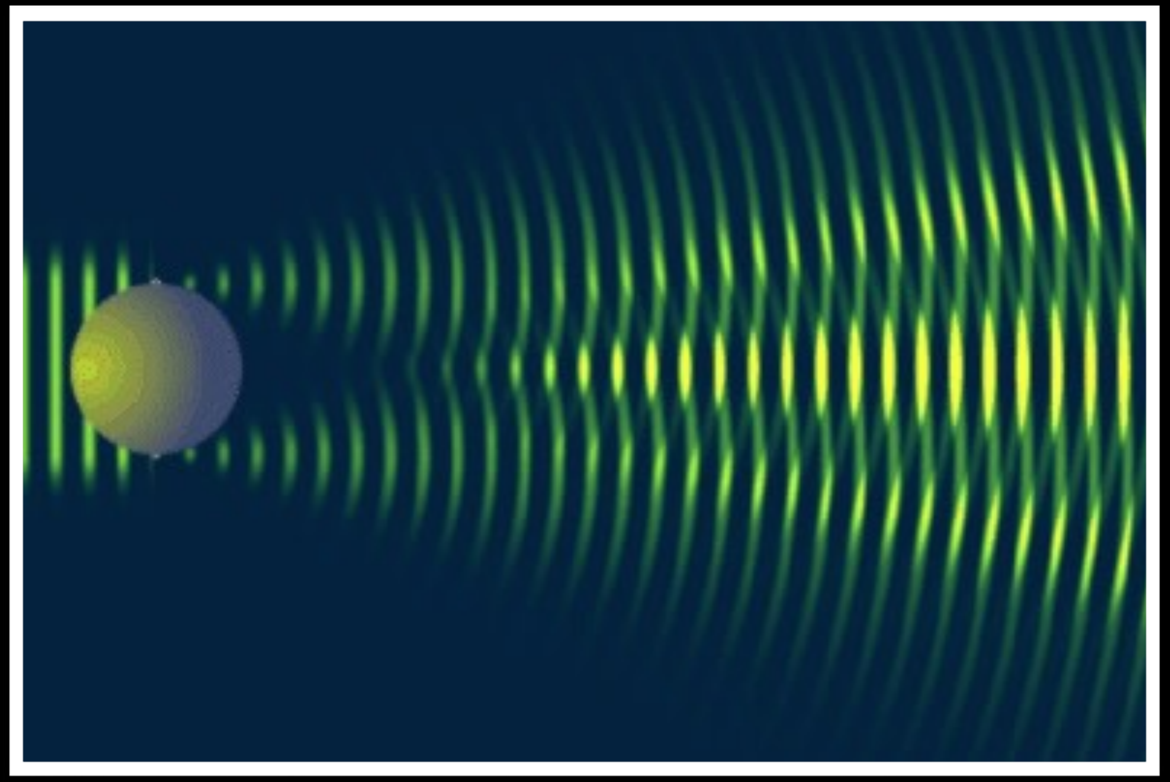
$$(Q^{(1)} \pm iU^{(1)})(\mathbf{k}) = \int d^3p \, \frac{P_{in}^{\phi}(|\mathbf{k}-\mathbf{p}|)}{|\mathbf{k}-\mathbf{p}|^3} \times \mathrm{I}^{(0)}(p) \times \left(\sqrt{\frac{40}{63}}Y_{\mp 4}^4(\theta,\varphi)\right) \times \left\{\begin{array}{c}\text{Deterministic evolution:}\\ \text{Transfer functions, Green's}\\ \text{function}\end{array}\right\}$$

Interaction of SGWB with cosmic web:

1. Can modulate amplitude of SGWB
2. Do not produce V
3. Can produce Q/U polarization modes

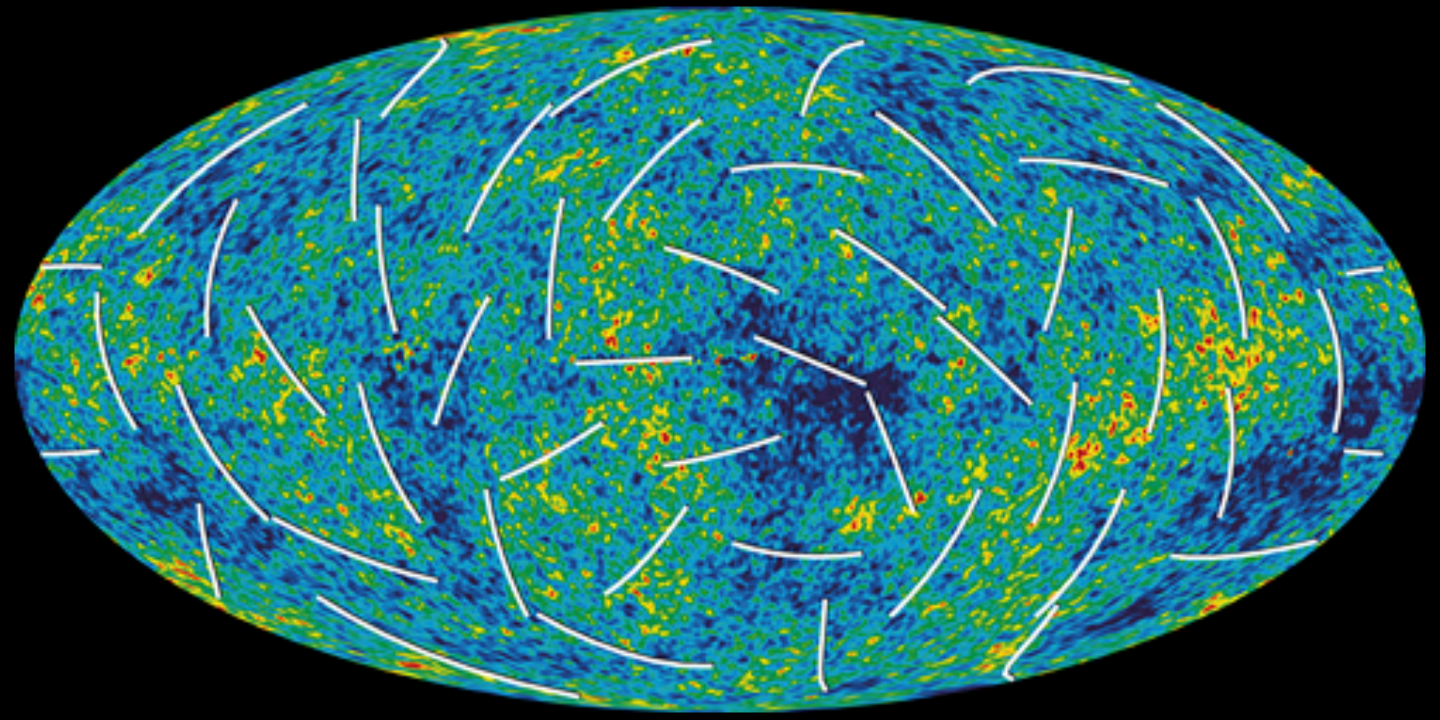


Conclusions



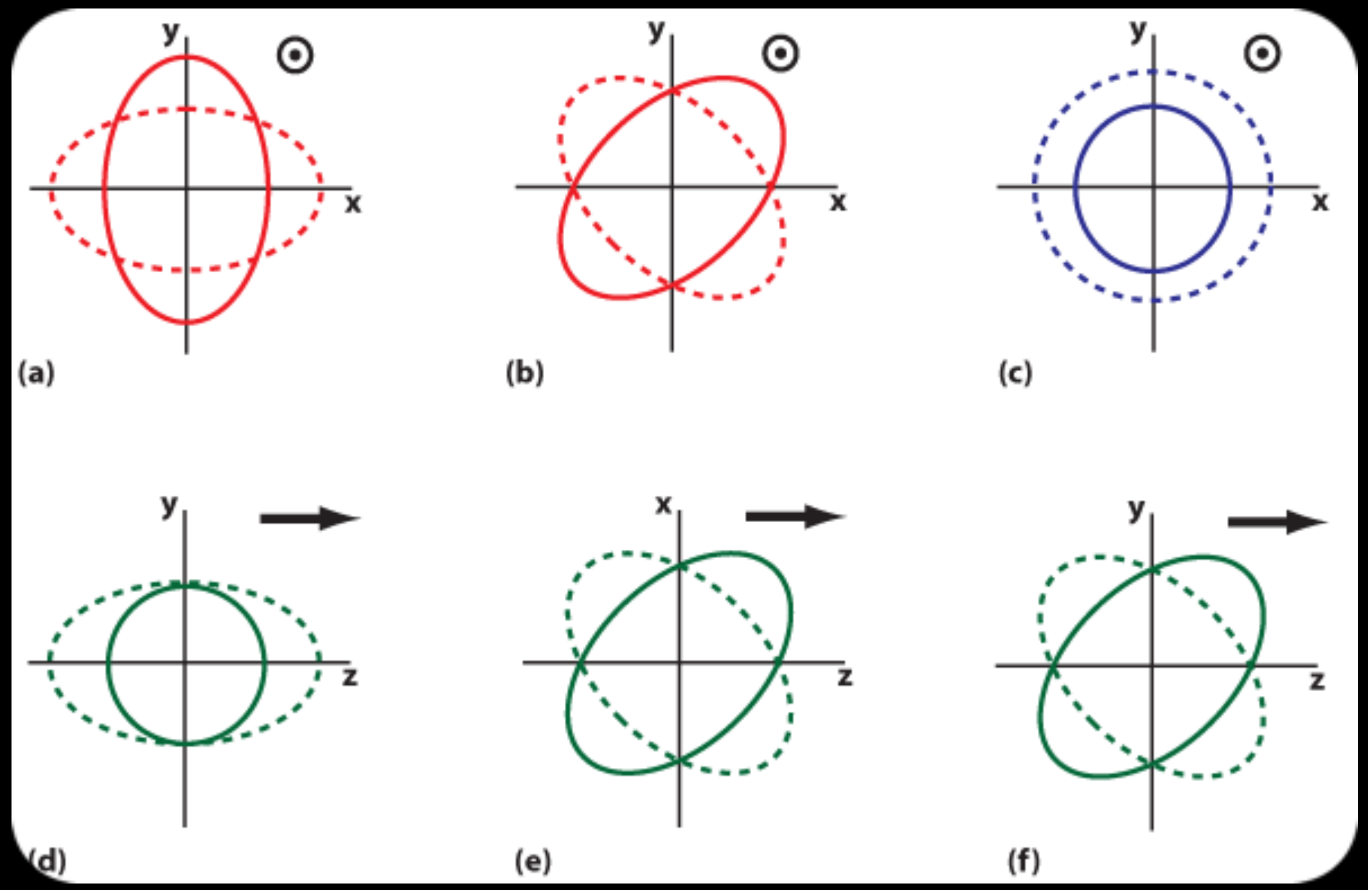
GWs are a new powerful window on the Universe.

They offer new theoretical challenges so we must learn how to use them before drawing any conclusions.



Why use AG, 2210.05718:

- Valid across the entire GW spectrum
- In the wave-optics limit propagation effects are frequency dependent: multi-band analysis to probe different scales
- New investigation channel: scalar and vector modes + Q/U



Thank you!